

## Intermediate Algebra

# Intermediate Algebra

Lumen Learning, NROC

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# MODULE 0: REVIEW

## WHY IT MATTERS: REVIEW

### Why review topics from earlier math classes?

You have been in class all day, running around, picking up your kids from daycare and after a long, brutal commute you get home only to realize you haven't thought about what to fix for dinner. You go to the fridge and all you see are leftovers, a jumble of unrelated bits and pieces from meals past. Despite your fatigue, you manage to cobble together a meal that everyone will eat willingly and to get on with your evening.

In this section, you may feel like the topics are a cobbled-together meal of leftovers. The goal is to give you a reminder of some of the skills and topics from earlier math classes that, like the box of leftovers in the back of your fridge, are easily forgotten by most people. Because sometimes we just forget how these concepts work . . . until we need them again.



*Leftovers again?*

As you work through the rest of the sections of this course, please return to this review if you feel like you need a reminder of the topics covered. These topics were chosen because they are often forgotten but are widely used throughout the course. Don't worry, just like ketchup, these concepts have a long shelf life.

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## INTRODUCTION TO USING THE ORDER OF OPERATIONS

### What you'll learn to do: Correctly use the order of operations

What is  $3 + 5 \times 2$  ? Is it 13 or 16 ? This may seem like a trick question, but there is actually only one correct answer.

Many years ago, mathematicians developed a standard **order of operations** that tells you which calculations to make first in an expression with more than one **operation**. In other words, order of operations

simply refers to the specific order of steps you should follow when you solve a math expression. Without a standard procedure for making calculations, two people could get two different answers to the same problem, like the one above. So which is it, 13 or 16 ? By the end of this module you'll know!

# SIMPLIFY EXPRESSIONS WITH THE ORDER OF OPERATIONS

## Learning Outcomes

- Recognize and combine like terms in an expression
- Use the order of operations to simplify expressions

## Introduction

Some important terminology before we begin:

- **operations/operators:** In mathematics we call things like multiplication, division, addition, and subtraction operations. They are the verbs of the math world, doing work on numbers and variables. The symbols used to denote operations are called operators, such as  $+$ ,  $-$ ,  $\times$ ,  $\div$ . As you learn more math, you will learn more operators.
- **term:** Examples of terms would be  $2x$ ,  $-\frac{3}{2}$ , or  $a^3$ . Even lone integers can be a term, like 0.
- **expression:** A mathematical expression is one that connects terms with mathematical operators. For example  $\frac{1}{2} + (2^2) - 9 \div \frac{6}{7}$  is an expression.

## Combining Like Terms

One way we can simplify expressions is to combine like terms. **Like terms** are terms where the variables match exactly (exponents included). Examples of like terms would be  $5xy$  and  $-3xy$ , or  $8a^2b$  and  $a^2b$ , or  $-3$  and  $8$ . If we have like terms we are allowed to add (or subtract) the numbers in front of the variables, then keep the variables the same. Kind of like saying four apples plus three apples equals seven apples. But two apples plus six oranges can't be combined and simplified because they are not "like terms". As we combine like terms we need to interpret subtraction signs as part of the following term. This means if we see a subtraction sign, we treat the following term like a negative term. The sign always stays with the term.

This is shown in the following examples:

In the following video you will be shown how to combine like terms using the idea of the distributive property. Note that this is a different method than is shown in the written examples on this page, but it obtains the same result.

In the video that follows, you will be shown another example of combining like terms. Pay attention to why you are not able to combine all three terms in the example.

## Order of Operations

## EXAMPLE

Combine like terms:  $5x - 2y - 8x + 7y$

### Answer

The like terms in this expression are:

$5x$  and  $-8x$

$-2y$  and  $7y$

Note how we kept the sign in front of each term.

Combine like terms:

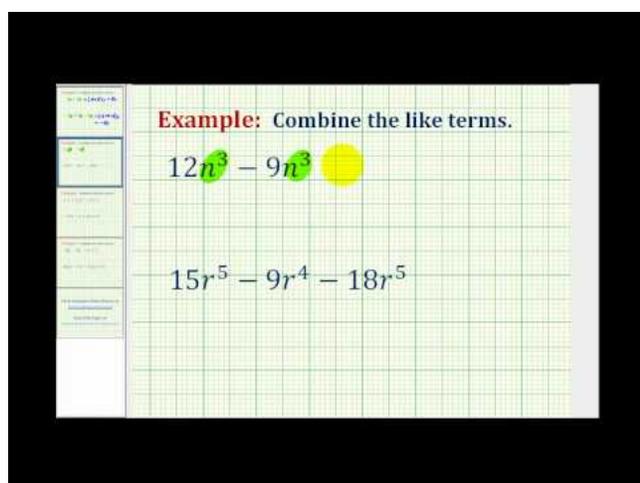
$$5x - 8x = -3x$$

$$-2y + 7y = 5y$$

Note how signs become operations when you combine like terms.

Simplified Expression:

$$5x - 2y - 8x + 7y = -3x + 5y$$



A YouTube element has been excluded from this version of the text. You can view it online here:

<https://courses.lumenlearning.com/intermediatealgebra/?p=271>

You may or may not recall the order of operations for applying several mathematical operations to one expression. Just as it is a social convention for us to drive on the right-hand side of the road, the order of operations is a set of conventions used to provide a formulaic outline to follow when you are required to use several mathematical operations for one expression.

## EXAMPLE

Combine like terms:  $x^2 - 3x + 9 - 5x^2 + 3x - 1$

### Answer

The like terms in this expression are:

$x^2$  and  $-5x^2$

$-3x$  and  $3x$

9 and  $-1$

Combine like terms:

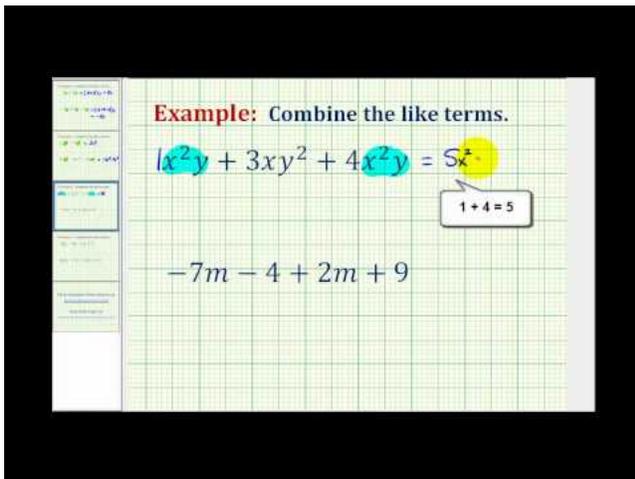
$$x^2 - 5x^2 = -4x^2$$

$$-3x + 3x = 0$$

$$9 - 1 = 8$$

Simplified Expression:

$$-4x^2 + 8$$



A YouTube element has been excluded from this version of the text. You can view it online here:  
<https://courses.lumenlearning.com/intermediatealgebra/?p=271>

## THE ORDER OF OPERATIONS

- Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), brackets [ ], braces { }, and fraction bars.
- Evaluate exponents or square roots.
- Multiply or divide, from left to right.
- Add or subtract, from left to right.

This order of operations is true for all real numbers.

### Example

Simplify  $7 - 5 + 3 \cdot 8$

### Answer

According to the order of operations, multiplication comes before addition and subtraction.

Multiply  $3 \cdot 8$

$$7 - 5 + 3 \cdot 8$$

$$7 - 5 + 24$$

Now, add and subtract from left to right.  $7 - 5$  comes first.

$$2 + 24$$

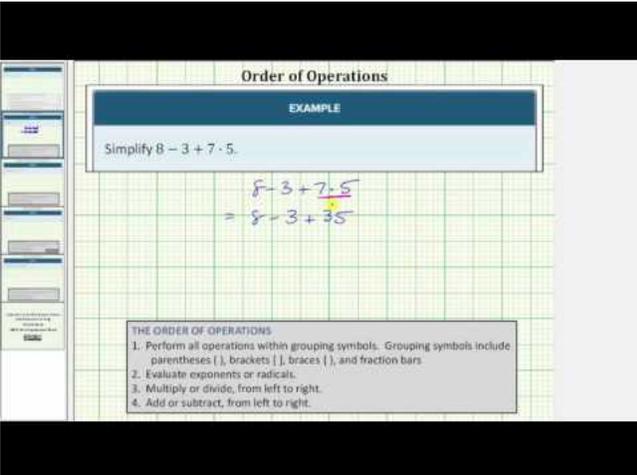
Finally, add.

$$2 + 24 = 26$$

**Answer**

$$7 - 5 + 3 \cdot 8 = 26$$

In the following example, you will be shown how to simplify an expression that contains both multiplication and subtraction using the order of operations.



**Order of Operations**

**EXAMPLE**

Simplify  $8 - 3 + 7 \cdot 5$ .

$$8 - 3 + 7 \cdot 5$$
$$= 8 - 3 + 35$$

**THE ORDER OF OPERATIONS**

1. Perform all operations within grouping symbols. Grouping symbols include parentheses ( ), brackets [ ], braces { }, and fraction bars.
2. Evaluate exponents or radicals.
3. Multiply or divide, from left to right.
4. Add or subtract, from left to right.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=271>

When you are applying the order of operations to expressions that contain fractions, decimals, and negative numbers, you will need to recall how to do these computations as well.

### Example

Simplify  $3 \cdot \frac{1}{3} - 8 \div \frac{1}{4}$

**Answer**

According to the order of operations, multiplication and division come before addition and subtraction. Sometimes it helps to add parentheses to help you know what comes first, so let's put parentheses around the multiplication and division since it will come before the subtraction.

$$\left(3 \cdot \frac{1}{3}\right) - \left(8 \div \frac{1}{4}\right)$$

Multiply  $3 \cdot \frac{1}{3}$  first.

$$\left(3 \cdot \frac{1}{3}\right) - \left(8 \div \frac{1}{4}\right)$$

$$(1) - \left(8 \div \frac{1}{4}\right)$$

Now, divide  $8 \div \frac{1}{4}$ .

$$8 \div \frac{1}{4} = \frac{8}{1} \cdot \frac{4}{1} = 32$$

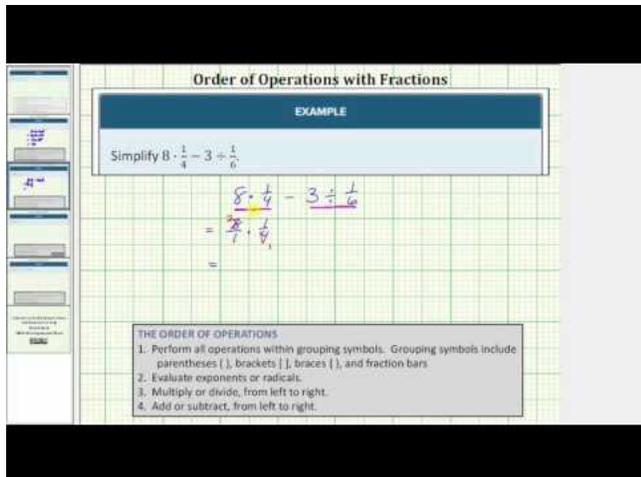
Subtract.

$$(1) - (32) = -31$$

Answer

$$3 \cdot \frac{1}{3} - 8 \div \frac{1}{4} = -31$$

In the following video you are shown how to use the order of operations to simplify an expression that contains multiplication, division, and subtraction with terms that contain fractions.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=271>

## Exponents and Square Roots

In this section, we expand our skills with applying the order of operation rules to expressions with exponents and square roots. If the expression has exponents or square roots, they are to be performed *after* parentheses and other grouping symbols have been simplified and *before* any multiplication, division, subtraction, and addition that are outside the parentheses or other grouping symbols.

Recall that an expression such as  $7^2$  is **exponential notation** for  $7 \cdot 7$ . Exponential notation has two parts: the **base** and the **exponent** or the **power**. In  $7^2$ , 7 is the base and 2 is the exponent — the exponent determines how many times the base is multiplied by itself.

Exponents are a way to represent repeated multiplication; the order of operations places it *before* any other multiplication, division, subtraction, and addition is performed.

### Example

Simplify  $3^2 \cdot 2^3$

#### Answer

This problem has exponents and multiplication in it. According to the order of operations, simplifying  $3^2$  and  $2^3$  comes before multiplication.

$$3^2 \cdot 2^3$$

$3^2$  is  $3 \cdot 3$ , which equals 9.

$$9 \cdot 2^3$$

$2^3$  is  $2 \cdot 2 \cdot 2$ , which equals 8.

$$9 \cdot 8$$

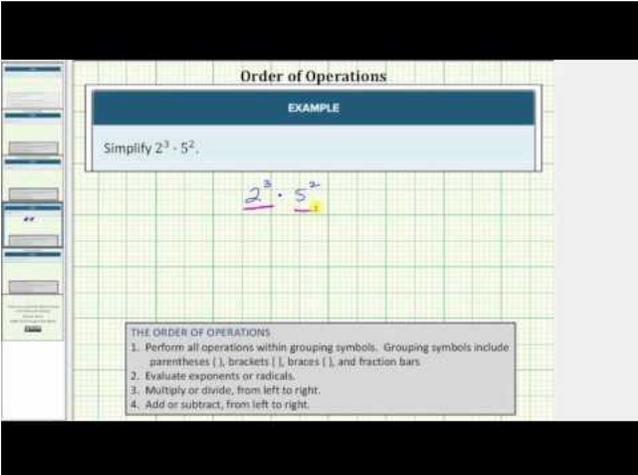
Multiply.

$$9 \cdot 8 = 72$$

#### Answer

$$3^2 \cdot 2^3 = 72$$

In the video that follows, an expression with exponents on its terms is simplified using the order of operations.



Order of Operations

EXAMPLE

Simplify  $2^3 - 5^2$ .

THE ORDER OF OPERATIONS

1. Perform all operations within grouping symbols. Grouping symbols include parentheses ( ), brackets [ ], braces { }, and fraction bars.
2. Evaluate exponents or radicals.
3. Multiply or divide, from left to right.
4. Add or subtract, from left to right.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=271>

When there are grouping symbols within grouping symbols, calculate from the inside to the outside. That is, begin simplifying within the innermost grouping symbols first.

Remember that parentheses can also be used to show multiplication. In the example that follows, both uses of parentheses—as a way to represent a group, as well as a way to express multiplication—are shown.

### Example

Simplify  $(3 + 4)^2 + (8)(4)$

#### Answer

This problem has parentheses, exponents, multiplication, and addition in it. The first set of parentheses is a grouping symbol. The second set indicates multiplication.

Grouping symbols are handled first. Add numbers in parentheses.

$$(3 + 4)^2 + (8)(4)$$

$$(7)^2 + (8)(4)$$

Simplify  $7^2$ .

$$7^2 + (8)(4)$$

$$49 + (8)(4)$$

Multiply.

$$49 + (8)(4)$$

$$49 + (32)$$

Add.

$$49 + 32 = 81$$

Answer

$$(3 + 4)^2 + (8)(4) = 81$$

In the following video, you are shown how to use the order of operations to simplify an expression with grouping symbols, exponents, multiplication, and addition.

**Order of Operations**

**EXAMPLE**

Simplify  $(2 + 4)^2 + (7)(8)$ .

$$(2 + 4)^2 + (7)(8)$$
$$= 6^2 + (7)(8)$$

**THE ORDER OF OPERATIONS**

1. Perform all operations within grouping symbols. Grouping symbols include parentheses ( ), brackets [ ], braces { }, and fraction bars.
2. Evaluate exponents or radicals.
3. Multiply or divide, from left to right.
4. Add or subtract, from left to right.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=271>

In the next example we will simplify an expression that has a square root.

Example

Simplify  $\frac{\sqrt{7 + 2} + 2^2}{(8)(4) - 11}$

## Answer

This problem has all the operations to consider with the order of operations.

Grouping symbols are handled first, in this case the fraction bar. We will simplify the numerator (top) and denominator (bottom) separately.

To simplify the top:

$$\sqrt{7+2} + 2^2$$

Add the numbers inside the square root (as they are essentially grouped by that symbol/operator), and the term  $2^2$

$$\sqrt{(7+2)} + (2^2)$$

$$= \sqrt{9} + 4$$

$$= 3 + 4 = 7$$

To simplify the bottom:

$$(8)(4) - 11$$

Multiply 8 and 4 first, then subtract 11.

$$(8)(4) - 11 =$$

$$32 - 11 = 21$$

Now put the fraction back together to see if any more simplifying needs to be done. The simplified numerator equaled 7, and the simplified denominator equaled 21.

so  $\frac{7}{21}$ , which can be reduced to  $\frac{1}{3}$

Answer

$$\frac{\sqrt{7+2} + 2^2}{(8)(4) - 11} = \frac{1}{3}$$

The screenshot shows a software interface titled "Simplifying Expressions with Square Roots". It features an "EXAMPLE" box with the expression  $\frac{\sqrt{16+9} + 3^2}{(7)(5) - 9}$ . Below this, handwritten-style work shows the simplification steps:  $\frac{\sqrt{16+9} + 3^2}{(7)(5) - 9} = \frac{\sqrt{25} + 3^2}{35 - 9}$ , followed by  $\sqrt{25} = \sqrt{5^2} = 5$ . At the bottom, a box titled "THE ORDER OF OPERATIONS" lists four steps: 1. Perform all operations within grouping symbols, 2. Evaluate exponents or radicals, 3. Multiply or divide, from left to right, and 4. Add or subtract, from left to right.

## THINK ABOUT IT

These problems are very similar to the examples given above. How are they different and what tools do you need to simplify them?

a) Simplify  $(1.5 + 3.5) - 2(0.5 \cdot 6)^2$ . This problem has parentheses, exponents, multiplication, subtraction, and addition in it, as well as decimals instead of integers.

Use the box below to write down a few thoughts about how you would simplify this expression with decimals and grouping symbols.

### Answer

Grouping symbols are handled first. Add numbers in the first set of parentheses.

$$(1.5 + 3.5) - 2(0.5 \cdot 6)^2$$

$$5 - 2(0.5 \cdot 6)^2$$

Multiply numbers in the second set of parentheses.

$$5 - 2(0.5 \cdot 6)^2$$

$$5 - 2(3)^2$$

Evaluate exponents.

$$5 - 2(3)^2$$

$$5 - 2 \cdot 9$$

Multiply.

$$5 - 2 \cdot 9$$

$$5 - 18$$

Subtract.

$$5 - 18 = -13$$

### Answer

$$(1.5 + 3.5) - 2(0.5 \cdot 6)^2 = -13$$

b) Simplify  $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^3 \cdot 32$

Use the box below to write down a few thoughts about how you would simplify this expression with fractions and grouping symbols.

### Answer

This problem has exponents, multiplication, and addition in it, as well as fractions instead of integers.

According to the order of operations, simplify the terms with the exponents first, then multiply, then add.

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^3 \cdot 32$$

Evaluate:  $\left(\frac{1}{2}\right)^2 = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

$$\frac{1}{4} + \left(\frac{1}{4}\right)^3 \cdot 32$$

Evaluate:  $\left(\frac{1}{4}\right)^3 = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64}$

$$\frac{1}{4} + \frac{1}{64} \cdot 32$$

Multiply.

$$\frac{1}{4} + \frac{32}{64}$$

Simplify.  $\frac{32}{64} = \frac{1}{2}$ , so you can add  $\frac{1}{4} + \frac{1}{2}$ .

$$\frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

Answer

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^3 \cdot 32 = \frac{3}{4}$$

Some people use a saying to help them remember the order of operations. This saying is called PEMDAS or **Please Excuse My Dear Aunt Sally**. The first letter of each word begins with the same letter of an arithmetic operation.

**Please** ⇒ **P**arentheses (and other grouping symbols)

**Excuse** ⇒ **E**xponents

**My Dear** ⇒ **M**ultiplication and **D**ivision (from left to right)

**Aunt Sally** ⇒ **A**ddition and **S**ubtraction (from left to right)

**Note:** Even though multiplication comes before division in the saying, division could be performed first. Which is performed first, between multiplication and division, is determined by which comes first when reading from left to right. The same is true of addition and subtraction. Don't let the saying confuse your understanding of this!

## Summary

The order of operations gives us a consistent sequence to use in computation. Without the order of operations, you could come up with different answers to the same computation problem. (Some of the early calculators, and some inexpensive ones, do NOT use the order of operations. In order to use these calculators, the user has to input the numbers in the correct order.)

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# USE THE DISTRIBUTIVE PROPERTY TO SIMPLIFY

## Learning Outcomes

- Simplify expressions with fraction bars, brackets, and parentheses
- Use the distributive property to simplify expressions with grouping symbols
- Simplify expressions containing absolute values

In this section, we will use the skills from the last section to simplify mathematical expressions that contain many grouping symbols and many operations. We will also use the distributive property to break up multiplication into addition. Additionally, you will see how to handle absolute value terms when you simplify expressions.

## Example

Simplify  $\frac{5 - [3 + (2 \cdot (-6))]}{3^2 + 2}$

### Answer

This problem has brackets, parentheses, fractions, exponents, multiplication, subtraction, and addition in it.

Grouping symbols are handled first. The parentheses around the  $-6$  aren't a grouping symbol; they are simply making it clear that the negative sign belongs to the 6. Start with the innermost set of parentheses that are a grouping symbol. In this example, the innermost set of parentheses would be in the numerator of the fraction,  $(2 \cdot (-6))$ . Begin working out from there. (The fraction line acts as a type of grouping symbol, too; you simplify the numerator and denominator independently, and then divide the numerator by the denominator at the end.)

$$\frac{5 - [3 + (2 \cdot (-6))]}{3^2 + 2}$$

$$\frac{5 - [3 + (-12)]}{3^2 + 2}$$

Add 3 and  $-12$ , which are in brackets, to get  $-9$ .

$$\frac{5 - [-9]}{3^2 + 2}$$

Subtract  $5 - [-9] = 5 + 9 = 14$ .

$$\frac{14}{3^2 + 2}$$

The top of the fraction is all set, but the bottom (denominator) has remained untouched. Apply the order of operations to that as well. Begin by evaluating  $3^2 = 9$ .

$$\frac{14}{9 + 2}$$

Now add.  $9 + 2 = 11$ .

$$\frac{14}{11}$$

Answer

$$\frac{5 - [3 + (2 \cdot (-6))]}{3^2 + 2} = \frac{14}{11}$$

The video that follows contains an example similar to the written one above. Note how the numerator and denominator of the fraction are simplified separately.

The screenshot shows a video player with a title "Simplify Compound Expressions With Real Numbers". Below the title is an "EXAMPLE" box containing the expression: 
$$\text{Simplify } \frac{8 - [5 + (3 \cdot (-4))]}{4^2 + 2} = \frac{(8 - [5 + (3 \cdot (-4))])}{(4^2 + 2)}$$
 Below this, the expression is simplified to: 
$$= \frac{8 - [5 + (-12)]}{16 + 2}$$
 At the bottom of the video frame, there is a box titled "THE ORDER OF OPERATIONS" with the following steps:

1. Perform all operations within grouping symbols. Grouping symbols include parentheses ( ), brackets [ ], braces { }, and fraction bars.
2. Evaluate exponents or radicals.
3. Multiply or divide, from left to right.
4. Add or subtract, from left to right.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=273>

## The Distributive Property

Parentheses are used to group, or combine expressions and terms in mathematics. You may see them used when you are working with formulas, and when you are translating a real situation into a mathematical problem so you can find a quantitative solution.

For example, you are on your way to hang out with your friends, and call them to ask if they want something from your favorite drive-through.

Three people want the same combo meal of 2 tacos and one drink. You can use the distributive property to find out how many total tacos and how many total drinks you should take to them.

$$\begin{aligned} & 3(2 \text{ tacos} + 1 \text{ drink}) \\ &= 3 \cdot 2 \text{ tacos} + 3 \text{ drinks} \\ &= 6 \text{ tacos} + 3 \text{ drinks} \end{aligned}$$

The diagram shows the equation  $3(2 \text{ tacos} + 1 \text{ drink}) = 6 \text{ tacos} + 3 \text{ drinks}$  with taco and drink icons. A red bracket above the parentheses is labeled "one combo meal". A blue arrow points from the 3 to both terms inside the parentheses. Below this, the algebraic equation  $3(2t + d) = 6t + 3d$  is shown with a blue arrow pointing from the 3 to both terms inside the parentheses.

*Combo Meal Distributive Property*

The distributive property allows us to explicitly describe a total that is a result of a group of groups. In the case of the combo meals, we have three groups of ( two tacos plus one drink). The following definition describes how to use the distributive property in general terms.

## THE DISTRIBUTIVE PROPERTY OF MULTIPLICATION

For real numbers  $a$ ,  $b$ , and  $c$ :

$$a(b + c) = ab + ac.$$

What this means is that when a number is multiplied by an expression inside parentheses, you can distribute the multiplier to each term of the expression individually.

To simplify  $3(3 + y) - y + 9$ , it may help to see the expression translated into words:

multiply three by (the sum of three and  $y$ ), then subtract  $y$ , then add 9

To multiply three by the sum of three and  $y$ , you use the distributive property –

$$\begin{aligned} & 3(3 + y) - y + 9 \\ &= \underbrace{3 \cdot 3} + \underbrace{3 \cdot y} - y + 9 \\ &= 9 + 3y - y + 9 \end{aligned}$$

Now you can combine the like-terms. Subtract  $y$  from  $3y$  and add 9 to 9.

$$\begin{aligned} & 9 + 3y - y + 9 \\ &= 18 + 2y \end{aligned}$$

The next example shows how to use the distributive property when one of the terms involved is negative.

### EXAMPLE

Simplify  $a + 2(5 - b) + 3(a + 4)$

**Answer**

This expression has two sets of parentheses with variables locked up in them. We will use the distributive property to remove the parentheses.

$$\begin{aligned} & a + 2(5 - b) + 3(a + 4) \\ &= a + (2 \cdot 5) + (2 \cdot -b) + (3 \cdot a) + (3 \cdot 4) \end{aligned}$$

Note that the sign belongs to the term it precedes, which is why the 2 is multiplied by a  $-b$ . Remember that when you multiply a negative by a positive the result is negative, so  $2 \cdot -b = -2b$ . It is important to be careful with negative signs when you are using the distributive property!

$$\begin{aligned} & a + (2 \cdot 5) + (2 \cdot -b) + (3 \cdot a) + (3 \cdot 4) \\ &= a + 10 - 2b + 3a + 12 \\ &= 4a + 22 - 2b \end{aligned}$$

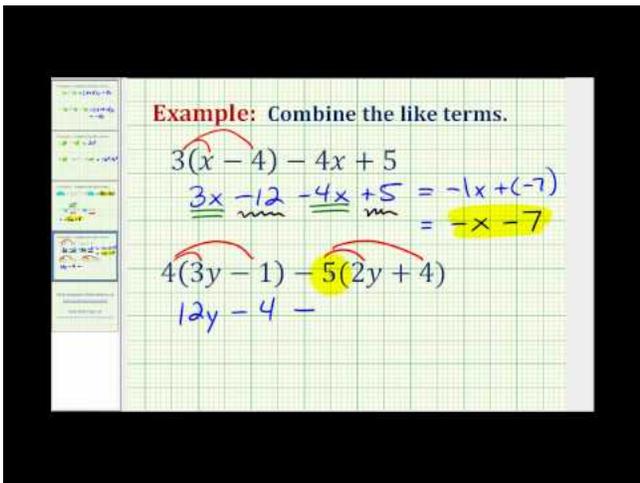
We combined all the terms we could to get our final result.

**Answer**

$$a + 2(5 - b) + 3(a + 4) = 4a + 22 - 2b$$

## Absolute Value

Absolute value expressions are another method of grouping that you may see. Recall that the absolute



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value of a quantity is always positive or 0.

When you see an absolute value expression included within a larger expression, treat the absolute value like a grouping symbol and evaluate the expression within the absolute value sign first. Then take the absolute value of that expression. The example below shows how this is done.

Example

Simplify  $\frac{3 + |2 - 6|}{2|3 \cdot 1.5| - (-3)}$

**Answer**

This problem has absolute values, decimals, multiplication, subtraction, and addition in it.

Grouping symbols, including absolute value, are handled first. Simplify the numerator, then the denominator.

Evaluate  $|2 - 6|$ .

$$\frac{3 + |2 - 6|}{2|3 \cdot 1.5| - (-3)}$$

$$\frac{3 + |-4|}{2|3 \cdot 1.5| - (-3)}$$

Take the absolute value of  $|-4|$ .

$$\frac{3 + 4}{2|3 \cdot 1.5| - (-3)}$$

Add the numbers in the numerator.

$$\frac{3 + 4}{2|3 \cdot 1.5| - (-3)}$$

$$\frac{7}{2|3 \cdot 1.5| - (-3)}$$

Now that the numerator is simplified, turn to the denominator.

Evaluate the absolute value expression first.  $3 \cdot 1.5 = 4.5$ , giving

$$\frac{7}{2|4.5| - (-3)}$$

Take the absolute value of  $|4.5|$ .

$$\frac{7}{2(4.5) - (-3)}$$

The expression “ $2|4.5|$ ” reads “2 times the absolute value of 4.5.” Multiply 2 times 4.5.

$$\frac{7}{9 - (-3)}$$

Subtract.  $9 - (-3) = 9 + 3 = 12$

$$\frac{7}{9 - (-3)}$$

$$\frac{7}{12}$$

Answer

$$\frac{3 + |2 - 6|}{2|3 \cdot 1.5| - 3(-3)} = \frac{7}{12}$$

The following video uses the order of operations to simplify an expression in fraction form that contains absolute value terms. Note how the absolute values are treated like parentheses and brackets when using the order of operations.

The screenshot shows a video player interface. The main content is a slide titled "Simplify Compound Expressions With Real Numbers" with an "EXAMPLE" section. The example shows the simplification of the fraction  $\frac{7 + |3 - 10|}{5|2 \cdot 1.2| - (-4)}$ . The numerator is simplified to  $7 + |-7|$  and the denominator to  $5|2.4| - (-4)$ . Below the example is a box titled "THE ORDER OF OPERATIONS" with the following steps: 1. Perform all operations within grouping symbols. Grouping symbols include parentheses ( ), brackets [ ], braces { }, and fraction bars. 2. Evaluate exponents or radicals. 3. Multiply or divide, from left to right. 4. Add or subtract, from left to right.

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# INTRODUCTION TO ARITHMETIC WITH FRACTIONS

What you'll learn to do: Perform arithmetic with fractions

One of the most common things people forget from arithmetic is how to add and subtract fractions. This section will remind you how to do this. You will also get to practice both adding and subtracting fractions as well as multiplying and dividing them. As you work through the rest of this course, remember to come back to this section for a quick reminder of these skills if the process becomes fuzzy.

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# ADDING AND SUBTRACTING FRACTIONS

## LEARNING OUTCOMES

- Find the common denominator of two or more fractions
- Use the common denominator to add or subtract fractions
- Simplify a fraction to its lowest terms

## Introduction

Before we get started, here is some important terminology that will help you understand the concepts about working with fractions in this section.

- **product:** the result of multiplication
- **factor:** something being multiplied — for  $3 \cdot 2 = 6$ , both 3 and 2 are factors of 6
- **numerator:** the top part of a fraction — the numerator in the fraction  $\frac{2}{3}$  is 2
- **denominator:** the bottom part of a fraction — the denominator in the fraction  $\frac{2}{3}$  is 3

# Math Question Instructions

Many different words are used by math textbooks and teachers to provide students with instructions on what they are to do with a given problem. For example, you may see instructions such as “Find” or “Simplify” in the example in this module. It is important to understand what these words mean so you can successfully work through the problems in this course. Here is a short list of the words you may see that can help you know how to work through the problems in this module.

Instruction	Interpretation
Find	Perform the indicated mathematical operations which may include addition, subtraction, multiplication, division.
Simplify	1) Perform the indicated mathematical operations including addition, subtraction, multiplication, division 2) Write a mathematical statement in smallest terms so there are no other mathematical operations that can be performed—often found in problems related to fractions and the order of operations
Evaluate	1) Perform the indicated mathematical operations including addition, subtraction, multiplication, division 2) Substitute a given value for a variable in an expression and then perform the indicated mathematical operations
Reduce	Write a mathematical statement in smallest or lowest terms so there are no other mathematical operations that can be performed—often found in problems related to fractions or division

## Adding Fractions

When you need to add or subtract fractions, you will need to first make sure that the fractions have the same denominator. The denominator tells you how many pieces the whole has been broken into, and the numerator tells you how many of those pieces you are using.

The “parts of a whole” concept can be modeled with pizzas and pizza slices. For example, imagine a pizza is cut into 4 pieces, and someone takes 1 piece. Now,  $\frac{1}{4}$  of the pizza is gone and  $\frac{3}{4}$  remains.



Note that both of these fractions have a denominator of 4, which refers to the number of slices the whole pizza has been cut into. What if you have another pizza that had been cut into 8 equal parts and 3 of those parts were gone, leaving  $\frac{5}{8}$ ?

How can you describe the total amount of pizza that is left between the two pizzas? There are now two very different fractions and it’s hard to compare the two. You will need a common

denominator, technically called the **least common multiple**. Remember that if a number is a multiple of another, you can divide them and have no remainder.

One way to find the least common multiple of two or more numbers is to first multiply each by 1, 2, 3, 4, etc. For example, find the least common multiple of 2 and 5.

First, list all the multiples of 2:	Then list all the multiples of 5:
$2 \cdot 1 = 2$	$5 \cdot 1 = 5$
$2 \cdot 2 = 4$	$5 \cdot 2 = 10$
$2 \cdot 3 = 6$	$5 \cdot 3 = 15$
$2 \cdot 4 = 8$	$5 \cdot 4 = 20$
$2 \cdot 5 = 10$	$5 \cdot 5 = 25$

The smallest multiple they have in common will be the common denominator for the two! In the example above, that is 10.

### EXAMPLE

Describe the amount of pizza left in the examples above using common terms.

**Answer**

Rewrite the fractions  $\frac{3}{4}$  and  $\frac{5}{8}$  as fractions with a least common denominator.

Find the least common multiple of the denominators. This is the least common denominator.

Multiples of 4 : 4, **8**, 12, 16, **24**

Multiples of 8 : **8**, 16, **24**

The least common denominator is 8 — the smallest multiple they have in common.

Rewrite  $\frac{3}{4}$  with a denominator of 8. You have to multiply both the top and bottom by 2 so you don't change the relationship between them (because  $\frac{2}{2} = 1$ ).

$$\frac{3}{4} \cdot \frac{2}{2} = \frac{6}{8}$$

We don't need to rewrite  $\frac{5}{8}$  since it already has the common denominator.

**Answer**

Both  $\frac{6}{8}$  and  $\frac{5}{8}$  have the same denominator, and you can describe how much pizza is left with common terms, and compare the values more easily.

To add fractions with unlike denominators, you will need to first rewrite them with common denominators. Then, you know what to do! The steps are shown below.

## ADDING FRACTIONS WITH UNLIKE DENOMINATORS

1. Find a common denominator.
2. Rewrite each fraction using the common denominator.
3. Now that the fractions have a common denominator, you can add the numerators.
4. Simplify by canceling out all common factors in the numerator and denominator.

### Simplifying a Fraction

Often, if the answer to a problem is a fraction, you will be asked to write it in lowest terms. This is a common convention used in mathematics, similar to starting a sentence with a capital letter and ending it with a period. In this course, we will not go into great detail about methods for reducing fractions because there are many. The process of simplifying a fraction is often called *reducing the fraction*. We can simplify by canceling (dividing) the common factors in a fraction's numerator and denominator. We can do this because a fraction represents division.

For example, to simplify  $\frac{6}{9}$  you can rewrite 6 and 9 using the smallest factors possible as follows:

$$\frac{6}{9} = \frac{2 \cdot 3}{3 \cdot 3}$$

Since there is a 3 in both the numerator and denominator, and fractions can be considered division, we can divide the 3 in the top by the 3 in the bottom to reduce to 1.

$$\frac{6}{9} = \frac{2 \cdot \cancel{3}}{3 \cdot \cancel{3}} = \frac{2}{3} \cdot 1 = \frac{2}{3}$$

Rewriting fractions with the smallest factors possible is often called prime factorization.

In the next example you are shown how to add two fractions with different denominators, then simplify the answer.

You can find a common denominator by finding the common multiples of the denominators. The least common multiple is the easiest to use.

In the following video you will see an example of how to add two fractions with different denominators.

You can also add more than two fractions as long as you first find a common denominator for all of them. An example of a sum of three fractions is shown below. In this example, you will use the prime factorization method to find the LCM.

### Subtracting Fractions

When you subtract fractions, you must think about whether they have a common denominator, just like with adding fractions. Below are some examples of subtracting fractions whose denominators are not alike.

## EXAMPLE

Add  $\frac{2}{3} + \frac{1}{5}$ . Simplify the answer.

### Answer

Since the denominators are not alike, find a common denominator by multiplying the denominators.

$$3 \cdot 5 = 15$$

Rewrite each fraction with a denominator of 15 by multiplying it by 1.

$$\frac{2}{3} \cdot \frac{5}{5} = \frac{10}{15}$$

$$\frac{1}{5} \cdot \frac{3}{3} = \frac{3}{15}$$

Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified.

$$\frac{10}{15} + \frac{3}{15} = \frac{13}{15}$$

### Answer

$$\frac{2}{3} + \frac{1}{5} = \frac{13}{15}$$

## EXAMPLE

Add  $\frac{3}{7} + \frac{2}{21}$ . Simplify the answer.

### Answer

Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 7 and 21.

Multiples of 7 : 7, 14, **21**

Multiples of 21 : **21**

Rewrite each fraction with a denominator of 21.

$$\frac{3}{7} \cdot \frac{3}{3} = \frac{9}{21}$$

$$\frac{2}{21}$$

Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified.

$$\frac{9}{21} + \frac{2}{21} = \frac{11}{21}$$

### Answer

$$\frac{3}{7} + \frac{2}{21} = \frac{11}{21}$$

The screenshot shows a math software interface with the title "Adding Fractions with Unlike Denominators". The main instruction is "Add and simplify the answer." for the problem  $\frac{1}{4} + \frac{1}{2}$ . A small box contains the formula  $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$ . The software shows the work:  $\frac{1}{4} + \frac{1 \cdot 2}{2 \cdot 2} = \frac{1}{4} + \frac{2}{4} = \frac{3}{4}$ . Below this, a diagram shows three bars representing the fractions: a bar for  $\frac{1}{4}$ , a bar for  $\frac{1}{2}$ , and a combined bar for  $\frac{3}{4}$ .

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=254>

## THINK ABOUT IT

Add  $\frac{3}{4} + \frac{1}{6} + \frac{5}{8}$ . Simplify the answer and write as a mixed number.

What makes this example different than the previous ones? Use the box below to write down a few thoughts about how you would add three fractions with different denominators together.

**Answer**

Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 4, 6, and 8.

$$4 = 2 \cdot 2$$

$$6 = 3 \cdot 2$$

$$8 = 2 \cdot 2 \cdot 2$$

$$\text{LCM: } 2 \cdot 2 \cdot 2 \cdot 3 = 24$$

Rewrite each fraction with a denominator of 24.

$$\frac{3}{4} \cdot \frac{6}{6} = \frac{18}{24}$$

$$\frac{1}{6} \cdot \frac{4}{4} = \frac{4}{24}$$

$$\frac{5}{8} \cdot \frac{3}{3} = \frac{15}{24}$$

Add the fractions by adding the numerators and keeping the denominator the same.

$$\frac{18}{24} + \frac{4}{24} + \frac{15}{24} = \frac{37}{24}$$

Write the improper fraction as a mixed number and simplify the fraction.

$$\frac{37}{24} = 1 \frac{13}{24}$$

**Answer**

$$\frac{3}{4} + \frac{1}{6} + \frac{5}{8} = 1 \frac{13}{24}$$

**Example**

Subtract  $\frac{1}{5} - \frac{1}{6}$ . Simplify the answer.

**Answer**

The fractions have unlike denominators, so you need to find a common denominator. Recall that a common denominator can be found by multiplying the two denominators together.

$$5 \cdot 6 = 30$$

Rewrite each fraction as an equivalent fraction with a denominator of 30.

$$\frac{1}{5} \cdot \frac{6}{6} = \frac{6}{30}$$

$$\frac{1}{6} \cdot \frac{5}{5} = \frac{5}{30}$$

Subtract the numerators. Simplify the answer if needed.

$$\frac{6}{30} - \frac{5}{30} = \frac{1}{30}$$

**Answer**

$$\frac{1}{5} - \frac{1}{6} = \frac{1}{30}$$

The example below shows how to use multiples to find the least common multiple, which will be the least common denominator.

## EXAMPLE

Subtract  $\frac{5}{6} - \frac{1}{4}$ . Simplify the answer.

### Answer

Find the least common multiple of the denominators—this is the least common denominator.

Multiples of 6 : 6, **12**, 18, 24

Multiples of 4 : 4, 8, **12**, 16, 20

12 is the least common multiple of 6 and 4.

Rewrite each fraction with a denominator of 12.

$$\frac{5}{6} \cdot \frac{2}{2} = \frac{10}{12}$$

$$\frac{1}{4} \cdot \frac{3}{3} = \frac{3}{12}$$

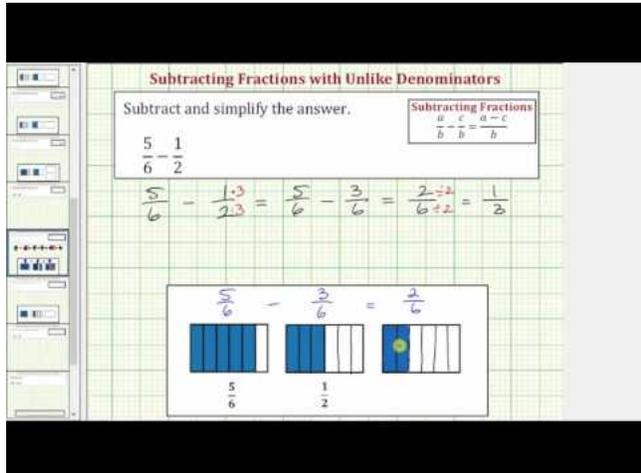
Subtract the fractions. Simplify the answer if needed.

$$\frac{10}{12} - \frac{3}{12} = \frac{7}{12}$$

### Answer

$$\frac{5}{6} - \frac{1}{4} = \frac{7}{12}$$

In the following video you will see an example of how to subtract fractions with unlike denominators.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=254>

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# MULTIPLYING FRACTIONS

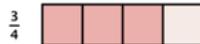
## LEARNING OUTCOMES

- Multiply two or more fractions
- Multiply a fraction by a whole number

## Multiply Fractions

Just as you add, subtract, multiply, and divide when working with whole numbers, you also use these operations when working with fractions. There are many times when it is necessary to multiply fractions. A model may help you understand multiplication of fractions.

When you multiply a fraction by a fraction, you are finding a “fraction of a fraction.” Suppose you have  $\frac{3}{4}$  of a candy bar and you want to find  $\frac{1}{2}$  of the  $\frac{3}{4}$ :



By dividing each fourth in half, you can divide the candy bar into eighths.



Then, choose half of those to get  $\frac{3}{8}$ .



In both of the above cases, to find the answer, you can multiply the numerators together and the denominators together.

## MULTIPLYING TWO FRACTIONS

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d} = \frac{\text{product of the numerators}}{\text{product of the denominators}}$$

## MULTIPLYING MORE THAN TWO FRACTIONS

$$\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f} = \frac{a \cdot c \cdot e}{b \cdot d \cdot f}$$

### EXAMPLE

Multiply  $\frac{2}{3} \cdot \frac{4}{5}$

**Answer**

Multiply the numerators and multiply the denominators.

$$\frac{2 \cdot 4}{3 \cdot 5}$$

Simplify, if possible. This fraction is already in lowest terms.

$$\frac{8}{15}$$

**Answer**

$$\frac{8}{15}$$

To review: if a fraction has common factors in the numerator and denominator, we can **reduce** the fraction to its simplified form by removing the common factors.

For example,

- Given  $\frac{8}{15}$ , the factors of 8 are: 1, 2, 4, 8 and the factors of 15 are: 1, 3, 5, 15.  $\frac{8}{15}$  is simplified because there are no common factors of 8 and 15.
- Given  $\frac{10}{15}$ , the factors of 10 are: 1, 2, 5, 10 and the factors of 15 are: 1, 3, 5, 15.  $\frac{10}{15}$  is not simplified because 5 is a common factor of 10 and 15.

Note that you can simplify the fractions first, before you multiplying them, to make your work easier. This allows you to work with smaller numbers when you multiply.

In the following video you will see an example of how to multiply two fractions, then simplify the answer.

The screenshot shows a video player interface with a grid background. At the top, it says "Multiplying Fractions". Below that, it says "Multiply. Write the answer in lowest terms." and shows two problems:  $\frac{1}{4} \cdot \frac{3}{5}$  and  $\frac{1}{8} \cdot \frac{4}{5}$ . The first problem is solved:  $\frac{1}{4} \cdot \frac{3}{5} = \frac{1 \cdot 3}{4 \cdot 5} = \frac{3}{20}$ . The second problem is partially solved:  $\frac{1}{8} \cdot \frac{4}{5}$ . Below the grid, there is a text box with the title "Multiplying Fractions" and the text: "To multiply fractions, multiply the numerators and multiply the denominators." followed by the formula  $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ . A note at the bottom says: "Note: The product needs to be simplified or written in lowest terms. We can simplify before or after multiplying."

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<https://courses.lumenlearning.com/intermediatealgebra/?p=258>

## THINK ABOUT IT

Multiply  $\frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{5}$ . Simplify the answer.

What makes this example different than the previous ones? Use the box below to write down a few thoughts about how you would multiply three fractions together.

### Answer

Multiply the numerators and multiply the denominators.

$$\frac{2 \cdot 1 \cdot 3}{3 \cdot 4 \cdot 5}$$

Simplify first by canceling (dividing) the common factors of 3 and 2. 3 divided by 3 is 1, and 2 divided by 2 is 1.

$$\frac{2 \cdot 1 \cdot 3}{3 \cdot (2 \cdot 2) \cdot 5}$$

$$\frac{\cancel{2} \cdot 1 \cdot \cancel{3}}{\cancel{3} \cdot (\cancel{2} \cdot 2) \cdot 5}$$

$$\frac{1}{10}$$

### Answer

$$\frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{5} = \frac{1}{10}$$

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## DIVIDING FRACTIONS

### LEARNING OUTCOMES

- Find the reciprocal of a number
- Divide a fraction by a whole number
- Divide a fraction by a fraction

## Divide Fractions

There are times when you need to use division to solve a problem. For example, if painting one coat of paint on the walls of a room requires 3 quarts of paint and you have a bucket that contains 6 quarts of paint, how many coats of paint can you paint on the walls? You divide 6 by 3 for an answer of 2 coats. There will also be times when you need to divide by a fraction. Suppose painting a closet with one coat only required  $\frac{1}{2}$  quart of paint. How many coats could be painted with the 6 quarts of paint? To find the answer, you need to divide 6 by the fraction,  $\frac{1}{2}$ .

Before we begin dividing fractions, let's cover some important terminology.

- **reciprocal:** two fractions are reciprocals if their product is 1 (Don't worry; we will show you examples of what this means.)
- **quotient:** the result of division

Dividing fractions requires using the reciprocal of a number or fraction. If you multiply two numbers together and get 1 as a result, then the two numbers are reciprocals. Here are some examples of reciprocals:

Original number	Reciprocal	Product
$\frac{3}{4}$	$\frac{4}{3}$	$\frac{3}{4} \cdot \frac{4}{3} = \frac{3 \cdot 4}{4 \cdot 3} = \frac{12}{12} = 1$
$\frac{1}{2}$	$\frac{2}{1}$	$\frac{1}{2} \cdot \frac{2}{1} = \frac{1 \cdot 2}{2 \cdot 1} = \frac{2}{2} = 1$
$3 = \frac{3}{1}$	$\frac{1}{3}$	$\frac{3}{1} \cdot \frac{1}{3} = \frac{3 \cdot 1}{1 \cdot 3} = \frac{3}{3} = 1$
$2\frac{1}{3} = \frac{7}{3}$	$\frac{3}{7}$	$\frac{7}{3} \cdot \frac{3}{7} = \frac{7 \cdot 3}{3 \cdot 7} = \frac{21}{21} = 1$

You can think of it as switching the numerator and denominator: swap the 2 with the 5 in  $\frac{2}{5}$  to get the reciprocal  $\frac{5}{2}$ . Make sure that if it's a negative fraction, the reciprocal is also negative. This is because the product of two negative numbers will give you the positive one that you are looking for.

## Division by Zero

You know what it means to divide by 2 or divide by 10, but what does it mean to divide a quantity by 0? Is this even possible? On the flip side, can you divide 0 by a number? Consider the fraction

$$\frac{0}{8}$$

We can read it as, "zero divided by eight." Since multiplication is the inverse of division, we could rewrite this as a multiplication problem. What number times 8 equals 0?

$$? \cdot 8 = 0$$

We can infer that the unknown must be 0 since that is the only number that will give a result of 0 when it is multiplied by 8.

Now let's consider the reciprocal of  $\frac{0}{8}$  which would be  $\frac{8}{0}$ . If we rewrite this as a multiplication problem, we will have "what times 0 equals 8?"

$$? \cdot 0 = 8$$

This doesn't make any sense. There are no numbers that you can multiply by zero to get a result of 8. In fact, any number divided by 0 is impossible, or better defined, all division by zero is undefined.



Caution! Division by zero is undefined and so is the reciprocal of any fraction that has a zero in the numerator. For any real number  $a$ ,  $\frac{a}{0}$  is undefined. Additionally, the reciprocal of  $\frac{0}{a}$  will always be undefined.

## Divide a Fraction by a Whole Number

When you divide by a whole number, you are also multiplying by the reciprocal. In the painting example where you need 3 quarts of paint for a coat and have 6 quarts of paint, you can find the total number of coats that can be painted by dividing 6 by 3,  $6 \div 3 = 2$ . You can also multiply 6 by the reciprocal of 3, which is  $\frac{1}{3}$ , so the multiplication problem becomes

$$\frac{6}{1} \cdot \frac{1}{3} = \frac{6}{3} = 2$$

### DIVIDING IS MULTIPLYING BY THE RECIPROCAL

For all division, you can turn the operation into multiplication by using the reciprocal. Dividing is the same as multiplying by the reciprocal.

The same idea will work when the divisor (the thing being divided) is a fraction.

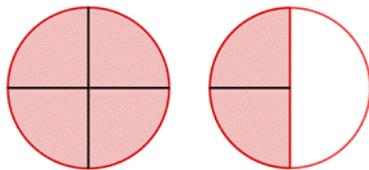
If you have a recipe that needs to be divided in half, you can divide each ingredient by 2, or you can multiply each ingredient by  $\frac{1}{2}$  to find the new amount.

If you have  $\frac{3}{4}$  of a candy bar and need to divide it among 5 people, each person gets  $\frac{1}{5}$  of the available candy:

$$\frac{1}{5} \text{ of } \frac{3}{4} = \frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20}$$

Each person gets  $\frac{3}{20}$  of a whole candy bar.

For example, dividing by 6 is the same as multiplying by the reciprocal of 6, which is  $\frac{1}{6}$ . Look at the diagram of two pizzas below. How can you divide what is left (the red shaded region) among 6 people fairly?



$$\frac{3}{2} \text{ divided by } 6 = \frac{3}{2} \times \frac{1}{6}$$

$$\frac{3}{2} \times \frac{1}{6} = \frac{1}{4}$$

Each person gets one piece, so each person gets  $\frac{1}{4}$  of a pizza.

Dividing a fraction by a whole number is the same as multiplying by the reciprocal, so you can always use multiplication of fractions to solve division problems.

### EXAMPLE

Find  $\frac{2}{3} \div 4$

**Answer**

Write your answer in lowest terms.

Dividing by 4 or  $\frac{4}{1}$  is the same as multiplying by the reciprocal of 4, which is  $\frac{1}{4}$ .

$$\frac{2}{3} \div 4 = \frac{2}{3} \cdot \frac{1}{4}$$

Multiply numerators and multiply denominators.

$$\frac{2 \cdot 1}{3 \cdot 4} = \frac{2}{12}$$

Simplify to lowest terms by dividing numerator and denominator by the common factor 4.

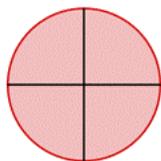
$$\frac{1}{6}$$

**Answer**

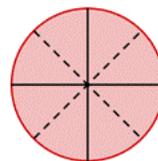
$$\frac{2}{3} \div 4 = \frac{1}{6}$$

## Divide a Fraction by a Fraction

Sometimes you need to solve a problem that requires dividing by a fraction. Suppose you have a pizza that is already cut into 4 slices. How many  $\frac{1}{2}$  slices are there?



number of slices = 4



number of  $\frac{1}{2}$  slices = 8

## EXAMPLE

Divide.  $9 \div \frac{1}{2}$

### Answer

Write your answer in lowest terms.

Dividing by  $\frac{1}{2}$  is the same as multiplying by the reciprocal of  $\frac{1}{2}$ , which is  $\frac{2}{1}$ .

$$9 \div \frac{1}{2} = \frac{9}{1} \cdot \frac{2}{1}$$

Multiply numerators and multiply denominators.

$$\frac{9 \cdot 2}{1 \cdot 1} = \frac{18}{1} = 18$$

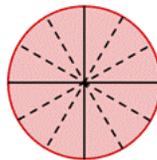
This answer is already simplified to lowest terms.

### Answer

$$9 \div \frac{1}{2} = 18$$

There are 8 slices. You can see that dividing 4 by  $\frac{1}{2}$  gives the same result as multiplying 4 by 2.

What would happen if you needed to divide each slice into thirds?



number of  $\frac{1}{3}$  slices = 12

You would have 12 slices, which is the same as multiplying 4 by 3.

## DIVIDING WITH FRACTIONS

1. Find the reciprocal of the number that follows the division symbol.
2. Multiply the first number (the one before the division symbol) by the reciprocal of the second number (the one after the division symbol).

Any easy way to remember how to divide fractions is the phrase “keep, change, flip.” This means to **KEEP** the first number, **CHANGE** the division sign to multiplication, and then **FLIP** (use the reciprocal) of the second number.

When solving a division problem by multiplying by the reciprocal, remember to write all whole numbers and mixed numbers as improper fractions before doing calculations (i.e.  $5 = \frac{5}{1}$  and  $1\frac{3}{4} = \frac{7}{4}$ ). The final answer should always be simplified and written as a mixed number if larger than 1.

In the following video you will see an example of how to divide an integer by a fraction, as well as an example of how to divide a fraction by another fraction.

## EXAMPLE

Divide  $\frac{2}{3} \div \frac{1}{6}$

**Answer**

Multiply by the reciprocal.

**KEEP**  $\frac{2}{3}$

**CHANGE**  $\div$  to  $\cdot$

**FLIP**  $\frac{1}{6}$

$$\frac{2}{3} \cdot \frac{6}{1}$$

Multiply numerators and multiply denominators.

$$\frac{2 \cdot 6}{3 \cdot 1} = \frac{12}{3}$$

Simplify.

$$\frac{12}{3} = 4$$

Answer

$$\frac{2}{3} \div \frac{1}{6} = 4$$

## EXAMPLE

Divide  $\frac{3}{5} \div \frac{2}{3}$

**Answer**

Multiply by the reciprocal. Keep  $\frac{3}{5}$ , change  $\div$  to  $\cdot$ , and flip  $\frac{2}{3}$ .

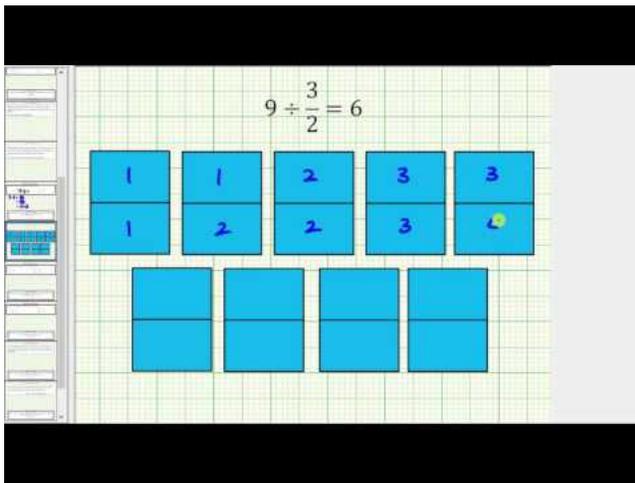
$$\frac{3}{5} \cdot \frac{3}{2}$$

Multiply numerators and multiply denominators.

$$\frac{3 \cdot 3}{5 \cdot 2} = \frac{9}{10}$$

Answer

$$\frac{3}{5} \div \frac{2}{3} = \frac{9}{10}$$



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=264>

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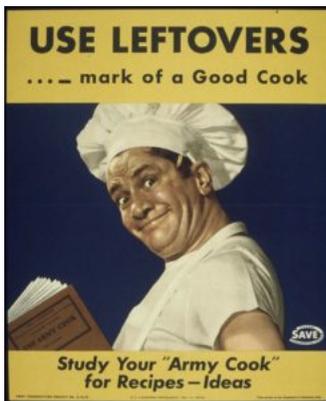
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## PUTTING IT TOGETHER: REVIEW



While the forgotten food in your fridge may not be what you spend your work day dreaming about devouring when you get home, those jumbled bits and pieces can be used to create something tasty enough to satisfy your hunger. Hopefully this section offered you enough tidbits to keep you full long enough to get to the next section, where you will learn how to put these forgotten skills to work as you continue to grow your mathematical prowess. Remember, you can come back to this review any time throughout the course for practice dividing fractions, simplifying with square roots, or whatever you might need to help you learn the concepts in this course more easily.

Oh, and now you know that  $3 + 5 \times 2$  is 13!

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# MODULE 1: REAL NUMBERS AND ALGEBRAIC EXPRESSIONS

## WHY IT MATTERS: REAL NUMBERS AND ALGEBRAIC EXPRESSIONS

### Why Solve Linear Equations and Inequalities?



You might be surprised to learn that applications of linear equations turn up in many places besides math classrooms. Knowing how to solve them is a basic math skill used in nearly every academic discipline and in many jobs. One of the fundamental principles of solving linear equations is that of reversing or undoing mathematical operations such as addition and subtraction. To see a linear equation in action, consider one that is used by forensic scientists to calculate blood alcohol content.

Blood alcohol content (BAC) is a measurement of how much alcohol is in someone's blood. It is usually measured in grams and as a percentage. For example, a BAC of 0.30% is three-tenths of 1%, and it indicates that there are 3 grams of alcohol for every 1,000 grams of blood—which is actually a lot. A BAC of 0.05% impairs reasoning and the ability to concentrate. A BAC of 0.30 can lead to a blackout, shortness of breath, and loss of bladder control. In most states, the legal limit for driving is a BAC of 0.08.

BAC is usually determined by the results of a breathalyzer, urinalysis, or blood test. Swedish physician E. M. P. Widmark developed an equation that works well for estimating BAC without using one of those tests. Widmark's formula is

widely used by forensic scientists:

$$B = -0.015t + \left( \frac{2.84N}{Wg} \right)$$

where

- B = percentage of BAC
- t = number of hours since the first drink
- N = number of "standard drinks" (a standard drink is one 12-ounce beer, one 5-ounce glass of wine, or one 1.5-ounce shot of liquor). N should be at least 1.
- W = weight in pounds
- g = gender constant: 0.68 for men and 0.55 for women

In the following table, the progressive effects of alcohol are defined for ranges of blood alcohol content.

<b>Progressive Effects of Alcohol</b>		
<b>BAC (% by vol.)</b>	<b>Behavior</b>	<b>Impairment</b>
0.001– 0.029	<ul style="list-style-type: none"> <li>• Average individual appears normal</li> </ul>	<ul style="list-style-type: none"> <li>• Subtle effects that can be detected with special tests</li> </ul>
0.030– 0.059	<ul style="list-style-type: none"> <li>• Mild euphoria</li> <li>• Relaxation</li> <li>• Joyousness</li> <li>• Talkativeness</li> <li>• Decreased inhibition</li> </ul>	<ul style="list-style-type: none"> <li>• Concentration</li> </ul>
0.060– 0.099	<ul style="list-style-type: none"> <li>• Blunted feelings</li> <li>• Reduced sensitivity to pain</li> <li>• Euphoria</li> <li>• Disinhibition</li> <li>• Extroversion</li> </ul>	<ul style="list-style-type: none"> <li>• Reasoning</li> <li>• Depth perception</li> <li>• Peripheral vision</li> <li>• Glare recovery</li> </ul>
0.100– 0.199	<ul style="list-style-type: none"> <li>• Overexpression</li> <li>• Boisterousness</li> <li>• Possibility of nausea and vomiting</li> </ul>	<ul style="list-style-type: none"> <li>• Reflexes</li> <li>• Reaction time</li> <li>• Gross motor control</li> <li>• Staggering</li> <li>• Slurred speech</li> <li>• Temporary erectile dysfunction</li> </ul>
0.200– 0.299	<ul style="list-style-type: none"> <li>• Nausea</li> <li>• Vomiting</li> <li>• Emotional swings</li> <li>• Anger or sadness</li> <li>• Partial loss of understanding</li> <li>• Impaired sensations</li> <li>• Decreased libido</li> <li>• Possibility of stupor</li> </ul>	<ul style="list-style-type: none"> <li>• Severe motor impairment</li> <li>• Loss of consciousness</li> <li>• Memory blackout</li> </ul>
0.300– 0.399	<ul style="list-style-type: none"> <li>• Stupor</li> <li>• Central nervous system depression</li> <li>• Loss of understanding</li> <li>• Lapses in and out of consciousness</li> <li>• Low possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Bladder function</li> <li>• Breathing</li> <li>• Dysequilibrium</li> <li>• Heart rate</li> </ul>
0.400– 0.500	<ul style="list-style-type: none"> <li>• Severe central nervous system depression</li> <li>• Coma</li> <li>• Possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Breathing</li> <li>• Heart rate</li> <li>• Positional Alcohol Nystagmus</li> </ul>
> 0.50	<ul style="list-style-type: none"> <li>• High risk of poisoning</li> <li>• High possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Life</li> </ul>

Joan likes to party and believes she is “just fine” when it comes to driving. At a party, though, she downs three standard drinks, one after the other, and then decides to leave. If Joan weighs 135 pounds, where would she be on the table of the progressive effects of alcohol after 1.5 hours? Would she be within the legal limit to drive home after this amount of time? Given any amount that she drinks, can you figure out how long she must wait before she can drive safely and legally?

As you will discover, these are all questions that can be answered by solving linear equations and inequalities. Read on to learn more. At the end of the module we will revisit Joan and see how she fared.

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# INTRODUCTION TO CLASSIFYING AND DEFINING PROPERTIES OF REAL NUMBERS

The real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or –). In this section we will further define real numbers and use their properties to solve linear equations and inequalities.

The classes of numbers we will explore include:

## Natural numbers

The most familiar numbers are the natural numbers (sometimes called counting numbers): 1, 2, 3, and so on. The mathematical symbol for the set of all natural numbers is written as  $\mathbb{N}$ .

## Whole numbers

The set of whole numbers includes all natural numbers as well as 0.

## Integers

When the set of negative numbers is combined with the set of natural numbers (including 0), the result is defined as the set of integers,  $\mathbb{Z}$ .

## Rational numbers

A rational number,  $\mathbb{Q}$ , is a number that can be expressed as a fraction with an integer numerator and a positive integer denominator.

## Real numbers

The real numbers include all the measuring numbers. The symbol for the real numbers is  $\mathbb{R}$ . Real numbers are usually represented by using decimal numerals.

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## CLASSIFY REAL NUMBERS

### LEARNING OUTCOME

- Classify a real number as a natural, whole, integer, rational, or irrational number

The numbers we use for counting, or enumerating items, are the **natural numbers**: 1, 2, 3, 4, 5, and so on. We describe them in set notation as  $\{1, 2, 3, \dots\}$  where the ellipsis (...) indicates that the numbers continue to infinity. The natural numbers are, of course, also called the *counting numbers*. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of **whole numbers** is the set of natural numbers and zero:  $\{0, 1, 2, 3, \dots\}$ .

The set of **integers** adds the opposites of the natural numbers to the set of whole numbers:  $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ . It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

negative integers    zero    positive integers

$\dots, -3, -2, -1, \quad 0, \quad 1, 2, 3, \dots$

The set of **rational numbers** is written as  $\left\{\frac{m}{n} \mid m \text{ and } n \text{ are integers and } n \neq 0\right\}$ . Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0. We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal:  $\frac{15}{8} = 1.875$ , or

2. a repeating decimal:  $\frac{4}{11} = 0.36363636\dots = 0.\overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

## Irrational Numbers

## EXAMPLE

Write each of the following as a rational number.

- a. 7
- b. 0
- c. -8

### Answer

Write a fraction with the integer in the numerator and 1 in the denominator.

- a.  $7 = \frac{7}{1}$
- b.  $0 = \frac{0}{1}$
- c.  $-8 = -\frac{8}{1}$

## EXAMPLE

Write each of the following rational numbers as either a terminating or repeating decimal.

- a.  $-\frac{5}{7}$
- b.  $\frac{15}{5}$
- c.  $\frac{13}{25}$

### Answer

Write each fraction as a decimal by dividing the numerator by the denominator.

- a.  $-\frac{5}{7} = -0.\overline{714285}$ , a repeating decimal
- b.  $\frac{15}{5} = 3$  (or 3.0), a terminating decimal
- c.  $\frac{13}{25} = 0.52$ , a terminating decimal

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even  $\frac{3}{2}$ , but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3, but still not a rational number. Such numbers are said to be *irrational* because they cannot be written as fractions. These numbers make up the set of **irrational numbers**. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. So we write this as shown.

{h | h is not a rational number}

## Real Numbers

Given any number  $n$ , we know that  $n$  is either rational or irrational. It cannot be both. The sets of rational and irrational numbers together make up the set of **real numbers**. As we saw with integers, the real numbers

## EXAMPLE

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

- a.  $\sqrt{25}$
- b.  $\frac{33}{9}$
- c.  $\sqrt{11}$
- d.  $\frac{17}{34}$
- e. 0.303303330333...

### Answer

a.  $\sqrt{25}$  : This can be simplified as  $\sqrt{25} = 5$ . Therefore,  $\sqrt{25}$  is rational.

b.  $\frac{33}{9}$  : Because it is a fraction,  $\frac{33}{9}$  is a rational number. Next, simplify and divide.

$$\frac{33}{9} = \frac{11 \cdot 3}{3 \cdot 3} = \frac{11}{3} = 3.\bar{6}$$

So,  $\frac{33}{9}$  is rational and a repeating decimal.

c.  $\sqrt{11}$  : This cannot be simplified any further. Therefore,  $\sqrt{11}$  is an irrational number.

d.  $\frac{17}{34}$  : Because it is a fraction,  $\frac{17}{34}$  is a rational number. Simplify and divide.

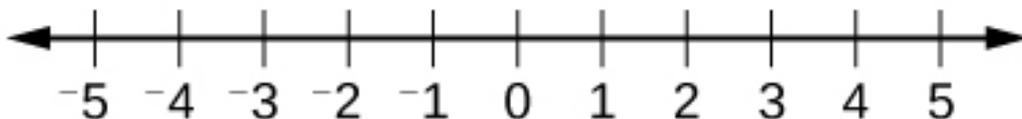
$$\frac{17}{34} = \frac{1}{2} = 0.5$$

So,  $\frac{17}{34}$  is rational and a terminating decimal.

e. 0.303303330333... is not a terminating decimal. Also note that there is no repeating pattern because the group of 3s increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or -). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0, with negative numbers to the left of 0 and positive numbers to the right of 0. A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0. Any real number corresponds to a unique position on the number line. The converse is also true: Each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the **real number line** as shown below.



The real number line.

## Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is

## EXAMPLE

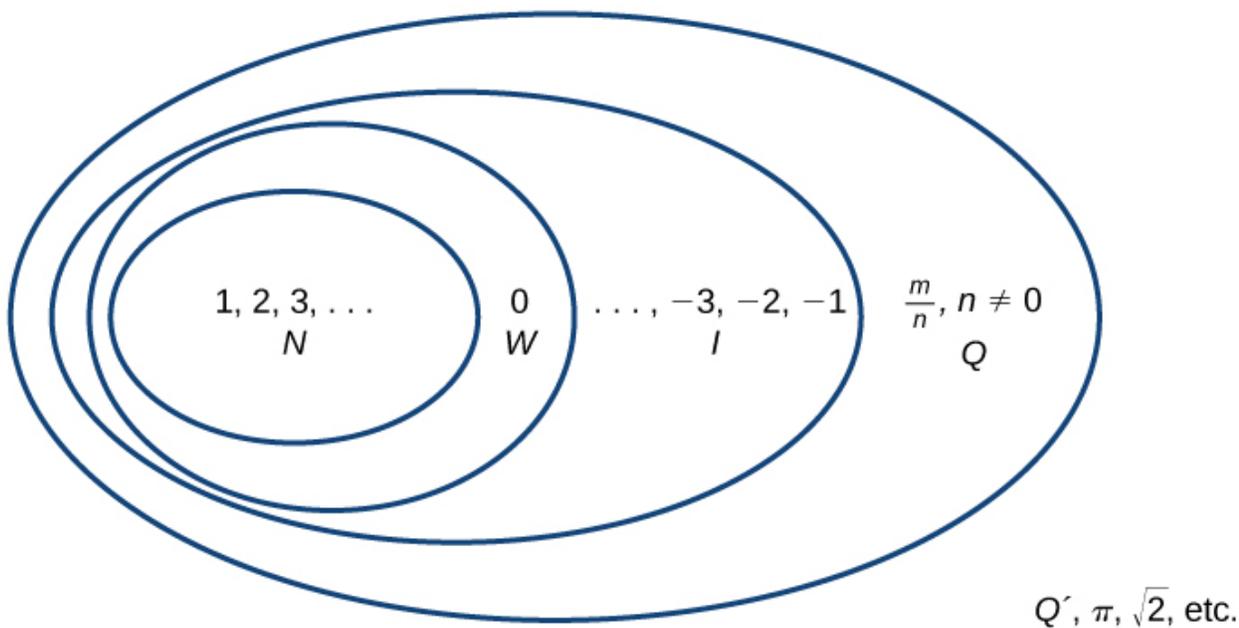
Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

- a.  $-\frac{10}{3}$
- b.  $\sqrt{5}$
- c.  $-\sqrt{289}$
- d.  $-6\pi$
- e.  $0.615384615384\dots$

### Answer

- a.  $-\frac{10}{3}$  is negative and rational. It lies to the left of 0 on the number line.
- b.  $\sqrt{5}$  is positive and irrational. It lies to the right of 0.
- c.  $-\sqrt{289} = -\sqrt{17^2} = -17$  is negative and rational. It lies to the left of 0.
- d.  $-6\pi$  is negative and irrational. It lies to the left of 0.
- e.  $0.615384615384\dots$  is a repeating decimal so it is rational and positive. It lies to the right of 0.

a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram.



Sets of numbers. *N*: the set of natural numbers *W*: the set of whole numbers *I*: the set of integers *Q*: the set of rational numbers *Q'*: the set of irrational numbers

## A GENERAL NOTE: SETS OF NUMBERS

The set of **natural numbers** includes the numbers used for counting:  $\{1, 2, 3, \dots\}$ .

The set of **whole numbers** is the set of natural numbers and zero:  $\{0, 1, 2, 3, \dots\}$ .

The set of **integers** adds the negative natural numbers to the set of whole numbers:  $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ .

The set of **rational numbers** includes fractions written as  $\{\frac{m}{n} | m \text{ and } n \text{ are integers and } n \neq 0\}$ .

The set of **irrational numbers** is the set of numbers that are not rational. They are nonrepeating and nonterminating decimals:  $\{h | h \text{ is not a rational number}\}$ .

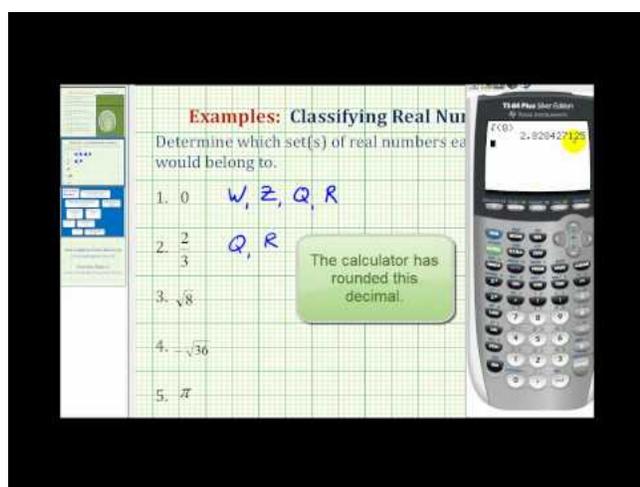
## EXAMPLE

Classify each number as being a natural number ( $N$ ), whole number ( $W$ ), integer ( $I$ ), rational number ( $Q$ ), and/or irrational number ( $Q'$ ).

- a.  $\sqrt{36}$
- b.  $\frac{8}{3}$
- c.  $\sqrt{73}$
- d.  $-6$
- e.  $3.2121121112\dots$

**Answer**

		<b>N</b>	<b>W</b>	<b>I</b>	<b>Q</b>	<b>Q'</b>
1.	$\sqrt{36} = 6$	X	X	X	X	
2.	$\frac{8}{3} = 2.\bar{6}$				X	
3.	$\sqrt{73}$					X
4.	$-6$			X	X	
5.	$3.2121121112\dots$					X



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# USE PROPERTIES OF REAL NUMBERS

## LEARNING OUTCOMES

- Define and use the commutative property of addition and multiplication
- Define and use the associative property of addition and multiplication
- Define and use the distributive property
- Define and use the identity property of addition and multiplication
- Define and use the inverse property of addition and multiplication

For some activities we perform, the order of certain processes does not matter, but the order of others do. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for addition and multiplication.

## Commutative Properties

The **commutative property of addition** states that numbers may be added in any order without affecting the sum.

$$a + b = b + a$$

We can better see this relationship when using real numbers.

### EXAMPLE

Show that numbers may be added in any order without affecting the sum.  $(-2) + 7 = 5$

**Answer**

$$7 + (-2) = 5$$

Similarly, the **commutative property of multiplication** states that numbers may be multiplied in any order without affecting the product.

$$a \cdot b = b \cdot a$$

Again, consider an example with real numbers.

## Associative Properties – Grouping

The **associative property of multiplication** tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the

## EXAMPLE

Show that numbers may be multiplied in any order without affecting the product.  $(-11) \cdot (-4) = 44$

**Answer**

$$(-4) \cdot (-11) = 44$$



Caution! It is important to note that neither subtraction nor division is commutative. For example,  $17 - 5$  is not the same as  $5 - 17$ . Similarly,  $20 \div 5 \neq 5 \div 20$ .

same.

$$a(bc) = (ab)c$$

Consider this example.

## EXAMPLE

Show that you can regroup numbers that are multiplied together and not affect the product.  $(3 \cdot 4) \cdot 5 = 60$

**Answer**

$$3 \cdot (4 \cdot 5) = 60$$

The **associative property of addition** tells us that numbers may be grouped differently without affecting the sum.

$$a + (b + c) = (a + b) + c$$

This property can be especially helpful when dealing with negative integers. Consider this example.

## EXAMPLE

Show that regrouping addition does not affect the sum.  $[15 + (-9)] + 23 = 29$

**Answer**

$$15 + [(-9) + 23] = 29$$

Are subtraction and division associative? Review these examples.

## Distributive Property

The **distributive property** states that the product of a factor times a sum is the sum of the factor times each term in the sum.

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.

To be more precise when describing this property, we say that multiplication distributes over addition.

The reverse is not true as we can see in this example.

## EXAMPLE

Use the associative property to explore whether subtraction and division are associative.

$$1) 8 - (3 - 15) \stackrel{?}{=} (8 - 3) - 15$$

$$2) 64 \div (8 \div 4) \stackrel{?}{=} (64 \div 8) \div 4$$

**Answer**

$$1) 8 - (3 - 15) \stackrel{?}{=} (8 - 3) - 15$$

$$8 - (-12) = 5 - 15$$

$$20 \neq -10$$

$$2) 64 \div (8 \div 4) \stackrel{?}{=} (64 \div 8) \div 4$$

$$64 \div 2 \stackrel{?}{=} 8 \div 4$$

$$32 \neq 2$$

As we can see, neither subtraction nor division is associative.

## EXAMPLE

Use the distributive property to show that  $4 \cdot [12 + (-7)] = 20$

**Answer**

$$\begin{aligned} 4 \cdot [12 + (-7)] &= 4 \cdot 12 + 4 \cdot (-7) \\ &= 48 + (-28) \\ &= 20 \end{aligned}$$

Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12, multiplying it by  $-7$ , and adding the products.

$$6 + (3 \cdot 5) \stackrel{?}{=} (6 + 3) \cdot (6 + 5)$$

$$6 + (15) \stackrel{?}{=} (9) \cdot (11)$$

$$21 \neq 99$$

A special case of the distributive property occurs when a sum of terms is subtracted.

$$a - b = a + (-b)$$

For example, consider the difference  $12 - (5 + 3)$ . We can rewrite the difference of the two terms 12 and  $(5 + 3)$  by turning the subtraction expression into addition of the opposite. So instead of subtracting  $(5 + 3)$ , we add the opposite.

$$12 + (-1) \cdot (5 + 3)$$

Now, distribute  $-1$  and simplify the result.

$$12 - (5 + 3) = 12 + (-1) \cdot (5 + 3)$$

$$= 12 + [(-1) \cdot 5 + (-1) \cdot 3]$$

$$= 12 + (-8)$$

$$= 4$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we

## EXAMPLE

Rewrite the last example by changing the sign of each term and adding the results.

**Answer**

$$\begin{aligned}12 - (5 + 3) &= 12 + (-5 - 3) \\ &= 12 + (-8) \\ &= 4\end{aligned}$$

introduce algebraic terms.

## Identity Properties

The **identity property of addition** states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$$a + 0 = a$$

The **identity property of multiplication** states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.

$$a \cdot 1 = a$$

## EXAMPLE

Show that the identity property of addition and multiplication are true for  $-6$  and  $23$ .

**Answer**

$$\begin{aligned}(-6) + 0 &= -6 \\ 23 + 0 &= 23 \\ -6 \cdot 1 &= -6 \\ 23 \cdot 1 &= 23\end{aligned}$$

There are no exceptions for these properties; they work for every real number, including 0 and 1.

## Inverse Properties

The **inverse property of addition** states that, for every real number  $a$ , there is a unique number, called the additive inverse (or opposite), denoted  $-a$ , that, when added to the original number, results in the additive identity, 0.

$$a + (-a) = 0$$

For example, if  $a = -8$ , the additive inverse is 8, since  $(-8) + 8 = 0$ .

The **inverse property of multiplication** holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number  $a$ , there is a unique number, called the multiplicative inverse (or reciprocal), denoted  $\frac{1}{a}$ , that, when multiplied by the original number, results in the multiplicative identity, 1.

$$a \cdot \frac{1}{a} = 1$$

## EXAMPLE

- 1) Define the additive inverse of  $a = -8$ , and use it to illustrate the inverse property of addition.
- 2) Write the reciprocal of  $a = -\frac{2}{3}$ , and use it to illustrate the inverse property of multiplication.

### Answer

- 1) The additive inverse is 8, and  $(-8) + 8 = 0$
- 2) The reciprocal is  $-\frac{3}{2}$  and  $\left(-\frac{2}{3}\right) \cdot \left(-\frac{3}{2}\right) = 1$

## A GENERAL NOTE: PROPERTIES OF REAL NUMBERS

The following properties hold for real numbers  $a$ ,  $b$ , and  $c$ .

	Addition	Multiplication
<b>Commutative Property</b>	$a + b = b + a$	$a \cdot b = b \cdot a$
<b>Associative Property</b>	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
<b>Distributive Property</b>	$a \cdot (b + c) = a \cdot b + a \cdot c$	
<b>Identity Property</b>	There exists a unique real number called the additive identity, 0, such that, for any real number $a$ $a + 0 = a$	There exists a unique real number called the multiplicative identity, 1, such that, for any real number $a$ $a \cdot 1 = a$
<b>Inverse Property</b>	Every real number $a$ has an additive inverse, or opposite, denoted $-a$ , such that $a + (-a) = 0$	Every nonzero real number $a$ has a multiplicative inverse, or reciprocal, denoted $\frac{1}{a}$ , such that $a \cdot \left(\frac{1}{a}\right) = 1$

## EXAMPLE

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

- a.  $3(6 + 4)$
- b.  $(5 + 8) + (-8)$
- c.  $6 - (15 + 9)$
- d.  $\frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right)$
- e.  $100 \cdot [0.75 + (-2.38)]$

### Answer

a.

$$\begin{aligned} 3 \cdot (6 + 4) &= 3 \cdot 6 + 3 \cdot 4 && \text{Distributive property} \\ &= 18 + 12 && \text{Simplify} \\ &= 30 && \text{Simplify} \end{aligned}$$

b.

$$\begin{aligned} (5 + 8) + (-8) &= 5 + [8 + (-8)] && \text{Associative property of addition} \\ &= 5 + 0 && \text{Inverse property of addition} \\ &= 5 && \text{Identity property of addition} \end{aligned}$$

c.

$$\begin{aligned} 6 - (15 + 9) &= 6 + [(-15) + (-9)] && \text{Distributive property} \\ &= 6 + (-24) && \text{Simplify} \\ &= -18 && \text{Simplify} \end{aligned}$$

d.

$$\begin{aligned} \frac{4}{7} \cdot \left(\frac{2}{3} \cdot \frac{7}{4}\right) &= \frac{4}{7} \cdot \left(\frac{7}{4} \cdot \frac{2}{3}\right) && \text{Commutative property of multiplication} \\ &= \left(\frac{4}{7} \cdot \frac{7}{4}\right) \cdot \frac{2}{3} && \text{Associative property of multiplication} \\ &= 1 \cdot \frac{2}{3} && \text{Inverse property of multiplication} \\ &= \frac{2}{3} && \text{Identity property of multiplication} \end{aligned}$$

e.

$$\begin{aligned} 100 \cdot [0.75 + (-2.38)] &= 100 \cdot 0.75 + 100 \cdot (-2.38) && \text{Distributive property} \\ &= 75 + (-238) && \text{Simplify} \end{aligned}$$

$$= -163$$

Simplify

The screenshot shows a video player interface. The video title is "Properties of Real Numbers". The content includes a blue box with the text "EXAMPLE Use the properties of real numbers to rewrite and simplify each expression. State which properties apply." followed by a list of five expressions: 1.  $4(7+5)$ , 2.  $(3+9)+(-9)$ , 3.  $8-(11+6)$ , 4.  $\frac{2}{4} \cdot (\frac{7}{9} - \frac{4}{3})$ , and 5.  $100\{0.23 + (-1.78)\}$ . A blue callout box says: "We will use the simplifying of these expressions to demonstrate the properties of real numbers. In some of the examples, we will show more steps than are needed if we were just simplifying to demonstrate the properties of real numbers." Below the list, a handwritten solution for expression 1 is shown:  $4(7+5) = 4 \cdot 7 + 4 \cdot 5$  (labeled "distributive property") and  $= 28 + 20$ .

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# EVALUATE ALGEBRAIC EXPRESSIONS

## LEARNING OUTCOMES

- Define and identify constants in an algebraic expression
- Evaluate algebraic expressions for different values

In mathematics, we may see expressions such as  $x + 5$ ,  $\frac{4}{3}\pi r^3$ , or  $\sqrt{2m^3n^2}$ . In the expression  $x + 5$ , 5 is called a **constant** because it does not vary and  $x$  is called a **variable** because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An **algebraic expression** is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

The examples in this section include exponents. Recall that an exponent is shorthand for writing repeated multiplication of the same number. When variables have exponents, it means repeated multiplication of the same variable. The base of an exponent is the number or variable being multiplied, and the exponent tells us how many times to multiply.

For example,

$$(-3)^5 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) \cdot (-3) \quad x^5 = x \cdot x \cdot x \cdot x \cdot x$$

$$(2 \cdot 7)^3 = (2 \cdot 7) \cdot (2 \cdot 7) \cdot (2 \cdot 7)$$

$$(yz)^3 = (yz) \cdot (yz) \cdot (yz)$$

In each case, the exponent tells us how many factors of the base to use regardless of whether the base consists of constants or variables.

In the following example, we will practice identifying constants and variables in mathematical expressions.

### EXAMPLE

List the constants and variables for each algebraic expression.

1.  $x + 5$
2.  $\frac{4}{3}\pi r^3$
3.  $\sqrt{2m^3n^2}$

**Answer**

Expression	Constants	Variables
1. $x + 5$	5	$x$
2. $\frac{4}{3}\pi r^3$	$\frac{4}{3}, \pi$	$r$
3. $\sqrt{2m^3n^2}$	2	$m, n$

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before. In the next example we show how to substitute various types of numbers into a mathematical expression.

Now we will show more examples of evaluating a variety of mathematical expressions for various values.

In the following video we present more examples of evaluating a variety of expressions for given values.

## EXAMPLE

Evaluate the expression  $2x - 7$  for each value for  $x$ .

a.  $x = 0$

b.  $x = 1$

c.  $x = \frac{1}{2}$

d.  $x = -4$

### Answer

a. Substitute 0 for  $x$ .

$$\begin{aligned}2x - 7 &= 2(0) - 7 \\ &= 0 - 7 \\ &= -7\end{aligned}$$

b. Substitute 1 for  $x$ .

$$\begin{aligned}2x - 7 &= 2(1) - 7 \\ &= 2 - 7 \\ &= -5\end{aligned}$$

c. Substitute  $\frac{1}{2}$  for  $x$ .

$$\begin{aligned}2x - 7 &= 2\left(\frac{1}{2}\right) - 7 \\ &= 1 - 7 \\ &= -6\end{aligned}$$

d. Substitute  $-4$  for  $x$ .

$$\begin{aligned}2x - 7 &= 2(-4) - 7 \\ &= -8 - 7 \\ &= -15\end{aligned}$$

## EXAMPLE

Evaluate each expression for the given values.

a.  $x + 5$  for  $x = -5$

b.  $\frac{t}{2t-1}$  for  $t = 10$

c.  $\frac{4}{3}\pi r^3$  for  $r = 5$

d.  $a + ab + b$  for  $a = 11, b = -8$

e.  $\sqrt{2m^3n^2}$  for  $m = 2, n = 3$

### Answer

a. Substitute  $-5$  for  $x$ .

$$\begin{aligned}x + 5 &= (-5) + 5 \\ &= 0\end{aligned}$$

b. Substitute 10 for  $t$ .

$$\begin{aligned}\frac{t}{2t-1} &= \frac{(10)}{2(10)-1} \\ &= \frac{10}{20-1} \\ &= \frac{10}{19}\end{aligned}$$

c. Substitute 5 for  $r$ .

$$\begin{aligned}\frac{4}{3}\pi r^3 &= \frac{4}{3}\pi(5)^3 \\ &= \frac{4}{3}\pi(125) \\ &= \frac{500}{3}\pi\end{aligned}$$

d. Substitute 11 for  $a$  and  $-8$  for  $b$ .

$$\begin{aligned}a + ab + b &= (11) + (11)(-8) + (-8) \\ &= 11 - 88 - 8 \\ &= -85\end{aligned}$$

e. Substitute 2 for  $m$  and 3 for  $n$ .

$$\begin{aligned}\sqrt{2m^3n^2} &= \sqrt{2(2)^3(3)^2} \\ &= \sqrt{2(8)(9)} \\ &= \sqrt{144} \\ &= 12\end{aligned}$$

**Evaluating Algebraic Expressions**

**EXAMPLE**

Evaluate each expression for the given values.

1.  $2x - 5$ , for  $x = -6$
2.  $\frac{4}{3t-4}$ , for  $t = 4$
3.  $\frac{2}{3}\pi r^2 h$  for  $r = 3$ ,  $h = 7$
4.  $a^2 + 2ab + b^2$  for  $a = -3$ ,  $b = 6$
5.  $\sqrt{5a^2bc}$ , for  $a = 2$ ,  $b = 5$ ,  $c = 1$

**THE ORDER OF OPERATIONS**

1. Perform all operations within grouping symbols. Grouping symbols include parentheses ( ), brackets [ ], braces { }, and fraction bars.
2. Evaluate exponents or radicals.
3. Multiply or divide, from left to right.
4. Add or subtract, from left to right.

$2x - 5$ , for  $x = -6$

$$2(-6) - 5$$

$$= -12 - 5$$

$$= \boxed{-17}$$

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# INTRODUCTION TO SOLVING MULTI-STEP LINEAR EQUATIONS

There are some equations that you can solve in your head quickly, but other equations are more complicated. Multi-step equations, ones that take several steps to solve, can still be simplified and solved by applying basic algebraic rules such as the multiplication and addition properties of equality.

In this section we will explore methods for solving multi-step equations that contain grouping symbols and several mathematical operations. We will also learn techniques for solving multi-step equations that contain absolute values. Finally, we will learn that some equations have no solutions, while others have an infinite number of solutions.

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## MULTI-STEP EQUATIONS

Learning Outcomes

- Use properties of equality to isolate variables and solve algebraic equations
- Solve equations containing absolute value



*Steps with an end in sight.*

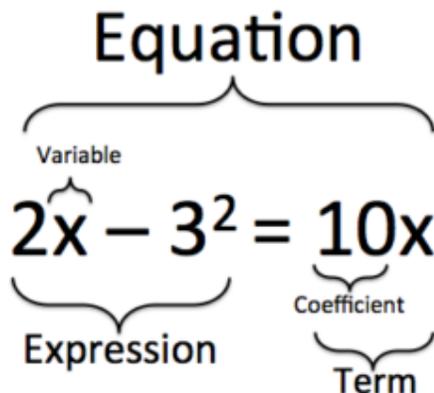
## Use Properties of Equality to Isolate Variables and Solve Algebraic Equations

First, let us define some important terminology:

- **variables:** variables are symbols that stand for an unknown quantity; they are often represented with letters, like  $x$ ,  $y$ , or  $z$ .
- **coefficient:** Sometimes a variable is multiplied by a number. This number is called the coefficient of the variable. For example, the coefficient of  $3x$  is 3.

- **term:** a single number, or variables and numbers connected by multiplication.  $-4$ ,  $6x$  and  $x^2$  are all terms.
- **expression:** groups of terms connected by addition and subtraction.  $2x^2 - 5$  is an expression.
- **equation:** an equation is a mathematical statement that two expressions are equal. An equation will always contain an equal sign with an expression on each side. Think of an equal sign as meaning “the same as.” Some examples of equations are  $y = mx + b$ ,  $\frac{3}{4}r = v^3 - r$ , and  $2(6 - d) + f(3 + k) = \frac{1}{4}d$ .

The following figure shows how coefficients, variables, terms, and expressions all come together to make equations. In the equation  $2x - 3^2 = 10x$ , the variable is  $x$ , a coefficient is 10, a term is  $10x$ , an expression is  $2x - 3^2$ .



There are some **equations** that you can solve in your head quickly. For example, what is the value of  $y$  in the equation  $2y = 6$ ? Chances are you didn't need to get out a pencil and paper to calculate that  $y = 3$ . You only needed to do one thing to get the answer: divide 6 by 2.

Other equations are more complicated. Solving  $4\left(\frac{1}{3}t + \frac{1}{2}\right) = 6$  without writing anything down is difficult!

That is because this equation contains not just a **variable** but also fractions and **terms** inside parentheses. This is a **multi-step equation**, one that takes several steps to solve. Although multi-step equations take more time and more operations, they can still be simplified and solved by applying basic algebraic rules.

Remember that you can think of an equation as a balance scale, with the goal being to rewrite the equation so that it is easier to solve but still balanced. The **addition property of equality** and the **multiplication property of equality** explain how you can keep the scale, or the equation, balanced. Whenever you perform an operation to one side of the equation, if you perform the same exact operation to the other side, you will keep both sides of the equation equal.

If the equation is in the form  $ax + b = c$ , where  $x$  is the variable, you can solve the equation as before. First “undo” the addition and subtraction and then “undo” the multiplication and division.

#### Example

Solve  $3y + 2 = 11$ .

**Answer**

Subtract 2 from both sides of the equation to get the term with the variable by itself.

$$\begin{array}{r} 3y + 2 = 11 \\ -2 \quad -2 \\ \hline 3y = 9 \end{array}$$

Divide both sides of the equation by 3 to get a coefficient of 1 for the variable.

$$\begin{array}{r} \underline{3y} = \underline{9} \\ 3 \quad 3 \\ y = 3 \end{array}$$

### Example

Solve  $3x + 5x + 4 - x + 7 = 88$ .

**Answer**

There are three like terms involving a variable:  $3x$ ,  $5x$ , and  $-x$ . Combine these like terms. 4 and 7 are also like terms and can be added.

$$\begin{aligned} 3x + 5x + 4 - x + 7 &= 88 \\ 7x + 11 &= 88 \end{aligned}$$

The equation is now in the form  $ax + b = c$ , so we can solve as before.

Subtract 11 from both sides.

$$\begin{aligned} 7x + 11 &= 88 \\ \underline{-11} \quad \underline{-11} & \\ 7x &= 77 \end{aligned}$$

Divide both sides by 7.

$$\begin{aligned} \underline{7x} &= \underline{77} \\ 7 \quad 7 & \\ x &= 11 \end{aligned}$$

Some equations may have the variable on both sides of the equal sign, as in this equation:

$$4x - 6 = 2x + 10.$$

To solve this equation, we need to “move” one of the variable terms. This can make it difficult to decide which side to work with. It does not matter which term gets moved,  $4x$  or  $2x$ ; however, to avoid negative coefficients, you can move the smaller term.

In this video, we show an example of solving equations that have variables on both sides of the equal sign.

## Solving Multi-Step Equations With Absolute Value

We can apply the same techniques we used for solving a one-step equation which contained absolute value to an equation that will take more than one step to solve. Let us start with an example where the first step is to write two equations, one equal to positive 26 and one equal to negative 26.

## EXAMPLES

Solve:  $4x - 6 = 2x + 10$

**Answer**

Choose the variable term to move—to avoid negative terms choose  $2x$

$$\begin{array}{r} 4x - 6 = 2x + 10 \\ -2x \quad -2x \\ \hline 4x - 6 = 10 \end{array}$$

Now add 6 to both sides to isolate the term with the variable.

$$\begin{array}{r} 4x - 6 = 10 \\ +6 \quad +6 \\ \hline 2x = 16 \end{array}$$

$$2x = 16$$

Now divide each side by 2 to isolate the variable  $x$ .

$$\frac{2x}{2} = \frac{16}{2}$$

$$x = 8$$

**Solving Multi-Step Equations**

EXAMPLE

Solve  $7x - 2 = 3x + 18$ .

$$\begin{array}{r} 7x - 2 = 3x + 18 \\ -3x \quad -3x \\ \hline 4x - 2 = 18 \\ +2 \quad +2 \\ \hline 4x = 20 \end{array}$$

**SOLVING MULTI-STEP EQUATIONS**

1. (Optional) Multiply to clear any fractions or decimals.
2. Simplify each side by clearing parentheses and combining like terms.
3. Add or subtract to isolate the variable term – you may have to move a term with a variable.
4. Multiply or divide to isolate the variable.
5. Check the solution.

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### Example

Solve for  $p$ .  $|2p - 4| = 26$

**Answer**

Write the two equations that will give an absolute value of 26.

$$2p - 4 = 26 \quad \text{or} \quad 2p - 4 = -26$$

Solve each equation for  $p$  by isolating the variable.

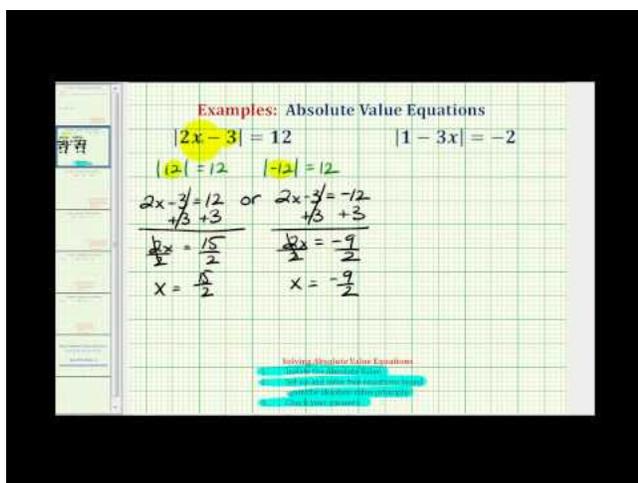
$$\begin{array}{r}
 2p - 4 = 26 \\
 \underline{+4 \quad +4} \\
 2p = 30 \\
 \underline{\quad \quad 2} \\
 p = 15
 \end{array}
 \quad \text{or} \quad
 \begin{array}{r}
 2p - 4 = -26 \\
 \underline{+4 \quad +4} \\
 2p = -22 \\
 \underline{\quad \quad 2} \\
 p = -11
 \end{array}$$

Check the solutions in the original equation.

$$\begin{array}{l}
 |2p - 4| = 26 \\
 |2(15) - 4| = 26 \\
 |30 - 4| = 26 \\
 |26| = 26
 \end{array}
 \quad
 \begin{array}{l}
 |2p - 4| = 26 \\
 |2(-11) - 4| = 26 \\
 |-22 - 4| = 26 \\
 |-26| = 26
 \end{array}$$

Both solutions check!

In the next video, we show more examples of solving a simple absolute value equation.



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Now let us look at an example where you need to do an algebraic step or two before you can write your two equations. The goal here is to get the absolute value on one side of the equation by itself. Then we can proceed as we did in the previous example.

### Example

Solve for  $w$ .  $3|4w - 1| - 5 = 10$

**Answer**

Isolate the term with the absolute value by adding 5 to both sides.

$$\begin{array}{r}
 3|4w - 1| - 5 = 10 \\
 \underline{+5 \quad +5} \\
 3|4w - 1| = 15
 \end{array}$$

Divide both sides by 3. Now the absolute value is isolated.

$$\underline{3|4w - 1| = 15}$$

3      3

$$|4w - 1| = 5$$

Write the two equations that will give an absolute value of 5 and solve them.

$$4w - 1 = 5 \quad \text{or} \quad 4w - 1 = -5$$

$$\begin{array}{r} +1 \quad +1 \\ \hline 4w = 6 \end{array} \quad \begin{array}{r} +1 \quad +1 \\ \hline 4w = -4 \end{array}$$

$$\frac{4w}{4} = \frac{6}{4} \quad \frac{4w}{4} = \frac{-4}{4}$$

$$w = \frac{3}{2} \quad w = -1$$

$$w = \frac{3}{2} \quad \text{or} \quad -1$$

Check the solutions in the original equation.

$$3|4w - 1| - 5 = 10 \quad 3|4w - 1| - 5 = 10$$

$$3 \left| 4 \left( \frac{3}{2} \right) - 1 \right| - 5 = 10 \quad 3|4w - 1| - 5 = 10$$

$$3 \left| \frac{12}{2} - 1 \right| - 5 = 10 \quad 3|4(-1) - 1| - 5 = 10$$

$$3|6 - 1| - 5 = 10 \quad 3|-4 - 1| - 5 = 10$$

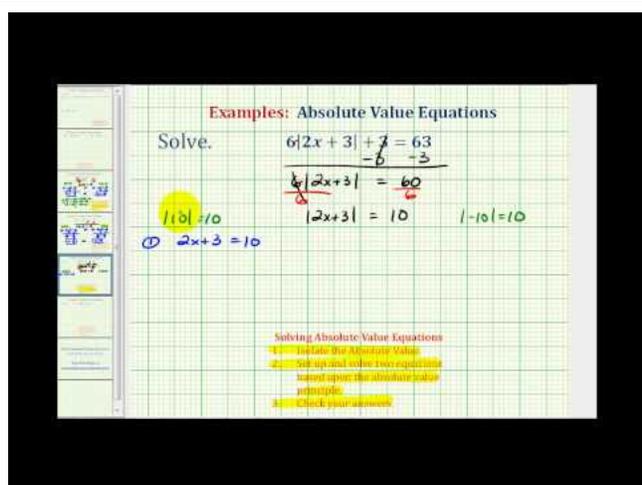
$$3(5) - 5 = 10 \quad 3|-5| - 5 = 10$$

$$15 - 5 = 10 \quad 15 - 5 = 10$$

$$10 = 10 \quad 10 = 10$$

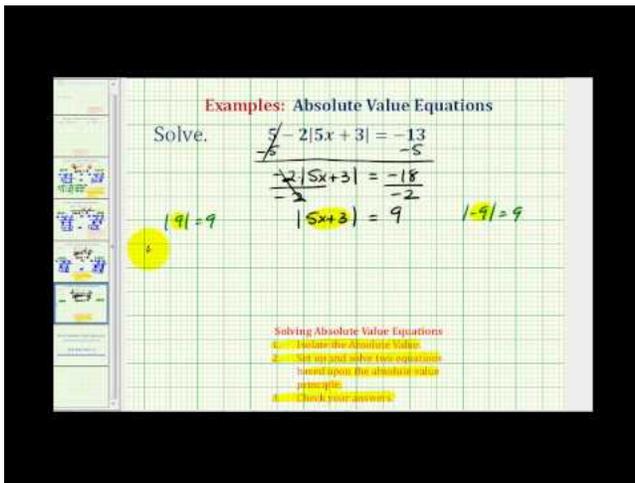
Both solutions check!

In the two videos that follow, we show examples of how to solve an absolute value equation that requires you to isolate the absolute value first using mathematical operations.



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# SOLVE EQUATIONS WITH THE DISTRIBUTIVE PROPERTY

## Learning Outcomes

- Use the properties of equality and the distributive property to solve equations containing parentheses
- Clear fractions and decimals from equations to make them easier to solve

## The Distributive Property

As we solve linear equations, we often need to do some work to write the linear equations in a form we are familiar with solving. This section will focus on manipulating an equation we are asked to solve in such a way that we can use the skills we learned for solving multi-step equations to ultimately arrive at the solution.

Parentheses can make solving a problem difficult, if not impossible. To get rid of these unwanted parentheses we have the distributive property. Using this property we multiply the number in front of the parentheses by each term inside of the parentheses.

## THE DISTRIBUTIVE PROPERTY OF MULTIPLICATION

For all real numbers  $a$ ,  $b$ , and  $c$ ,  $a(b + c) = ab + ac$ .

What this means is that when a number multiplies an expression inside parentheses, you can distribute the multiplication to each term of the expression individually. Then, you can follow the steps we have already practiced to **isolate the variable** and solve the equation.

### Example

Solve for  $a$ .  $4(2a + 3) = 28$

**Answer**

Apply the distributive property to expand  $4(2a + 3)$  to  $8a + 12$

$$4(2a + 3) = 28$$

$$8a + 12 = 28$$

Subtract 12 from both sides to isolate the variable term.

$$8a + 12 = 28$$

$$\begin{array}{r} -12 \\ -12 \\ \hline \end{array}$$

$$8a = 16$$

Divide both terms by 8 to get a coefficient of 1.

$$\frac{8a}{8} = \frac{16}{8}$$

$$8 \quad 8$$

$$a = 2$$

In the video that follows, we show another example of how to use the distributive property to solve a multi-step linear equation.

**Solving Multi-Step Equations**

EXAMPLE

Solve  $4(2m - 8) = 24$ .

$$4 \cdot 2m - 4 \cdot 8 = 24$$
$$8m - 32 = 24$$
$$\begin{array}{r} +32 \\ +32 \\ \hline \end{array}$$

**Solving Multi-Step Equations**

1. (Optional) Multiply to clear any fractions or decimals.
2. Simplify each side by clearing parentheses and combining like terms.
3. Add or subtract to isolate the variable term – you may have to move a term with a variable.
4. Multiply or divide to isolate the variable.
5. Check the solution.

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In the next example, you will see that there are parentheses on both sides of the equal sign, so you will need to use the distributive property twice. Notice that you are going to need to distribute a negative number, so be careful with negative sign!

### Example

Solve for  $t$ .  $2(4t - 5) = -3(2t + 1)$

#### Answer

Apply the distributive property to expand  $2(4t - 5)$  to  $8t - 10$  and  $-3(2t + 1)$  to  $-6t - 3$ . Be careful in this step—you are distributing a negative number, so keep track of the sign of each number after you multiply.

$$2(4t - 5) = -3(2t + 1)$$

$$8t - 10 = -6t - 3$$

Add  $6t$  to both sides to begin combining like terms.

$$8t - 10 = -6t - 3$$

$$\begin{array}{r} +6t \quad \quad \quad +6t \\ \hline \end{array}$$

$$14t - 10 = -3$$

Add 10 to both sides of the equation to isolate  $t$ .

$$14t - 10 = -3$$

$$\begin{array}{r} +10 \quad +10 \\ \hline \end{array}$$

$$14t = 7$$

The last step is to divide both sides by 14 to completely isolate  $t$ .

$$\frac{14t}{14} = \frac{7}{14}$$

We simplify the fraction  $\frac{7}{14}$  into the final answer of  $t = \frac{1}{2}$

In the following video, we solve another multi-step equation with two sets of parentheses.

**Solving Multi-Step Equations**

EXAMPLE

Solve  $3(2t - 8) = -2(t + 2)$ .

$$6t - 24 = -2t - 4$$
$$\begin{array}{r} +2t \quad \quad \quad +2t \\ \hline 8t - 24 = -4 \end{array}$$
$$\begin{array}{r} +24 \quad \quad \quad +24 \\ \hline 8t = 20 \end{array}$$
$$t = \frac{20}{8}$$

Solving Multi-Step Equations

1. (Optional) Multiply to clear any fractions or decimals.
2. Simplify each side by clearing parentheses and combining like terms.
3. Add or subtract to isolate the variable term — you may have to move a term with a variable.
4. Multiply or divide to isolate the variable.
5. Check the solution.

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Sometimes, you will encounter a multi-step equation with fractions. If you prefer not working with fractions, you can use the multiplication property of equality to multiply both sides of the equation by a common denominator of all of the fractions in the equation. This will clear all the fractions out of the equation. See the example below.

### Example

Solve  $\frac{1}{2}x - 3 = 2 - \frac{3}{4}x$  by clearing the fractions in the equation first.

**Answer**

Multiply both sides of the equation by 4, the common denominator of the fractional coefficients.

$$\frac{1}{2}x - 3 = 2 - \frac{3}{4}x$$

$$4\left(\frac{1}{2}x - 3\right) = 4\left(2 - \frac{3}{4}x\right)$$

Use the distributive property to expand the expressions on both sides. Multiply.

$$4\left(\frac{1}{2}x\right) - 4(3) = 4(2) - 4\left(-\frac{3}{4}x\right)$$

$$\frac{4}{2}x - 12 = 8 - \frac{12}{4}x$$

$$2x - 12 = 8 - 3x$$

Add  $3x$  to both sides to move the variable terms to only one side.

$$2x - 12 = 8 - 3x$$

$$\begin{array}{r} +3x \qquad \qquad +3x \\ \hline \end{array}$$

$$5x - 12 = 8$$

Add 12 to both sides to move the **constant** terms to the other side.

$$5x - 12 = 8$$

$$\begin{array}{r} +12 +12 \\ \hline \end{array}$$

$$5x = 20$$

Divide to isolate the variable.

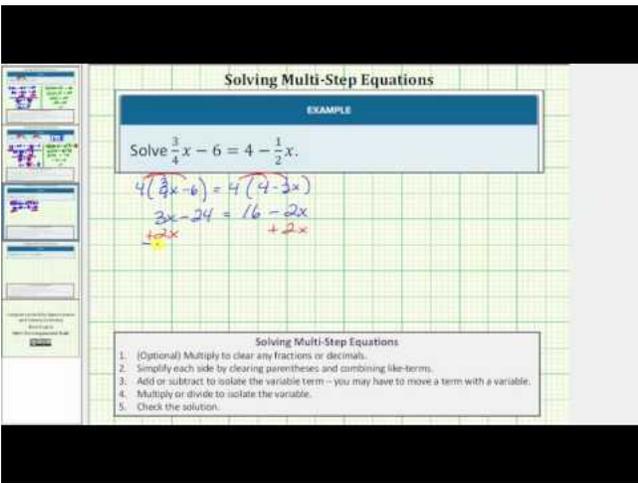
$$\underline{5x = 20}$$

$$\begin{array}{r} 5 \quad 5 \\ \hline \end{array}$$

$$x = 4$$

Of course, if you like to work with fractions, you can just apply your knowledge of operations with fractions and solve.

In the following video, we show how to solve a multi-step equation with fractions.



The screenshot shows a video player interface. The title is "Solving Multi-Step Equations". Below the title, there is a blue box labeled "EXAMPLE" containing the equation: "Solve  $\frac{3}{4}x - 6 = 4 - \frac{1}{2}x$ ". Below the equation, there are handwritten steps:  $4(\frac{3}{4}x - 6) = 4(4 - \frac{1}{2}x)$ ,  $3x - 24 = 16 - 2x$ , and  $+2x$  on both sides. Below the steps, there is a box titled "Solving Multi-Step Equations" with a list of five steps: 1. (Optional) Multiply to clear any fractions or decimals. 2. Simplify each side by clearing parentheses and combining like terms. 3. Add or subtract to isolate the variable term -- you may have to move a term with a variable. 4. Multiply or divide to isolate the variable. 5. Check the solution.

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Regardless of which method you use to solve equations containing variables, you will get the same answer. You can choose the method you find the easiest! Remember to check your answer by substituting your solution into the original equation.

Sometimes, you will encounter a multi-step equation with decimals. If you prefer not working with decimals, you can use the multiplication property of equality to multiply both sides of the equation by a factor of 10 that will help clear the decimals. See the example below.

**Example**

Solve  $3y + 10.5 = 6.5 + 2.5y$  by clearing the decimals in the equation first.

**Answer**

Since the smallest decimal place value represented in the equation is 0.10, we want to multiply by 10 to clear the decimals from the equation.

$$3y + 10.5 = 6.5 + 2.5y$$
$$10(3y + 10.5) = 10(6.5 + 2.5y)$$

Use the distributive property to expand the expressions on both sides.

$$10(3y) + 10(10.5) = 10(6.5) + 10(2.5y)$$

Multiply.

$$30y + 105 = 65 + 25y$$

Move the smaller variable term,  $25y$ , by subtracting it from both sides.

$$\begin{array}{r} 30y + 105 = 65 + 25y \\ -25y \qquad \qquad -25y \\ \hline 5y + 105 = 65 \end{array}$$

Subtract 105 from both sides to isolate the term with the variable.

$$5y + 105 = 65$$

$$\underline{-105 \quad -105}$$

$$5y = -40$$

Divide both sides by 5 to isolate the  $y$ .

$$\underline{5y = -40}$$

$$5 \quad 5$$

$$y = -8$$

In the following video, we show another example of clearing decimals first to solve a multi-step linear equation.

**Solving Multi-Step Equations**

EXAMPLE

Solve  $8a - 3.2 = 1.6 + 6.8a$ .

$10(8a - 3.2) = 10(1.6 + 6.8a)$   
 $80a - 32 = 16 + 68a$   
 $-68a - 32 = 16 + 68a$   
 $-68a - 32 = 16 + 68a$   
 $12a = 48$   
 $a = 4$

**Solving Multi-Step Equations**

1. (Optional) Multiply to clear any fractions or decimals.
2. Simplify each side by clearing parentheses and combining like terms.
3. Add or subtract to isolate the variable term—you may have to move a term with the variable.
4. Multiply or divide to isolate the variable.
5. Check the solution.

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Here are some steps to follow when you solve multi-step equations.

## SOLVING MULTI-STEP EQUATIONS

1. (Optional) Multiply to clear any fractions or decimals.
2. Simplify each side by clearing parentheses and combining like terms.
3. Add or subtract to isolate the variable term—you may have to move a term with the variable.
4. Multiply or divide to isolate the variable.
5. Check the solution.

## Summary

Complex, multi-step equations often require multi-step solutions. Before you can begin to isolate a variable, you may need to simplify the equation first. This may mean using the distributive property to remove parentheses or multiplying both sides of an equation by a common denominator to get rid of fractions. Sometimes it requires both techniques. If your multi-step equation has an absolute value, you will need to solve two equations, sometimes isolating the absolute value expression first.

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# CLASSIFY SOLUTIONS TO LINEAR EQUATIONS

## Learning Outcomes

- Solve equations that have one solution, no solution, or an infinite number of solutions
- Recognize when a linear equation that contains absolute value does not have a solution

There are three cases that can come up as we are solving linear equations. We have already seen one, where an equation has one solution. Sometimes we come across equations that do not have any solutions and even some that have an infinite number of solutions. The case where an equation has no solution is illustrated in the next example.

## Equations with No Solutions

### Example

Solve for  $x$ .  $12 + 2x - 8 = 7x + 5 - 5x$

**Answer**

Combine **like terms** on both sides of the equation.

$$\begin{aligned}12 + 2x - 8 &= 7x + 5 - 5x \\2x + 4 &= 2x + 5\end{aligned}$$

Isolate the  $x$  term by subtracting  $2x$  from both sides.

$$\begin{array}{r}2x + 4 = 2x + 5 \\ -2x \quad -2x \\ \hline 4 = 5\end{array}$$

This false statement implies there are **no solutions** to this equation. Sometimes, we say the solution does not exist, or DNE for short.

This is *not* a solution! You did *not* find a value for  $x$ . Solving for  $x$  the way you know how, you arrive at the false statement  $4 = 5$ . Surely 4 cannot be equal to 5!

This may make sense when you consider the second line in the solution where like terms were combined. If you multiply a number by 2 and add 4 you would never get the same answer as when you multiply that same

number by 2 and add 5. Since there is no value of  $x$  that will ever make this a true statement, the solution to the equation above is “no solution.”

Be careful that you do not confuse the solution  $x = 0$  with “no solution.” The solution  $x = 0$  means that the value 0 satisfies the equation, so there *is* a solution. “No solution” means that there is no value, not even 0, which would satisfy the equation.

Also, be careful not to make the mistake of thinking that the equation  $4 = 5$  means that 4 and 5 are values for  $x$  that are solutions. If you substitute these values into the original equation, you’ll see that they do not satisfy the equation. This is because there is truly *no solution*—there are no values for  $x$  that will make the equation  $12 + 2x - 8 = 7x + 5 - 5x$  true.

### Think About It

Try solving these equations. How many steps do you need to take before you can tell whether the equation has no solution or one solution?

a) Solve  $8y = 3(y + 4) + y$

Use the textbox below to record how many steps you think it will take before you can tell whether there is no solution or one solution.

#### Answer

Solve  $8y = 3(y + 4) + y$

First, distribute the 3 into the parentheses on the right-hand side.

$$8y = 3y + 12 + y$$

Next, begin combining like terms.

$$8y = 4y + 12$$

Now move the variable terms to one side. Moving the  $4y$  will help avoid a negative sign.

$$8y = 4y + 12$$

$$\underline{-4y \quad -4y}$$

$$4y = 12$$

Now, divide each side by  $4y$ .

$$\frac{4y}{4} = \frac{12}{4}$$

$$y = 3$$

Because we were able to isolate  $y$  on one side and a number on the other side, we have one solution to this equation.

b) Solve  $2(3x - 5) - 4x = 2x + 7$

Use the textbox below to record how many steps you think it will take before you can tell whether there is no solution or one solution.

#### Answer

Solve  $2(3x - 5) - 4x = 2x + 7$ .

First, distribute the 2 into the parentheses on the left-hand side.

$$6x - 10 - 4x = 2x + 7$$

Now begin simplifying. You can combine the  $x$  terms on the left-hand side.

$$2x - 10 = 2x + 7$$

Now, take a moment to ponder this equation. It says that  $2x - 10$  is equal to  $2x + 7$ . Can some number times two minus 10 be equal to that same number times two plus seven?

Pretend  $x = 3$ .

Is it true that  $2(3) - 10 = -4$  is equal to  $2(3) + 7 = 13$ . NO! We do not even really need to continue solving the equation, but we can just to be thorough.

Add 10 to both sides.

$$2x - 10 = 2x + 7$$

$$\begin{array}{r} +10 \quad +10 \\ \hline \end{array}$$

$$2x = 2x + 17$$

Now subtract  $2x$  from both sides.

$$2x = 2x + 17$$

$$\begin{array}{r} -2x \quad -2x \\ \hline \end{array}$$

$$0 = 17$$

We know that 0 and 17 are not equal, so there is no number that  $x$  could be to make this equation true.

This false statement implies there are **no solutions** to this equation, or DNE (does not exist) for short.

## Algebraic Equations with an Infinite Number of Solutions

You have seen that if an equation has no solution, you end up with a false statement instead of a value for  $x$ . It is possible to have an equation where any value for  $x$  will provide a solution to the equation. In the example below, notice how combining the terms  $5x$  and  $-4x$  on the left leaves us with an equation with exactly the same terms on both sides of the equal sign.

### Example

Solve for  $x$ .  $5x + 3 - 4x = 3 + x$

**Answer**

Combine like terms on both sides of the equation.

$$5x + 3 - 4x = 3 + x$$

$$x + 3 = 3 + x$$

Isolate the  $x$  term by subtracting  $x$  from both sides.

$$x + 3 = 3 + x$$

$$\begin{array}{r} -x \quad -x \\ \hline \end{array}$$

$$3 = 3$$

This true statement implies there are an infinite number of solutions to this equation, or we can also write the solution as “All Real Numbers”

You arrive at the true statement “ $3 = 3$ .” When you end up with a true statement like this, it means that the solution to the equation is “all real numbers.” Try substituting  $x = 0$  into the original equation—you will get a true statement! Try  $x = -\frac{3}{4}$  and it will also check!

This equation happens to have an infinite number of solutions. Any value for  $x$  that you can think of will make this equation true. When you think about the context of the problem, this makes sense—the equation  $x + 3 = 3 + x$  means “some number plus 3 is equal to 3 plus that same number.” We know that this is always true—it is the commutative property of addition!

### Example

Solve for  $x$ .  $3(2x - 5) = 6x - 15$

**Answer**

Distribute the 3 through the parentheses on the left-hand side.

$$3(2x - 5) = 6x - 15$$

$$6x - 15 = 6x - 15$$

Wait! This looks just like the previous example. You have the same expression on both sides of an equal sign. No matter what number you choose for  $x$ , you will have a true statement. We can finish the algebra:

$$6x - 15 = 6x - 15$$

$$\underline{-6x} \quad \quad \underline{-6x}$$

$$-15 = -15$$

This true statement implies there are an infinite number of solutions to this equation.

In the following video, we show more examples of attempting to solve a linear equation with either no solution or many solutions.

The screenshot shows a video player with a green grid background. At the top, the title is "Classifying Solutions to Linear Equations". Below that, it says "EXAMPLE" and "Solve  $9 + 4x - 5 = 8x - 4x + 14$ ". The video shows handwritten work:  $4x + 4 = 4x + 14$ , then  $-4x$  is subtracted from both sides, resulting in  $4 = 14$ , which is labeled "False". Below this, it says "The equation has no solutions. (DNE)". At the bottom, there is a list of steps for "Solving Multi-Step Equations":

1. (Optional) Multiply to clear any fractions or decimals.
2. Simplify each side by clearing parentheses and combining like terms.
3. Add or subtract to isolate the variable term – you may have to move a term with a variable.
4. Multiply or divide to isolate the variable.
5. Check the solution.

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In the following video, we show more examples of solving linear equations with parentheses that have either no solution or many solutions.

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## Absolute Value Equations with No Solutions

As we are solving absolute value equations, it is important to be aware of special cases. An absolute value is defined as the distance from 0 on a number line, so it must be a positive number. When an absolute value expression is equal to a negative number, we say the equation has no solution, or DNE. Notice how this happens in the next two examples.

### Example

Solve for  $x$ .  $7 + |2x - 5| = 4$

**Answer**

Notice absolute value is not alone. Subtract 7 from each side to isolate the absolute value.

$$\begin{array}{r} 7 + |2x - 5| = 4 \\ -7 \qquad \qquad -7 \\ \hline |2x - 5| = -3 \end{array}$$

Result of absolute value is negative! The result of an absolute value must always be positive, so we say there is no solution to this equation, or DNE.

### Example

Solve for  $x$ .  $-\frac{1}{2}|x + 3| = 6$

**Answer**

Notice absolute value is not alone. Multiply both sides by the reciprocal of  $-\frac{1}{2}$ , which is  $-2$ .

$$\begin{array}{r} -\frac{1}{2}|x + 3| = 6 \\ (-2) - \frac{1}{2}|x + 3| = (-2) 6 \end{array}$$

$$|x + 3| = -12$$

Again, we have a result where an absolute value is negative!

There is no solution to this equation, or DNE.

In this last video, we show more examples of absolute value equations that have no solutions.

**Classifying Solutions to Absolute Value Equations**

**EXAMPLE**

Solve  $11 + |3x + 2| = 8$

$11 + |3x + 2| = 8$   
 $-11$   
 $|3x + 2| = -3$

the equation has  
no solution.

$-3$   
 $3$

The **absolute value** of a number or expression describes its distance from 0 on a number line. Since the absolute value expresses only the distance, not the direction of the number on a number line, it is always expressed as a positive number or 0.

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## Summary

We have seen that solutions to equations can fall into three categories:

- One solution
- No solution, DNE (does not exist)
- Many solutions, also called infinitely many solutions or All Real Numbers

Keep in mind that sometimes we do not need to do much algebra to see what the outcome will be.

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# INTRODUCTION TO PROBLEM SOLVING

What You Will Learn to Do: Develop and Use a Method for Solving Mathematical Problems

Many real-world applications can be modeled by linear equations. For example, a cell phone package may include a monthly service fee plus an additional charge if you exceed your data plan; a car rental company charges a daily fee plus an amount per mile driven; a restaurant offers a base price for a burrito plus additional charges for extra toppings, like guacamole and sour cream. These are examples of applications we come across every day that are modeled by linear equations. In this section, we will set up and use linear equations to solve such problems.

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# SET UP A LINEAR EQUATION TO SOLVE AN APPLICATION

## LEARNING OUTCOMES

- Translate words into algebraic expressions and equations
- Define a process for solving word problems

To set up or model a linear equation to fit a real-world application, we must first determine the known quantities and define the unknown quantity as a variable. Then, we begin to interpret the words as mathematical expressions using mathematical symbols. Let us use an example of a car rental company. The company charges \$0.10/*mi* in addition to a flat rate. In this case, a known cost, such as \$0.10/*mi*, is multiplied by an unknown quantity, the number of miles driven. Therefore, we can write  $0.10x$ . This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it according to the problem. As these amounts do not change, we call them fixed costs. Consider a car rental agency that charges \$0.10/*mi* plus a daily fee of \$50. We can use these quantities to model an equation that can be used to find the daily car rental cost  $C$ .

$$C = 0.10x + 50$$

When dealing with real-world applications, there are certain expressions that we can translate directly into math. The table lists some common verbal expressions and their equivalent mathematical expressions.

Verbal	Translation to Math Operations
One number exceeds another by $a$	$x, x + a$
Twice a number	$2x$
One number is $a$ more than another number	$x, x + a$
One number is $a$ less than twice another number	$x, 2x - a$
The product of a number and $a$ , decreased by $b$	$ax - b$
The quotient of a number and the number plus $a$ is three times the number	$\frac{x}{x + a} = 3x$
The product of three times a number and the number decreased by $b$ is $c$	$3x(x - b) = c$

## HOW TO: GIVEN A REAL-WORLD PROBLEM, MODEL A LINEAR EQUATION TO FIT IT

1. Identify known quantities.
2. Assign a variable to represent the unknown quantity.
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words as mathematical operations.
5. Solve the equation. Be sure to explain the solution in words including the units of measure.

### EXAMPLE

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31. Find the two numbers.

#### Answer

Let  $x$  equal the first number. Then, as the second number exceeds the first by 17, we can write the second number as  $x + 17$ . The sum of the two numbers is 31. We usually interpret the word *is* as an equal sign.

$$x + (x + 17) = 31$$

$$2x + 17 = 31 \quad \text{Simplify and solve.}$$

$$2x = 14$$

$$x = 7$$

The second number would then be  $x + 17 = 7 + 17 = 24$

The two numbers are 7 and 24.

In the following video, we show another example of how to translate an expression in English into a mathematical equation that can then be solved.

In the next example we will write equations that will help us compare cell phone plans.

The following video shows another example of writing two equations that will allow you to compare two different cell phone plans.

**Writing and Solving Linear Equation**

**EXAMPLE**

Find a linear equation to solve for the following unknown quantities. One number is less than another by 21. Their sum is 51. Find the numbers.

$x =$  the larger number  
 $x - 21 =$  the number 21 less than the larger number

$$x + (x - 21) = 51$$
$$x + x - 21 = 51$$
$$2x - 21 = 51$$
$$\quad \quad \quad +21 \quad +21$$

---

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## EXAMPLE

There are two cell phone companies that offer different packages. Company A charges a monthly service fee of \$34 plus \$.05/*min* talk-time. Company B charges a monthly service fee of \$40 plus \$.04/*min* talk-time.

1. Write a linear equation that models the packages offered by both companies.
2. If the average number of minutes used each month is 1,160, which company offers the better plan?
3. If the average number of minutes used each month is 420, which company offers the better plan?
4. How many minutes of talk-time would yield equal monthly statements from both companies?

### Answer

1. The model for Company A can be written as  $A = 0.05x + 34$ . This includes the variable cost of  $0.05x$  plus the monthly service charge of \$34. Company B's package charges a higher monthly fee of \$40, but a lower variable cost of  $0.04x$ . Company B's model can be written as  $B = 0.04x + 40$ .
2. If the average number of minutes used each month is 1,160, we have the following:

$$\begin{aligned}\text{Company A} &= 0.05(1,160) + 34 \\ &= 58 + 34 \\ &= 92\end{aligned}$$

$$\begin{aligned}\text{Company B} &= 0.04(1,160) + 40 \\ &= 46.4 + 40 \\ &= 86.4\end{aligned}$$

So, Company B offers the lower monthly cost of \$86.40 as compared with the \$92 monthly cost offered by Company A when the average number of minutes used each month is 1,160.

3. If the average number of minutes used each month is 420, we have the following:

$$\begin{aligned}\text{Company A} &= 0.05(420) + 34 \\ &= 21 + 34 \\ &= 55\end{aligned}$$

$$\begin{aligned}\text{Company B} &= 0.04(420) + 40 \\ &= 16.8 + 40 \\ &= 56.8\end{aligned}$$

If the average number of minutes used each month is 420, then Company A offers a lower monthly cost of \$55 compared to Company B's monthly cost of \$56.80.

4. To answer the question of how many talk-time minutes would yield the same bill from both companies, we should think about the problem in terms of  $(x, y)$  coordinates: At what point are both the  $x$ -value and the  $y$ -value equal? We can find this point by setting the equations equal to each other and solving for  $x$ .

$$0.05x + 34 = 0.04x + 40$$

$$0.01x = 6$$

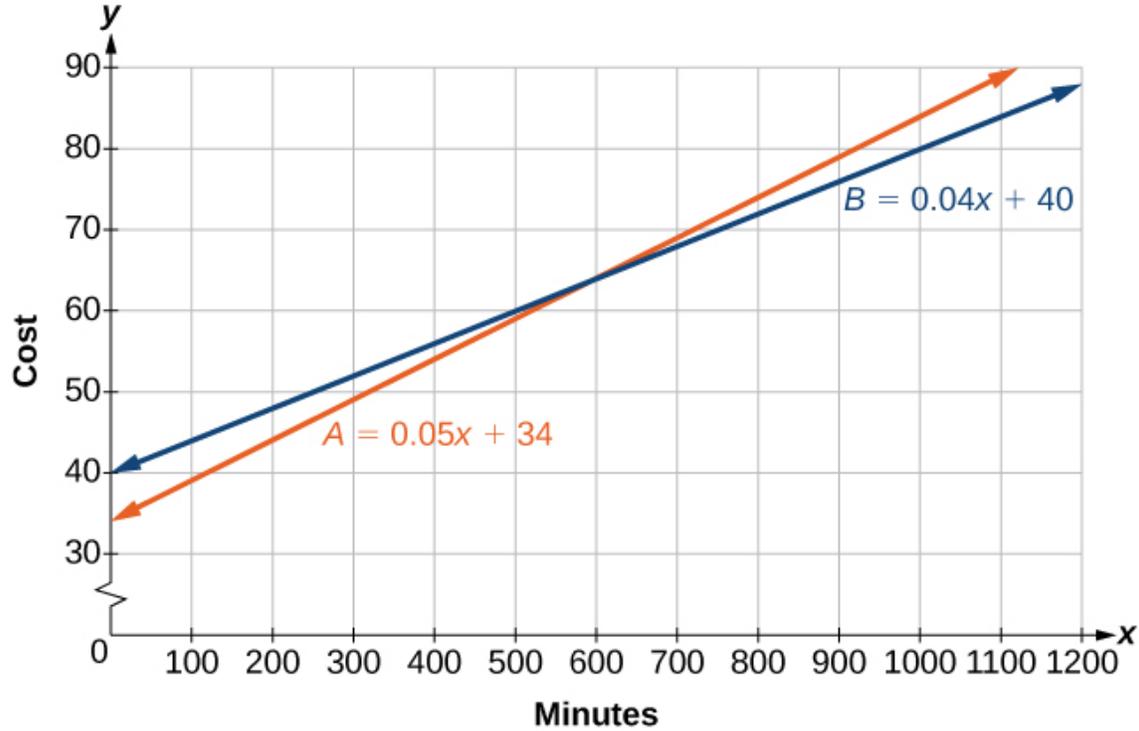
$$x = 600$$

Check the  $x$ -value in each equation.

$$0.05(600) + 34 = 64$$

$$0.04(600) + 40 = 64$$

Therefore, a monthly average of 600 talk-time minutes renders the plans equal.



A graph representing the relationship between cell phone plan cost and minutes used.

**Writing and Solving Linear Equation**

**EXAMPLE**

Find a linear equation to solve for the following unknown quantities. One number is less than another by 21. Their sum is 51. Find the numbers.

$x =$  the larger number  
 $x - 21 =$  the number 21 less than the larger number

$$x + (x - 21) = 51$$

$$x + x - 21 = 51$$

$$2x - 21 = 51$$

$$\underline{\quad + 21 \quad + 21}$$

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# SOLVE AN APPLICATION USING A FORMULA

## LEARNING OUTCOMES

- Solve distance, rate, and time problems
- Solve area, volume, and perimeter problems
- Solve temperature conversion problems
- Rearrange formulas to isolate specific variables
- Identify an unknown given a formula

Many applications are solved using known formulas. The problem is stated, a formula is identified, the known quantities are substituted into the formula, the equation is solved for the unknown, and the problem's question is answered. Typically, these problems involve two equations representing two trips, two investments, two areas, and so on. Examples of formulas include the **area** of a rectangular region,  $A = LW$ ; the **perimeter** of a rectangle,  $P = 2L + 2W$ ; and the **volume** of a rectangular solid,  $V = LWH$ . When there are two unknowns, we find a way to write one in terms of the other because we can solve for only one variable at a time.

## Distance, Rate, and Time

In the next example, we will find the length and width of a rectangular field given its perimeter and a relationship between its sides.

The following video shows an example of finding the dimensions of a rectangular field given its perimeter.

The following video shows another example of finding the area of a rectangle given its perimeter and the relationship between its side lengths.

## Volume

In the following video we show how to find the volume of a right circular cylinder formed from a given rectangle.

## Isolate Variables in Formulas

Sometimes, it is easier to isolate the variable you are solving for when you are using a formula. This is especially helpful if you have to perform the same calculation repeatedly, or you are having a computer perform the calculation repeatedly. In the next examples, we will use algebraic properties to isolate a variable in a formula.

## Temperature

Let us look at another formula that includes parentheses and fractions, the formula for converting from the Fahrenheit temperature scale to the Celsius scale.

$$C = (F - 32) \cdot \frac{5}{9}$$

## EXAMPLE

It takes Andrew 30 min to drive to work in the morning. He drives home using the same route, but it takes 10 min longer and he averages  $10mi/h$  less than in the morning. How far does Andrew drive to work?

### Answer

This is a distance problem, so we can use the formula  $d = rt$ , where distance equals rate multiplied by time. Note that when rate is given in  $mi/h$ , time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.

First, we identify the known and unknown quantities. Andrew's morning drive to work takes 30 min, or  $\frac{1}{2}$  h at rate  $r$ . His drive home takes 40 min, or  $\frac{2}{3}$  h, and his speed averages  $10mi/h$  less than the morning drive. Both trips cover distance  $d$ . A table, such as the one below, is often helpful for keeping track of information in these types of problems.

	$d$	$r$	$t$
<b>To Work</b>	$d$	$r$	$\frac{1}{2}$
<b>To Home</b>	$d$	$r - 10$	$\frac{2}{3}$

Write two equations, one for each trip.

$$d = r \left( \frac{1}{2} \right) \quad \text{To work}$$

$$d = (r - 10) \left( \frac{2}{3} \right) \quad \text{To home}$$

As both equations equal the same distance, we set them equal to each other and solve for  $r$ .

$$r \left( \frac{1}{2} \right) = (r - 10) \left( \frac{2}{3} \right)$$

$$\frac{1}{2}r = \frac{2}{3}r - \frac{20}{3}$$

$$\frac{1}{2}r - \frac{2}{3}r = -\frac{20}{3}$$

$$-\frac{1}{6}r = -\frac{20}{3}$$

$$r = -\frac{20}{3}(-6)$$

$$r = 40$$

We have solved for the rate of speed to work,  $40mph$ . Substituting 40 into the rate on the return trip yields  $30mi/h$ . Now we can answer the question. Substitute the rate back into either equation and solve for  $d$ .

$$d = 40 \left( \frac{1}{2} \right)$$

$$= 20$$

The distance between home and work is  $20mi$ . Note that we could have cleared the fractions in the equation by multiplying both sides of the equation by the LCD to solve for  $r$ .

$$r \left( \frac{1}{2} \right) = (r - 10) \left( \frac{2}{3} \right)$$

$$6 \times r \left( \frac{1}{2} \right) = 6 \times (r - 10) \left( \frac{2}{3} \right)$$

$$3r = 4(r - 10)$$

$$3r = 4r - 40$$

$$-r = -40$$

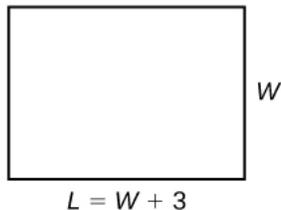
$$r = 40$$

## EXAMPLE

The perimeter of a rectangular outdoor patio is 54 ft. The length is 3 ft greater than the width. What are the dimensions of the patio?

### Answer

The perimeter formula is standard:  $P = 2L + 2W$ . We have two unknown quantities, length and width. However, we can write the length in terms of the width as  $L = W + 3$ . Substitute the perimeter value and the expression for length into the formula. It is often helpful to make a sketch and label the sides.



Now we can solve for the width and then calculate the length.

$$P = 2L + 2W$$

$$54 = 2(W + 3) + 2W$$

$$54 = 2W + 6 + 2W$$

$$54 = 4W + 6$$

$$48 = 4W$$

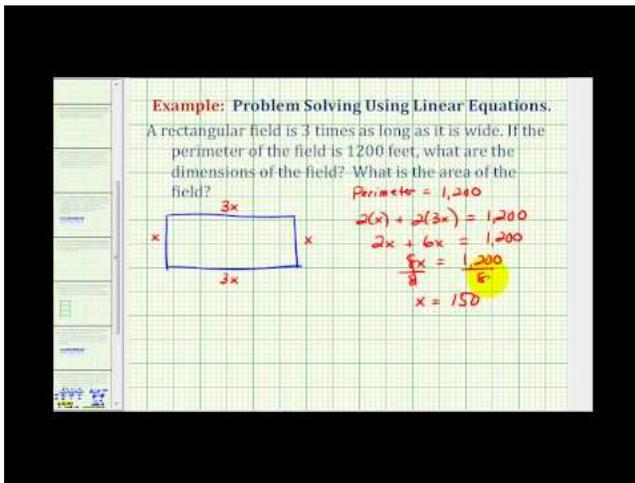
$$12 = W$$

$$(12 + 3) = L$$

$$15 = L$$

The dimensions are  $L = 15$  ft and  $W = 12$  ft.

As with the other formulas we have worked with, we could have isolated the variable  $F$  first then substituted in the given temperature in Celsius.



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## EXAMPLE

The perimeter of a tablet of graph paper is 48 in. The length is 6 in. more than the width. Find the area of the graph paper.

### Answer

The standard formula for area is  $A = LW$ ; however, we will solve the problem using the perimeter formula. The reason we use the perimeter formula is because we know enough information about the perimeter that the formula will allow us to solve for one of the unknowns. As both perimeter and area use length and width as dimensions, they are often used together to solve a problem such as this one.

We know that the length is 6 in. more than the width, so we can write length as  $L = W + 6$ . Substitute the value of the perimeter and the expression for length into the perimeter formula and find the length.

$$P = 2L + 2W$$

$$48 = 2(W + 6) + 2W$$

$$48 = 2W + 12 + 2W$$

$$48 = 4W + 12$$

$$36 = 4W$$

$$9 = W$$

$$(9 + 6) = L$$

$$15 = L$$

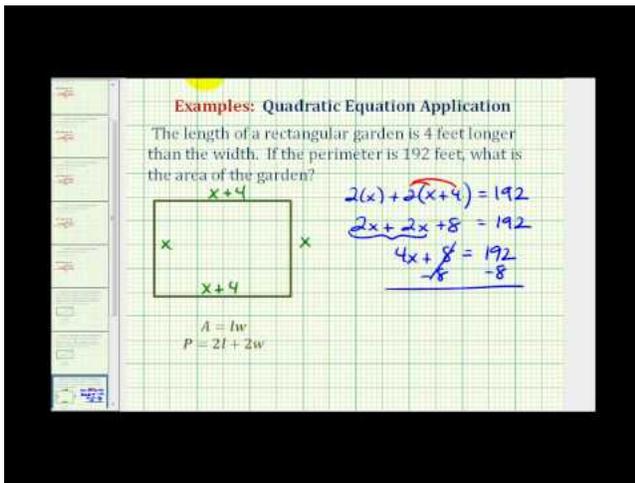
Now, we find the area given the dimensions of  $L = 15$  in. and  $W = 9$  in.

$$A = LW$$

$$A = 15(9)$$

$$= 135 \text{ in}^2$$

The area is  $135 \text{ in}^2$ .



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## EXAMPLE

Find the dimensions of a shipping box given that the length is twice the width, the height is 8 inches, and the volume is 1,600 in.<sup>3</sup>.

### Answer

The formula for the volume of a box is given as  $V = LWH$ , the product of length, width, and height. We are given that  $L = 2W$ , and  $H = 8$ . The volume is 1,600 cubic inches.

$$V = LWH$$

$$1,600 = (2W)W(8)$$

$$1,600 = 16W^2$$

$$100 = W^2$$

$$10 = W$$

The dimensions are  $L = 20$  in.,  $W = 10$  in., and  $H = 8$  in. Note that the square root of  $W^2$  would result in a positive and a negative value. However, because we are describing width, we can use only the positive result.

## THINK ABOUT IT

Express the formula for the surface area of a cylinder,  $s = 2\pi rh + 2\pi r^2$ , in terms of the height,  $h$ .

In this example, the variable  $h$  is buried pretty deeply in the formula for surface area of a cylinder. Using the order of operations, it can be isolated. Before you look at the solution, use the box below to write down what you think is the best first step to take to isolate  $h$ .

### Answer

Isolate the term containing the variable  $h$  by subtracting  $2\pi r^2$  from both sides.

$$S = 2\pi rh + 2\pi r^2$$

$$\begin{array}{r} -2\pi r^2 \\ \hline \end{array} \quad \begin{array}{r} -2\pi r^2 \\ \hline \end{array}$$

$$S - 2\pi r^2 = 2\pi rh$$

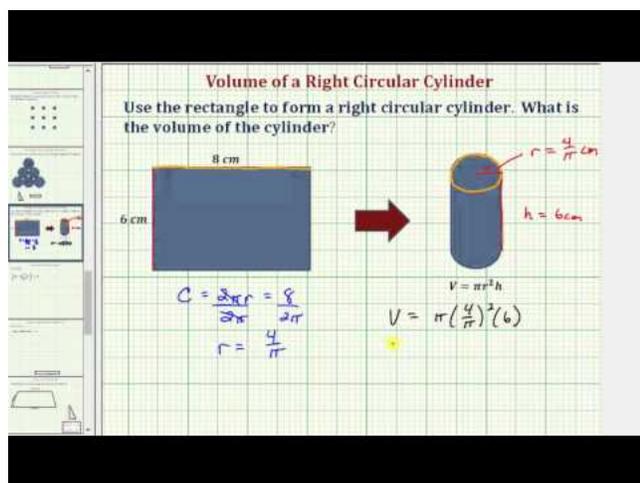
Next, isolate the variable  $h$  by dividing both sides of the equation by  $2\pi r$ .

$$\frac{S - 2\pi r^2}{2\pi r} = \frac{2\pi r h}{2\pi r}$$

$$\frac{S - 2\pi r^2}{2\pi r} = h$$

You can rewrite the equation so the isolated variable is on the left side.

$$h = \frac{S - 2\pi r^2}{2\pi r}$$



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## EXAMPLE

Isolate the term containing the variable  $w$  from the formula for the perimeter of a rectangle:

$$P = 2(L) + 2(W).$$

**Answer**

First, isolate the term with  $w$  by subtracting  $2l$  from both sides of the equation.

$$P = 2l + 2w$$

$$\frac{-2l \quad -2l}{P - 2l = \quad 2w}$$

Next, clear the coefficient of  $w$  by dividing both sides of the equation by 2.

$$\frac{P - 2l}{2} = \frac{2w}{2}$$

$$\frac{P - 2l}{2} = w$$

$$w = \frac{P - 2l}{2}$$

You can rewrite the equation so the isolated variable is on the left side.

$$w = \frac{P - 2l}{2}$$

## EXAMPLE

Use the multiplication and division properties of equality to isolate the variable  $b$  given  $A = \frac{1}{2}bh$

**Answer**

$$A = \frac{1}{2}bh$$

$$(2) A = (2) \frac{1}{2}bh$$

$$2A = bh$$

$$\frac{2A}{h} = \frac{bh}{h}$$

$$\frac{2A}{h} = \frac{b\cancel{h}}{\cancel{h}}$$

Write the equation with the desired variable on the left-hand side as a matter of convention:

$$b = \frac{2A}{h}$$

Use the multiplication and division properties of equality to isolate the variable  $h$  given  $A = \frac{1}{2}bh$

**Answer**

$$A = \frac{1}{2}bh$$

$$(2) A = (2) \frac{1}{2}bh$$

$$2A = bh$$

$$\frac{2A}{b} = \frac{bh}{b}$$

$$\frac{2A}{b} = \frac{h\cancel{b}}{\cancel{b}}$$

Write the equation with the desired variable on the left-hand side as a matter of convention:

$$h = \frac{2A}{b}$$

## EXAMPLE

Given a temperature of  $12^{\circ}C$ , find the equivalent in  $^{\circ}F$ .

**Answer**

Substitute the given temperature in  $^{\circ}C$  into the conversion formula:

$$12 = (F - 32) \cdot \frac{5}{9}$$

Isolate the variable  $F$  to obtain the equivalent temperature.

$$12 = (F - 32) \cdot \frac{5}{9}$$

$$\left(\frac{9}{5}\right) 12 = F - 32$$

$$\left(\frac{108}{5}\right) 12 = F - 32$$

$$21.6 = F - 32$$

$$\begin{array}{r} +32 \quad \quad + 32 \\ \hline \end{array}$$

$$53.6 = ^{\circ}F$$

## EXAMPLE

Solve the formula shown below for converting from the Fahrenheit scale to the Celsius scale for F.

$$C = (F - 32) \cdot \frac{5}{9}$$

### Answer

To isolate the variable F, it would be best to clear the fraction involving F first. Multiply both sides of the equation by  $\frac{9}{5}$ .

$$\left(\frac{9}{5}\right) C = (F - 32) \left(\frac{5}{9}\right) \left(\frac{9}{5}\right)$$

$$\frac{9}{5} C = F - 32$$

Add 32 to both sides.

$$\frac{9}{5} C + 32 = F - 32 + 32$$

$$\frac{9}{5} C + 32 = F$$

$$F = \frac{9}{5} C + 32$$

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# PUTTING IT TOGETHER: REAL NUMBERS AND ALGEBRAIC EXPRESSIONS

At the start of this module, you were presented with a formula commonly used by forensic scientists to calculate blood alcohol content:

$$B = -0.015t + \left(\frac{2.84N}{Wg}\right)$$

where

- B = percentage of BAC
- t = number of hours since the first drink
- N = number of “standard drinks” (a standard drink is one 12-ounce beer, one 5-ounce glass of wine, or one 1.5-ounce shot of liquor). N should be at least 1.
- W = weight in pounds
- g = gender constant: 0.68 for men and 0.55 for women

Remember Joan? When we left her, she was at a party. She had three drinks and then wanted to leave. This was worrisome. We posed the following questions:

1. Where would she fall on the table of the progressive effects of alcohol after 1.5 hours?
2. Would she be within the legal limit to drive after this amount of time?

The table showing the progressive effects of alcohol appears at the bottom of the page.

The information needed to answer the first question includes BAC, N, W, t and g. Of these, we know N, W, t, and g—we will need to solve for BAC.

- N = 3 standard drinks
- W = 135 pounds
- t = 1.5 hours
- g = 0.55

Using these values, we can calculate Joan’s BAC, which is the first step in answering question 1.

$$B = -0.015t + \left( \frac{2.84N}{Wg} \right)$$

$$B = -0.015(1.5) + \left( \frac{2.84(3)}{(135)(0.55)} \right)$$

$$B = -0.0225 + \left( \frac{8.52}{74.25} \right)$$

$$B = -0.0225 + (0.1147)$$

$$B = 0.092$$

The units of B are a percentage, so in this scenario after 1.5 hours and after consuming three standard drinks, Joan’s BAC is 0.092%.

How is Joan doing? From the table below, we can predict that she’s experiencing blunted feelings, reduced sensitivity to pain, euphoria, disinhibition, and extroversion. In addition, her reasoning, depth perception, peripheral vision, and glare recovery are probably impaired.



Question 2 asks whether she would be within the legal limit to drive. If we assume that the legal limit is 0.08%, then NO, she would not be within the legal limit to drive. One of her friends should take her car keys and help her get home.

<b>Progressive effects of alcohol</b>		
<b>BAC (% by vol.)</b>	<b>Behavior</b>	<b>Impairment</b>
0.001– 0.029	<ul style="list-style-type: none"> <li>• Average individual appears normal</li> </ul>	<ul style="list-style-type: none"> <li>• Subtle effects that can be detected with special tests</li> </ul>
0.030– 0.059	<ul style="list-style-type: none"> <li>• Mild euphoria</li> <li>• Relaxation</li> <li>• Joyousness</li> <li>• Talkativeness</li> <li>• Decreased inhibition</li> </ul>	<ul style="list-style-type: none"> <li>• Concentration</li> </ul>
0.060– 0.099	<ul style="list-style-type: none"> <li>• Blunted feelings</li> <li>• Reduced sensitivity to pain</li> <li>• Euphoria</li> <li>• Disinhibition</li> <li>• Extroversion</li> </ul>	<ul style="list-style-type: none"> <li>• Reasoning</li> <li>• Depth perception</li> <li>• Peripheral vision</li> <li>• Glare recovery</li> </ul>
0.100– 0.199	<ul style="list-style-type: none"> <li>• Over-expression</li> <li>• Boisterousness</li> <li>• Possibility of nausea and vomiting</li> </ul>	<ul style="list-style-type: none"> <li>• Reflexes</li> <li>• Reaction time</li> <li>• Gross motor control</li> <li>• Staggering</li> <li>• Slurred speech</li> <li>• Temporary erectile dysfunction</li> </ul>
0.200– 0.299	<ul style="list-style-type: none"> <li>• Nausea</li> <li>• Vomiting</li> <li>• Emotional swings</li> <li>• Anger or sadness</li> <li>• Partial loss of understanding</li> <li>• Impaired sensations</li> <li>• Decreased libido</li> <li>• Possibility of stupor</li> </ul>	<ul style="list-style-type: none"> <li>• Severe motor impairment</li> <li>• Loss of consciousness</li> <li>• Memory blackout</li> </ul>
0.300– 0.399	<ul style="list-style-type: none"> <li>• Stupor</li> <li>• Central nervous system depression</li> <li>• Loss of understanding</li> <li>• Lapses in and out of consciousness</li> <li>• Low possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Bladder function</li> <li>• Breathing</li> <li>• Dysequilibrium</li> <li>• Heart rate</li> </ul>
0.400– 0.500	<ul style="list-style-type: none"> <li>• Severe central nervous system depression</li> <li>• Coma</li> <li>• Possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Breathing</li> <li>• Heart rate</li> <li>• Positional Alcohol Nystagmus</li> </ul>
> 0.50	<ul style="list-style-type: none"> <li>• High risk of poisoning</li> <li>• High possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Life</li> </ul>

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# MODULE 2: LINEAR INEQUALITIES

## WHY IT MATTERS: LINEAR INEQUALITIES

### Why Learn about Linear Inequalities?

Previously, we introduced Widmark's formula for calculating blood alcohol content:

$$B = -0.015t + \left( \frac{2.84N}{Wg} \right)$$

where

- B = percentage of BAC
- t = number of hours since the first drink
- N = number of "standard drinks" (a standard drink is one 12-ounce beer, one 5-ounce glass of wine, or one 1.5-ounce shot of liquor). N should be at least 1.
- W = weight in pounds
- g = gender constant: 0.68 for men and 0.55 for women

Remember Joan? When we left her, she was at a party. She had three drinks and then wanted to leave. This was worrisome. We posed the following questions:

1. Where would she fall on the table of the progressive effects of alcohol after 1.5 hours?
2. Would she be within the legal limit to drive after this amount of time?

We found that Joan's BAC was 0.092% and by using the table below, she's probably experiencing blunted feelings, reduced sensitivity to pain, euphoria, disinhibition, and extroversion. In addition, her reasoning, depth perception, peripheral vision, and glare recovery are probably impaired.

Additionally, we determined that she would not be within the legal limit to drive assuming the legal limit is 0.08%. Joan needs a ride home, or she needs to wait until she has metabolized the alcohol in her system so she can safely drive.

Now we will pose another question that may help her plan how long she needs to wait before she drives after drinking at a party.

1. Given any amount that she drinks, can you figure out how much time should pass before she can drive safely and legally?

We solved for BAC given three standard drinks, 1.5 hours, and Joan's weight. What is different about this question? What are we solving, an equation or an inequality?

After working through the readings, videos and problems sets in this module, you will hopefully realize that you can answer this question with an inequality. We will revisit the answer to this question at the end of the module.

In the following table, the progressive effects of alcohol are defined for ranges of blood alcohol content.

<b>Progressive effects of alcohol</b>		
<b>BAC (% by vol.)</b>	<b>Behavior</b>	<b>Impairment</b>
0.001– 0.029	<ul style="list-style-type: none"> <li>• Average individual appears normal</li> </ul>	<ul style="list-style-type: none"> <li>• Subtle effects that can be detected with special tests</li> </ul>
0.030– 0.059	<ul style="list-style-type: none"> <li>• Mild euphoria</li> <li>• Relaxation</li> <li>• Joyousness</li> <li>• Talkativeness</li> <li>• Decreased inhibition</li> </ul>	<ul style="list-style-type: none"> <li>• Concentration</li> </ul>
0.060– 0.099	<ul style="list-style-type: none"> <li>• Blunted feelings</li> <li>• Reduced sensitivity to pain</li> <li>• Euphoria</li> <li>• Disinhibition</li> <li>• Extroversion</li> </ul>	<ul style="list-style-type: none"> <li>• Reasoning</li> <li>• Depth perception</li> <li>• Peripheral vision</li> <li>• Glare recovery</li> </ul>
0.100– 0.199	<ul style="list-style-type: none"> <li>• Overexpression</li> <li>• Boisterousness</li> <li>• Possibility of nausea and vomiting</li> </ul>	<ul style="list-style-type: none"> <li>• Reflexes</li> <li>• Reaction time</li> <li>• Gross motor control</li> <li>• Staggering</li> <li>• Slurred speech</li> <li>• Temporary erectile dysfunction</li> </ul>
0.200– 0.299	<ul style="list-style-type: none"> <li>• Nausea</li> <li>• Vomiting</li> <li>• Emotional swings</li> <li>• Anger or sadness</li> <li>• Partial loss of understanding</li> <li>• Impaired sensations</li> <li>• Decreased libido</li> <li>• Possibility of stupor</li> </ul>	<ul style="list-style-type: none"> <li>• Severe motor impairment</li> <li>• Loss of consciousness</li> <li>• Memory blackout</li> </ul>
0.300– 0.399	<ul style="list-style-type: none"> <li>• Stupor</li> <li>• Central nervous system depression</li> <li>• Loss of understanding</li> <li>• Lapses in and out of consciousness</li> <li>• Low possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Bladder function</li> <li>• Breathing</li> <li>• Dysequilibrium</li> <li>• Heart rate</li> </ul>
0.400– 0.500	<ul style="list-style-type: none"> <li>• Severe central nervous system depression</li> <li>• Coma</li> <li>• Possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Breathing</li> <li>• Heart rate</li> <li>• Positional Alcohol Nystagmus</li> </ul>
> 0.50	<ul style="list-style-type: none"> <li>• High risk of poisoning</li> <li>• High possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Life</li> </ul>

Joan likes to party and believes she is “just fine” when it comes to driving. At a party, though, she downs three standard drinks, one after the other, and then decides to leave. If Joan weighs 135 pounds, where would she be on the table of the progressive effects of alcohol after 1.5 hours? Would she be within the legal limit to drive home after this amount of time? Given any amount that she drinks, can you figure out how long she must wait before she can drive safely and legally?

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# INTRODUCTION TO SOLVING SINGLE-STEP AND MULTI-STEP INEQUALITIES

## What You Will Learn to Do: Solve Single-Step and Multi-Step Inequalities

Sometimes there is a range of possible values to describe a situation. When you see a sign that says “Speed Limit 25,” you know that it doesn’t mean that you have to drive exactly at a speed of 25 miles per hour (mph). This sign means that you are not supposed to go faster than 25 mph, but there are many legal speeds you could drive, such as 22 mph, 24.5 mph or 19 mph. In a situation like this, which has more than one acceptable value, inequalities are used to represent the situation rather than equations.

Solving multi-step inequalities is very similar to solving equations—what you do to one side you need to do to the other side in order to maintain the “balance” of the inequality. The **Properties of Inequality** can help you understand how to add, subtract, multiply, or divide within an inequality.

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# DESCRIBE SOLUTIONS TO INEQUALITIES

## Learning Outcomes

- Represent inequalities on a number line
- Represent inequalities using interval notation

# Represent Inequalities on a Number Line

First, let us define some important terminology. An inequality is a mathematical statement that compares two expressions using the ideas of greater than or less than. Special symbols are used in these statements. When you read an inequality, read it from left to right—just like reading text on a page. In algebra, inequalities are used to describe large sets of solutions. Sometimes there are an infinite amount of numbers that will satisfy an inequality, so rather than try to list off an infinite amount of numbers, we have developed some ways to describe very large lists in succinct ways.

The first way you are probably familiar with—the basic inequality. For example:

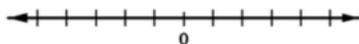
- $x < 9$  indicates the list of numbers that are less than 9. Would you rather write  $x < 9$  or try to list all the possible numbers that are less than 9? (Most people would answer with the former)
- $-5 \leq t$  indicates all the numbers that are greater than or equal to  $-5$ .

Note how placing the variable on the left or right of the inequality sign can change whether you are looking for greater than or less than.

For example:

- $x < 5$  means all the real numbers that are less than 5, whereas;
- $5 < x$  means that 5 is less than  $x$ , or we could rewrite this with the  $x$  on the left:  $x > 5$ . Note how the inequality is still pointing the same direction relative to  $x$ . This statement represents all the real numbers that are greater than 5 which is easier to interpret than 5 is less than  $x$ .

The second way is with a graph using a number line:



And the third way is with an interval.

We will explore the second and third ways in depth in this section. Again, the three ways to write solutions to inequalities are:

- an inequality
- a graph
- an interval

## Inequality Signs

The box below shows the symbol, meaning, and an example for each inequality sign. Sometimes it is easy to get tangled up in inequalities, just remember to read them from left to right.

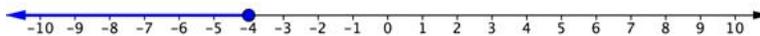
Symbol	Words	Example
$\neq$	not equal to	$2 \neq 8$ , 2 is not equal to 8.
$>$	greater than	$5 > 1$ , 5 is greater than 1
$<$	less than	$2 < 11$ , 2 is less than 11
$\geq$	greater than or equal to	$4 \geq 4$ , 4 is greater than or equal to 4
$\leq$	less than or equal to	$7 \leq 9$ , 7 is less than or equal to 9

The inequality  $x > y$  can also be written as  $y < x$ . The sides of any inequality can be switched as long as the inequality symbol between them is also reversed.

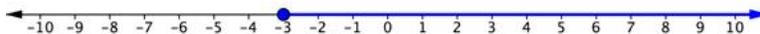
# Graphing an Inequality

Inequalities can also be graphed on a number line. Below are three examples of inequalities and their graphs. Graphs are a very helpful way to visualize information, especially when that information represents an infinite list of numbers!

$x \leq -4$ . This translates to all the real numbers on a number line that are less than or equal to 4.



$x \geq -3$ . This translates to all the real numbers on the number line that are greater than or equal to -3.



Each of these graphs begins with a circle—either an open or closed (shaded) circle. This point is often called the *end point* of the solution. A closed, or shaded, circle is used to represent the inequalities *greater than or equal to* ( $\geq$ ) or *less than or equal to* ( $\leq$ ). The point is part of the solution. An open circle is used for *greater than* ( $>$ ) or *less than* ( $<$ ). The point is *not* part of the solution.

The graph then extends endlessly in one direction. This is shown by a line with an arrow at the end. For example, notice that for the graph of  $x \geq -3$  shown above, the end point is  $-3$ , represented with a closed circle since the inequality is *greater than or equal to*  $-3$ . The blue line is drawn to the right on the number line because the values in this area are greater than  $-3$ . The arrow at the end indicates that the solutions continue infinitely.

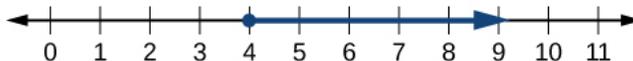
## EXAMPLE

Graph the inequality  $x \geq 4$

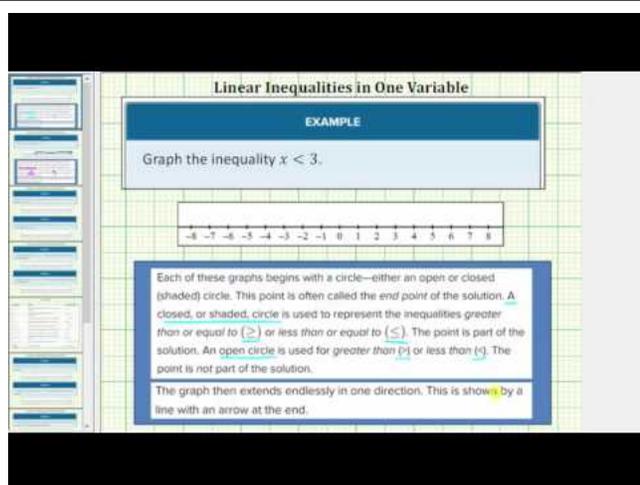
### Answer

We can use a number line as shown. Because the values for  $x$  include 4, we place a solid dot on the number line at 4.

Then we draw a line that begins at  $x = 4$  and, as indicated by the arrowhead, continues to positive infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4.



This video shows an example of how to draw the graph of an inequality.



A YouTube element has been excluded from this version of the text. You can view it online here:  
<https://courses.lumenlearning.com/intermediatealgebra/?p=380>

## EXAMPLE

Write an inequality describing all the real numbers on the number line that are less than 2. Then draw the corresponding graph.

### Answer

We need to start from the left and work right, so we start from negative infinity and end at 2. We will not include either because infinity is not a number, and the inequality does not include 2.

Inequality:  $x < 2$

To draw the graph, place an open dot on the number line first, and then draw a line extending to the left. Draw an arrow at the leftmost point of the line to indicate that it continues for infinity.



## Represent Inequalities Using Interval Notation

Another commonly used, and arguably the most concise, method for describing inequalities and solutions to inequalities is called **interval notation**. With this convention, sets are built with parentheses or brackets, each having a distinct meaning. The solutions to  $x \geq 4$  are represented as  $[4, \infty)$ . This method is widely used and will be present in other math courses you may take.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be “equaled.” A few examples of an **interval**, or a set of numbers in which a solution falls, are  $[-2, 6)$ , or all numbers between  $-2$  and  $6$ , including  $-2$ , but not including  $6$ ;  $(-1, 0)$ , all real numbers between, but not including  $-1$  and  $0$ ; and  $(-\infty, 1]$ , all real numbers less than and including  $1$ . The table below outlines the possibilities. Remember to read inequalities from left to right, just like text.

The table below describes all the possible inequalities that can occur and how to write them using interval notation, where  $a$  and  $b$  are real numbers.

Inequality	Words	Interval Notation
$a < x < b$	all real numbers between $a$ and $b$ , not including $a$ and $b$	$(a, b)$
$x > a$	All real numbers greater than $a$ , but not including $a$	$(a, \infty)$
$x < b$	All real numbers less than $b$ , but not including $b$	$(-\infty, b)$
$x \geq a$	All real numbers greater than $a$ , including $a$	$[a, \infty)$
$x \leq b$	All real numbers less than $b$ , including $b$	$(-\infty, b]$
$a \leq x < b$	All real numbers between $a$ and $b$ , including $a$	$[a, b)$
$a < x \leq b$	All real numbers between $a$ and $b$ , including $b$	$(a, b]$
$a \leq x \leq b$	All real numbers between $a$ and $b$ , including $a$ and $b$	$[a, b]$
$x < a$ or $x > b$	All real numbers less than $a$ or greater than $b$	$(-\infty, a) \cup (b, \infty)$
All real numbers	All real numbers	$(-\infty, \infty)$

## EXAMPLE

Describe the inequality  $x \geq 4$  using interval notation

**Answer**

The solutions to  $x \geq 4$  are represented as  $[4, \infty)$ .

Note the use of a bracket on the left because 4 is included in the solution set.

## EXAMPLE

Use interval notation to indicate all real numbers greater than or equal to  $-2$ .

**Answer**

Use a bracket on the left of  $-2$  and parentheses after infinity:  $[-2, \infty)$ . The bracket indicates that  $-2$  is included in the set with all real numbers greater than  $-2$  to infinity.

## THINK ABOUT IT

In the previous examples you were given an inequality or a description of one with words and asked to draw the corresponding graph and write the interval. In this example you are given an interval and asked to write the inequality and draw the graph.

Given  $(-\infty, 10)$ , write the associated inequality and draw the graph.

In the box below, write down whether you think it will be easier to draw the graph first or write the inequality first.

**Answer**

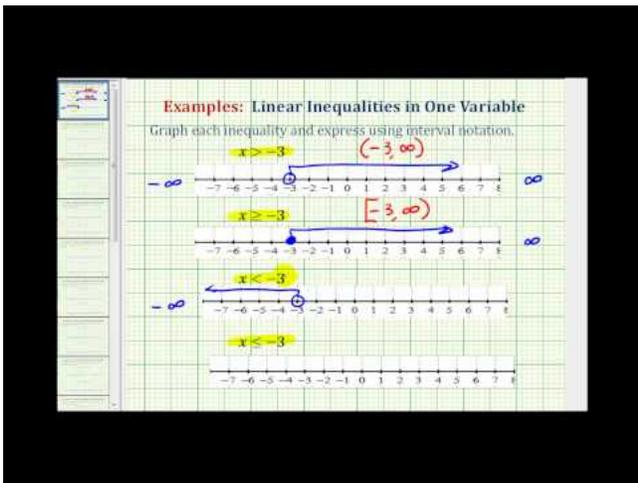
We will draw the graph first.

The interval reads "all real numbers less than 10," so we will start by placing an open dot on 10 and drawing a line to the left with an arrow indicating the solution continues to negative infinity.



To write the inequality, we will use  $<$  since the parentheses indicate that 10 is not included.  $x < 10$

In the following video, you will see examples of how to write inequalities in the three ways presented here: as an inequality, in interval notation, and with a graph.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=380>

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# SOLVE INEQUALITIES

## LEARNING OUTCOMES

- Solve one-step inequalities
- Solve multi-step inequalities

## Multiplication and Division Properties of Inequality

Solving inequalities is very similar to solving equations, except you have to reverse the inequality symbols when you multiply or divide both sides of an inequality by a negative number. There are three ways to represent solutions to inequalities: an interval, a graph, and an inequality. Because there is usually more than one solution to an inequality, when you check your answer, you should check the end point and one other value to check the direction of the inequality. When we work with inequalities, we can usually treat them similar to but not exactly as we treat equations. We can use the **addition property** and the **multiplication property** to help us solve them. The one exception is when we multiply or divide by a negative number; doing so reverses the inequality symbol.

The following table illustrates how the multiplication property is applied to inequalities, and how multiplication by a negative reverses the inequality:

Start With	Multiply By	Final Inequality
$a > b$	$c$	$ac > bc$
$5 > 3$	$3$	$15 > 9$
$a > b$	$-c$	$-ac < -bc$
$5 > 3$	$-3$	$-15 < -9$

The following table illustrates how the division property is applied to inequalities, and how dividing by a negative reverses the inequality:

Start With	Divide By	Final Inequality
$a > b$	$c$	$\frac{a}{c} > \frac{b}{c}$
$4 > 2$	$2$	$\frac{4}{2} > \frac{2}{2}$
$a > b$	$-c$	$-\frac{a}{c} < -\frac{b}{c}$
$4 > 2$	$-2$	$-\frac{4}{2} < -\frac{2}{2}$

In the first example, we will show how to apply the multiplication and division properties of equality to solve some inequalities.

## Solve Inequalities Using the Addition Property

When we solve equations we may need to add or subtract in order to isolate the variable, the same is true for inequalities. There are no special behaviors to watch out for when using the addition property to solve inequalities.

The following table illustrates how the addition property applies to inequalities.

## EXAMPLE

Illustrate the multiplication property for inequalities by solving each of the following:

- a.  $3x < 6$
- b.  $-2x - 1 \geq 5$
- c.  $5 - x > 10$

### Answer

a.

$$3x < 6$$

$$\frac{1}{3}(3x) < (6) \frac{1}{3}$$

$$x < 2$$

b.

$$-2x - 1 \geq 5$$

$$-2x \geq 6$$

Multiply both sides by  $-\frac{1}{2}$ .

$$\left(-\frac{1}{2}\right)(-2x) \geq (6)\left(-\frac{1}{2}\right)$$

Reverse the inequality.

$$x \leq -3$$

c.

$$5 - x > 10$$

$$-x > 5$$

Multiply both sides by  $-1$ .

$$(-1)(-x) > (5)(-1)$$

Reverse the inequality

$$x < -5$$

Start With	Add	Final Inequality
$a > b$	$c$	$a + c > b + c$
$5 > 3$	$3$	$8 > 6$
$a > b$	$-c$	$a - c > b - c$
$5 > 3$	$-3$	$2 > 0$

These properties also apply to  $a \leq b$ ,  $a > b$ , and  $a \geq b$ .

In our next example we will use the addition property to solve inequalities.

## EXAMPLE

Illustrate the addition property for inequalities by solving each of the following:

- a.  $x - 15 < 4$
- b.  $6 \geq x - 1$
- c.  $x + 7 > 9$

### Answer

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.

a.

$$x - 15 < 4$$

$$x - 15 + 15 < 4 + 15 \quad \text{Add 15 to both sides.}$$

$$x < 19$$

b.

$$6 \geq x - 1$$

$$6 + 1 \geq x - 1 + 1 \quad \text{Add 1 to both sides.}$$

$$7 \geq x$$

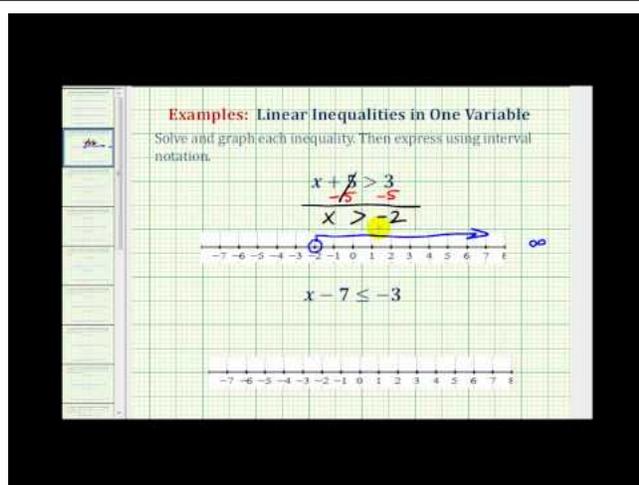
c.

$$x + 7 > 9$$

$$x + 7 - 7 > 9 - 7 \quad \text{Subtract 7 from both sides.}$$

$$x > 2$$

The following video shows examples of solving single-step inequalities using the multiplication and addition properties.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=384>

The following video shows examples of solving inequalities with the variable on the right side.

**Examples: Linear Inequalities in One Variable**  
Solve and graph each inequality. Then express using interval notation.

$-8 > x - 3$   
 $+3 \quad +3$   
 $-5 > x$   
 $x < -5$

$9 \leq x + 4$

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## Solve Multi-Step Inequalities

As the previous examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations. To isolate the variable and solve, we combine like terms and perform operations with the multiplication and addition properties.

### EXAMPLE

Solve the inequality:  $13 - 7x \geq 10x - 4$ .

**Answer**

Solving this inequality is similar to solving an equation up until the last step.

$$13 - 7x \geq 10x - 4$$

$$13 - 17x \geq -4 \quad \text{Move variable terms to one side of the inequality.}$$

$$-17x \geq -17 \quad \text{Isolate the variable term.}$$

$$x \leq 1 \quad \begin{array}{l} \text{Dividing both sides by} \\ -17 \text{ reverses the inequality.} \end{array}$$

The solution set is given by the interval  $(-\infty, 1]$ , or all real numbers less than and including 1.

In the next example we solve an inequality that contains fractions, not how we need to reverse the inequality sign at the end because we multiply by a negative.

## EXAMPLE

Solve the following inequality and write the answer in interval notation:  $-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$ .

### Answer

We begin solving in the same way we do when solving an equation.

$$-\frac{3}{4}x \geq -\frac{5}{8} + \frac{2}{3}x$$

$$-\frac{3}{4}x - \frac{2}{3}x \geq -\frac{5}{8} \quad \text{Put variable terms on one side.}$$

$$-\frac{9}{12}x - \frac{8}{12}x \geq -\frac{5}{8} \quad \text{Write fractions with common denominator.}$$

$$-\frac{17}{12}x \geq -\frac{5}{8}$$

$$x \leq -\frac{5}{8} \left(-\frac{12}{17}\right) \quad \text{Multiplying by a negative number reverses the inequality.}$$

$$x \leq \frac{15}{34}$$

The solution set is the interval  $\left(-\infty, \frac{15}{34}\right]$ .

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# INTRODUCTION TO SOLVING COMPOUND INEQUALITIES

## What You Will Learn to Do: Solve Compound Inequalities

Many times, solutions to inequalities lie between two quantities rather than continuing endlessly in one direction. Your blood pressure reading is one example of this. When you go to the doctor, the nurse takes your blood pressure reading. Blood pressure measures the pressure of blood in the circulatory system and is composed of two numbers, systolic and diastolic. When the nurse takes your blood pressure, the nurse is looking to see what range your blood pressure falls into. For example, systolic (top number) blood pressure that is between 120 and 139 mm Hg is called borderline high blood pressure. This can be described using a compound inequality,  $b < 139$  and  $b > 120$ . In this section we will explore how to solve compound inequalities and represent solutions using standard conventions.

# DESCRIBE SETS AS INTERSECTIONS OR UNIONS

## Learning Outcomes

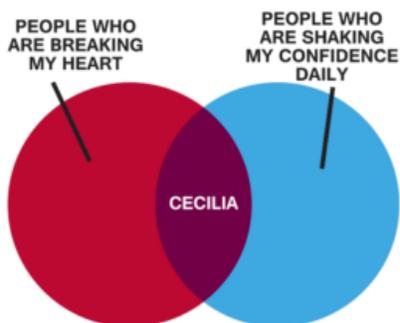
- Use interval notation to describe intersections and unions
- Use graphs to describe intersections and unions

## Use Interval Notation to Describe Sets of Numbers as Intersections and Unions

When two inequalities are joined by the word *and*, the solution of the compound inequality occurs when *both* inequalities are true at the same time. It is the overlap, or intersection, of the solutions for each inequality. When the two inequalities are joined by the word *or*, the solution of the compound inequality occurs when *either* of the inequalities is true. The solution is the combination, or union, of the two individual solutions.

In this section we will learn how to solve compound inequalities that are joined with the words AND and OR. First, it will help to see some examples of inequalities, intervals, and graphs of compound inequalities. This will help you describe the solutions to compound inequalities properly.

Venn diagrams use the concept of intersections and unions to compare two or more things. For example, this Venn diagram shows the intersection of people who are breaking your heart and those who are shaking your confidence daily. Apparently Cecilia has both of these qualities; therefore, she is the intersection of the two.



In mathematical terms, consider the inequality  $x < 6$  and  $x > 2$ . How would we interpret what numbers  $x$  can be, and what would the interval look like?

In words,  $x$  must be less than 6, and at the same time, it must be greater than 2. This is much like the Venn diagram above where Cecilia is at once breaking your heart and shaking your confidence daily. Now look at a graph to see what numbers are possible with these constraints.

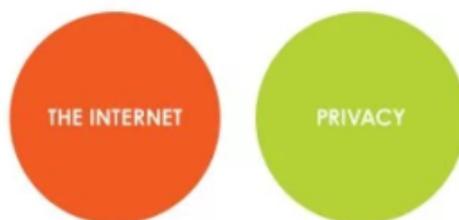


$2 < x < 6$

The numbers that are shared by both lines on the graph are called the intersection of the two inequalities  $x < 6$  and  $x > 2$ . This is called a *bounded* inequality and is written as  $2 < x < 6$ . Think about that one for a minute.  $x$  must be less than 6 and greater than two—the values for  $x$  will fall *between two numbers*. In interval notation, this looks like  $(2, 6)$ . The graph would look like this:

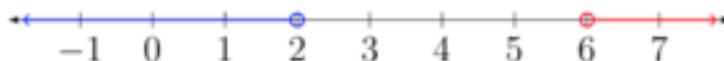


On the other hand, if you need to represent two things that don't share any common elements or traits, you can use a union. The following Venn diagram shows two things that share no similar traits or elements but are often considered in the same application, such as online shopping or banking.



In mathematical terms, for example,  $x > 6$  *or*  $x < 2$  is an inequality joined by the word *or*. Using interval notation, we can describe each of these inequalities separately:

$x > 6$  is the same as  $(6, \infty)$  and  $x < 2$  is the same as  $(-\infty, 2)$ . If we are describing solutions to inequalities, what effect does the *or* have? We are saying that solutions are either real numbers less than two *or* real numbers greater than 6. Can you see why we need to write them as two separate intervals? Let us look at a graph to get a clear picture of what is going on.



When you place both of these inequalities on a graph, we can see that they share no numbers in common. As mentioned above, this is what we call a union. The interval notation associated with a union is a big U, so instead of writing *or*, we join our intervals with a big U, like this:

$$(-\infty, 2) \cup (6, \infty)$$

It is common convention to construct intervals starting with the value that is furthest left on the number line as the left value, such as  $(2, 6)$ , where 2 is less than 6. The number on the right should be greater than the number on the left.

In the following video you will see two examples of how to express inequalities involving *or* graphically and as an interval.

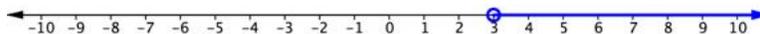
The following video presents two examples of how to draw inequalities involving *and* as well as write the corresponding intervals.

## EXAMPLE

Draw the graph of the compound inequality  $x > 3$  or  $x \leq 4$  and describe the set of  $x$ -values that will satisfy it with an interval.

### Answer

The graph of  $x > 3$  has an open circle on 3 and a blue arrow drawn to the right to contain all the numbers greater than 3.



The graph of  $x \leq 4$  has a closed circle at 4 and a red arrow to the left to contain all the numbers less than 4.



What do you notice about the graph that combines these two inequalities?



Since this compound inequality is an *or* statement, it includes all of the numbers in each of the solutions. In this case, the solution is all the numbers on the number line. (The region of the line greater than 3 and less than or equal to 4 is shown in purple because it lies on both of the original graphs.) The solution to the compound inequality  $x > 3$  or  $x \leq 4$  is the set of all real numbers and can be described in interval notation as  $(-\infty, \infty)$

A screenshot of a YouTube video titled "Compound Inequalities: Describe Sets as Intersections or Unions". The video content includes:

- An "EXAMPLE" section with the text: "Draw the graph of the compound inequality  $x \leq -4$  or  $x > 3$  and describe the set using interval notation."
- A number line graph showing a closed blue circle at -4 with a blue arrow pointing left, and an open pink circle at 3 with a pink arrow pointing right. The region between -4 and 3 is shaded purple.
- Text explaining that for "and" (intersection), the solution is the overlap of the two inequalities. For "or" (union), the solution is the combination of both.
- A small box with the text: "and  $\rightarrow$  intersection" and "or  $\rightarrow$  union".

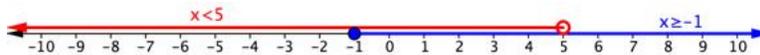
A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=424>

## EXAMPLES

Draw a graph of the compound inequality:  $x < 5$  and  $x \geq -1$ , and describe the set of  $x$ -values that will satisfy it with an interval.

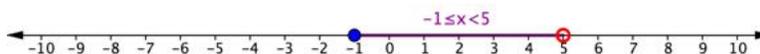
### Answer

The graph of each individual inequality is shown in color.



Since the word *and* joins the two inequalities, the solution is the overlap of the two solutions. This is where both of these statements are true at the same time.

The solution to this compound inequality is shown below.



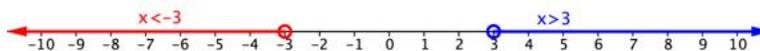
Notice that this is a bounded inequality. You can rewrite  $x \geq -1$  and  $x \leq 5$  as  $-1 \leq x \leq 5$  since the solution is between  $-1$  and  $5$ , including  $-1$ . You read  $-1 \leq x < 5$  as “ $x$  is greater than or equal to  $-1$  and less than  $5$ .” You can rewrite an *and* statement this way only if the answer is *between* two numbers. The set of solutions to this inequality can be written in interval notation like this:  $[-1, 5)$

## EXAMPLES

Draw the graph of the compound inequality  $x < -3$  and  $x > 3$ , and describe the set of  $x$ -values that will satisfy it with an interval.

### Answer

First, draw a graph. We are looking for values for  $x$  that will satisfy **both** inequalities since they are joined with the word *and*.



In this case, there are no shared  $x$ -values, and therefore there is no intersection for these two inequalities. We can write “no solution,” or DNE.

**Compound Inequalities: Describe Sets as Intersections or Unions**

**EXAMPLE**

Draw the graph of the compound inequality  $x > -2$  and  $x \leq 6$  and describe the set using interval notation.

When two inequalities are joined by the word *and*, the solution of the compound inequality occurs when both inequalities are true at the same time. It is the overlap, or **intersection**, of the solutions for each inequality.

When the two inequalities are joined by the word *or*, the solution of the compound inequality occurs when either of the inequalities is true. The solution is the combination, or **union**, of the two individual solutions.

*and* → **intersection**  
*or* → **union**

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# SOLVE COMPOUND INEQUALITIES—OR

## Learning Outcome

- Solve compound inequalities in the form of *or* and express the solution graphically and with interval notation

## Solve Compound Inequalities in the Form of “*or*”

As we saw in the last section, the solution of a compound inequality that consists of two inequalities joined with the word *or* is the union of the solutions of each inequality. Unions allow us to create a new set from two that may or may not have elements in common.

In this section, you will see that some inequalities need to be simplified before their solution can be written or graphed.

In the following example, you will see an example of how to solve a one-step inequality in the *or* form. Note how each inequality is treated independently until the end where the solution is described in terms of both inequalities. You will use the same properties to solve compound inequalities that you used to solve regular inequalities.

### Example

Solve for  $x$ .  $3x - 1 < 8$  or  $x - 5 > 0$

**Answer**

Solve each inequality by isolating the variable.

$$\begin{array}{rcl} x - 5 > 0 & \text{or} & 3x - 1 < 8 \\ \hline +5 & +5 & \hline x > 5 & & \frac{3x}{3} < \frac{9}{3} \\ & & x < 3 \end{array}$$

$$x > 5 \text{ or } x < 3$$

Inequality notation:  $x > 5$  or  $x < 3$

Interval notation:  $(-\infty, 3) \cup (5, \infty)$

The solution to this compound inequality can also be shown graphically. Sometimes it helps to draw the graph first before writing the solution using interval notation.



Remember to apply the properties of inequalities when you are solving compound inequalities. The next example involves dividing by a negative to isolate a variable.

## EXAMPLE

Solve for  $y$ .  $2y + 7 < 13$  or  $-3y - 2 < 10$

**Answer**

Solve each inequality separately.

$$2y + 7 < 13 \quad \text{or} \quad -3y - 2 < 10$$

$$\begin{array}{r} -7 \quad -7 \\ \hline 2y < 6 \end{array} \qquad \begin{array}{r} +2 \quad +2 \\ \hline -3y < 12 \end{array}$$

$$\begin{array}{r} 2 \quad 2 \\ \hline y < 3 \end{array} \qquad \begin{array}{r} -3 \quad -3 \\ \hline y > -4 \end{array}$$

$$y < 3 \quad \text{or} \quad y > -4$$

The inequality sign is reversed with division by a negative number.

Since  $y$  could be less than 3 or greater than  $-4$ ,  $y$  could be any number. Graphing the inequality helps with this interpretation.

Inequality notation:  $y < 3$  or  $y > -4$

Interval notation:  $(-\infty, \infty)$

Graph:



Even though the graph shows empty dots at  $y = 3$  and  $y = -4$ , they are included in the solution.

In the last example, the final answer included solutions whose intervals overlapped. This caused the answer to include all the numbers on the number line. In words, we call this solution “all real numbers.” Any real number will produce a true statement for either  $y < 3$  or  $y > -4$  when it is substituted for  $y$ .

The following video contains an example of solving a compound inequality involving *or* and drawing the associated graph.

In the next section you will see examples of how to solve compound inequalities containing *and*.

## EXAMPLE

Solve for  $z$ .

$$5z - 3 > -18 \text{ or } -2z - 1 > 15$$

**Answer**

Solve each inequality separately. Combine the solutions.

$$5z - 3 > -18 \quad \text{or} \quad -2z - 1 > 15$$

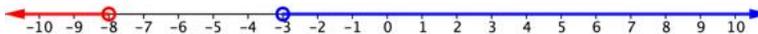
$$\begin{array}{r} +3 \quad +3 \\ \hline 5z > -15 \\ \hline 5 \quad 5 \\ \hline z > -3 \end{array} \quad \text{or} \quad \begin{array}{r} +1 \quad +1 \\ \hline -2z > 16 \\ \hline -2 \quad -2 \\ \hline z < -8 \end{array}$$

$$z > -3 \text{ or } z < -8$$

Inequality notation:  $z > -3$  or  $z < -8$

Interval notation:  $(-\infty, -8) \cup (-3, \infty)$  Note how we write the intervals with the one containing the most negative solutions first then move to the right on the number line.  $z < -8$  has solutions that continue all the way to the left on the number line, whereas  $x > -3$  has solutions that continue all the way to the right.

Graph:



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# SOLVE COMPOUND INEQUALITIES—AND

## LEARNING OUTCOMES

- Express solutions to inequalities graphically and using interval notation
- Identify solutions for compound inequalities in the form  $a < x < b$  including cases with no solution

## Solve Compound Inequalities in the Form of “and” and Express the Solution Graphically

The solution of a compound inequality that consists of two inequalities joined with the word *and* is the intersection of the solutions of each inequality. In other words, both statements must be true at the same time. The solution to an *and* compound inequality are all the solutions that the two inequalities have in common. As we saw in previous sections, this is where the two graphs overlap.

In this section we will see more examples where we have to simplify the compound inequalities before we can express their solutions graphically or with an interval.

### Example

Solve for  $x$ .  $1 - 4x \leq 21$  and  $5x + 2 \geq 22$

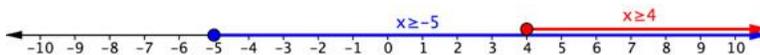
**Answer**

Solve each inequality for  $x$ . Determine the intersection of the solutions.

$$\begin{array}{rcl}
 1 - 4x \leq 21 & \text{and} & 5x + 2 \geq 22 \\
 \hline
 -1 & & -2 \\
 \hline
 -4x \leq 20 & & 5x \geq 20 \\
 \hline
 -4 & & 5 \\
 \hline
 x \geq -5 & & x \geq 4
 \end{array}$$

$$x \geq -5 \text{ and } x \geq 4$$

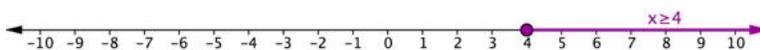
The number line below shows the graphs of the two inequalities in the problem. The solution to the compound inequality is  $x \geq 4$  since this is where the two graphs overlap.



Inequality notation:  $x \geq 4$

Interval notation:  $[4, \infty)$

Graph:



## Compound Inequalities in the Form $a < x < b$

## EXAMPLE

Solve for  $x$ :  $5x - 2 \leq 3$  and  $4x + 7 > 3$

**Answer**

Solve each inequality separately. Find the overlap between the solutions.

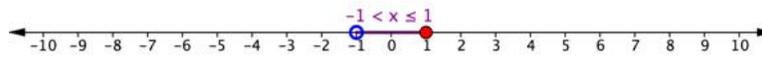
$$\begin{array}{rcl} 5x - 2 \leq 3 & \text{and} & 4x + 7 > 3 \\ \hline +2 & +2 & \hline -7 & -7 & \hline \underline{5x \leq 5} & & \underline{4x > -4} \\ 5 & 5 & 4 & 4 \\ \underline{x \leq 1} & & \underline{x > -1} \end{array}$$

$$x \leq 1 \text{ and } x > -1$$

Inequality notation:  $-1 < x \leq 1$

Interval notation:  $(-1, 1]$

Graph:



Rather than splitting a compound inequality in the form of  $a < x < b$  into two inequalities  $x < b$  and  $x > a$ , you can more quickly solve the inequality by applying the properties of inequalities to all three segments of the compound inequality.

## Example

Solve for  $x$ .  $3 < 2x + 3 \leq 7$

**Answer**

Isolate the variable by subtracting 3 from all 3 parts of the inequality then dividing each part by 2.

$$\begin{array}{rcl} 3 < 2x + 3 \leq 7 \\ \hline -3 & -3 & -3 \\ \hline \underline{0 < 2x \leq 4} \\ 2 & 2 & 2 \\ \hline 0 < x \leq 2 \end{array}$$

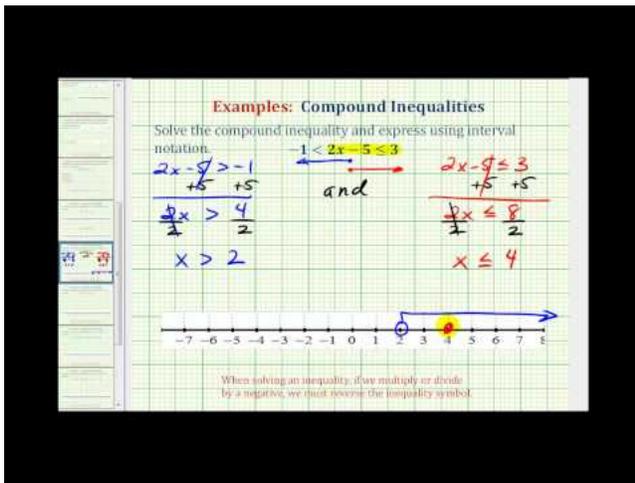
Inequality notation:  $0 < x \leq 2$

Interval notation:  $(0, 2]$

Graph:



In the video below, you will see another example of how to solve an inequality in the form  $a < x < b$ .



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=456>

## EXAMPLE

Solve the compound inequality with variables in all three parts:  $3 + x > 7x - 2 > 5x - 10$ .

### Answer

Let us try the first method: write two inequalities.

$$3 + x > 7x - 2 \quad \text{and} \quad 7x - 2 > 5x - 10$$

$$3 > 6x - 2 \qquad 2x - 2 > -10$$

$$5 > 6x \qquad 2x > -8$$

$$\frac{5}{6} > x \qquad x > -4$$

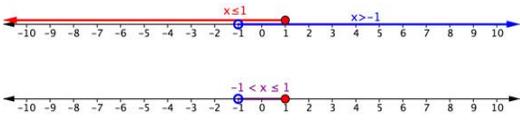
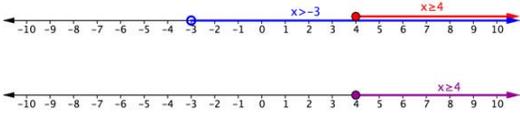
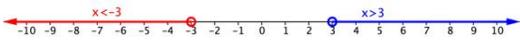
$$x < \frac{5}{6} \qquad -4 < x$$

The solution set is  $-4 < x < \frac{5}{6}$  or in interval notation  $\left(-4, \frac{5}{6}\right)$ . Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right as they appear on a number line.



To solve inequalities like  $a < x < b$ , use the addition and multiplication properties of inequality to solve the inequality for  $x$ . Whatever operation you perform on the middle portion of the inequality, you must also perform to each of the outside sections as well. Pay particular attention to division or multiplication by a negative.

The solution to a compound inequality with *and* is always the overlap between the solution to each inequality. There are three possible outcomes for compound inequalities joined by the word *and*:

<b>Case 1:</b>	
Description	The solution could be all the values between two endpoints
Inequalities	$x \leq 1$ and $x > -1$ , or as a bounded inequality: $-1 < x \leq 1$
Interval	$(-1, 1]$
Graphs	
<b>Case 2:</b>	
Description	The solution could begin at a point on the number line and extend in one direction.
Inequalities	$x > 3$ and $x \geq 4$
Interval	$[4, \infty)$
Graphs	
<b>Case 3:</b>	
Description	In cases where there is no overlap between the two inequalities, there is no solution to the compound inequality
Inequalities	$x < -3$ and $x > 3$
Intervals	$(-\infty, -3)$ and $(3, \infty)$
Graph	

In the example below, there is no solution to the compound inequality because there is no overlap between the inequalities.

### Example

Solve for  $x$ .  $x + 2 > 5$  and  $x + 4 < 5$

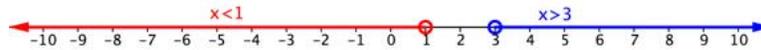
**Answer**

Solve each inequality separately.

$$\begin{array}{r}
 x + 2 > 5 \\
 \underline{-2 \quad -2} \\
 x > 3
 \end{array}
 \quad \text{and} \quad
 \begin{array}{r}
 x + 4 < 5 \\
 \underline{-4 \quad -4} \\
 x < 1
 \end{array}$$

$$x > 3 \quad \text{and} \quad x < 1$$

Find the overlap between the solutions.



There is no overlap between  $x > 3$  and  $x < 1$ , so there is no solution.

## Summary

A compound inequality is a statement of two inequality statements linked together either by the word *or* or by the word *and*. Sometimes, an *and* compound inequality is shown symbolically, like  $a < x < b$  and does not even need the word *and*. Because compound inequalities represent either a union or intersection of the individual inequalities, graphing them on a number line can be a helpful way to see or check a solution. Compound inequalities can be manipulated and solved in much the same way any inequality is solved, by paying attention to the properties of inequalities and the rules for solving them.

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# SOLVE ABSOLUTE VALUE INEQUALITIES

## Learning Outcomes

- Express solutions to inequalities containing absolute value
- Identify solutions for absolute inequalities where there are no solutions

## Solve Inequalities Containing Absolute Value

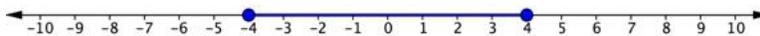
Let us apply what you know about solving equations that contain absolute value and what you know about inequalities to solve inequalities that contain absolute value. Let us start with a simple inequality.

$$|x| \leq 4$$

This inequality is read, “the absolute value of  $x$  is less than or equal to 4.” If you are asked to solve for  $x$ , you want to find out what values of  $x$  are 4 units or less away from 0 on a number line. You could start by thinking about the number line and what values of  $x$  would satisfy this equation.

4 and  $-4$  are both four units away from 0, so they are solutions. 3 and  $-3$  are also solutions because each of these values is less than 4 units away from 0. So are 1 and  $-1$ , 0.5 and  $-0.5$ , and so on—there are an infinite number of values for  $x$  that will satisfy this inequality.

The graph of this inequality will have two closed circles, at 4 and  $-4$ . The distance between these two values on the number line is colored in blue because all of these values satisfy the inequality.

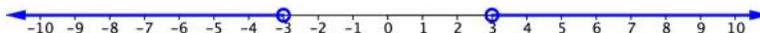


The solution can be written this way:

Inequality notation:  $-4 \leq x \leq 4$

Interval notation:  $[-4, 4]$

The situation is a little different when the inequality sign is “greater than” or “greater than or equal to.” Consider the simple inequality  $|x| > 3$ . Again, you could think of the number line and what values of  $x$  are greater than 3 units away from zero. This time, 3 and  $-3$  are not included in the solution, so there are open circles on both of these values. 2 and  $-2$  would not be solutions because they are not more than 3 units away from 0. But 5 and  $-5$  would work and so would all of the values extending to the left of  $-3$  and to the right of 3. The graph would look like the one below.



The solution to this inequality can be written this way:

Inequality notation:  $x < -3$  or  $x > 3$ .

Interval notation:  $(-\infty, -3) \cup (3, \infty)$

In the following video, you will see examples of how to solve and express the solution to absolute value inequalities involving both *and* and *or*.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=468>

## Writing Solutions to Absolute Value Inequalities

For any positive value of  $a$  and  $x$ , a single variable, or any algebraic expression:

Absolute Value Inequality	Equivalent Inequality	Interval Notation
$ x  \leq a$	$-a \leq x \leq a$	$[-a, a]$
$ x  < a$	$-a < x < a$	$(-a, a)$
$ x  \geq a$	$x \leq -a$ or $x \geq a$	$(-\infty, -a] \cup [a, \infty)$
$ x  > a$	$x < -a$ or $x > a$	$(-\infty, -a) \cup (a, \infty)$

Let us look at a few more examples of inequalities containing absolute value.

### Example

Solve for  $x$ .  $|x + 3| > 4$

**Answer**

Since this is a “greater than” inequality, the solution can be rewritten according to the “greater than” rule.

$$x + 3 < -4 \quad \text{or} \quad x + 3 > 4$$

Solve each inequality.

$$\begin{array}{r} x + 3 < -4 & x + 3 > 4 \\ \underline{-3 \quad -3} & \underline{-3 \quad -3} \\ x < -7 & x > 1 \end{array}$$

$$x < -7 \quad \text{or} \quad x > 1$$

Check the solutions in the original equation to be sure they work. Check the end point of the first related equation,  $-7$  and the end point of the second related equation,  $1$ .

$$\begin{array}{r} |x + 3| > 4 & |x + 3| > 4 \\ |-7 + 3| = 4 & |1 + 3| = 4 \\ |-4| = 4 & |4| = 4 \\ 4 = 4 & 4 = 4 \end{array}$$

Try  $-10$ , a value less than  $-7$ , and  $5$ , a value greater than  $1$ , to check the inequality.

$$\begin{array}{r} |x + 3| > 4 & |x + 3| > 4 \\ |-10 + 3| > 4 & |5 + 3| > 4 \\ |-7| > 4 & |8| > 4 \\ 7 > 4 & 8 > 4 \end{array}$$

Both solutions check!

Inequality notation:  $x < -7$  or  $x > 1$

Interval notation:  $(-\infty, -7) \cup (1, \infty)$

Graph:





**Examples: Absolute Value Inequalities**

Solve. Express the solution as a compound inequality, a graph, and using interval notation.

$|2x + 3| < 7$

$2x + 3 < 7$  and  $2x + 3 > -7$

$\frac{2x}{2} < \frac{4}{2}$  and  $\frac{2x}{2} > \frac{-10}{2}$

$x < 2$  and  $x > -5$

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**Examples: Absolute Value Inequalities**

Solve. Express the solution as a compound inequality, a graph, and using interval notation.

$|2x - 7| \leq 4$

$2x - 7 \leq 4$  and  $2x - 7 \geq -4$

$\frac{2x}{2} \leq \frac{11}{2}$  and  $\frac{2x}{2} \geq \frac{3}{2}$

$x \leq \frac{11}{2}$  and  $x \geq \frac{3}{2}$

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**Examples: Absolute Value Inequalities**

Solve. Express the solution as a compound inequality, a graph, and using interval notation.

~~$|x - 5| \leq -4$~~

$|x - 5| \geq 1$

$x - 5 \geq 1$  or  $x - 5 \leq -1$

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## Identify Cases of Inequalities Containing Absolute Value That Have No Solutions

As with equations, there may be instances where there is no solution to an inequality.

### Example

Solve for  $x$ .  $|2x + 3| + 9 \leq 7$

**Answer**

Isolate the absolute value by subtracting 9 from both sides of the inequality.

$$\begin{array}{r} |2x + 3| + 9 \leq 7 \\ \underline{-9 \quad -9} \\ |2x + 3| \leq -2 \end{array}$$

The absolute value of a quantity can never be a negative number, so there is no solution to the inequality.

There is no solution

## Summary

Inequalities containing absolute value can be solved by rewriting them using compound inequalities. The first step to solving absolute inequalities is to isolate the absolute value. The next step is to decide whether you are working with an *or* inequality or an *and* inequality. If the inequality is greater than a number, we will use *or*. If the inequality is less than a number, we will use *and*. Remember that if we end up with an absolute value greater than or less than a negative number, there is no solution.

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- Ex 2: Solve and Graph Absolute Value inequalities Mathispower4u. **Authored by:** James Sousa (Mathispower4u.com) for Lumen Learning. **Located at:** <https://youtu.be/d-hUviSkmqE>. **License:** CC BY: Attribution
- Ex 3: Solve and Graph Absolute Value inequality. **Authored by:** James Sousa (Mathispower4u.com) for Lumen Learning. **Located at:** <https://youtu.be/rtUaRf-Gzpm>. **License:** CC BY: Attribution
- Ex 4: Solve and Graph Absolute Value inequalities (Requires Isolating Abs. Value). **Authored by:** James Sousa (Mathispower4u.com) for Lumen Learning. **Located at:** <https://youtu.be/5JRUiMuxWQ>. **License:** CC BY: Attribution

## PUTTING IT TOGETHER: LINEAR INEQUALITIES

At the start of this module, we continued the story of Joan who had been drinking at a party. She was ready to drive home when we posed a question:

- Given any amount that she drinks, can you figure out how much time should pass before she can drive safely and legally?

Previously, we used the same formula commonly used by forensic scientists to calculate blood alcohol content to find her blood alcohol content to determine whether she was within the legal limit to drive:

$$B = -0.015t + \left( \frac{2.84N}{Wg} \right)$$

where

- B = percentage of BAC

- $t$  = number of hours since the first drink
- $N$  = number of “standard drinks” (a standard drink is one 12-ounce beer, one 5-ounce glass of wine, or one 1.5-ounce shot of liquor).  $N$  should be at least 1.
- $W$  = weight in pounds
- $g$  = gender constant: 0.68 for men and 0.55 for women

We calculated Joan’s BAC as 0.092% which meant that she’s probably experiencing blunted feelings, reduced sensitivity to pain, euphoria, disinhibition, and extroversion. In addition, her reasoning, depth perception, peripheral vision, and glare recovery are probably impaired.

We also found that if the legal limit were 0.08%, she would not be within the legal limit to drive.

Now that we know about the algebra of inequalities, we can answer the new question about Joan.

- Given any amount that she drinks, can you figure out how much time should pass before she can drive safely and legally?

This question takes a little more thinking. Let’s continue with our example of three standard drinks to answer it. In the first question, we solved for BAC given a specific number of drinks, amount of time, and Joan’s weight. What is different about this question? Are we solving an equation or an inequality?

Let’s start with what we know. We want to know *how long* it will take to be *within* the legal limit to drive, and we know that 1.5 hours is not enough. The word *within* is often associated with inequalities. We can translate this as “When will this person’s BAC be less than 0.08%?”

When writing an inequality based on a situation, one of the trickiest parts is knowing which direction the inequality sign will go. In other words, which of the following inequalities will answer our question correctly?

$$0.08\% < -0.015t + \left(\frac{2.84N}{Wg}\right)$$

or

$$0.08\% > -0.015t + \left(\frac{2.84N}{Wg}\right)$$

Using words to translate what these expressions mean is often helpful:

$$0.08\% < -0.015t + \left(\frac{2.84N}{Wg}\right)$$

Translation: For a limit of 0.08%, what time will make this side greater than 0.08%?

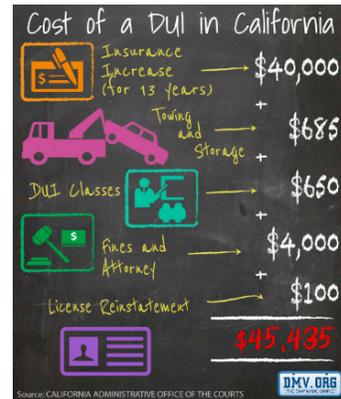
That doesn’t seem right. We want to know how much time will make the right-hand side *less than* 0.08%, so we would want to choose the other one:

$$0.08\% > -0.015t + \left(\frac{2.84N}{Wg}\right)$$

Solving for time gives us the following:

$$0.08 > -0.015t + 0.1147$$

$$-0.0347 > -0.015t$$



Cost of a DUI in California.

Divide both sides by  $-0.015$  and remember to change the direction of the inequality sign. That gives us the following:

$$2.31 < t$$

We have our answer to this new question. According to the expression above, our answer is “any time greater than 2.3 hours will be long enough for Joan to be within the legal limit for driving after she’s consumed three standard drinks.” We hope Joan takes our advice and gets home okay.

<b>Progressive Effects of Alcohol</b>		
<b>BAC (% by vol.)</b>	<b>Behavior</b>	<b>Impairment</b>
0.001– 0.029	<ul style="list-style-type: none"> <li>• Average individual appears normal</li> </ul>	<ul style="list-style-type: none"> <li>• Subtle effects that can be detected with special tests</li> </ul>
0.030– 0.059	<ul style="list-style-type: none"> <li>• Mild euphoria</li> <li>• Relaxation</li> <li>• Joyousness</li> <li>• Talkativeness</li> <li>• Decreased inhibition</li> </ul>	<ul style="list-style-type: none"> <li>• Concentration</li> </ul>
0.060– 0.099	<ul style="list-style-type: none"> <li>• Blunted feelings</li> <li>• Reduced sensitivity to pain</li> <li>• Euphoria</li> <li>• Disinhibition</li> <li>• Extroversion</li> </ul>	<ul style="list-style-type: none"> <li>• Reasoning</li> <li>• Depth perception</li> <li>• Peripheral vision</li> <li>• Glare recovery</li> </ul>
0.100– 0.199	<ul style="list-style-type: none"> <li>• Over-expression</li> <li>• Boisterousness</li> <li>• Possibility of nausea and vomiting</li> </ul>	<ul style="list-style-type: none"> <li>• Reflexes</li> <li>• Reaction time</li> <li>• Gross motor control</li> <li>• Staggering</li> <li>• Slurred speech</li> <li>• Temporary erectile dysfunction</li> </ul>
0.200– 0.299	<ul style="list-style-type: none"> <li>• Nausea</li> <li>• Vomiting</li> <li>• Emotional swings</li> <li>• Anger or sadness</li> <li>• Partial loss of understanding</li> <li>• Impaired sensations</li> <li>• Decreased libido</li> <li>• Possibility of stupor</li> </ul>	<ul style="list-style-type: none"> <li>• Severe motor impairment</li> <li>• Loss of consciousness</li> <li>• Memory blackout</li> </ul>
0.300– 0.399	<ul style="list-style-type: none"> <li>• Stupor</li> <li>• Central nervous system depression</li> <li>• Loss of understanding</li> <li>• Lapses in and out of consciousness</li> <li>• Low possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Bladder function</li> <li>• Breathing</li> <li>• Dysequilibrium</li> <li>• Heart rate</li> </ul>
0.400– 0.500	<ul style="list-style-type: none"> <li>• Severe central nervous system depression</li> <li>• Coma</li> <li>• Possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Breathing</li> <li>• Heart rate</li> <li>• Positional Alcohol Nystagmus</li> </ul>
> 0.50	<ul style="list-style-type: none"> <li>• High risk of poisoning</li> <li>• High possibility of death</li> </ul>	<ul style="list-style-type: none"> <li>• Life</li> </ul>



# MODULE 3: GRAPH LINEAR EQUATIONS AND INEQUALITIES

## WHY IT MATTERS: GRAPH LINEAR EQUATIONS AND INEQUALITIES

### Why Graph Linear Relations and Inequalities?

Just as the road sign at the right can help you quickly understand a “falling rocks” hazard ahead, graphs are a useful way of conveying information visually. Graphs of linear equations are especially effective for representing relationships between things that change at a constant rate, and they often do a better job than words or mathematical equations alone.



Remember Joan? Joan was smart and got a ride home from the party. She picked up her car the next day and is happy that she made the choice not to drive intoxicated.

This week at work, Joan’s manager has asked her to create and present a financial report at the next company meeting. She wants to know how the amount of revenue Joan’s team brought to the company changed during the last year. Her job is to get the information and choose the best method for communicating it.

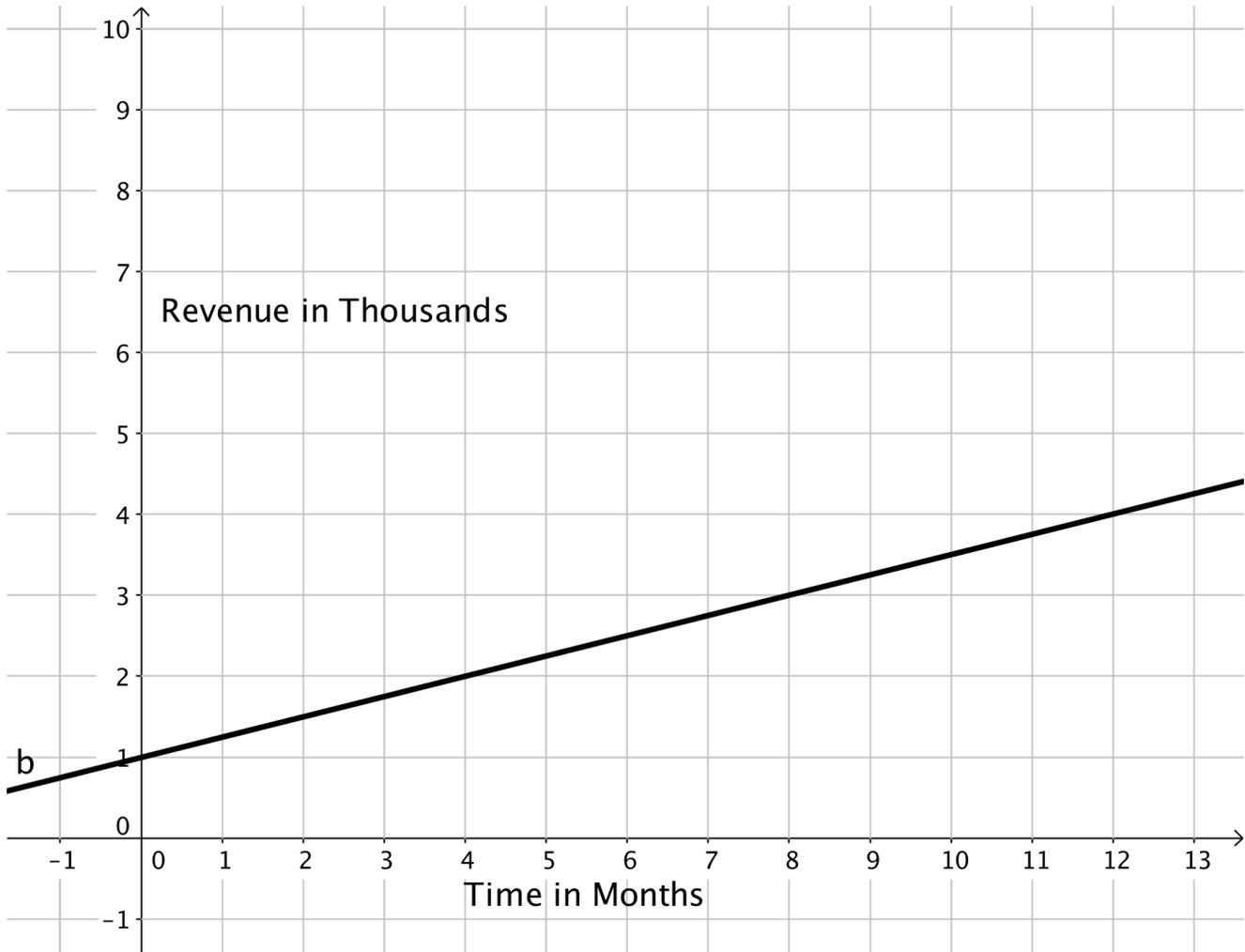
Joan starts with a table and adds the values:

Month	Revenue
0	1000
1	1250
3	1750
5	2250
7	2750
9	3250
11	3750

The table is nicely organized, and its values are very precise, but it does not readily show how the values change over time. So she studies the numbers and tries to describe this change in words: “Our team started the year with \$1000 per month in revenue. For each additional month in the year 2015, our team produced an increase in \$250 of revenue.”

Next, Joan tries expressing the information as a linear equation:  $y = \frac{1}{4}x + 1000$

Again, both are very accurate, but neither reading two sentences out loud nor putting an equation on a poster board will go over very well with her manager or team members. She has not given them a way to picture the information. So, she tries graphing the equation which gives the following:



A graph showing the linear relationship between time and the change in revenue.

All four—the table, the words, the equation, and the graph—are representations of the same thing. And, depending on the purpose, one representation may get Joan's point across more clearly than another.

As you can see, graphs are especially good at conveying information about the relationship between things (in this case, *time* and *revenue*) with very little explanation. Joan's graph shows that the revenue her team produced during the last year steadily increased—a very encouraging trend! Joan decides that the graph is definitely what she will present at the meeting.

In this module, you will learn how to create Cartesian graphs (like the one above) and why graphs are an especially useful means of conveying information.

## INTRODUCTION TO GRAPHING LINEAR EQUATIONS IN TWO VARIABLES

# What You Will Learn to Do: Graph Linear Equations on the Coordinate Plane

The **coordinate plane** was developed centuries ago and refined by the French mathematician René Descartes. In his honor, the system is sometimes called the Cartesian coordinate system. The coordinate plane can be used to plot points and graph lines. This system allows us to describe algebraic relationships in a visual sense and also helps us create and interpret algebraic concepts.

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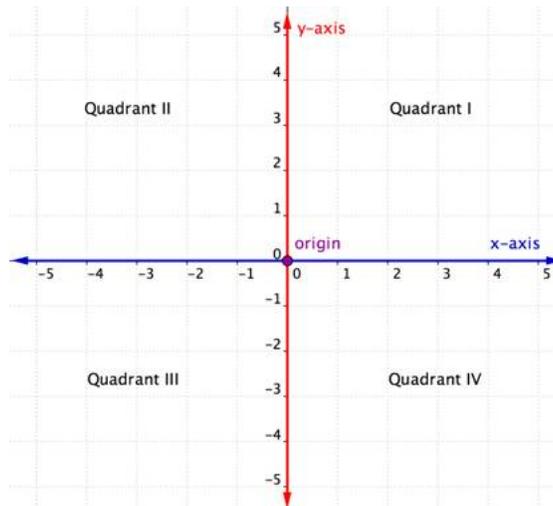
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## GRAPH LINEAR EQUATIONS

### Learning Outcomes

- Create a table of ordered pairs from a two-variable linear equation
- Graph a two-variable linear equation using a table of ordered pairs
- Determine whether an ordered pair is a solution of an equation

You have likely used a coordinate plane before. The coordinate plane consists of a horizontal axis and a vertical axis, number lines that intersect at right angles. (They are perpendicular to each other.)



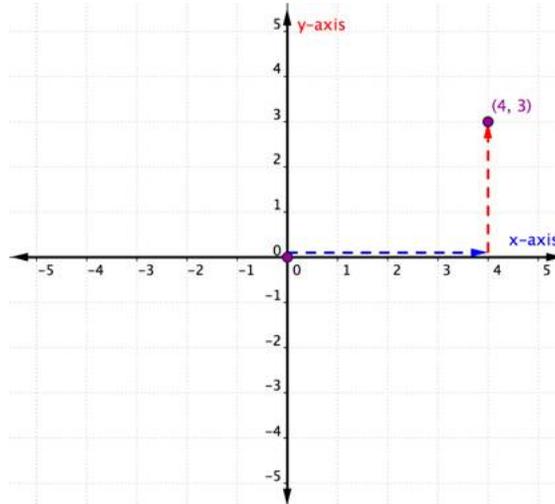
The horizontal axis in the coordinate plane is called the **x-axis**. The vertical axis is called the **y-axis**. The point at which the two axes intersect is called the **origin**. The origin is at 0 on the x-axis and 0 on the y-axis.

Locations on the coordinate plane are described as **ordered pairs**. An ordered pair tells you the location of a point by relating the point's location along the x-axis (the first value of the ordered pair) and along the y-axis (the second value of the ordered pair).

In an ordered pair, such as  $(x, y)$ , the first value is called the **x-coordinate** and the second value is the **y-coordinate**. Note that the x-coordinate is listed before the y-coordinate. Since the origin has an x-coordinate

of 0 and a  $y$ -coordinate of 0, its ordered pair is written as  $(0, 0)$ .

Consider the point below.

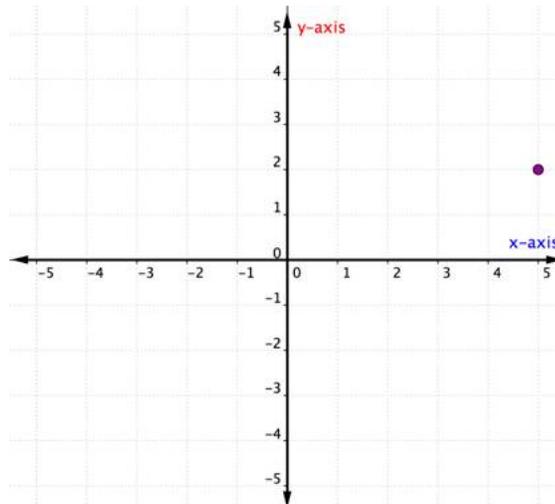


To identify the location of this point, start at the origin  $(0, 0)$  and move right along the  $x$ -axis until you are under the point. Look at the label on the  $x$ -axis. The 4 indicates that, from the origin, you have traveled four units to the right along the  $x$ -axis. This is the  $x$ -coordinate, the first number in the ordered pair.

From 4 on the  $x$ -axis move up to the point and notice the number with which it aligns on the  $y$ -axis. The 3 indicates that, after leaving the  $x$ -axis, you traveled 3 units up in the vertical direction, the direction of the  $y$ -axis. This number is the  $y$ -coordinate, the second number in the ordered pair. With an  $x$ -coordinate of 4 and a  $y$ -coordinate of 3, you have the ordered pair  $(4, 3)$ .

### Example

Describe the point shown as an ordered pair.



### Answer

Begin at the origin and move along the  $x$ -axis. This is the  $x$ -coordinate and is written first in the ordered pair.

$(5, y)$

Move from 5 up to the ordered pair and read the number on the  $y$ -axis. This is the  $y$ -coordinate and is written second in the ordered pair.

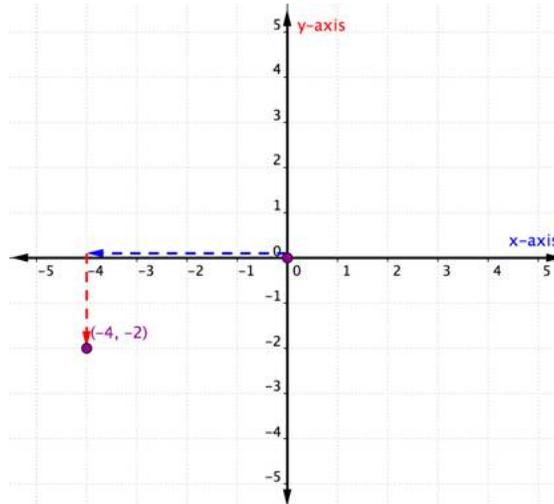
(5, 2)

The ordered pair of the point is (5, 2).

## EXAMPLE

Plot the point  $(-4, -2)$ .

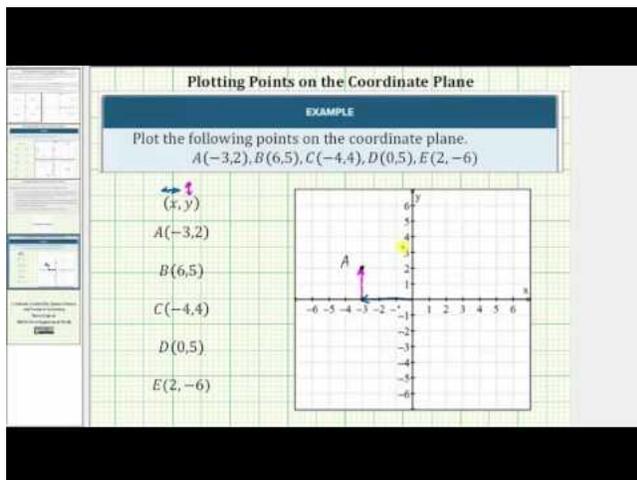
**Answer**



The x-coordinate is  $-4$  because it comes first in the ordered pair. Start at the origin and move 4 units in a negative direction (left) along the x-axis.

The y-coordinate is  $-2$  because it comes second in the ordered pair. Now move 2 units in a negative direction (down). If you look over to the y-axis, you should be lined up with  $-2$  on the axis.

Draw a point at this location and label the point  $(-4, -2)$ .



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Graphing ordered pairs is only the beginning of the story. Once you know how to place points on a grid, you can use them to make sense of all kinds of mathematical relationships.

You can use a **coordinate plane** to plot points and to map various relationships such as the relationship between an object's distance and the elapsed time. Many mathematical relationships are **linear relationships**. Let us look at what a linear relationship is.

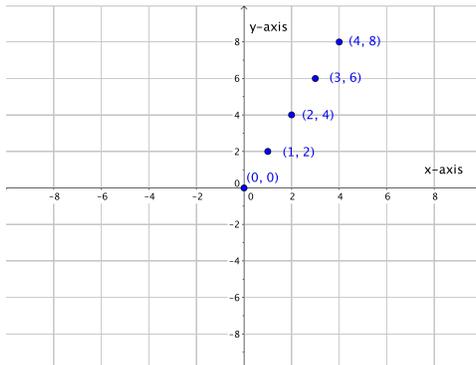
## Plotting Points to Graph Linear Relationships

A linear relationship is a relationship between variables such that when plotted on a coordinate plane, the points lie on a line. Start by looking at a series of points in Quadrant I on the coordinate plane.

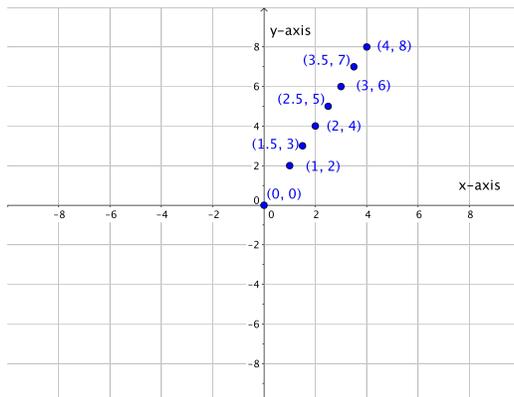
These series of points can also be represented in a table. In the table below, the  $x$ - and  $y$ -coordinates of each ordered pair on the graph is recorded.

<b>x-coordinate</b>	<b>y-coordinate</b>
0	0
1	2
2	4
3	6
4	8

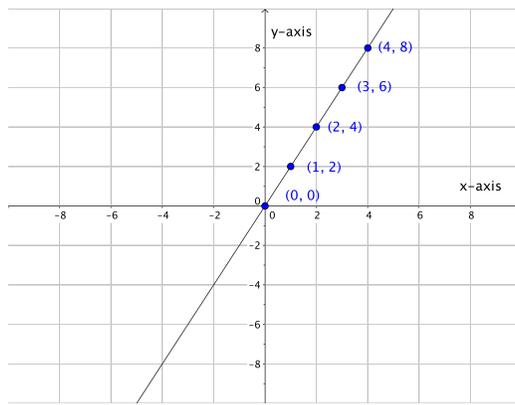
Notice that each  $y$ -coordinate is twice the corresponding  $x$ -value. All of these  $x$ - and  $y$ -values follow the same pattern, and, when placed on a coordinate plane, they all line up as shown below.



Once you know the pattern that relates the  $x$ - and  $y$ -values, you can find a  $y$ -value for any  $x$ -value that lies on the line. So if the rule of this pattern is that each  $y$ -coordinate is *twice* the corresponding  $x$ -value, then the ordered pairs  $(1.5, 3)$ ,  $(2.5, 5)$ , and  $(3.5, 7)$  should all appear on the line too, correct? Look to see what happens below.



If you were to keep adding ordered pairs  $(x, y)$  where the  $y$ -value was twice the  $x$ -value, you would end up with the graph seen below.



Look at how all of the points blend together to create a line. You can think of a line, then, as a collection of an infinite number of individual points that share the same mathematical relationship. In this case, the relationship is that the  $y$ -value is twice the  $x$ -value.

There are multiple ways to represent a linear relationship—a table, a linear graph, and there is also a **linear equation**. A linear equation is an equation with two variables whose ordered pairs graph a straight line.

There are several ways to create a graph from a linear equation. One way is to create a table of values for  $x$  and  $y$  and then plot these ordered pairs on the coordinate plane. Two points are enough to determine a line. However, it is always a good idea to plot more than two points to avoid possible errors.

After graphing the points, draw a line through the points to show that all of the points are on the line. The arrows at each end of the graph indicate that the line continues endlessly in both directions. Every point on this line is a solution to the linear equation.

### Example

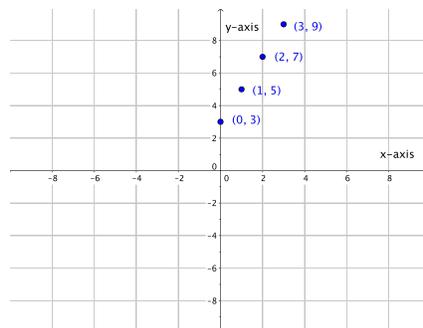
Graph the linear equation  $y = 2x + 3$ .

**Answer**

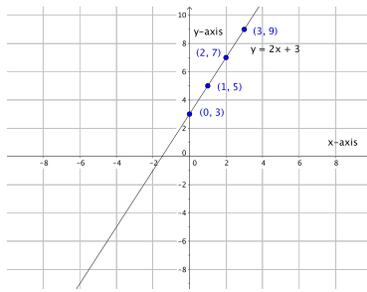
Evaluate  $y = 2x + 3$  for different values of  $x$ , and create a table of corresponding  $x$  and  $y$  values.

$x$ values	$2x + 3$	$y$ values	<b>Ordered Pairs</b>
0	$2(0) + 3$	3	(0, 3)
1	$2(1) + 3$	5	(1, 5)
2	$2(2) + 3$	7	(2, 7)
3	$2(3) + 3$	9	(3, 9)

Plot the ordered pairs.

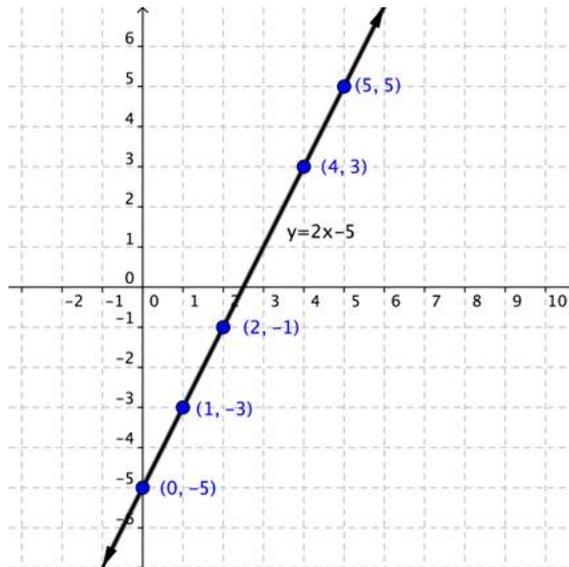


Draw a line through the points to indicate all of the points on the line to obtain the graph seen below.



## Ordered Pairs as Solutions

So far, you have considered the following ideas about lines: a line is a visual representation of a linear equation, and the line itself is made up of an infinite number of points (or ordered pairs). The picture below shows the line of the linear equation  $y = 2x - 5$  with some of the specific points on the line.



Every point on the line is a solution to the equation  $y = 2x - 5$ . You can plug the ordered pair  $(1, -3)$  into the equation and the equation will be satisfied.

$$y = 2x - 5$$

$$-3 = 2(1) - 5$$

$$-3 = 2 - 5$$

$$-3 = -3$$

This is true.

You can also plug ANY of the other points on the line into the equation. Every point on the line is a solution to the equation  $y = 2x - 5$ . All this means is that determining whether an ordered pair is a solution of an equation is pretty straightforward. If the ordered pair is on the line created by the linear equation, then it is a solution to the equation. But if the ordered pair is not on the line—no matter how close it may look—then it is not a solution to the equation.

To find out whether an ordered pair is a solution of a linear equation, you can do the following:

- Graph the linear equation and graph the ordered pair. If the ordered pair appears to be on the graph of a line, then it is a possible solution of the linear equation. If the ordered pair does not lie on the graph of a line, then it is not a solution.

- Substitute the  $(x, y)$  values into the equation. If the equation yields a true statement, then the ordered pair is a solution of the linear equation. If the ordered pair does not yield a true statement then it is not a solution.

### Example

Determine whether  $(-2, 4)$  is a solution to the equation  $4y + 5x = 3$ .

#### Answer

For this problem, you will use the substitution method. Substitute  $x = -2$  and  $y = 4$  into the equation.

$$4y + 5x = 3$$

$$4(4) + 5(-2) = 3$$

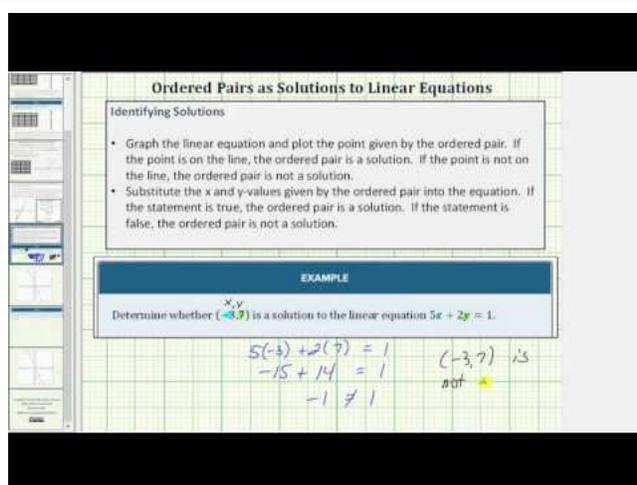
Evaluate.

$$16 + (-10) = 3$$

$$6 = 3$$

The statement is not true, so  $(-2, 4)$  is not a solution to the equation  $4y + 5x = 3$ .

Below is a video further explaining how to determine if an ordered pair is a solution to a linear equation.



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# GRAPH USING INTERCEPTS

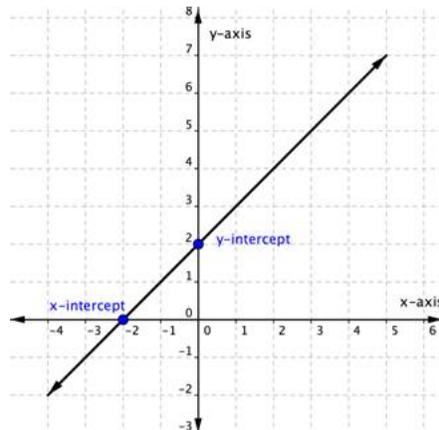
## Learning Outcomes

- Recognize when an ordered pair is a  $y$ -intercept or an  $x$ -intercept
- Graph a linear equation using  $x$  and  $y$ -intercepts

## Intercepts

The intercepts of a line are the points where the line intersects or crosses the horizontal and vertical axes. To help you remember what “intercept” means, think about the word “intersect.” The two words sound alike and in this case mean the same thing.

The straight line on the graph below intersects the two coordinate axes. The point where the line crosses the  $x$ -axis is called the  **$x$ -intercept**. The  **$y$ -intercept** is the point where the line crosses the  $y$ -axis.



The  $x$ -intercept above is the point  $(-2, 0)$ . The  $y$ -intercept above is the point  $(0, 2)$ .

Notice that the  $y$ -intercept always occurs where  $x = 0$ , and the  $x$ -intercept always occurs where  $y = 0$ .

To find the  $x$  and  $y$ -intercepts of a linear equation, you can substitute 0 for  $y$  and for  $x$  respectively.

For example, the linear equation  $3y + 2x = 6$  has an  $x$  intercept when  $y = 0$ , so  $3(0) + 2x = 6$ .

$$2x = 6$$

$$x = 3$$

The  $x$ -intercept is  $(3, 0)$ .

Likewise, the  $y$ -intercept occurs when  $x = 0$ .

$$3y + 2(0) = 6$$

$$3y = 6$$

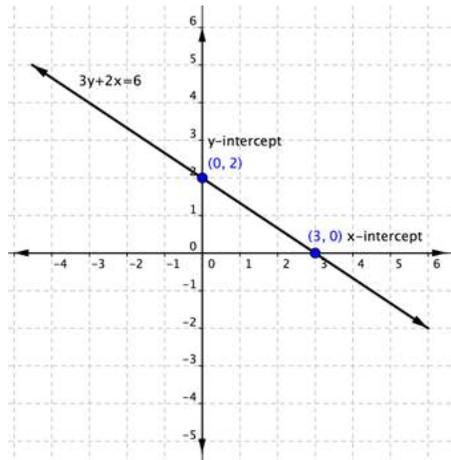
$$y = 2$$

The  $y$ -intercept is  $(0, 2)$ .

## Using Intercepts to Graph Lines

You can use intercepts to graph linear equations. Once you have found the two intercepts, draw a line through them.

Do this with the equation  $3y + 2x = 6$ . You figured out that the intercepts of the line this equation represents are  $(0, 2)$  and  $(3, 0)$ . That is all you need to know.



### Example

Graph  $5y + 3x = 30$  using the  $x$  and  $y$ -intercepts.

**Answer**

When an equation is in  $Ax + By = C$  form, you can easily find the  $x$ - and  $y$ -intercepts and then graph.

To find the  $y$ -intercept, set  $x = 0$  and solve for  $y$ .

$$5y + 3x = 30$$

$$5y + 3(0) = 30$$

$$5y + 0 = 30$$

$$5y = 30$$

$$y = 6$$

$y$ -intercept  $(0, 6)$

To find the  $x$ -intercept, set  $y = 0$  and solve for  $x$ .

$$5y + 3x = 30$$

$$5(0) + 3x = 30$$

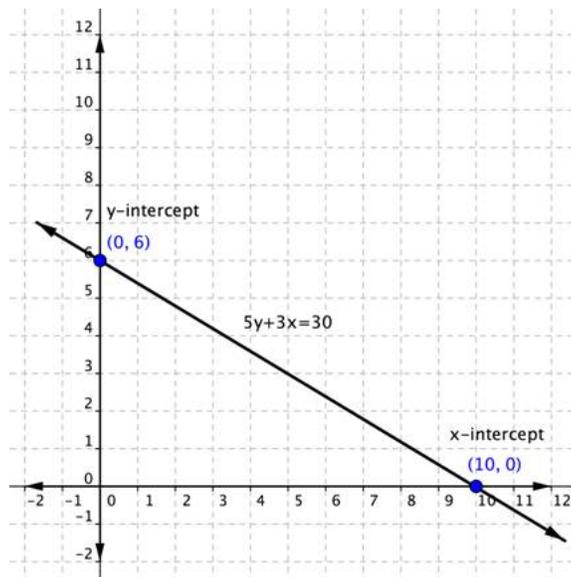
$$0 + 3x = 30$$

$$3x = 30$$

$$x = 10$$

$x$ -intercept  $(10, 0)$

The graph can be seen below.



**EXAMPLE**  
Graph  $4y - 6x = 24$  using x and y-intercepts.

x	y	(x, y)
-4	0	$(-4, 0)$
0	6	$(0, 6)$

x-intercept  $\rightarrow$  set  $y = 0$   
 $4y - 6x = 24$   
 $-6x = 24$   
 $\div -6$   
 $x = -4 \rightarrow (-4, 0)$

y-intercept  $\rightarrow$  set  $x = 0$   
 $4y - 6x = 24$   
 $4y = 24$   
 $\div 4$   
 $y = 6 \rightarrow (0, 6)$

To determine the x-intercept (horizontal intercept), set  $y = 0$  and solve for  $x$ .

To determine the y-intercept (vertical intercept), set  $x = 0$  and solve for  $y$ .

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=699>

### Example

Graph  $y = 2x - 4$  using the x and y-intercepts.

**Answer**

First, find the y-intercept. Set  $x$  equal to zero and solve for  $y$ .

$$y = 2x - 4$$

$$y = 2(0) - 4$$

$$y = 0 - 4$$

$$y = -4$$

y-intercept  $(0, -4)$

To find the x-intercept, set  $y = 0$  and solve for  $x$ .

$$y = 2x - 4$$

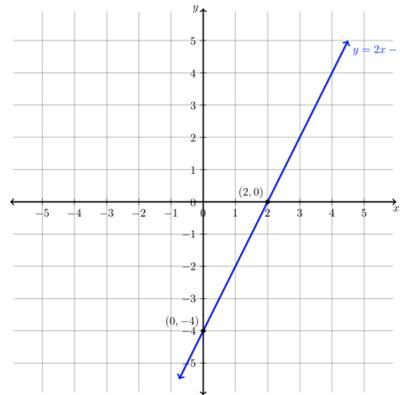
$$0 = 2x - 4$$

$$4 = 2x$$

$$x = 2$$

$x$ -intercept  $(2, 0)$

The graph can be seen below.



## INTRODUCTION TO GRAPHING LINEAR INEQUALITIES

### What You Will Learn to Do: Graph Linear Inequalities on the Coordinate Plane

Did you know that you also use linear equations when you shop online? When you use the option to view items within a specific price range, you are asking the search engine to use a linear inequality based on price. Essentially, you are saying, "Show me all the items for sale between \$50 and \$100," which can be written as  $50 \leq x \leq 100$ , where  $x$  is price. In this section, you will apply what you know about graphing linear equations to graphing linear inequalities.

## GRAPH AN INEQUALITY IN TWO VARIABLES

### Learning Outcome

- Identify and follow steps for graphing a linear inequality with two variables

So how do you get from the algebraic form of an inequality, like  $y > 3x + 1$ , to a graph of that inequality? Plotting inequalities is fairly straightforward if you follow a couple steps.

To graph an inequality:

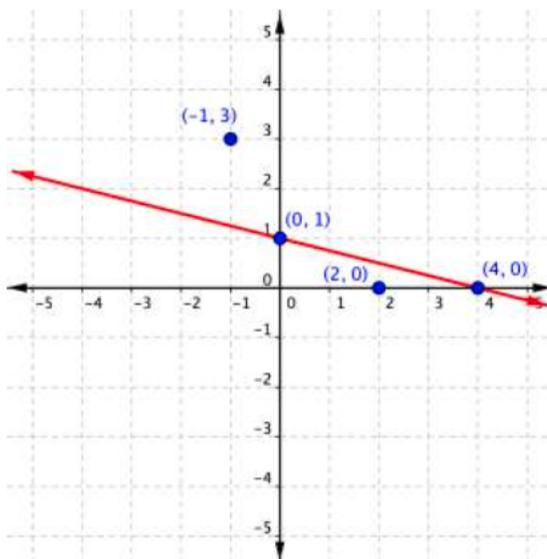
- Graph the related boundary line. Replace the  $<$ ,  $>$ ,  $\leq$  or  $\geq$  sign in the inequality with  $=$  to find the equation of the boundary line.
- Identify at least one ordered pair on either side of the boundary line and substitute those  $(x, y)$  values into the inequality. Shade the region that contains the ordered pairs that make the inequality a true statement.
- If points on the boundary line are solutions, then use a solid line for drawing the boundary line. This will happen for  $\leq$  or  $\geq$  inequalities.
- If points on the boundary line are not solutions, then use a dotted line for the boundary line. This will happen for  $<$  or  $>$  inequalities.

Graph the inequality  $x + 4y \leq 4$ .

To graph the boundary line, find at least two values that lie on the line  $x + 4y = 4$ . You can use the  $x$  and  $y$ -intercepts for this equation by substituting 0 in for  $x$  first and finding the value of  $y$ ; then substitute 0 in for  $y$  and find  $x$ .

$x$	$y$
0	1
4	0

Plot the points  $(0, 1)$  and  $(4, 0)$ , and draw a line through these two points for the boundary line. The line is solid because  $\leq$  means “less than or equal to,” so all ordered pairs along the line are included in the solution set.



The next step is to find the region that contains the solutions. Is it above or below the boundary line? To identify the region where the inequality holds true, you can test a couple of ordered pairs, one on each side of the boundary line.

If you substitute  $(-1, 3)$  into  $x + 4y \leq 4$ :

$$-1 + 4(3) \leq 4$$

$$-1 + 12 \leq 4$$

$$11 \leq 4$$

This is a false statement since 11 is not less than or equal to 4.

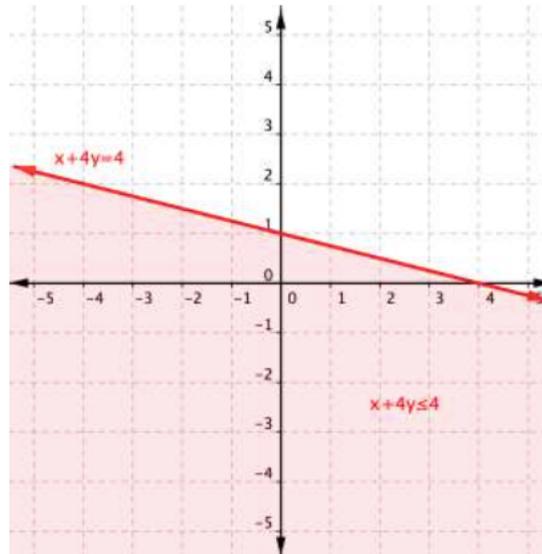
On the other hand, if you substitute (2, 0) into  $x + 4y \leq 4$ :

$$2 + 4(0) \leq 4$$

$$2 + 0 \leq 4$$

$$2 \leq 4$$

This is true! The region that includes (2, 0) should be shaded, as this is the region of solutions for the inequality.



And there you have it, the graph of the set of solutions for  $x + 4y \leq 4$ .

Below is a video about how to graph inequalities with two variables.

Example: Inequalities in Two Variables  
Graph  $x - 4y < 4$   
 $x - 4y = 4$  (dashed)  
Table:  

x	y
4	0
0	-1

  
Solving for y:  
 $x - 4y = 4$   
 $-4y = 4 - x$   
 $y = -1 + \frac{x}{4}$

A YouTube element has been excluded from this version of the text. You can view it online here:  
<https://courses.lumenlearning.com/intermediatealgebra/?p=739>

### Example

Graph the inequality  $2y > 4x - 6$ .

**Answer**

Solve for  $y$ .

$$2y > 4x - 6$$

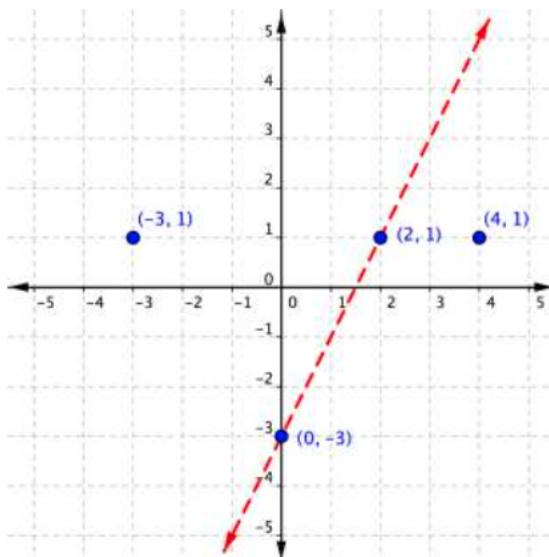
$$\frac{2y}{2} > \frac{4x}{2} - \frac{6}{2}$$

$$y > 2x - 3$$

Create a table of values to find two points on the line  $y = 2x - 3$ .

<b>x</b>	<b>y</b>
0	-3
2	1

Plot the points and graph the line. The line is dotted because the sign in the inequality is  $>$ , not  $\geq$  and therefore points on the line are not solutions to the inequality.



The inequality is  $2y > 4x - 6$ . Find an ordered pair on either side of the boundary line. Insert the  $x$  and  $y$  values into the inequality. If the simplified result is true, then shade on the side of the line the point is located.

Test 1 :  $(-3, 1)$

$$2(1) > 4(-3) - 6$$

$$2 > -12 - 6$$

$$2 > -18$$

TRUE

Test 2 :  $(4, 1)$

$$2(1) > 4(4) - 6$$

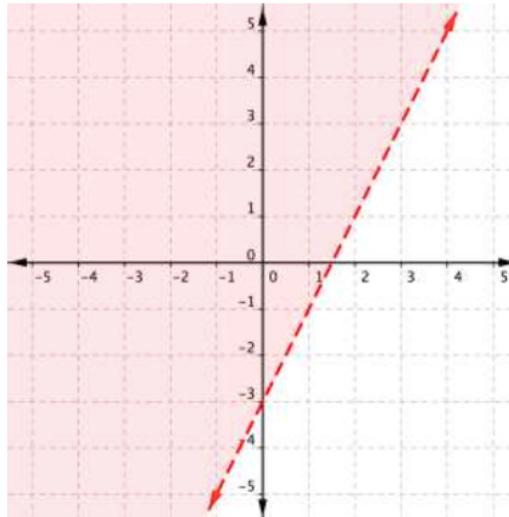
$$2 > 16 - 6$$

$$2 > 10$$

FALSE

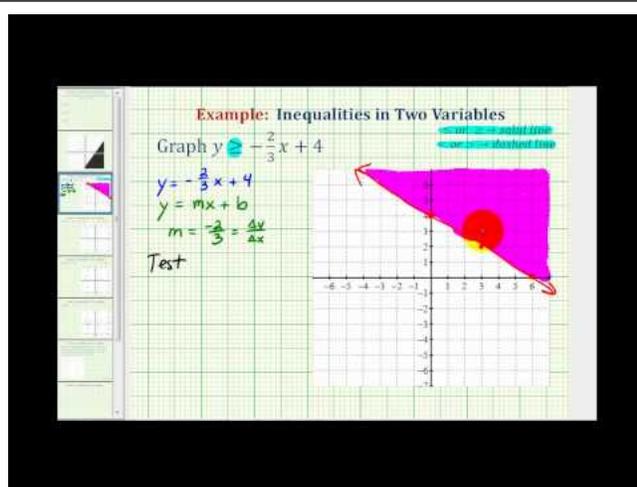
Since  $(-3, 1)$  results in a true statement, the region that includes  $(-3, 1)$  should be shaded.

The graph of the inequality  $2y > 4x - 6$  is:



A quick note about the problem above: notice that you can use the points  $(0, -3)$  and  $(2, 1)$  to graph the boundary line, but these points are not included in the region of solutions since the region does not include the boundary line!

Below is a video about how to graph inequalities with two variables when the equation is in what is known as slope-intercept form.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=739>

## Summary

When inequalities are graphed on a coordinate plane, the solutions are located in a region of the coordinate plane which is represented as a shaded area on the plane. The boundary line for the inequality is drawn as a solid line if the points on the line itself do satisfy the inequality, as in the cases of  $\leq$  and  $\geq$ . It is drawn as a

dashed line if the points on the line do not satisfy the inequality, as in the cases of  $<$  and  $>$ . You can tell which region to shade by testing some points in the inequality. Using a coordinate plane is especially helpful for visualizing the region of solutions for inequalities with two variables.

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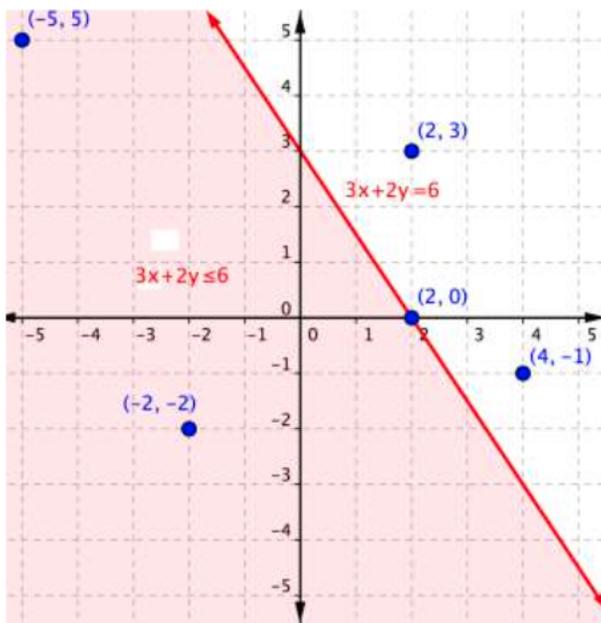
- Ex 2: Graphing Linear Inequalities in Two Variables (Standard Form). **Authored by:** James Sousa (Mathispower4u.com). **Located at:** <https://youtu.be/2VgFg2ztspl>. **License:** CC BY: Attribution
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## SOLUTION SETS OF INEQUALITIES

### LEARNING OUTCOME

- Identify whether an ordered pair is in the solution set of a linear inequality

The graph below shows the region of values that makes the inequality  $3x + 2y \leq 6$  true (shaded red), the boundary line  $3x + 2y = 6$ , as well as a handful of ordered pairs. The boundary line is solid because points on the boundary line  $3x + 2y = 6$  will make the inequality  $3x + 2y \leq 6$  true.



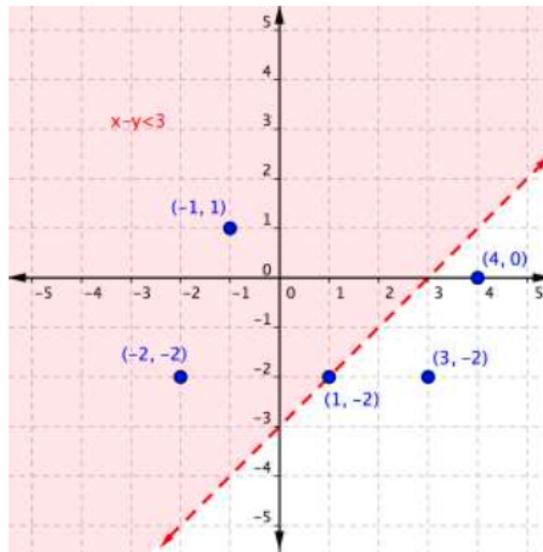
You can substitute the  $x$  and  $y$ -values of each of the  $(x, y)$  ordered pairs into the inequality to find solutions. Sometimes making a table of values makes sense for more complicated inequalities.

Ordered Pair	Makes the inequality $3x + 2y \leq 6$ a true statement	Makes the inequality $3x + 2y \leq 6$ a false statement
$(-5, 5)$	$3(-5) + 2(5) \leq 6$ $-15 + 10 \leq 6$ $-5 \leq 6$	
$(-2, -2)$	$3(-2) + 2(-2) \leq 6$ $-6 + (-4) \leq 6$ $-10 \leq 6$	
$(2, 3)$		$3(2) + 2(3) \leq 6$ $6 + 6 \leq 6$ $12 \leq 6$
$(2, 0)$	$3(2) + 2(0) \leq 6$ $6 + 0 \leq 6$ $6 \leq 6$	
$(4, -1)$		$3(4) + 2(-1) \leq 6$ $12 + (-2) \leq 6$ $10 \leq 6$

If substituting  $(x, y)$  into the inequality yields a true statement, then the ordered pair is a solution to the inequality, and the point will be plotted within the shaded region or the point will be part of a solid boundary line. A false statement means that the ordered pair is not a solution, and the point will graph outside the shaded region, or the point will be part of a dotted boundary line.

#### Example

Use the graph to determine which ordered pairs plotted below are solutions of the inequality  $x - y < 3$ .



**Answer**

Solutions will be located in the shaded region. Since this is a “less than” problem, ordered pairs on the boundary line are not included in the solution set.

The values below are located in the shaded region so they are solutions. When substituted into the inequality  $x - y < 3$ , they produce true statements.

$(-1, 1)$

$(-2, -2)$

These values are not located in the shaded region, so are not solutions. When substituted into the inequality  $x - y < 3$ , they produce false statements.

$(1, -2)$

$(3, -2)$

$(4, 0)$

$(-1, 1)$  and  $(-2, -2)$  are the ordered pairs indicated in the graph that are solutions to the inequality.

The following video shows an example of determining whether an ordered pair is a solution to an inequality.

A YouTube element has been excluded from this version of the text. You can view it online here:  
<https://courses.lumenlearning.com/intermediatealgebra/?p=512>

### Example

Is  $(2, -3)$  a solution of the inequality  $y < -3x + 1$ ?

**Answer**

If  $(2, -3)$  is a solution, then it will yield a true statement when substituted into the inequality  $y < -3x + 1$ .

$$y < -3x + 1$$

Substitute  $x = 2$  and  $y = -3$  into inequality.

$$-3 < -3(2) + 1$$

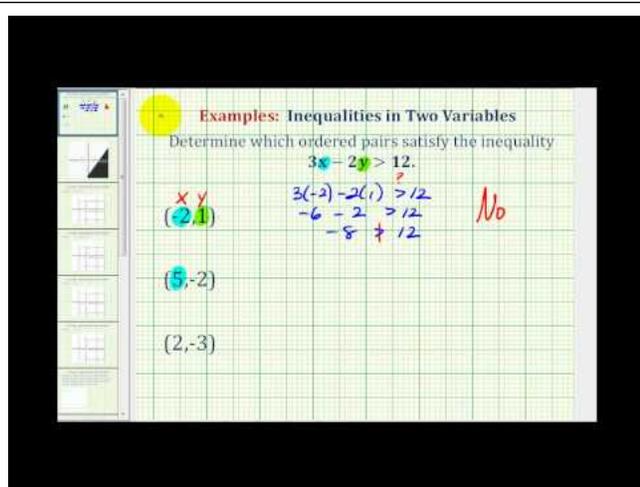
Evaluate.

$$-3 < -6 + 1$$

$$-3 < -5$$

This statement is **not** true, so the ordered pair  $(2, -3)$  is **not** a solution.

The following video shows another example of determining whether an ordered pair is a solution to an inequality.



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- Unit 13: Graphing, from Developmental Math: An Open Program. **Provided by:** Monterey Institute of Technology. **Located at:** <http://nrocnetwork.org/dm-opentext>. **License:** CC BY: Attribution

## PUTTING IT TOGETHER: GRAPH LINEAR EQUATIONS AND INEQUALITIES

Joan’s uncle Devon is a semiprofessional blogger who writes about media (he would really like to be a paid film critic, but no one has offered yet). Recently, Devon posted a short blog piece complaining about the number of ads on TV these days compared to when he was younger. He was not very scientific about it, and a couple of his readers disputed a claim he made and tried to start an argument.



Devon asks Joan, who is taking an algebra class at night, to help him gather and cite some data to support his claim that there are “way more” TV ads now than when he was a kid. Together, they get online and do a little research and find a website that reports some interesting data on the number of minutes of TV commercials per hour since 2009. The data is shown below.

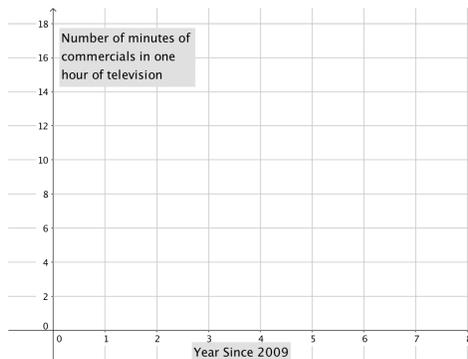
Years Since 2009	Minutes
0	8.5
1	9.25
2	10
3	10.75
4	11.5
5	12.25
6	13

The data show that, yes, there are more commercials now than in 2009. Joan decides the table is not very exciting because of the success she had with her manager when she reported her increase in sales using a graph. Also, Devon doubts that his readers will actually read a table. He agrees that a graph is the perfect thing to post on his blog to convince his skeptical readers!

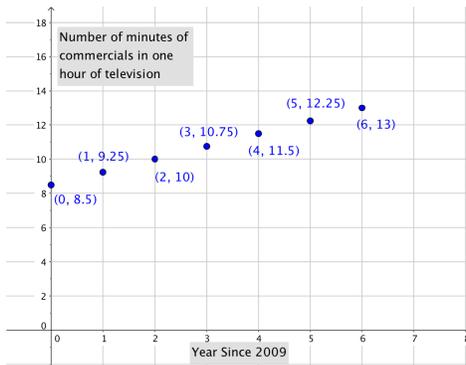
They get to work.

First, they figure out what information they need to draw a graph of the line that represents the change in the number of minutes of commercials in one hour of TV since 2009?

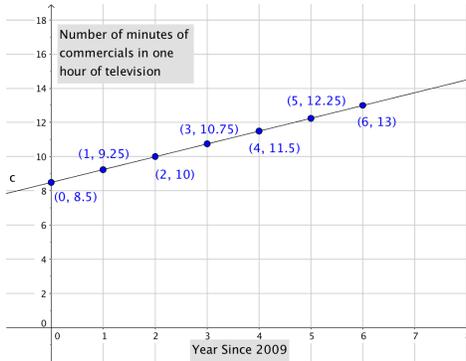
Joan remembers that the coordinate plane gives graphs structure and meaning. A straight line on a page will not tell Devon’s readers much. Joan draws the axes and labels the horizontal one “Years Since 2009,” because that is the first data point they have. Then Devon labels the vertical axis from 1 to 18, because the minute data ranges from 8.5 to 13 minutes.



They plot the ordered pairs from the table of values on the coordinate plane as shown below.



The points give them a guide for drawing the line which is shown below.



Devon is excited to post the graph on his blog to show people how much more time they are being exposed to commercials in one hour of watching TV.

# MODULE 4: FUNCTIONS AND FUNCTION NOTATION

## WHY IT MATTERS: FUNCTIONS AND FUNCTION NOTATION

### Why Learn About Functions and Function Notation?

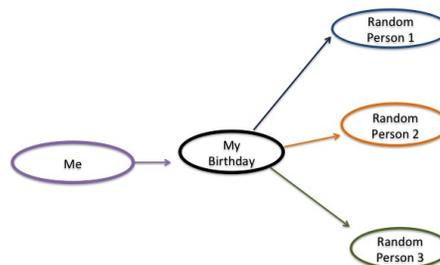


It is Joan's birthday, and she has decided to spend it with her friend Hazel. They are going to the city to see an art exhibit of paintings by artists from the Northwest School, including some pieces by Mark Tobey, her favorite artist. Joan has been looking forward to it for a long time.

As Joan and Hazel are driving to the city, Joan wonders how many other people in the world share her birthday. Joan remembers following some click-bait on the internet about birthdays and reading that September 16 is the most common birthday in the U.S., while December 25 and February 29 are the least common. (Note: <http://www.dailymail.co.uk/femail/article-2145471/How-common-birthday-Chart-reveals-date-rates.html#ixzz4Hnj1TAyd> Accessed on 8/19/2016).

She considers how it is interesting that each person only has one unique birthday, while many people might share the same birthday, and doodles the following drawing on the back of an envelope.

Without realizing it, Joan has discovered the definition of a mathematical function. Think of each individual person on Earth as a variable,  $p$ . Now imagine that all the birthdays are a function,  $B$ . For each individual person you place in the Birthday function, you will get out one unique birthday for that person. If you were to go backward, though, you could have many people with the same birthday.



In this module we will introduce the definition of a function and the formal mathematical notation that is used to express functions. You will see that there are many mathematical relationships that you are already familiar with that fit the definition of a function. Probably the most familiar of these relationships are linear equations.

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# INTRODUCTION TO FUNCTIONS AND THEIR NOTATION

Algebra gives us a way to explore and describe relationships. Imagine tossing a ball straight up in the air and watching it rise to reach its highest point before dropping back down into your hands. As time passes, the height of the ball changes. There is a relationship between the amount of time that has elapsed since the toss and the height of the ball. In mathematics, a correspondence between variables that change together (such as time and height) is called a relation. Some, but not all relations, can also be described as functions.

## DEFINE A FUNCTION

### LEARNING OUTCOMES

- Define a function using tables
- Determine if a set of ordered pairs creates a function
- Define the domain and range of a function given as a table or a set of ordered pairs

There are many kinds of relations. A relation is simply a correspondence between sets of values or information. Think about members of your family and their ages. The pairing of each member of your family and their age is a relation. Each family member can be paired with an age in the set of ages of your family members. Another example of a relation is the pairing of a state with its U.S. senators. Each state can be matched with two individuals who have each been elected to serve as a senator. In turn, each senator can be matched with one specific state that he or she represents. Both of these are real-life examples of relations.

The first value of a relation is an input value and the second value is the output value. A **function** is a specific type of relation in which each input value has one and only one output value. An input is the *independent* value, and the output is the *dependent* value, as it depends on the value of the input.

Notice in the first table below, where the input is “name” and the output is “age,” each input matches with exactly one output. This is an example of a function.

Family Member's Name (Input)	Family Member's Age (Output)
Nellie	13
Marcos	11
Esther	46
Samuel	47
Nina	47
Paul	47
Katrina	21
Andrew	16
Maria	13
Ana	81

Compare this with the next table where the input is “age” and the output is “name.” Some of the inputs result in more than one output. This is an example of a correspondence that is *not* a function.

Family Member's Age (Input)	Family Member's Name (Output)
11	Marcos
13	Nellie, Maria
16	Andrew
21	Katrina
46	Esther
47	Samuel, Nina, Paul
81	Ana

Now let us look at some other examples to determine whether the relations are functions or not and under what circumstances. Remember that a relation is a function if there is only **one** output for each input.

Relations can be written as ordered pairs of numbers or as numbers in a table of values. By examining the inputs ( $x$ -coordinates) and outputs ( $y$ -coordinates), you can determine whether or not the relation is a function. Remember, in a function, each input has only one output.

There is a name for the set of input values and another name for the set of output values for a function. The set of input values is called the **domain of the function**. The set of output values is called the **range of the function**.

If you have a set of ordered pairs, you can find the domain by listing all of the input values, which are the  $x$ -coordinates. To find the range, list all of the output values, which are the  $y$ -coordinates.

Consider the following set of ordered pairs:

$$\{(-2, 0), (0, 6), (2, 12), (4, 18)\}$$

You have the following:

## EXAMPLE

Fill in the table.

Input	Output	Function?	Why or why not?
Name of senator	Name of state		
Name of state	Name of senator		
Time elapsed	Height of a tossed ball		
Height of a tossed ball	Time elapsed		
Number of cars	Number of tires		
Number of tires	Number of cars		

**Answer**

Input	Output	Function?	Why or why not?
Name of senator	Name of state	Yes	For each input, there will only be one output because a senator only represents one state.
Name of state	Name of senator	No	For each state that is an input, 2 names of senators would result because each state has two senators.
Time elapsed	Height of a tossed ball	Yes	At a specific time, the ball has one specific height.
Height of a tossed ball	Time elapsed	No	Remember that the ball was tossed up and fell down. So for a given height, there could be two different times when the ball was at that height. The input height can result in more than one output.
Number of cars	Number of tires	Yes	For any input of a specific number of cars, there is one specific output representing the number of tires.
Number of tires	Number of cars	Yes	For any input of a specific number of tires, there is one specific output representing the number of cars.

Domain:  $\{-2, 0, 2, 4\}$

Range:  $\{0, 6, 12, 18\}$

Now try it yourself.

In the following video we provide another example of identifying whether a table of values represents a function as well as determining the domain and range of each.

In the following video, we show how to determine whether a relation is a function and how to find the domain and range.

## Summary: Determining Whether a Relation is a Function

## EXAMPLE

List the domain and range for the following table of values where  $x$  is the input and  $y$  is the output.

$x$	$y$
-3	4
-2	4
-1	4
2	4
3	4

### Answer

The domain describes all the inputs, and we can use set notation with brackets  $\{ \}$  to make the list.

Domain :  $\{-3, -2, -1, 2, 3\}$

The range describes all the outputs.

Range :  $\{4\}$

We only listed 4 once because it is not necessary to list it every time it appears in the range.

**Example: Functions and Tables**  
Determine which tables are functions.

A function is a relation in which every input has exactly one output.  
Every  $x$  has exactly one corresponding  $y$ .

**Not a function**

$x$	$y$
2	1
-1	4
-4	0
2	3
5	7

**function**

$x$	$y$
4	-5
-3	-2
1	-2
2	-2
5	0

$x$	$y$
0	3
-1	4
4	0
2	-1
-1	1

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<https://courses.lumenlearning.com/intermediatealgebra/?p=73>

1. Identify the input values – this is your domain.
2. Identify the output values – this is your range.
3. If each value in the domain leads to only one value in the range, classify the relationship as a function. If any value in the domain leads to two or more values in the range, do not classify the relationship as a function.

## EXAMPLE

Define the domain and range for the following set of ordered pairs, and determine whether the relation given is a function.

$\{(-3, -6), (-2, -1), (1, 0), (1, 5), (2, 0)\}$

### Answer

We list all of the input values as the domain. The input values are represented first in the ordered pair as a matter of convention.

Domain:  $\{-3, -2, 1, 2\}$

Note how we did not enter repeated values more than once; it is not necessary.

The range is the list of outputs for the relation; they are entered second in the ordered pair.

Range:  $\{-6, -1, 0, 5\}$

Organizing the ordered pairs in a table can help you tell whether this relation is a function. By definition, the inputs in a function have only one output.

$x$	$y$
-3	-6
-2	-1
1	0
1	5
2	0

The relation is not a function because the input 1 has two outputs: 0 and 5.

The screenshot shows a digital learning interface with a grid background. At the top, the title "Domain, Range, and Determining Functions" is displayed in red. Below the title, a problem asks to find the domain and range for the relation  $\{(5,-7), (4,-7), (3,-7), (2,-7)\}$  and to determine if it is a function. The user has entered the domain as  $\{2, 3, 4, 5\}$  and the range as  $\{-7\}$ . There are two radio buttons for "Is the relation a function?": "Yes" (selected) and "No". To the right of the radio buttons are two empty ovals. At the bottom, a definition box states: "Domain: Set of all possible inputs (x-values)", "Range: Set of all possible outputs (y-value or function values)", "A function is a relation in which each input has exactly one output.", and "Ordered Pair: (input, output)".

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## EXAMPLE

Find the domain and range of the relation and determine whether it is a function.

$\{(-3, 4), (-2, 4), (-1, 4), (2, 4), (3, 4)\}$

**Answer**

Domain:  $\{-3, -2, -1, 2, 3\}$

Range:  $\{4\}$

To help you determine whether this is a function, you could reorganize the information by creating a table.

$x$	$y$
-3	4
-2	4
-1	4
2	4
3	4

Each input has only one output, and the fact that it is the same output (4) does not matter.

This relation is a function.

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# FUNCTION NOTATION

## LEARNING OUTCOMES

- Write functions using algebraic notation
- Use the vertical line test to determine whether a graph represents a function

Some people think of functions as “mathematical machines.” Imagine you have a machine that changes a number according to a specific rule such as “multiply by 3 and add 2” or “divide by 5, add 25, and multiply by  $-1$ .” If you put a number into the machine, a new number will pop out the other end having been changed according to the rule. The number that goes in is called the input, and the number that is produced is called the output.

You can also call the machine “ $f$ ” for function. If you put  $x$  into the box,  $f(x)$ , comes out. Mathematically speaking,  $x$  is the input, or the “independent variable,” and  $f(x)$  is the output, or the “dependent variable,” since it depends on the value of  $x$ .

$f(x) = 4x + 1$  is written in function notation and is read “ $f$  of  $x$  equals  $4x$  plus 1.” It represents the following situation: A function named  $f$  acts upon an input,  $x$ , and produces  $f(x)$  which is equal to  $4x + 1$ . This is the same as the equation  $y = 4x + 1$ .

Function notation gives you more flexibility because you do not have to use  $y$  for every equation. Instead, you could use  $f(x)$  or  $g(x)$  or  $c(x)$ . This can be a helpful way to distinguish equations of functions when you are dealing with more than one at a time.

## Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationship so that we can understand it, use it, and possibly even program it into a computer. There are various ways of representing functions. A standard **function notation** is one representation that facilitates working with functions.

Now you try it.

### EXAMPLE

Represent height as a function of age using function notation.

#### Answer

To represent “height as a function of age,” we start by identifying the variables:  $h$  for height and  $a$  for age.

$h$  is  $f$  of  $a$       We name the function  $f$ ;  
height is a function of age.

$h = f(a)$       We use parentheses to indicate the function input.

$f(a)$       We name the function  $f$ ;  
the expression is read as “ $f$  of  $a$ .”

Note: We can use any letter to name the function; the notation  $h = f(a)$  shows us that  $h$  depends on, or is a function of,  $a$ . The value  $a$  must be put into the function  $f$  to get a result (height). The parentheses indicate that age is the input for the function; they do not indicate multiplication.

Now try another.

### EXAMPLE

1. Write the formula for perimeter of a square,  $P = 4s$ , as a function.
2. Write the formula for area of a square,  $A = l^2$ , as a function.

#### Answer

1. Name the function  $P$ .  $P$  is a function of the length of the sides,  $s$ . Perimeter, as a function of side length, is equal to 4 times side length.  $P(s) = 4s$
2. Name the function  $A$ . Area, as a function of the length of the sides, is equal to the length squared.  $A(l) = l^2$ .

This would make it easy to graph both functions on the same graph without confusion about the variables.

We can also use an algebraic expression as the input to a function. For example  $f(a + b)$  means “first add  $a$  and  $b$ , and the result is the input for the function  $f$ .” The operations must be performed in this order to obtain the correct result.

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels

## A GENERAL NOTE: FUNCTION NOTATION

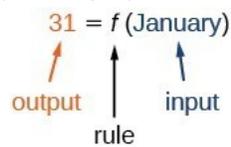
The notation  $y = f(x)$  defines a function named  $f$ . This is read as “ $y$  is a function of  $x$ .” The letter  $x$  represents the input value, or independent variable. The letter  $y$  or  $f(x)$ , represents the output value, or dependent variable.

### EXAMPLE

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

#### Answer

The number of days in a month is a function of the name of the month, so if we name the function  $f$ , we write  $\text{days} = f(\text{month})$  or  $d = f(m)$ . The name of the month is the input to a “rule” that associates a specific number (the output) with each input.



For example,  $f(\text{March}) = 31$ , because March has 31 days. The notation  $d = f(m)$  reminds us that the number of days,  $d$  (the output), is dependent on the name of the month,  $m$  (the input).

of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

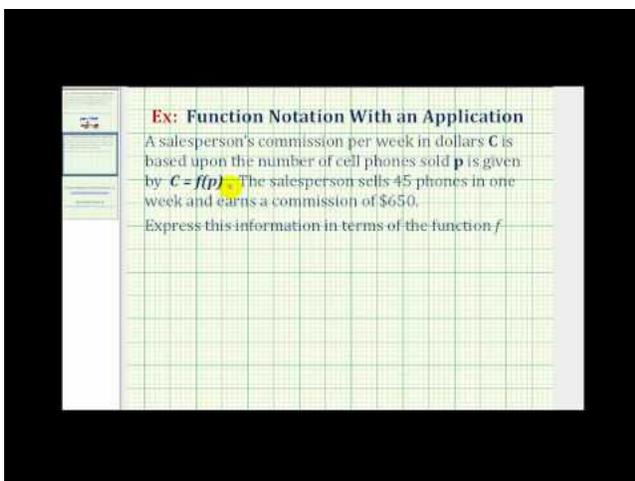
### EXAMPLE

A function  $N = f(y)$  gives the number of police officers,  $N$ , in a town in year  $y$ . What does  $f(2005) = 300$  represent?

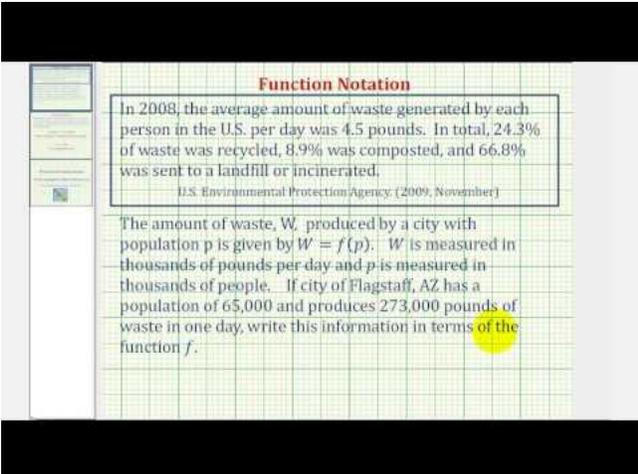
#### Answer

When we read  $f(2005) = 300$ , we see that the input year is 2005. The value for the output, the number of police officers,  $N$ , is 300. Remember,  $N = f(y)$ . The statement  $f(2005) = 300$  tells us that in the year 2005 there were 300 police officers in the town.

In the following videos we show two more examples of how to express a relationship using function notation.



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**Function Notation**

In 2008, the average amount of waste generated by each person in the U.S. per day was 4.5 pounds. In total, 24.3% of waste was recycled, 8.9% was composted, and 66.8% was sent to a landfill or incinerated.

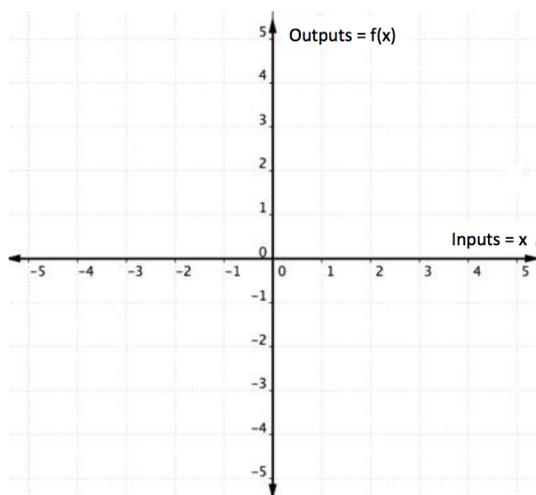
U.S. Environmental Protection Agency (2009, November)

The amount of waste,  $W$ , produced by a city with population  $p$  is given by  $W = f(p)$ .  $W$  is measured in thousands of pounds per day and  $p$  is measured in thousands of people. If city of Flagstaff, AZ has a population of 65,000 and produces 273,000 pounds of waste in one day, write this information in terms of the function  $f$ .

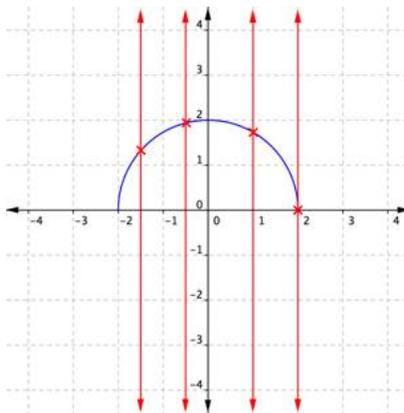
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## Graphs of Functions

When both the independent quantity (input) and the dependent quantity (output) are real numbers, a function can be represented by a graph in the coordinate plane. The independent value is plotted on the  $x$ -axis and the dependent value is plotted on the  $y$ -axis. The fact that each input value has exactly one output value means graphs of functions have certain characteristics. For each input on the graph, there will be exactly one output. For a function defined as  $y = f(x)$ , or  $y$  is a function of  $x$ , we would write ordered pairs  $(x, f(x))$  using function notation instead of  $(x, y)$  as you may have seen previously.



We can identify whether the graph of a relation represents a function because for each  $x$ -coordinate there will be exactly one  $y$ -coordinate.



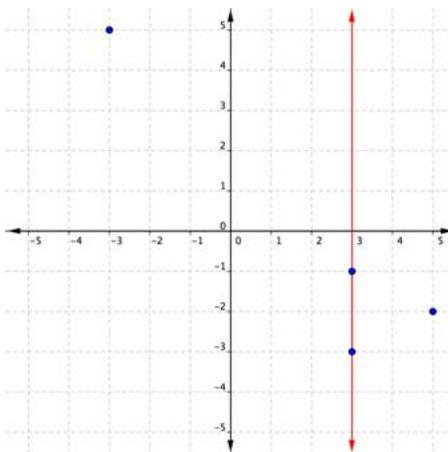
When a vertical line is placed across the plot of this relation, it does not intersect the graph more than once for any values of  $x$ .

If, on the other hand, a graph shows two or more intersections with a vertical line, then an input ( $x$ -coordinate) can have more than one output ( $y$ -coordinate), and  $y$  is not a function of  $x$ . Examining the graph of a relation to determine if a vertical line would intersect with more than one point is a quick way to determine if the relation shown by the graph is a function. This method is often called the “vertical line test.”

You try it.

The vertical line method can also be applied to a set of ordered pairs plotted on a coordinate plane to determine if the relation is a function.

Now consider the set of ordered pairs  $\{(3, -1), (5, -2), (3, -3), (-3, 5)\}$ . One of the inputs, 3, can produce two different outputs,  $-1$  and  $-3$ . You know what that means—this set of ordered pairs is not a function. A plot confirms this.

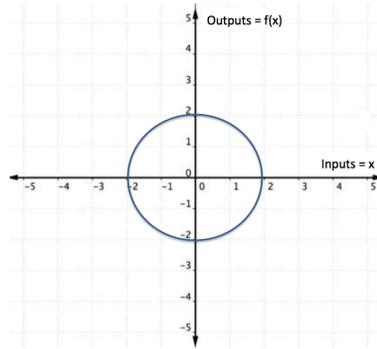


Notice that a vertical line passes through two plotted points. One  $x$ -coordinate has multiple  $y$ -coordinates. This relation is not a function.

In the following video, we show another example of determining whether a graph represents a function using the vertical line test.

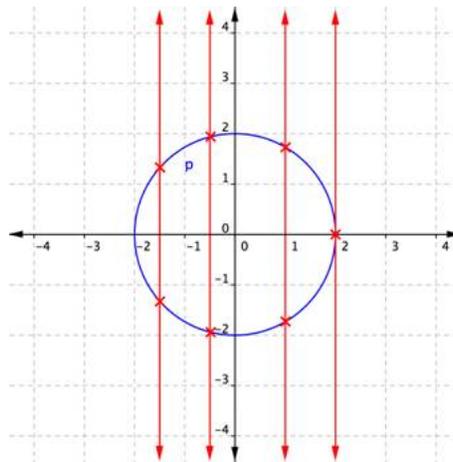
## EXAMPLE

Use the vertical line test to determine whether the relation plotted on this graph is a function.



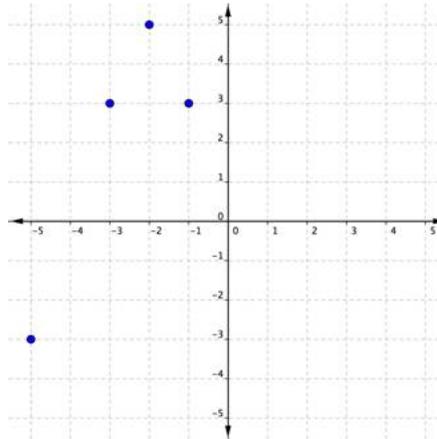
### Answer

This relationship cannot be a function, because some of the  $x$ -coordinates have two corresponding  $y$ -coordinates.



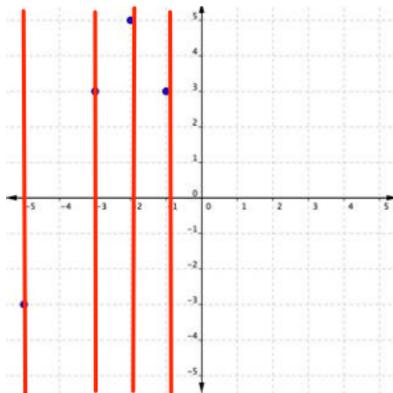
## EXAMPLE

Consider the ordered pairs  $\{(-1, 3), (-2, 5), (-3, 3), (-5, -3)\}$  plotted on the graph below. Use the vertical line test to determine whether the set of ordered pairs represents a function.



### Answer

Drawing vertical lines through each point results in each line only touching one point. This means that none of the  $x$ -coordinates have two corresponding  $y$ -coordinates, so this is a function.



**Example: Graphs as Function**

Which graphs represent functions?

A function is a relation in which every input or value in the domain is paired with exactly one output or value in the range.

Graphically, the vertical line test is an easy way to see if a relation is a function.

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# EVALUATE FUNCTIONS

## LEARNING OUTCOMES

- Given a function equation, find function values (outputs) for specified numbers and variables (inputs)

Throughout this course, you have been working with algebraic equations. Many of these equations are functions. For example,  $y = 4x + 1$  is an equation that represents a function. When you input values for  $x$ , you can determine a single output for  $y$ . In this case, if you substitute  $x = 10$  into the equation you will find that  $y$  must be 41; there is no other value of  $y$  that would make the equation true.

Rather than using the variable  $y$ , the equations of functions can be written using **function notation**. Function notation is very useful when you are working with more than one function at a time and substituting more than one value in for  $x$ .

Equations written using function notation can also be evaluated. With function notation, you might see the following:

Given  $f(x) = 4x + 1$ , find  $f(2)$ .

You read this problem like this: “given  $f$  of  $x$  equals  $4x$  plus one, find  $f$  of 2.” While the notation and wording is different, the process of evaluating a function is the same as evaluating an equation. In both cases, you substitute 2 for  $x$ , multiply it by 4 and add 1, simplifying to get 9. In both a function and an equation, an input of 2 results in an output of 9.

$$f(x) = 4x + 1$$

$$f(2) = 4(2) + 1 = 8 + 1 = 9$$

You can simply apply what you already know about evaluating expressions to evaluate a function. It is important to note that the parentheses that are part of function notation do not mean multiply. The notation  $f(x)$  does not mean  $f$  multiplied by  $x$ . Instead, the notation means “ $f$  of  $x$ ” or “the function of  $x$ .” To evaluate the function, take the value given for  $x$ , and substitute that value in for  $x$  in the expression. Let us look at a couple of examples.

Functions can be evaluated for negative values of  $x$ , too. Keep in mind the rules for integer operations.

You may also be asked to evaluate a function for more than one value as shown in the example that follows.

## Variable Inputs

So far, you have evaluated functions for inputs that have been constants. Functions can also be evaluated

## EXAMPLE

Given  $f(x) = 3x - 4$ , find  $f(5)$ .

**Answer**

Substitute 5 in for  $x$  in the function.

$$f(5) = 3(5) - 4$$

Simplify the expression on the right side of the equation.

$$f(5) = 15 - 4$$

$$f(5) = 11$$

## EXAMPLE

Given  $p(x) = 2x^2 + 5$ , find  $p(-3)$ .

**Answer**

Substitute  $-3$  in for  $x$  in the function.

$$p(-3) = 2(-3)^2 + 5$$

Simplify the expression on the right side of the equation.

$$p(-3) = 2(9) + 5$$

$$p(-3) = 18 + 5$$

$$p(-3) = 23$$

## EXAMPLE

Given  $f(x) = |4x - 3|$ , find  $f(0)$ ,  $f(2)$ , and  $f(-1)$ .

**Answer**

Treat each of these like three separate problems. In each case, you substitute the value in for  $x$  and simplify.

Start with  $x = 0$ .

$$f(0) = |4(0) - 3| = |-3| = 3$$

$$f(0) = 3$$

Evaluate for  $x = 2$ .

$$f(2) = |4(2) - 3| = |5| = 5$$

$$f(2) = 5$$

Evaluate for  $x = -1$ .

$$f(-1) = |4(-1) - 3| = |-7| = 7$$

$$f(-1) = 7$$

for inputs that are variables or expressions. The process is the same, but the simplified answer will contain a variable. The following examples show how to evaluate a function for a variable input.

### Example

Given  $f(x) = 3x^2 + 2x + 1$ , find  $f(b)$ .

**Answer**

This problem is asking you to evaluate the function for  $b$ . This means substitute  $b$  in the equation for  $x$ .

$$f(b) = 3b^2 + 2b + 1$$

(That is all—you are done.)

In the following example, you evaluate a function for an expression. So here you will substitute the entire expression in for  $x$  and simplify.

### Example

Given  $f(x) = 4x + 1$ , find  $f(h + 1)$ .

#### Answer

This time, you substitute  $(h + 1)$  into the equation for  $x$ .

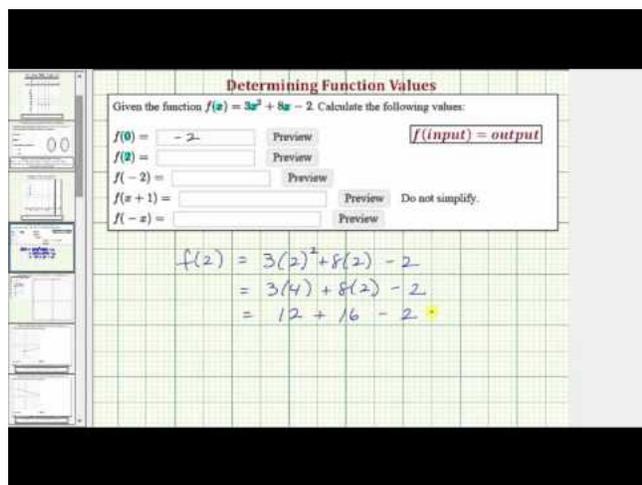
$$f(h + 1) = 4(h + 1) + 1$$

Use the distributive property on the right side, and then combine like terms to simplify.

$$f(h + 1) = 4h + 4 + 1 = 4h + 5$$

Given  $f(x) = 4x + 1$ ,  $f(h + 1) = 4h + 5$ .

In the following video, we show more examples of evaluating functions for both integer and variable inputs.



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## Summary

Function notation takes the form such as  $f(x) = 18x - 10$  and is read “ $f$  of  $x$  equals 18 times  $x$  minus 10.” Function notation can use letters other than  $f$ , such as  $c(x)$ ,  $g(x)$ , or  $h(x)$ . As you go further in your study of functions, this notation will provide you more flexibility, allowing you to examine and compare different functions more easily. Just as an algebraic equation written in  $x$  and  $y$  can be evaluated for different values of the input  $x$ , an equation written in function notation can also be evaluated for different values of  $x$ . To evaluate a function, substitute in values for  $x$  and simplify to find the related output.

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# INTRODUCTION TO GRAPHS OF FUNCTIONS

When both the input (independent variable) and output (dependent variable) are real numbers, a function can be represented by a coordinate graph. The input is plotted on the horizontal  $x$ -axis, and the output is plotted on the vertical  $y$ -axis.

## GRAPH LINEAR FUNCTIONS

### Learning Outcome

- Graph linear functions using tables
- Identify important features of graphs of linear functions

A helpful first step in graphing a function is to make a table of values. This is particularly useful when you do not know the general shape the function will have. You probably already know that a linear function will be a straight line, but let us make a table first to see how it can be helpful.

When making a table, it is a good idea to include negative values, positive values, and zero to ensure that you do have a linear function.

Make a table of values for  $f(x) = 3x + 2$ .

Make a two-column table. Label the columns  $x$  and  $f(x)$ .

$x$	$f(x)$

Choose several values for  $x$  and put them as separate rows in the  $x$  column. These are YOUR CHOICE – there is no “right” or “wrong” values to pick, just go for it.

*Tip:* It is always good to include 0, positive values, and negative values, if possible.

$x$	$f(x)$
-2	
-1	
0	
1	
3	

Evaluate the function for each value of  $x$ , and write the result in the  $f(x)$  column next to the  $x$  value you used.

When  $x = 0$ ,  $f(0) = 3(0) + 2 = 2$ ,

$$f(1) = 3(1) + 2 = 5,$$

$$f(-1) = 3(-1) + 2 = -3 + 2 = -1, \text{ and so on.}$$

$x$	$f(x)$
-2	-4
-1	-1
0	2
1	5
3	11

(Note that your table of values may be different from someone else's. You may each choose different numbers for  $x$ .)

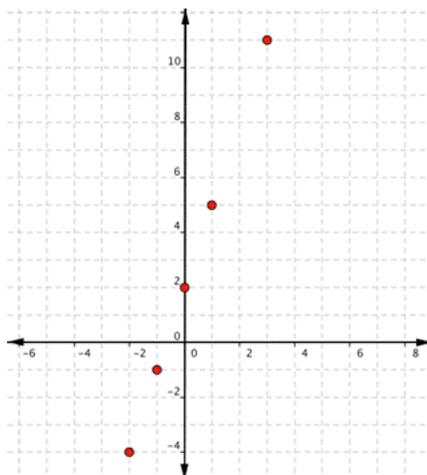
Now that you have a table of values, you can use them to help you draw both the shape and location of the function. *Important:* The graph of the function will show all possible values of  $x$  and the corresponding values of  $y$ . This is why the graph is a line and not just the dots that make up the points in our table.

Now graph  $f(x) = 3x + 2$ .

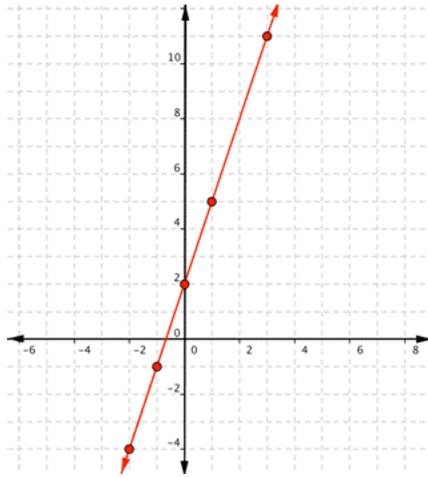
Using the table of values we created above, you can think of  $f(x)$  as  $y$ . Each row forms an ordered pair that you can plot on a coordinate grid.

$x$	$f(x)$
-2	-4
-1	-1
0	2
1	5
3	11

Plot the points.



Since the points lie on a line, use a straight edge to draw the line. Try to go through each point without moving the straight edge.



Let us try another one. Before you look at the answer, try to make the table yourself and draw the graph on a piece of paper.

### Example

Graph  $f(x) = -x + 1$ .

#### Answer

Start with a table of values. You can choose different values for  $x$ , but once again, it is helpful to include 0, some positive values, and some negative values.

If you think of  $f(x)$  as  $y$ , each row forms an ordered pair that you can plot on a coordinate grid.

$$f(-2) = -(-2) + 1 = 2 + 1 = 3$$

$$f(-1) = -(-1) + 1 = 1 + 1 = 2$$

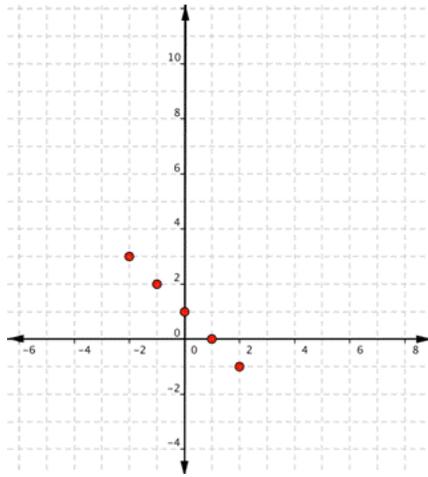
$$f(0) = -(0) + 1 = 0 + 1 = 1$$

$$f(1) = -(1) + 1 = -1 + 1 = 0$$

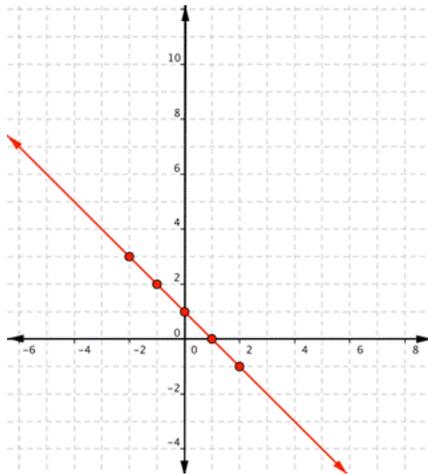
$$f(2) = -(2) + 1 = -2 + 1 = -1$$

$x$	$f(x)$
-2	3
-1	2
0	1
1	0
2	-1

Plot the points.



Since the points lie on a line, use a straight edge to draw the line. Try to go through each point without moving the straight edge.



In the following video, we show another example of how to graph a linear function on a set of coordinate axes.

**Graphing Lines by Plotting Points ( $m > 0$ )**

Graphing Linear Equations by plotting points.

Complete the table below for the equation  $f(x) = 4x - 5$ . Then use two of the ordered pairs to graph  $f(x)$ .

$f(-2) = 4(-2) - 5$   
 $= -8 - 5 = -13$   
 $f(-1) = 4(-1) - 5$   
 $= -4 - 5 = -9$   
 $f(x) = 4x - 5$

x	f(x)	Ordered Pair
-2	-13	$(-2, -13)$
-1	-9	$(-1, -9)$
0	-5	$(0, -5)$
1	-1	$(1, -1)$
2		

$f(2) = 4(2) - 5 = 8 - 5 = 3$   
 $f(1) = 4(1) - 5 = 4 - 5 = -1$   
 $f(x) = 4(x) - 5$

Clear All Done

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=98>

These graphs are representations of linear functions. Remember that a function is a correspondence between two variables such as  $x$  and  $y$ .

# GRAPH QUADRATIC FUNCTIONS

## LEARNING OUTCOMES

- Graph quadratic functions using tables and transformations
- Identify important features of the graphs of quadratic functions

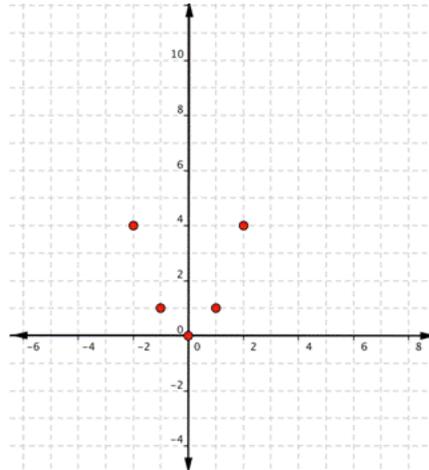
Quadratic functions can also be graphed. It is helpful to have an idea about what the shape should be so you can be sure that you have chosen enough points to plot as a guide. Let us start with the most basic quadratic function,  $f(x) = x^2$ .

Graph  $f(x) = x^2$ .

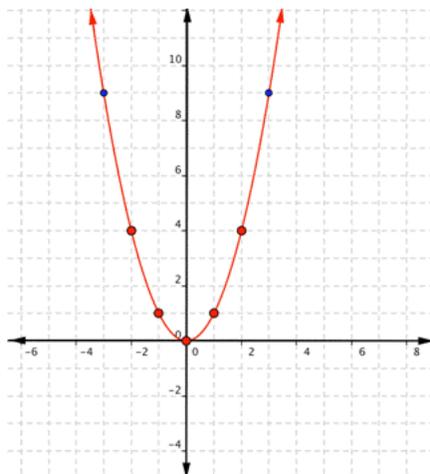
Start with a table of values. Then think of each row of the table as an ordered pair.

$x$	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

Plot the points  $(-2, 4)$ ,  $(-1, 1)$ ,  $(0, 0)$ ,  $(1, 1)$ ,  $(2, 4)$



Since the points are *not* on a line, you cannot use a straight edge. Connect the points as best as you can using a *smooth curve* (not a series of straight lines). You may want to find and plot additional points (such as the ones in blue here). Placing arrows on the tips of the lines implies that they continue in that direction forever.



Notice that the shape is similar to the letter U. This is called a parabola. One-half of the parabola is a mirror image of the other half. The lowest point on this graph is called the vertex. The vertical line that goes through the vertex is called the line of reflection. In this case, that line is the  $y$ -axis.

The equations for quadratic functions have the form  $f(x) = ax^2 + bx + c$  where  $a \neq 0$ . In the basic graph above,  $a = 1$ ,  $b = 0$ , and  $c = 0$ .

In the following video, we show an example of plotting a quadratic function using a table of values.

**Example: Graphing Functions**  
Complete the table using the function rule.

$$f(x) = \frac{1}{2}x^2$$

x	f(x) = y
-3	$\frac{1}{2}(-3)^2 = \frac{9}{2}$ (-3, 4.5)
-2	$\frac{1}{2}(-2)^2 = 2$ (-2, 2)
0	$\frac{1}{2}(0)^2 = 0$ (0, 0)
2	
3	

A YouTube element has been excluded from this version of the text. You can view it online here:  
<https://courses.lumenlearning.com/intermediatealgebra/?p=112>

Changing  $a$  changes the width of the parabola and whether it opens up ( $a > 0$ ) or down ( $a < 0$ ). If  $a$  is positive, the vertex is the lowest point, if  $a$  is negative, the vertex is the highest point. In the following example, we show how changing the value of  $a$  will affect the graph of the function.

If there is no  $b$  term, changing  $c$  moves the parabola up or down so that the  $y$  intercept is  $(0, c)$ . In the next example, we show how changes to  $c$  affect the graph of the function.

Changing  $b$  moves the line of reflection, which is the vertical line that passes through the vertex (the high or low point) of the parabola. It may help to know how to calculate the vertex of a parabola to understand how changing the value of  $b$  in a function will change its graph.

To find the vertex of the parabola, use the formula  $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ .

For example, if the function being considered is  $f(x) = 2x^2 - 3x + 4$ , to find the vertex, first calculate  $\frac{-b}{2a}$

## EXAMPLE

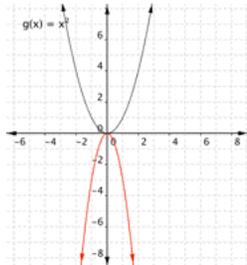
Match each function with its graph.

a)  $f(x) = 3x^2$

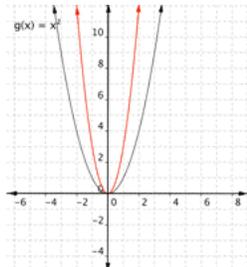
b)  $f(x) = -3x^2$

c)  $f(x) = \frac{1}{2}x^2$

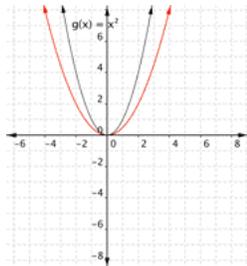
1)



2)



3)



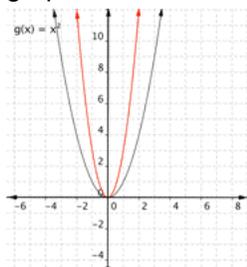
### Answer

Function a) matches graph 2

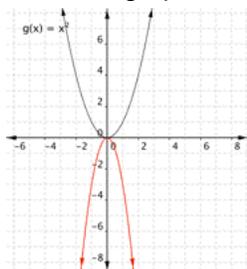
Function b) matches graph 1

Function c) matches graph 3

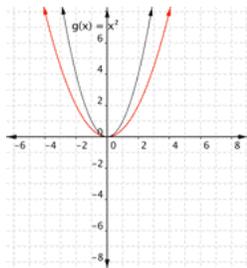
Function a)  $f(x) = 3x^2$  means that inputs are squared and then multiplied by three, so the outputs will be greater than they would have been for  $f(x) = x^2$ . This results in a parabola that has been squeezed, so graph 2 is the best match for this function.



Function b)  $f(x) = -3x^2$  means that inputs are squared and then multiplied by negative three, so the outputs will be farther away from the  $x$ -axis than they would have been for  $f(x) = x^2$ , but negative in value, so graph 1 is the best match for this function.



Function c)  $f(x) = \frac{1}{2}x^2$  means that inputs are squared then multiplied by  $\frac{1}{2}$ , so the outputs are less than they would be for  $f(x) = x^2$ . This results in a parabola that has been opened wider than  $f(x) = x^2$ . Graph 3 is the best match for this function.



$$a = 2, \text{ and } b = -3, \text{ therefore } \frac{-b}{2a} = \frac{-(-3)}{2(2)} = \frac{3}{4}.$$

This is the  $x$  value of the vertex.

Now evaluate the function at  $x = \frac{3}{4}$  to get the corresponding  $y$ -value for the vertex.

$$f\left(\frac{-b}{2a}\right) = 2\left(\frac{3}{4}\right)^2 - 3\left(\frac{3}{4}\right) + 4 = 2\left(\frac{9}{16}\right) - \frac{9}{4} + 4 = \frac{18}{16} - \frac{9}{4} + 4 = \frac{9}{8} - \frac{9}{4} + 4 = \frac{9}{8} - \frac{18}{8} + \frac{32}{8} =$$

The vertex is at the point  $\left(\frac{3}{4}, \frac{23}{8}\right)$ . This means that the vertical line of reflection passes through this point as well. It is not easy to tell how changing the values for  $b$  will change the graph of a quadratic function, but if you find the vertex, you can tell how the graph will change.

In the next example, we show how changing  $b$  can change the graph of the quadratic function.

Note that the vertex can change if the value for  $c$  changes because the  $y$ -value of the vertex is calculated by substituting the  $x$ -value into the function. Here is a summary of how the changes to the values for  $a$ ,  $b$ , and  $c$  of a quadratic function can change it is graph.

## Properties of a Parabola

For  $f(x) = ax^2 + bx + c$ , where  $a$ ,  $b$ , and  $c$  are real numbers,

- The parabola opens upward if  $a > 0$  and downward if  $a < 0$ .
- $a$  changes the width of the parabola. The parabola gets narrower if  $|a| > 1$  and wider if  $|a| < 1$ .
- The vertex depends on the values of  $a$ ,  $b$ , and  $c$ . The vertex is  $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ .

In the last example, we showed how you can use the properties of a parabola to help you make a graph

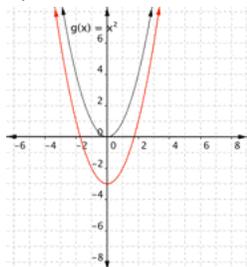
## EXAMPLE

Match each of the following functions with its graph.

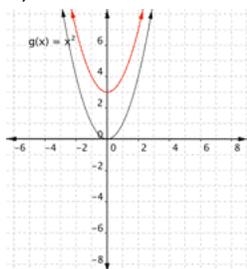
a)  $f(x) = x^2 + 3$

b)  $f(x) = x^2 - 3$

1)

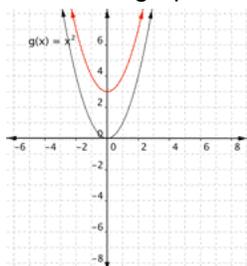


2)

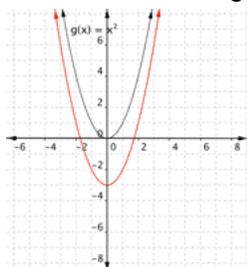


### Answer

Function a)  $f(x) = x^2 + 3$  means square the inputs then add three, so every output will be moved up 3 units. The graph that matches this function best is 2.



Function b)  $f(x) = x^2 - 3$  means square the inputs then subtract three, so every output will be moved down 3 units. The graph that matches this function best is 1.



without having to calculate an exhaustive table of values.

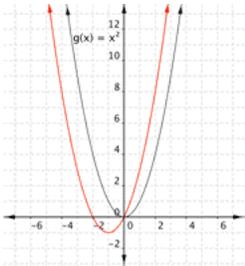
## EXAMPLE

Match each of the following functions with its graph.

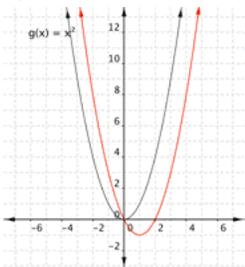
a)  $f(x) = x^2 + 2x$

b)  $f(x) = x^2 - 2x$

a)



b)



### Answer

Find the vertex of function a)

$$f(x) = x^2 + 2x.$$

$$a = 1, b = 2$$

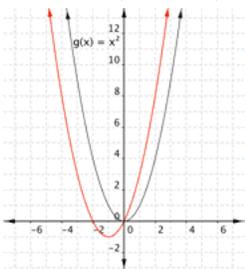
x-value:

$$\frac{-b}{2a} = \frac{-2}{2(1)} = -1$$

y-value:

$$f\left(\frac{-b}{2a}\right) = (-1)^2 + 2(-1) = 1 - 2 = -1.$$

Vertex =  $(-1, -1)$ , which means the graph that best fits this function is a)



Find the vertex of function b)

$$f(x) = x^2 - 2x.$$

$$a = 1, b = -2$$

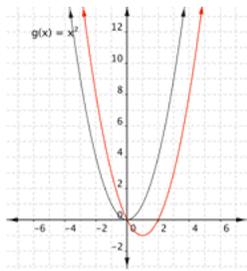
x-value:

$$\frac{-b}{2a} = \frac{2}{2(1)} = 1$$

y-value:

$$f\left(\frac{-b}{2a}\right) = (1)^2 - 2(1) = 1 - 2 = -1.$$

Vertex =  $(1, -1)$ , which means the graph that best fits this function is b)



### Example

Graph  $f(x) = -2x^2 + 3x - 3$ .

**Answer**

Before making a table of values, look at the values of  $a$  and  $c$  to get a general idea of what the graph should look like.

$a = -2$ , so the graph will open down and be thinner than  $f(x) = x^2$ .

$c = -3$ , so it will move to intersect the  $y$ -axis at  $(0, -3)$ .

To find the vertex of the parabola, use the formula  $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ . Finding the vertex may make graphing the parabola easier.

$$\text{Vertex formula} = \left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

x-coordinate of vertex:

$$\frac{-b}{2a} = \frac{-(3)}{2(-2)} = \frac{-3}{-4} = \frac{3}{4}$$

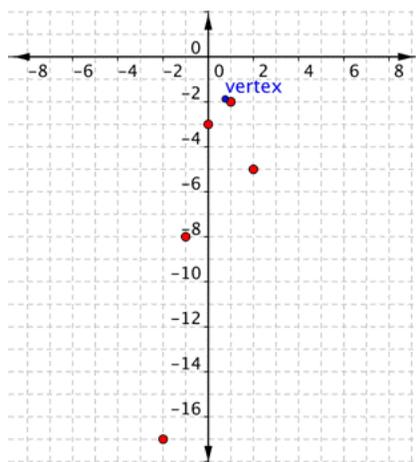
y-coordinate of vertex:

$$\begin{aligned} f\left(\frac{-b}{2a}\right) &= f\left(\frac{3}{4}\right) \\ f\left(\frac{3}{4}\right) &= -2\left(\frac{3}{4}\right)^2 + 3\left(\frac{3}{4}\right) - 3 \\ &= -2\left(\frac{9}{16}\right) + \frac{9}{4} - 3 \\ &= \frac{-18}{16} + \frac{9}{4} - 3 \\ &= \frac{-9}{8} + \frac{18}{8} - \frac{24}{8} \\ &= \frac{15}{8} \end{aligned}$$

$$\text{Vertex: } \left(\frac{3}{4}, -\frac{15}{8}\right)$$

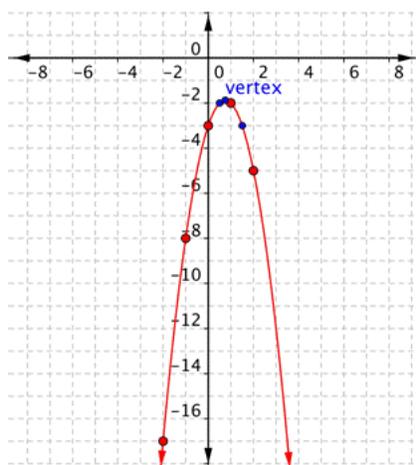
Use the vertex,  $\left(\frac{3}{4}, -\frac{15}{8}\right)$ , and the properties you described to get a general idea of the shape of the graph. You can create a table of values to verify your graph. Notice that in this table, the  $x$  values increase. The  $y$  values increase and then start to decrease again. This indicates a parabola.

$x$	$f(x)$
-2	-17
-1	-8
0	-3
1	-2
2	-5



Connect the points as best you can using a *smooth curve*. Remember that the parabola is two mirror images, so if your points do not have pairs with the same value, you may want to include additional points (such as the ones in blue shown below). Plot points on either side of the vertex.

$x = \frac{1}{2}$  and  $x = \frac{3}{2}$  are good values to include.



The following video shows another example of plotting a quadratic function using the vertex.

**The Graph of a Quadratic Function**

**EXAMPLE**

Graph the quadratic function by making a table of values and determining the vertex.  
 $f(x) = -\frac{1}{3}x^2 + 3x - 2$

$a = -\frac{1}{3} < 0$  The parabola opens downward.  
 $|a| = |-\frac{1}{3}| = \frac{1}{3} < 1$  The graph of the parabola will be wider than  $f(x) = x^2$ .

x	f(x)
0	-2
3	4
-3	-11
6	

$f(6) = -\frac{1}{3}(6)^2 + 3(6) - 2$   
 $= -\frac{1}{3}(36) + 18 - 2$

Vertex:  $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=112>

## Summary

Creating a graph of a function is one way to understand the relationship between the inputs and outputs of that function. Creating a graph can be done by choosing values for  $x$ , finding the corresponding  $y$  values, and plotting them. However, it helps to understand the basic shape of the function. Knowing how changes to the basic function equation affect the graph is also helpful.

The shape of a quadratic function is a parabola. Parabolas have the equation  $f(x) = ax^2 + bx + c$ , where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ . The value of  $a$  determines the width and the direction of the parabola, while the vertex depends on the values of  $a$ ,  $b$ , and  $c$ . The vertex is  $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$ .

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# GRAPH RADICAL FUNCTIONS

## LEARNING OUTCOME

- Graph radical functions using tables and transformations
- Identify important features of graphs of radical functions

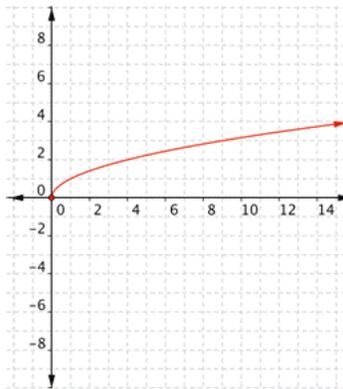
You can also graph radical functions (such as square root functions) by choosing values for  $x$  and finding points that will be on the graph. Again, it is helpful to have some idea about what the graph will look like.

Think about the basic square root function,  $f(x) = \sqrt{x}$ . Take a look at a table of values for  $x$  and  $y$  and then graph the function. Notice that all the values for  $x$  in the table are perfect squares. Since you are taking the square root of  $x$ , using perfect squares makes more sense than just finding the square roots of 0, 1, 2, 3, 4, etc.

$x$	$f(x)$
0	0
1	1
4	2
9	3
16	4

Recall that  $x$  can never be negative because when you square a real number, the result is always positive. For example,  $\sqrt{49}$ , this means “find the number whose square is 49.” Since there is no real number that we can square and get a negative, the function  $f(x) = \sqrt{x}$  will be defined for  $x > 0$ .

Take a look at the graph.



As with parabolas, multiplying and adding numbers makes some changes, but the basic shape is still the same. Here are some examples.

Multiplying  $\sqrt{x}$  by a positive value changes the width of the half-parabola. Multiplying  $\sqrt{x}$  by a negative number gives you the other half of a horizontal parabola.

In the following example, we will show how multiplying a radical function by a constant can change the shape of the graph.

Adding a value *outside* the radical moves the graph up or down. Think about it as adding the value to the basic  $y$  value of  $\sqrt{x}$ , so a positive value added moves the graph up.

Adding a value *inside* the radical moves the graph left or right. Think about it as adding a value to  $x$  before you take the square root—so the  $y$  value gets moved to a different  $x$  value. For example, for  $f(x) = \sqrt{x}$ , the square root is 3 if  $x = 9$ . For  $f(x) = \sqrt{x + 1}$ , the square root is 3 when  $x + 1$  is 9, so  $x$  is 8. Changing  $x$  to  $x + 1$  shifts the graph to the left by 1 unit (from 9 to 8). Changing  $x$  to  $x - 2$  shifts the graph to the right by 2 units.

Notice that as  $x$  gets greater, adding or subtracting a number inside the square root has less of an effect on the value of  $y$ .

In the next example, we will combine some of the changes that we have seen into one function.

## EXAMPLE

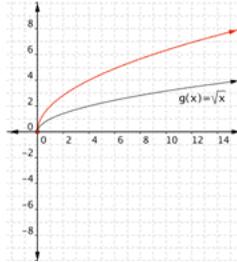
Match each of the following functions to the graph that it represents.

a)  $f(x) = -\sqrt{x}$

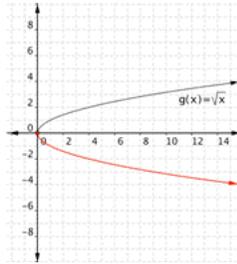
b)  $f(x) = 2\sqrt{x}$

c)  $f(x) = \frac{1}{2}\sqrt{x}$

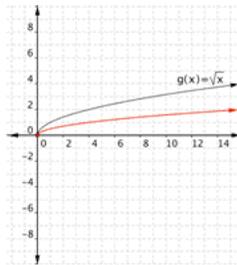
1)



2)



3)



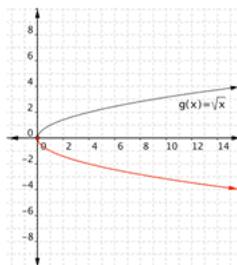
### Answer

Function a) goes with graph 2)

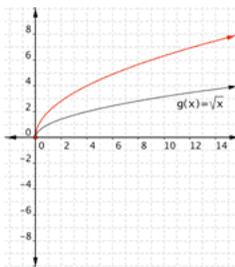
Function b) goes with graph 1)

Function c) goes with graph 3)

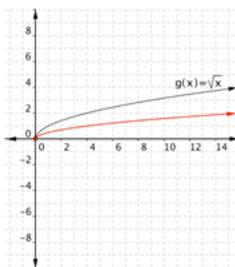
Function a)  $f(x) = -\sqrt{x}$  means that all the outputs will be negative – the function is the negative of the square roots of the input. This will give the other half of the parabola on its side. Therefore graph 2) goes with the function  $f(x) = -\sqrt{x}$ .



Function b)  $f(x) = 2\sqrt{x}$  means take the square root of all the inputs, then multiply by two, so the outputs will be larger than the outputs for  $\sqrt{x}$ . Graph 1) goes with the function  $f(x) = 2\sqrt{x}$ .



Function c)  $f(x) = \frac{1}{2}\sqrt{x}$  means take the square root of the inputs then multiply by  $\frac{1}{2}$ . The outputs will be smaller than the outputs for  $\sqrt{x}$ . Graph 3) goes with the function  $f(x) = \frac{1}{2}\sqrt{x}$ .



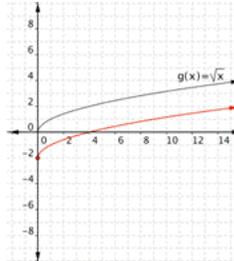
## EXAMPLE

Match each of the following functions to the graph that it represents.

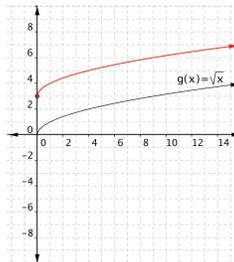
a)  $f(x) = \sqrt{x} + 3$

b)  $f(x) = \sqrt{x} - 2$

1)



2)

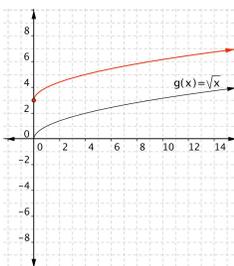


### Answer

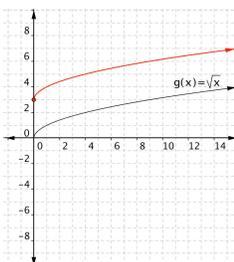
Function a) goes with graph 2)

Function b) goes with graph 1)

Function a)  $f(x) = \sqrt{x} + 3$  means take the square root of all the inputs and add three, so the outputs will be greater than those for  $\sqrt{x}$ , therefore graph 2) goes with this function.



Function b)  $f(x) = \sqrt{x} - 2$  means take the square root of the input then subtract two. The outputs will be less than those for  $\sqrt{x}$ , therefore graph 1) goes with this function.



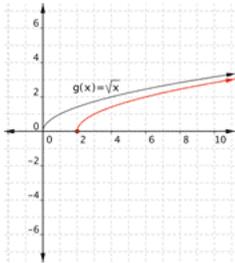
## EXAMPLE

Match each of the following functions to the graph that it represents.

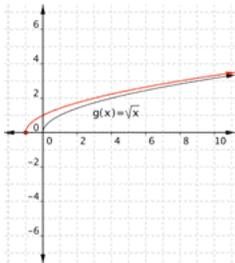
a)  $f(x) = \sqrt{x + 1}$

b)  $f(x) = \sqrt{x - 2}$

1)



2)

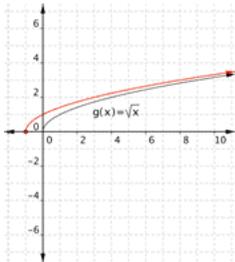


### Answer

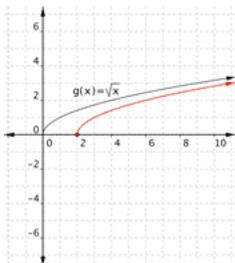
Function a) matches graph 2)

Function b) matches graph 1)

Function a)  $f(x) = \sqrt{x + 1}$  adds one to the inputs before the square root is taken. The outputs will be greater, so it ends up looking like a shift to the left. Graph 2) matches this function.



Function b)  $f(x) = \sqrt{x - 2}$  means subtract before the square root is taken. This makes the outputs less than they would be for the standard  $\sqrt{x}$ , and looks like a shift to the right. Graph 1) matches this function.



## EXAMPLE

Graph  $f(x) = -2 + \sqrt{x - 1}$ .

### Answer

Before making a table of values, look at the function equation to get a general idea about what the graph should look like.

Inside the square root, you are subtracting 1, so the graph will move to the right 1 from the basic  $f(x) = \sqrt{x}$  graph.

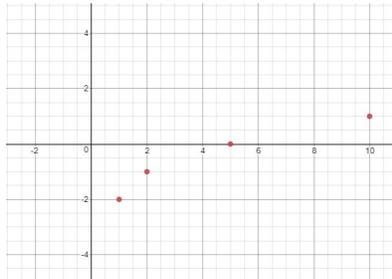
You are also adding  $-2$  outside the square root, so the graph will move down two from the basic  $f(x) = \sqrt{x}$  graph.

Create a table of values. Choose values that will make your calculations easy. You want  $x - 1$  to be a perfect square (0, 1, 4, 9, and so on) so you can take the square root.

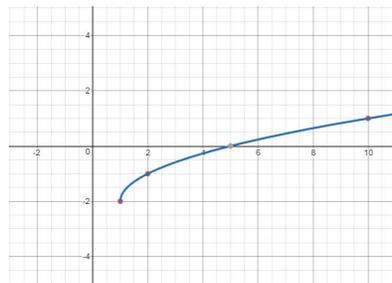
x	f(x)
1	-2
2	-1
5	0
10	1

Since values of  $x$  less than 1 makes the value inside the square root negative, there will be no points on the coordinate graph to the left of  $x = 1$ . There is no need to choose  $x$  values less than 1 for your table!

Use ordered pairs from each row of the table to plot points.



Connect the points as best you can using a smooth curve.



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# INTRODUCTION TO DOMAIN AND RANGE

Functions are a correspondence between two sets, called the **domain** and the **range**. When defining a function, you usually state what kind of numbers the domain ( $x$ ) and range ( $f(x)$ ) values can be. But even if you say they are real numbers, that does not mean that *all* real numbers can be used for  $x$ . It also does not mean that all real numbers can be function values,  $f(x)$ . There may be restrictions on the domain and range. The restrictions partly depend on the *type* of function.

## IDENTIFY A ONE-TO-ONE FUNCTION

### LEARNING OUTCOMES

- Define a one-to-one function
- Use the horizontal line test to determine whether a function is one-to-one

Remember that in a function, the input value must have one and only one value for the output. There is a name for the set of input values and another name for the set of output values for a function. The set of input values is called the **domain of the function**. The set of output values is called the **range of the function**.

In the first example, we remind you how to define domain and range using a table of values.

### EXAMPLE

Find the domain and range for the function.

$x$	$y$
-5	-6
-2	-1
-1	0
0	3
5	15

#### Answer

The domain is the set of inputs or  $x$ -coordinates.

$\{-5, -2, -1, 0, 5\}$

The range is the set of outputs or  $y$ -coordinates.

$\{-6, -1, 0, 3, 15\}$

In the following video, we show another example of finding domain and range from tabular data.

Some functions have a given output value that corresponds to two or more input values. For example, in the

**Domain and Range of a Discrete Function**

The function  $f(x)$  is defined in the table below. Using the table, identify the domain and range of  $f(x)$ .  
Write your answers as an ordered list enclosed in curly brackets.

Domain:  $\{ \}$

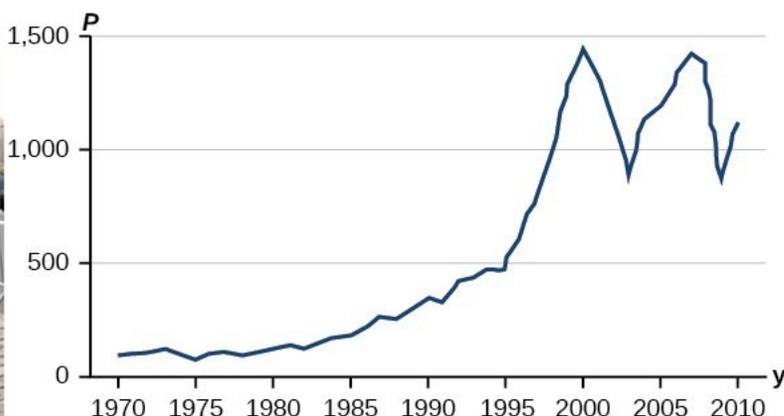
You can find the { and } to the right of the p on the keyboard. You do need to press Shift access these brackets.

inputs	outputs
$x$	$f(x)$
-3	11
-2	6
-1	3
0	2
1	3
2	6
3	11

The domain is the set of all possible inputs (x-values).  
The range is the set of all possible outputs (y-values).

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following stock chart the stock price was \$1000 on five different dates, meaning that there were five different input values that all resulted in the same output value of \$1000.

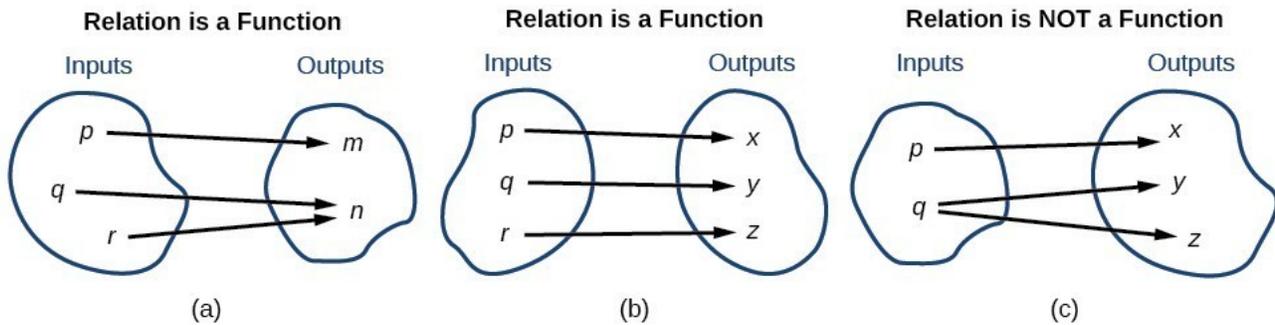


However, some functions have only one input value for each output value as well as having only one output value for each input value. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents as listed below.

Letter grade	Grade point average
A	4.0
B	3.0
C	2.0
D	1.0

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let us look again at the two simple functions sketched in (a) and (b) below. Note that (c) is not a function since the input  $q$  produces two outputs,  $y$  and  $z$ .



The function in part (a) shows a relationship that is not a one-to-one function because inputs  $q$  and  $r$  both give output  $n$ . The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

## A GENERAL NOTE: ONE-TO-ONE FUNCTION

A one-to-one function is a function in which each output value corresponds to exactly one input value.

### EXAMPLE

Determine whether each of the following tables represents a one-to-one function.

a)

Input	Output
1	5
12	2
0	-1
4	2
-5	0

b)

Input	Output
4	8
8	16
16	32
32	64
64	128

#### Answer

Table a) maps the output value 2 to two different input values, therefore this is NOT a one-to-one function.

Table b) maps each output to one unique input, therefore this IS a one-to-one function.

Only table b) is one-to-one.

In the following video, we show an example of using tables of values to determine whether a function is one-to-one.

**Example: One to One Functions**

- A function is a relation in which every input has exactly one output.  
Every  $x$  has exactly one corresponding  $y$ .
- A one-to-one function is a function in which each input has one unique output.  
Every  $y$  has exactly one corresponding  $x$ .

$x$	$y$
-4	-5
-2	2
1	0
3	2

Mapping:

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## Using the Horizontal Line Test

An easy way to determine whether a function is a one-to-one function is to use the **horizontal line test** on the graph of the function. To do this, draw horizontal lines through the graph. If any horizontal line intersects the graph more than once, then the graph does not represent a one-to-one function.

### HOW TO: GIVEN A GRAPH OF A FUNCTION, USE THE HORIZONTAL LINE TEST TO DETERMINE IF THE GRAPH REPRESENTS A ONE-TO-ONE FUNCTION

1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, determine that the function is not one-to-one.

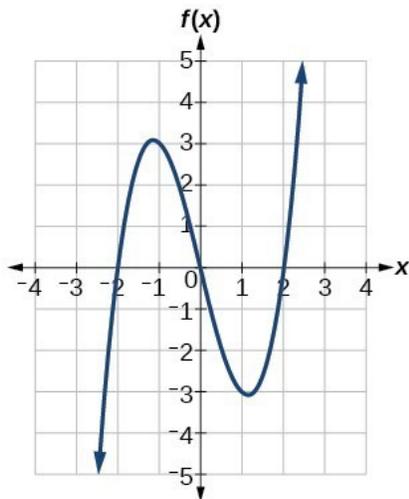
The following video provides another example of using the horizontal line test to determine whether a graph represents a one-to-one function.

## Summary

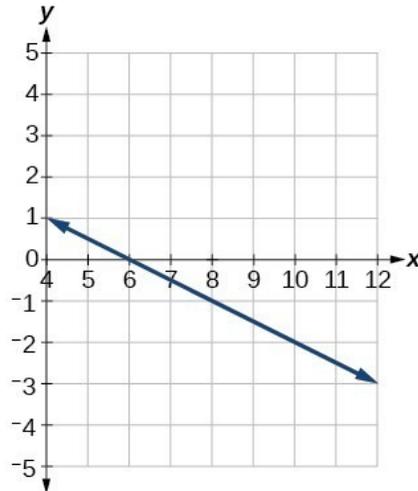
In real life and in algebra, different variables are often linked. When a change in value of one variable causes a change in the value of another variable, their interaction is called a relation. A relation has an input value which corresponds to an output value. When each input value has one and only one output value, the relation is a function. When each output value has one and only one input value, the function is one-to-one. Functions can be written as ordered pairs, tables, or graphs. The set of input values is called the domain, and the set of output values is called the range.

## EXAMPLE

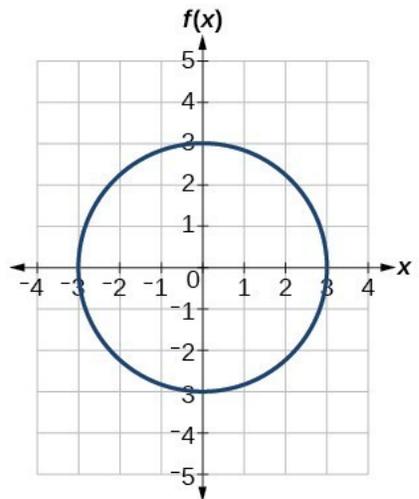
For the following graphs, determine which represent one-to-one functions.



(a)



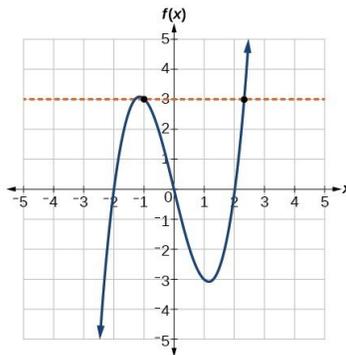
(b)



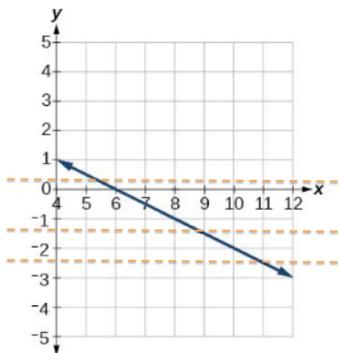
(c)

### Answer

The function in (a) is not one-to-one. Using the horizontal line test, as shown below, it intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)

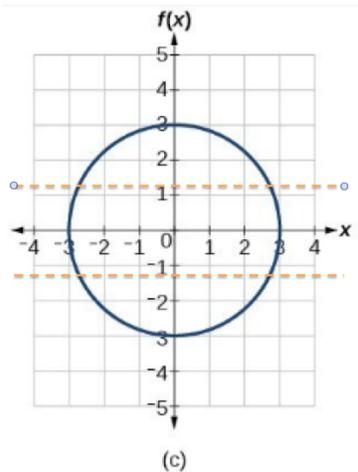


The function in (b) is one-to-one. Any horizontal line will intersect a diagonal line at most once.



(b)

The function (c) is not one-to-one and is in fact not a function.



**Ex: Graphs as One to One Function**  
Which graphs represent one-to-one functions?

Graphically, the vertical line test is a way to determine if a relation is a function.

Graphically, the horizontal line test is a way to determine if a function is a one-to-one function.

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<https://courses.lumenlearning.com/intermediatealgebra/?p=79>

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- Ex 1: Use the Vertical Line Test to Determine if a Graph Represents a Function. **Authored by:** James Sousa (Mathspower4u.com). **Located at:** <https://youtu.be/5z8DaZPJLKY>. **License:** *CC BY: Attribution*
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## DOMAIN RESTRICTIONS

### LEARNING OUTCOMES

- Find domain of a function algebraically

Functions are a correspondence between two sets, called the **domain** and the **range**. When defining a function, you usually state what kind of numbers the domain ( $x$ ) and range ( $f(x)$ ) values can be. But even if you say they are real numbers, that does not mean that *all* real numbers can be used for  $x$ . It also does not mean that all real numbers can be function values,  $f(x)$ . There may be restrictions on the domain and range. The restrictions partly depend on the *type* of function.

In this topic, all functions will be restricted to real number values. That is, only real numbers can be used in the domain, and only real numbers can be in the range.

There are two main reasons why domains are restricted.

- You cannot divide by 0.
- You cannot take the square (or other even) root of a negative number, as the result will not be a real number.

These two issues would be of concern when the function is either of the following:

- A rational function where the denominator could potentially become 0 for some value or values of  $x$ ,  $f(x) = \frac{x+1}{2-x}$  is a rational function.
- A radical function with an even index (such as a square root), where the radicand (quantity under the radical) could potentially be negative for some value or values of  $x$ .  $f(x) = \sqrt{7-x}$  is a radical function.

The following table gives examples of domain restrictions for several different rational functions. Note that a rational function has the variable present in the denominator.

Function	Restrictions to the Domain
$f(x) = \frac{1}{x}$	If $x = 0$ , you would be dividing by 0, so $x \neq 0$ .
$f(x) = \frac{2+x}{x-3}$	If $x = 3$ , you would be dividing by 0, so $x \neq 3$ .
$f(x) = \frac{2(x-1)}{x-1}$	Although you can simplify this function to $f(x) = 2$ , when $x = 1$ , the original function would include division by 0, so $x \neq 1$ .
$f(x) = \frac{x+1}{x^2-1}$	Both $x = 1$ and $x = -1$ would make the denominator 0. Again, this function can be simplified to $f(x) = \frac{1}{x-1}$ , but when $x = 1$ or $x = -1$ , the <i>original</i> function would include division by 0, so $x \neq 1$ and $x \neq -1$ .
$f(x) = \frac{2(x-1)}{x^2+1}$	This is an example with <i>no</i> domain restrictions, even though there is a variable in the denominator. Since $x^2 \geq 0$ , $x^2 + 1$ can never be 0. The least it can be is 1, so there is no danger of division by 0.

*Square roots of negative numbers* could happen whenever the function has a variable under a radical with an even root. Look at the following examples and note that “square root of a negative variable” doesn’t necessarily mean that the value under the radical sign is negative. For example, if  $x = -4$ , then  $-x = -(-4) = 4$ , a positive number.

Function	Restrictions to the Domain
$f(x) = \sqrt{x}$	If $x < 0$ , you would be taking the square root of a negative number, so $x \geq 0$ .
$f(x) = \sqrt{x + 10}$	If $x < -10$ , you would be taking the square root of a negative number, so $x \geq -10$ .
$f(x) = \sqrt{-x}$	When is $-x$ negative? Only when $x$ is positive. (For example, if $x = 1$ , then $-x = -1$ . But if $x = -3$ , then $-x = 3$ . This means $x \leq 0$ .)
$f(x) = \sqrt{x^2 - 1}$	$x^2 - 1$ must be positive, $x^2 - 1 > 0$ . So $x^2 > 1$ . This happens only when $x$ is greater than 1 or less than $-1$ : $x \leq -1$ or $x \geq 1$ .
$f(x) = \sqrt{x^2 + 10}$	There are no domain restrictions even though there is a variable under the radical. Since $x^2 \geq 0$ , $x^2 + 10$ can never be negative. The least it can be is $\sqrt{10}$ , so there is no danger of taking the square root of a negative number.

So, how exactly do you define the domain of a function?

## HOW TO: GIVEN A FUNCTION WRITTEN IN EQUATION FORM, FIND THE DOMAIN

1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

## EXAMPLE

Find the domain of the function  $f(x) = x^2 - 1$ .

### Answer

The input value, shown by the variable  $x$  in the equation, is squared and then the result is lowered by one. Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.

In interval form, the domain of  $f$  is  $(-\infty, \infty)$ .

## HOW TO: GIVEN A FUNCTION WRITTEN IN EQUATION FORM THAT INCLUDES A FRACTION, FIND THE DOMAIN

1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for  $x$ . If the function's formula contains an even root, set the radicand greater than or equal to 0 and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

The domain and range of a function are not necessarily the same. For example, the function  $f(x) = -\frac{1}{\sqrt{x}}$  has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

## EXAMPLE

Find the domain of the function  $f(x) = \frac{x+1}{2-x}$ .

### Answer

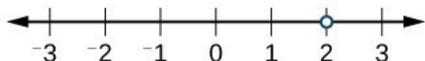
When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for  $x$ .

$$2 - x = 0$$

$$-x = -2$$

$$x = 2$$

Now, we will exclude 2 from the domain. The answers are all real numbers where  $x < 2$  or  $x > 2$ . We can use a symbol known as the union,  $\cup$ , to combine the two sets. In interval notation, we write the solution:  $(-\infty, 2) \cup (2, \infty)$ .

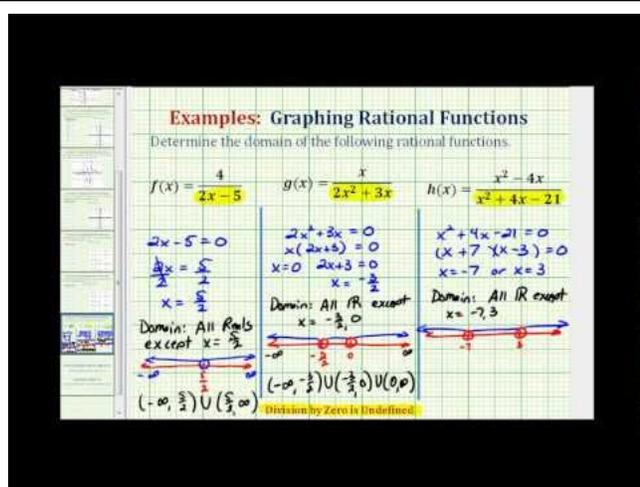


$$x < 2 \text{ or } x > 2$$

$$\downarrow \qquad \qquad \downarrow$$

$$(-\infty, 2) \cup (2, \infty)$$

In interval form, the domain of  $f$  is  $(-\infty, 2) \cup (2, \infty)$ .



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## HOW TO: GIVEN A FUNCTION WRITTEN IN EQUATION FORM INCLUDING AN EVEN ROOT, FIND THE DOMAIN

1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for  $x$ .
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

## EXAMPLE

Find the domain of the function  $f(x) = \sqrt{7-x}$ .

### Answer

When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.

Set the radicand greater than or equal to zero and solve for  $x$ .

$$7 - x \geq 0$$

$$-x \geq -7$$

$$x \leq 7$$

Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7, or  $(-\infty, 7]$ .

The screenshot shows a video titled "Ex: Domain and Range of Square Root Functions". It lists three functions and their domains and ranges:

Function	Domain	Range
$f(x) = \sqrt{x}$	$x \geq 0$   $[0, \infty)$	$y \geq 0$   $[0, \infty)$
$g(x) = \sqrt{x+3}$	$x+3 \geq 0$ $x \geq -3$   $[-3, \infty)$	
$h(x) = \sqrt{3-2x} - 4$		

At the bottom, definitions are provided: Domain: Set of all possible inputs (x-values); Range: Set of all possible outputs (y-value or function values).

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## FIND DOMAIN AND RANGE FROM A GRAPH

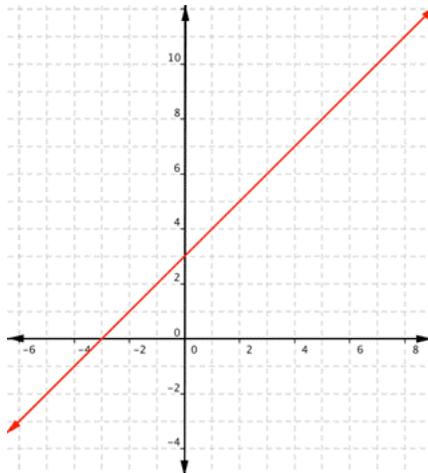
### LEARNING OUTCOME

- Find the domain and range from the graph of a function

Finding domain and range of different functions is often a matter of asking yourself, what values can this function *not* have? Pictures make it easier to visualize what the domain and range are, so we will show how to define the domain and range of functions given their graphs.

What are the domain and range of the real-valued function  $f(x) = x + 3$ ?

This is a *linear* function. Remember that linear functions are lines that continue forever in each direction.



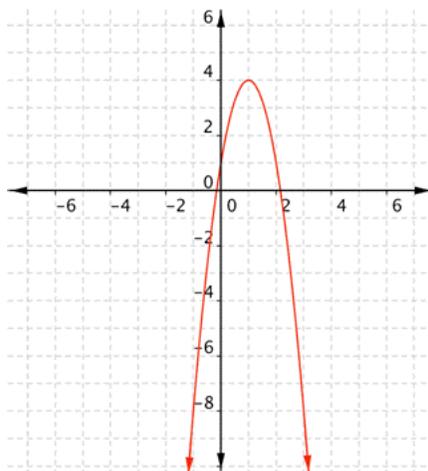
Any real number can be substituted for  $x$  and get a meaningful output. For *any* real number, you can always find an  $x$  value that gives you that number for the output. Unless a linear function is a constant, such as  $f(x) = 2$ , there is no restriction on the range.

The domain and range are all real numbers.

For the examples that follow, try to figure out the domain and range of the graphs before you look at the answer.

#### Example

What are the domain and range of the real-valued function  $f(x) = -3x^2 + 6x + 1$ ?



#### Answer

This is a *quadratic* function. There are no rational (divide by zero) or radical (negative number under a root) expressions, so there is nothing to restrict from the domain. Any real number can be used for  $x$  to get a meaningful output.

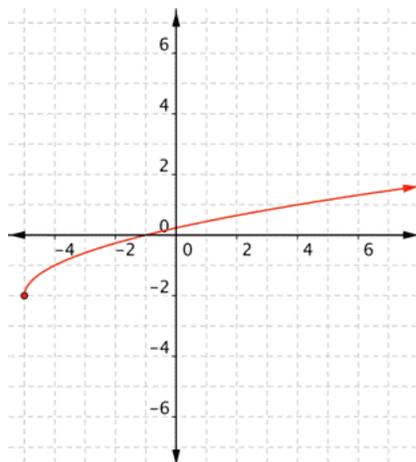
Because the coefficient of  $x^2$  is negative, it will open downward. With quadratic functions, remember that there is either a maximum (greatest) value or a minimum (least) value. In this case, there is a maximum value.

The vertex, or high point, is at  $(1, 4)$ . From the graph, you can see that  $f(x) \leq 4$ .

The domain is all real numbers, and the range is all real numbers  $f(x)$  such that  $f(x) \leq 4$ .

### Example

What is the domain and range of the real-valued function  $f(x) = -2 + \sqrt{x + 5}$ ?



### Answer

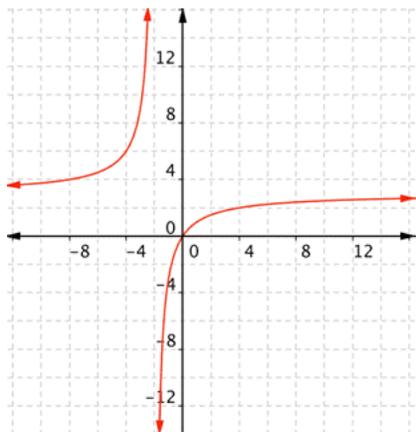
This is a *radical* function. The domain of a radical function is any  $x$  value for which the radicand (the value under the radical sign) is not negative. That means  $x + 5 \geq 0$ , so  $x \geq -5$ .

Since the square root must always be positive or 0,  $\sqrt{x + 5} \geq 0$ . That means  $-2 + \sqrt{x + 5} \geq -2$ .

The domain is all real numbers  $x$  where  $x \geq -5$ , and the range is all real numbers  $f(x)$  such that  $f(x) \geq -2$ .

### Example

What is the domain of the real-valued function  $f(x) = \frac{3x}{x + 2}$ ?

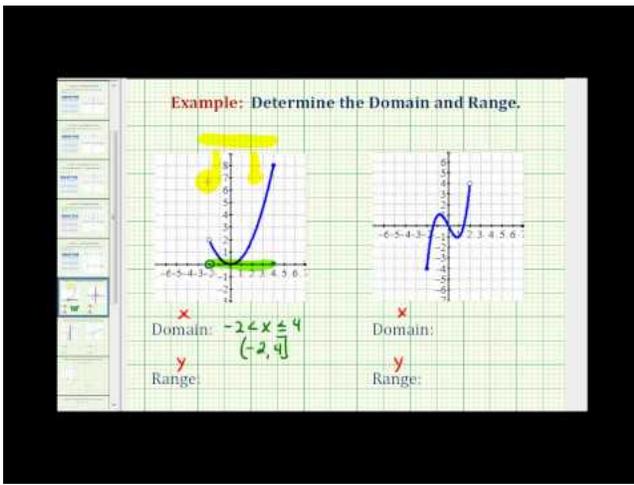


### Answer

This is a *rational* function. The domain of a rational function is restricted where the denominator is 0. In this case,  $x + 2$  is the denominator, and this is 0 only when  $x = -2$ .

The domain is all real numbers except  $-2$

In the following video we show how to define the domain and range of functions from their graphs.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=144>

## Summary

Although a function may be given as “real valued,” it may be that the function has restrictions to its domain and range. There may be some real numbers that cannot be part of the domain or part of the range. This is particularly true with rational and radical functions which can have restrictions to their domain, range, or both. Other functions, such as quadratic functions and polynomial functions of even degree, can also have restrictions to their range.

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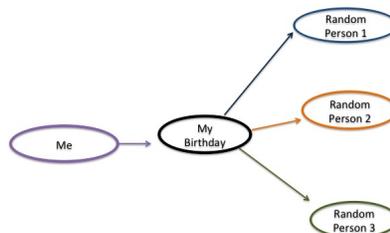
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# PUTTING IT TOGETHER: FUNCTIONS

At the beginning of the module, we left Joan pondering the idea that even though her birthday is unique to her – we all have only one birthday – each possible birth date is not unique with respect to how many people have it.



We proposed the following idea:

Think of each individual person on the earth as a variable,  $p$ . Now imagine that all the birthdays are a function,  $B$ . For each individual person you place in the Birthday function, you will get out one unique birthday for that person.

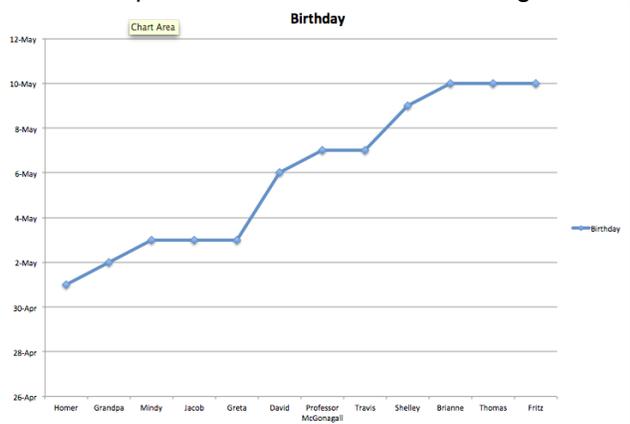
As we have shown in this module, we can write this relationship as a function in the following way:

$$B(p) = \text{Person } p\text{'s birthdate}$$

Joan decides to survey people so she can make a plot of her birthday function. She asks her Facebook friends (of which she has many) to send their birthdays. Some of the data she collected for May is in the table below.

Name	Birthday
Homer	May 1
Grandpa	May 2
Mindy, Edward	May 3
Jacob	May 4
Greta , Maureen	May 5
David	May 6
Professor McGonagall , January Jones	May 7
Travis	May 8
Shelley, Shannon, Kelvin	May 9
Brianne	May 10
Thomas, Monica	May 11
Fritz, Liam	May 12

Joan then plotted the data with Excel and got the following graph:



Notice how some dates, like May 3, have multiple people associated with them. All the people, on the other hand, have only one birthdate that they are associated with.

Functions are a very useful tool in mathematics, and we will continue to use them throughout the rest of this course. You will most definitely study more functions if you take more math courses in the future.

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# MODULE 5: LINEAR FUNCTIONS

## WHY IT MATTERS: LINEAR FUNCTIONS



### Why Learn About Linear Functions?

Joan wants to travel but does not have a lot of extra spending cash. She read on several travel blogs that a good way to travel is to get a credit card with air miles and use it for all the things you normally budget for. This way, a big chunk of your airfare can be paid for with air miles.

She applied for and got a credit card with air miles. Joan has planned to use her air miles card for the following things each month:

- gas = \$30
- car insurance = \$40
- mobile phone bill = \$65
- groceries = \$150
- entertainment – restaurants, movies, and live music = \$50

Each dollar she spends gives her 1.25 air miles, and she expects to need about 1200 air miles to buy the ticket she wants. Joan would like to plan for how long to save so she knows when to start looking for a good deal on airfare, but is not sure how to make sense of all the details of her budget and airfare needs.

In this module, you will learn how to take information like Joan’s and translate it into mathematical functions that can help answer questions like, “How long do I need to save to take a trip?” We will revisit Joan at the end of this module and help her figure out how long she will need to use her air miles card so she can go on her trip.

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## INTRODUCTION TO CHARACTERISTICS OF LINEAR FUNCTIONS AND THEIR GRAPHS



Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain types of bamboo species. These members of the grass family are the fastest-growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour. In a twenty-four hour period, this bamboo plant grows about 36 inches or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo species, is a linear function.

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## CALCULATE AND INTERPRET SLOPE

### LEARNING OUTCOMES

- Define slope for a linear function
- Calculate slope given two points

One well known form for writing linear functions is known as **slope-intercept form**, where  $x$  is the input value,  $m$  is the rate of change or slope, and  $b$  is the initial value of the dependant variable.

Equation form  $y = mx + b$

Function notation  $f(x) = mx + b$

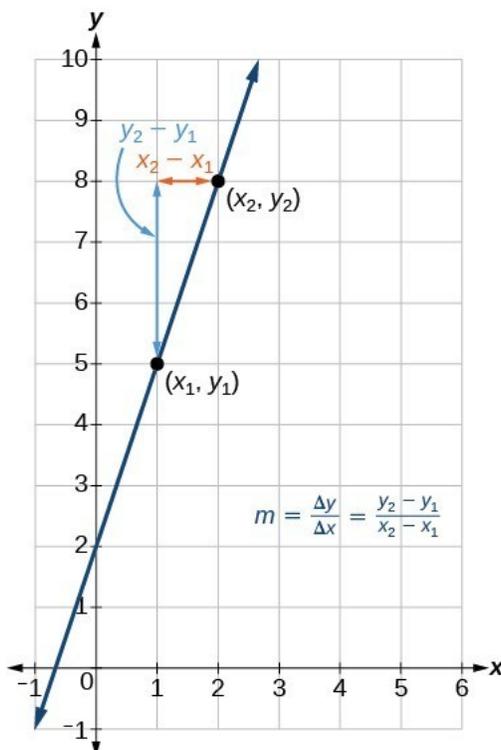
We often need to calculate the **slope** given input and output values. Given two values for the input,  $x_1$  and  $x_2$ , and two corresponding values for the output,  $y_1$  and  $y_2$  —which can be represented by a set of points,  $(x_1, y_1)$  and  $(x_2, y_2)$ —we can calculate the slope  $m$ , as follows

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where  $\Delta y$  is the vertical displacement and  $\Delta x$  is the horizontal displacement. Note in function notation two corresponding values for the output  $y_1$  and  $y_2$  for the function  $f$  are  $y_1 = f(x_1)$  and  $y_2 = f(x_2)$ , so we could equivalently write

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

The graph below indicates how the slope of the line between the points,  $(x_1, y_1)$  and  $(x_2, y_2)$  is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.



The slope of a function is calculated by the change in  $y$  divided by the change in  $x$ . It does not matter which coordinate is used as the  $(x_2, y_2)$  and which is the  $(x_1, y_1)$ , as long as each calculation is started with the elements from the same coordinate pair.

The units for slope are always  $\frac{\text{units for the output}}{\text{units for the input}}$ . Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output per units for the input.

## CALCULATING SLOPE

The slope, or rate of change, of a function  $m$  can be calculated using the following formula:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where  $x_1$  and  $x_2$  are input values,  $y_1$  and  $y_2$  are output values.

When the slope of a linear function is positive, the line is moving in an uphill direction from left to right across the coordinate axes. This is also called an increasing linear function. Likewise, a decreasing linear function

is one whose slope is negative. The graph of a decreasing linear function is a line moving in a downhill direction from left to right across the coordinate axes.

In mathematical terms,

For a linear function  $f(x) = mx + b$ , if  $m > 0$ , then  $f(x)$  is an increasing function.

For a linear function  $f(x) = mx + b$ , if  $m < 0$ , then  $f(x)$  is a decreasing function.

For a linear function  $f(x) = mx + b$ , if  $m = 0$ , then  $f(x)$  is a constant function. Sometimes we say this is neither increasing nor decreasing.

In the following example, we will first find the slope of a linear function through two points then determine whether the line is increasing, decreasing, or neither.

## EXAMPLE

If  $f(x)$  is a linear function and  $(3, -2)$  and  $(8, 1)$  are points on the line, find the slope. Is this function increasing or decreasing?

### Answer

The coordinate pairs are  $(3, -2)$  and  $(8, 1)$ . To find the rate of change, we divide the change in output by the change in input.

$$m = \frac{\text{change in output}}{\text{change in input}} = \frac{1 - (-2)}{8 - 3} = \frac{3}{5}$$

We could also write the slope as  $m = 0.6$ . The function is increasing because  $m > 0$ .

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value, or  $y$ -coordinate, used corresponds with the first input value, or  $x$ -coordinate, used.

In the following video we show examples of how to find the slope of a line passing through two points and then determine whether the line is increasing, decreasing or neither.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1988>

In the next video, we show an example where we determine the increase in cost for producing solar panels given two data points.

The following video provides an example of how to write a function that will give the cost in dollars for a given number of credit hours taken,  $x$ .

## EXAMPLE

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change of population per year if we assume the change was constant from 2008 to 2012.

### Answer

The rate of change relates the change in population to the change in time. The population increased by  $27,800 - 23,400 = 4400$  people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.

$$\frac{4,400 \text{ people}}{4 \text{ years}} = 1,100 \frac{\text{people}}{\text{year}}$$

So the population increased by 1,100 people per year.

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

The screenshot shows a green grid background with handwritten text and formulas. At the top, it says "Example: Application of Slope". Below that is the slope formula:  $m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$ . The text then describes a problem: "A company discovers that to produce 400 new solar panels, it will cost \$5450. To produce 650 new solar panels, it will cost \$7850. Determine the amount at which the cost is increasing or decreasing per panel." Two points are written:  $(400, 5450)$  and  $(650, 7850)$ , with  $x_1, y_1$  and  $x_2, y_2$  labels. At the bottom, the slope formula is written as  $m = \frac{\Delta y}{\Delta x} = \frac{\Delta \text{cost}}{\Delta x}$ .

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1988>

The screenshot shows a green grid background with handwritten text and a table. At the top, it says "Introduction to Linear Functions". Below that is the linear function:  $C(x) = 85x + 50$ . A table is shown with columns for  $x$ ,  $C(x)$ , and  $(x, C(x))$ . The table has four rows:  $x=0, C(x)=50, (0, 50)$ ;  $x=3, C(x)=305, (3, 305)$ ;  $x=6, C(x)=560, (6, 560)$ ; and  $x=12$ . Below the table, three calculations are shown:  $C(0) = 85(0) + 50 = 0 + 50 = 50$ ;  $C(3) = 85(3) + 50 = 255 + 50 = 305$ ; and  $C(6) = 85(6) + 50 = 510 + 50 = 560$ .

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- Write and Graph a Linear Function by Making a Table of Values (Intro). **Authored by:** James Sousa (Mathispower4u.com) for Lumen Learning. **Located at:** <https://youtu.be/X35x2TxH-J0>. **License:** CC BY: Attribution

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# GRAPH LINEAR FUNCTIONS USING SLOPE AND Y-INTERCEPT

## LEARNING OUTCOME

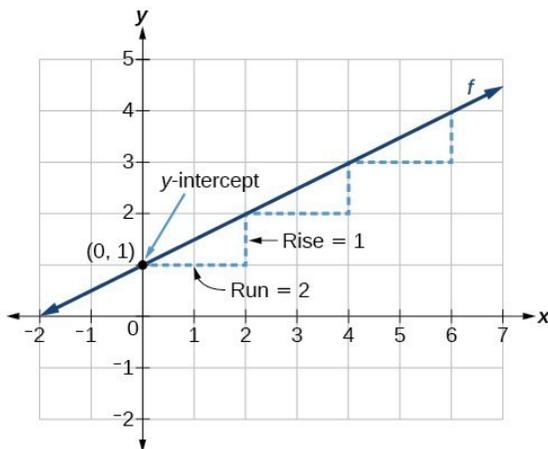
- Graph a linear function using the slope and y-intercept

Another way to graph a linear function is by using its slope  $m$  and y-intercept.

Let us consider the following function.

$$f(x) = \frac{1}{2}x + 1$$

The function is in slope-intercept form, so the slope is  $\frac{1}{2}$ . Because the slope is positive, we know the graph will slant upward from left to right. The y-intercept is the point on the graph when  $x = 0$ . The graph crosses the y-axis at  $(0, 1)$ . Now we know the slope and the y-intercept. We can begin graphing by plotting the point  $(0, 1)$ . We know that the slope is rise over run,  $m = \frac{\text{rise}}{\text{run}}$ . From our example, we have  $m = \frac{1}{2}$ , which means that the rise is 1 and the run is 2. So starting from our y-intercept  $(0, 1)$ , we can rise 1 and then run 2, or run 2 and then rise 1. We repeat until we have a few points and then we draw a line through the points as shown in the graph below.



All linear functions cross the y-axis and therefore have y-intercepts. (Note: A vertical line parallel to the y-axis does not have a y-intercept, but it is not a function.)

## A GENERAL NOTE: GRAPHICAL INTERPRETATION OF A LINEAR FUNCTION

In the equation  $f(x) = mx + b$

- $b$  is the  $y$ -intercept of the graph and indicates the point  $(0, b)$  at which the graph crosses the  $y$ -axis.
- $m$  is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

## HOW TO: GIVEN THE EQUATION FOR A LINEAR FUNCTION, GRAPH THE FUNCTION USING THE $y$ -INTERCEPT AND SLOPE

1. Evaluate the function at an input value of zero to find the  $y$ -intercept.
2. Identify the slope.
3. Plot the point represented by the  $y$ -intercept.
4. Use  $\frac{\text{rise}}{\text{run}}$  to determine at least two more points on the line.
5. Sketch the line that passes through the points.

In the following video we show another example of how to graph a linear function given the  $y$ -intercepts and the slope.

In the last example, we will show how to graph another linear function using the slope and  $y$ -intercept.

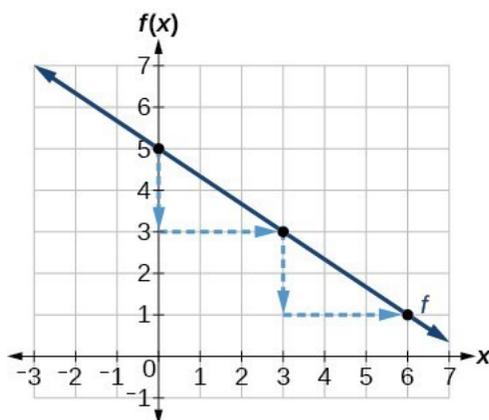
## EXAMPLE

Graph  $f(x) = -\frac{2}{3}x + 5$  using the  $y$ -intercept and slope.

### Answer

Evaluate the function at  $x = 0$  to find the  $y$ -intercept. The output value when  $x = 0$  is 5, so the graph will cross the  $y$ -axis at  $(0, 5)$ .

According to the equation for the function, the slope of the line is  $-\frac{2}{3}$ . This tells us that for each vertical decrease in the “rise” of  $-2$  units, the “run” increases by 3 units in the horizontal direction. We can now graph the function by first plotting the  $y$ -intercept in the graph below. From the initial value  $(0, 5)$ , we move down 2 units and to the right 3 units. We can extend the line to the left and right by using this relationship to plot additional points and then drawing a line through the points.



The graph slants downward from left to right, which means it has a negative slope as expected.

## TRY IT

An interactive or media element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1990>

Graphing a Line in Slope Intercept Form / Give Intercepts

Identify the Slope and Intercepts of the equation  $f(x) = -\frac{1}{2}x - 3$ . Then draw an accurate graph of the equation. Intercepts must be in ordered pairs.

$f(x) = m \cdot x + b$   
 $m = -\frac{1}{2}$   
 $b = -3 = (0, -3)$   
 $f(x) = -\frac{1}{2}x - 3$

Slope:  $-\frac{1}{2}$   
Intercept:  $(0, -3)$   
Vertical Intercept:  $(0, -3)$   
Horizontal Intercept:

Slope Intercept Form of a Line  
 $y = mx + b \rightarrow f(x) = mx + b$   
 $m = \text{slope}$   $b = \text{vertical intercept}$

Clear All Draw

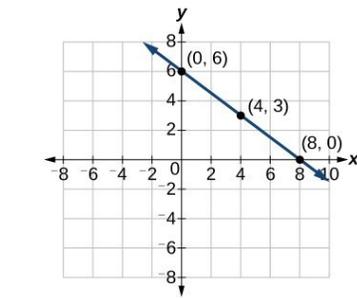
A YouTube element has been excluded from this version of the text. You can view it online here:  
<https://courses.lumenlearning.com/intermediatealgebra/?p=1990>

## EXAMPLE

Graph  $f(x) = -\frac{3}{4}x + 6$  using the slope and y-intercept.

### Answer

The slope of this function is  $-\frac{3}{4}$  and the y-intercept is  $(0, 6)$ . We can start graphing by plotting the y-intercept and counting down three units and right 4 units. The first stop would be  $(4, 3)$ , and the next stop would be  $(8, 0)$ .



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- QID 79774: Graph linear eq. in slope-intercept form, give slope and intercept.. **Authored by:** Day, Alyson. **License:** [CC BY: Attribution](#)

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# TRANSFORM LINEAR FUNCTIONS

## LEARNING OUTCOMES

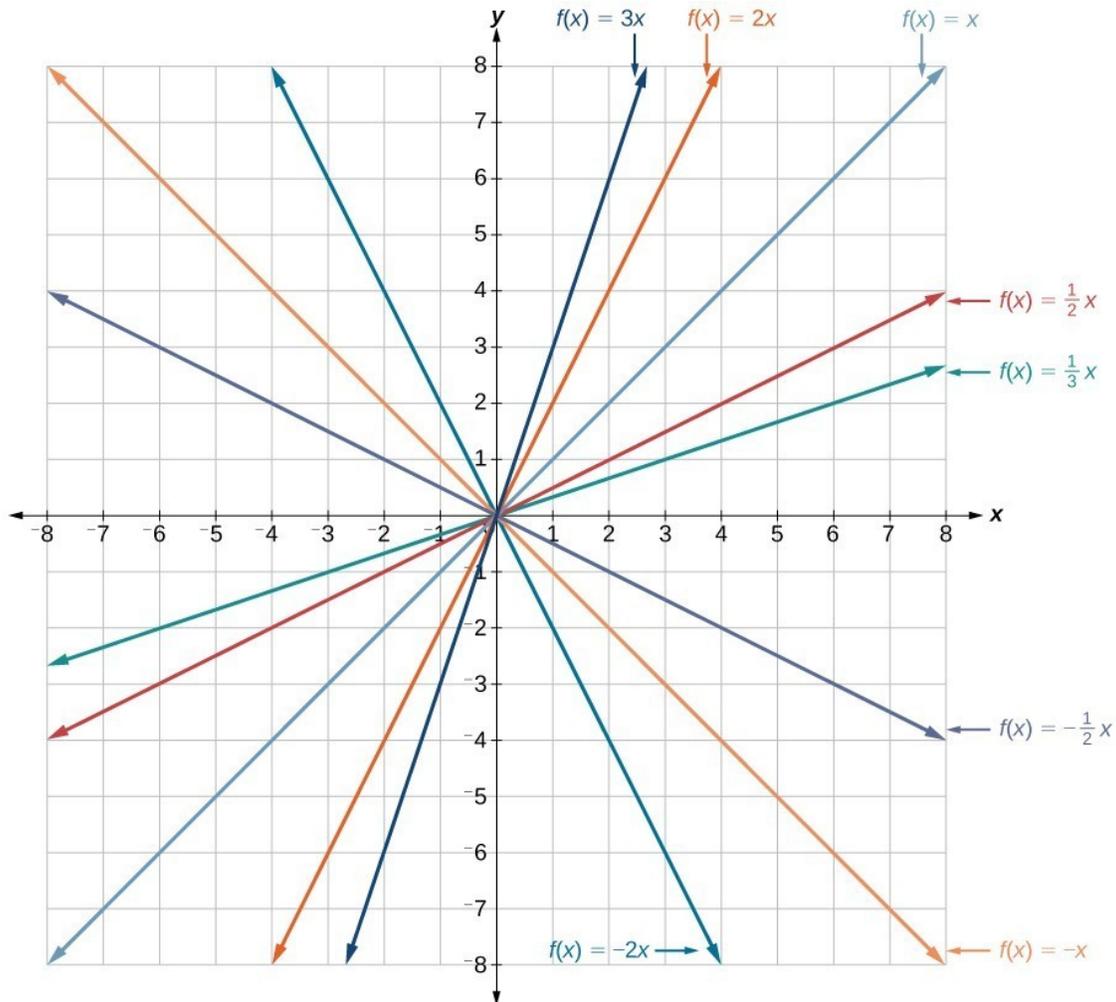
- Use simple transformations to graph linear functions

## Graphing a Linear Function Using Transformations

Another option for graphing linear functions is to use **transformations** of the identity function  $f(x) = x$ . A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

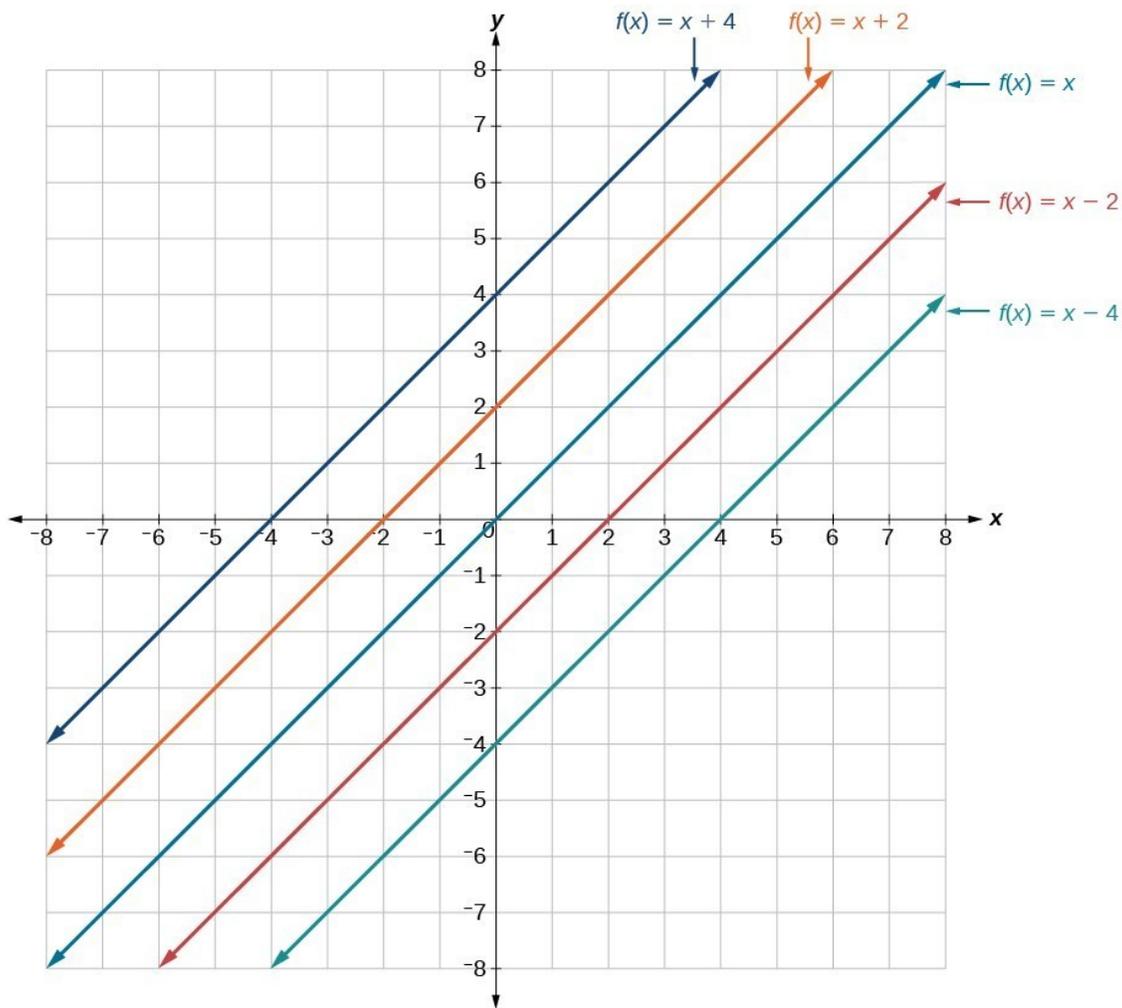
## Vertical Stretch or Compression

In the equation  $f(x) = mx$ , the  $m$  is acting as the **vertical stretch** or **compression** of the identity function. When  $m$  is negative, there is also a vertical reflection of the graph. Notice in the figure below that multiplying the equation of  $f(x) = x$  by  $m$  vertically stretches the graph of  $f$  by a factor of  $m$  units if  $m > 1$  and vertically compresses the graph of  $f$  by a factor of  $m$  units if  $0 < m < 1$ . This means the larger the absolute value of  $m$ , the steeper the slope. Below you can see a variety of vertical stretches, compressions, and/or reflections on the function  $f(x) = x$ .



## Vertical Shift

In  $f(x) = mx + b$ , the  $b$  acts as the **vertical shift**, moving the graph up and down without affecting the slope of the line. Notice in the figure below that adding a value of  $b$  to the equation of  $f(x) = x$  shifts the graph of  $f$  a total of  $b$  units up if  $b$  is positive and  $|b|$  units down if  $b$  is negative. The graph below illustrates vertical shifts of the function  $f(x) = x$ .



Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.

**HOW TO: GIVEN THE EQUATION OF A LINEAR FUNCTION, USE TRANSFORMATIONS TO GRAPH A LINEAR FUNCTION OF THE FORM**

$$f(x) = mx + b$$

1. Graph  $f(x) = x$ .
2. Vertically stretch or compress the graph by a factor of  $|m|$ .
3. Shift the graph up or down  $b$  units.

In the first example, we will see how a vertical compression changes the graph of the identity function.

In the next example we will vertically stretch the identity by a factor of 2.

Our next example shows how making the slope negative reflects the identity across the  $y$ -axis.

In our last example, we will combine a vertical compression and a vertical shift to transform  $f(x) = x$  into  $f(x) = \frac{1}{2}x - 3$  and draw the graph.

The following video example describes another linear transformation of the identity and its corresponding graph.

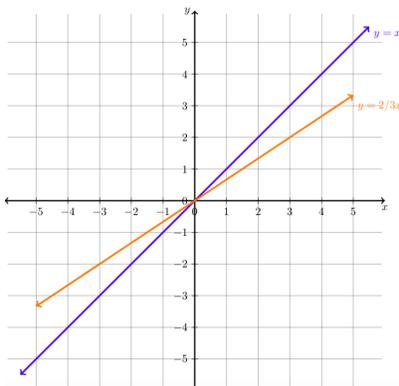
## EXAMPLE

Describe the transformations to the function  $f(x) = \frac{2}{3}x$  and draw a graph.

### Answer

In this case,  $m = \frac{2}{3}$ , so this is a vertical compression since  $0 < m < 1$ .

The graph of  $f(x) = \frac{2}{3}x$  is plotted below with the identity:



Note how the identity is compressed because the rate of change is “slowed” due to the vertical compression of  $\frac{2}{3}$

## Summary

- Vertical compressions of the identity happen when the slope is between 0 and 1.
- Vertical stretches of the identity happen when the slope is greater than 1.
- Reflections happen when the slope is negative.
- Vertical shifts happen when the intercept is not equal to 0.
- Multiple transformations can be made to a function.

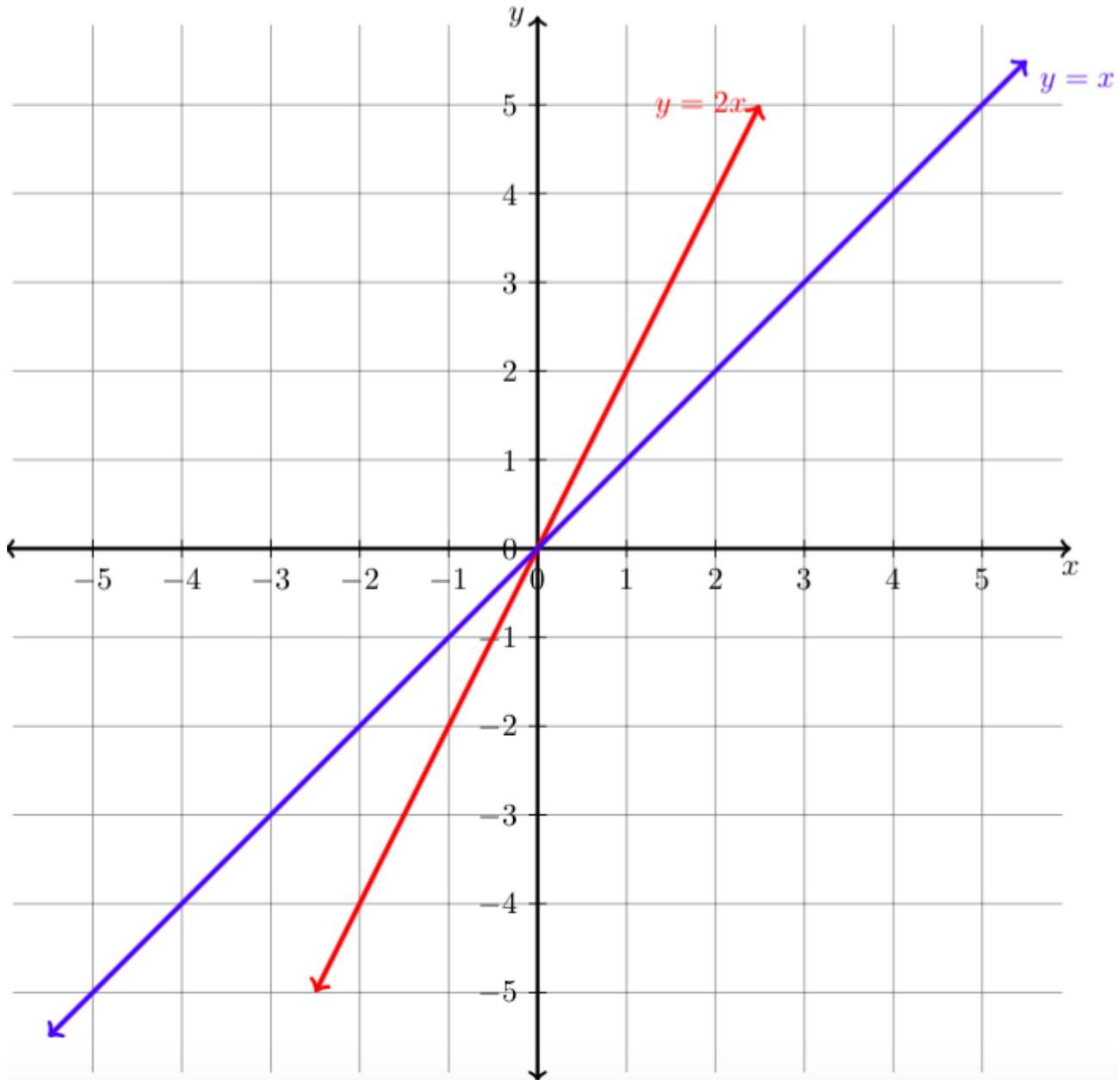
## EXAMPLE

Describe the transformations to the function  $f(x) = 2x$  and draw a graph.

### Answer

In this case,  $m = 2$ , so this is a vertical stretch since  $m > 1$ .

The graph of  $f(x) = 2x$  is plotted below with the identity:



$y=2x$  and  $y=x$

Note how the identity is more steep because the rate of change is “faster” due to the vertical stretch of 2

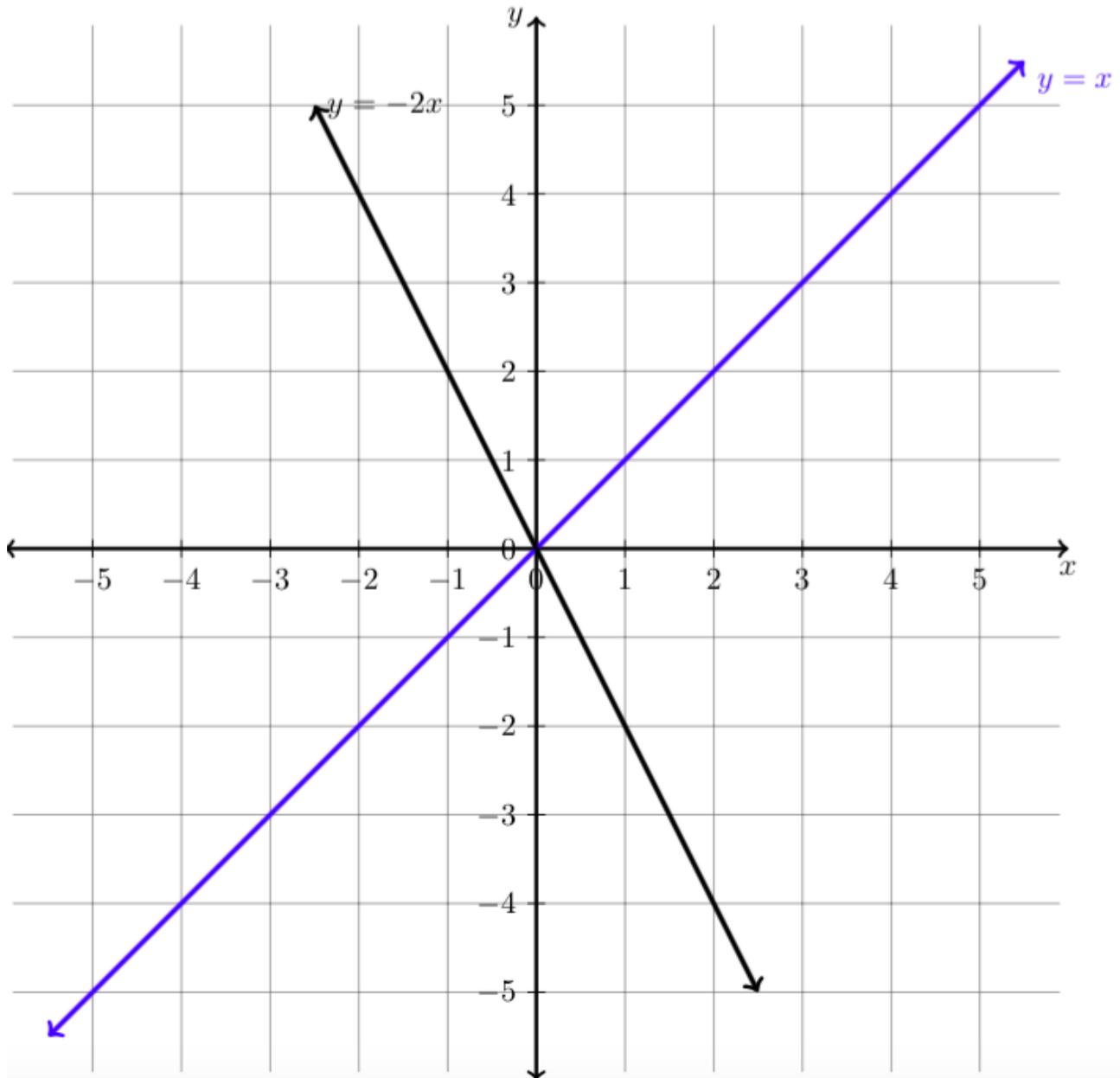
## EXAMPLE

Describe the transformations to the function  $f(x) = -2x$  and draw a graph.

### Answer

In this case,  $m = -2$ , so this is a vertical stretch since  $|m| > 1$ . The negative sign reflects the graph across the y-axis.

The graph of  $f(x) = -2x$  is plotted below with the identity:



$y = x$  and  $y = -2x$

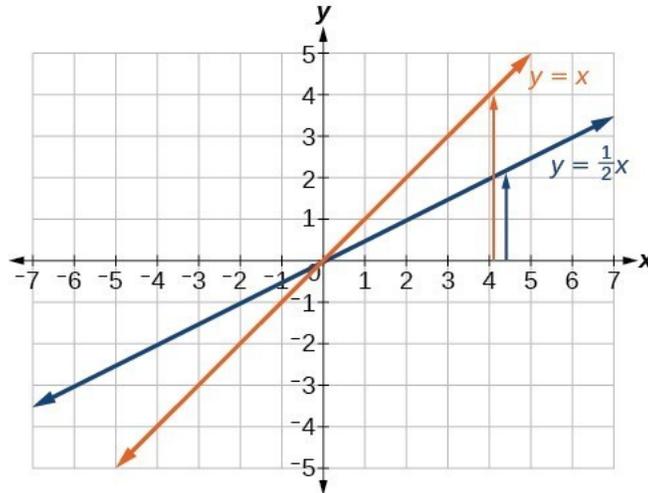
Note how the steepness of the graph of  $f(x) = -2x$  is similar to  $f(x) = 2x$  but it points in the opposite direction because of the negative.

## EXAMPLE

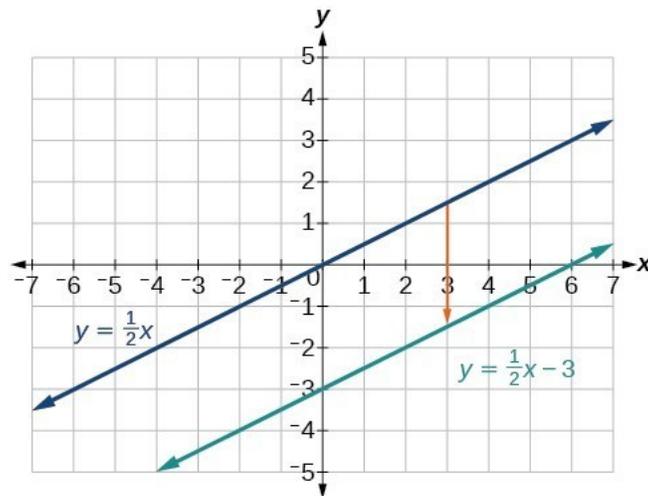
Graph  $f(x) = \frac{1}{2}x - 3$  using transformations.

### Answer

The equation for the function shows that  $m = \frac{1}{2}$ , so the identity function is vertically compressed by  $\frac{1}{2}$ . The equation for the function also shows that  $b = -3$ , so the identity function is vertically shifted down 3 units. First, graph the identity function and show the vertical compression. Below is the function  $y = x$ , compressed by a factor of  $\frac{1}{2}$ .



Now show the vertical shift. The function  $y = \frac{1}{2}x$  shifted down 3 units is shown below.



**Graphing a Linear Function as a Transformation of  $f(x) = x$**

**EXAMPLE**

Describe the transformation of the identity function  $f(x) = x$ , given  $g(x) = \frac{2}{3}x - 3$ . Then graph  $g(x)$ .

$g(x) = mx + b$

$|\frac{2}{3}| = \frac{2}{3} < 1$   
 $g(x)$  is compressed by a factor of  $\frac{2}{3}$ .

$b = -3$   
 $g(x)$  is shifted down 3 units.

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## Q & A

In the example above, could we have sketched the graph by reversing the order of the transformations? No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first.

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# INTRODUCTION TO EQUATIONS OF LINEAR FUNCTIONS

## Why Study Equations of Linear Functions?

In this section, we will learn how to write equations for linear functions given different pieces of information, including two points on the line and the graph of the line. We will also identify the equations of horizontal and vertical lines and write linear equations from written information.

# WRITE THE EQUATION OF A LINEAR FUNCTION PART I

## LEARNING OUTCOMES

- Define the equation of a line given the slope and a point
- Define the equation of a line given two points
- Write a linear function in standard form
- Write the equation for linear functions whose graphs are horizontal and vertical lines

## Point-Slope Form

We have seen that we can define the slope of a line given two points on the line and use that information along with the y-intercept to graph the line. If you do not know the y-intercept or the equation for the line, you can use two points to define the equation of the line using point-slope form.

$$y - y_1 = m(x - x_1)$$

This is an important formula, as it will be used in other algebra courses and often in calculus to find the equation of a tangent line. We need only one point and the slope of the line to use the formula. After substituting the slope and the coordinates of one point into the formula, we simplify it and write it in slope-intercept form.

## POINT-SLOPE FORM

Given a point  $(x_1, y_1)$  and slope  $m$ , point-slope form will give the following equation of a line:

$$y - y_1 = m(x - x_1)$$

In our first example, we will start with the slope. Then, we will show how to find the equation of a line when the slope is not given.

In our next example, we will start with two points and find the equation of the line that passes through them.

The following video shows how to write the equation for a line given its slope and a point on the line.

## Standard Form of a Line

Another way that we can represent the equation of a line is in **standard form**. Standard form is given as

$$Ax + By = C$$

where  $A$ ,  $B$ , and  $C$  are integers. The  $x$  and  $y$  terms are on one side of the equal sign, and the constant term is on the other side.

## Vertical and Horizontal Lines

## EXAMPLE

Write the equation of the line with slope  $m = -3$  that passes through the point  $(4, 8)$ . Write the final equation in slope-intercept form.

### Answer

Using point-slope form, substitute  $-3$  for  $m$  and the point  $(4, 8)$  for  $(x_1, y_1)$ .

$$y - y_1 = m(x - x_1)$$

$$y - 8 = -3(x - 4)$$

$$y - 8 = -3x + 12$$

$$y = -3x + 20$$

Note that any point on the line can be used to find the equation. If done correctly, the same final equation will be obtained.

The equations of vertical and horizontal lines do not require any of the preceding formulas although we can use the formulas to prove that the equations are correct. The equation of a **vertical line** is given as

$$x = c$$

where  $c$  is a constant. The slope of a vertical line is undefined, and regardless of the  $y$ -value of any point on the line, the  $x$ -coordinate of the point will be  $c$ .

Suppose that we want to find the equation of a line containing the following points:  $(-3, -5)$ ,  $(-3, 1)$ ,  $(-3, 3)$ , and  $(-3, 5)$ . First, we will find the slope.

$$m = \frac{5 - 3}{-3 - (-3)} = \frac{2}{0}$$

Zero in the denominator means that the slope is undefined which makes using point-slope form impossible. However, we can plot the points. Notice that all of the  $x$ -coordinates are the same and we find a vertical line through  $x = -3$ .

The equation of a **horizontal line** is given as

$$y = c$$

where  $c$  is a constant. The slope of a horizontal line is zero, and for any  $x$ -value of a point on the line, the corresponding  $y$ -coordinate will be  $c$ .

Suppose we want to find the equation of a line that contains the following set of points:

$(-2, -2)$ ,  $(0, -2)$ ,  $(3, -2)$ , and  $(5, -2)$ . We can use point-slope form. First, we find the slope using any two points on the line.

$$\begin{aligned} m &= \frac{-2 - (-2)}{0 - (-2)} \\ &= \frac{0}{2} \\ &= 0 \end{aligned}$$

Use any point for  $(x_1, y_1)$  in point-slope form.

## EXAMPLE

Find the equation of the line that passes through the points  $(3, 4)$  and  $(0, -3)$ . Write the final equation in slope-intercept form.

### Answer

First, we calculate the slope using the slope formula and two points.

$$\begin{aligned}m &= \frac{-3 - 4}{0 - 3} \\ &= \frac{-7}{-3} \\ &= \frac{7}{3}\end{aligned}$$

Next, we use point-slope form with the slope of  $\frac{7}{3}$  and either point. Let us pick the point  $(3, 4)$  for  $(x_1, y_1)$ .

$$y - 4 = \frac{7}{3}(x - 3)$$

$$y - 4 = \frac{7}{3}x - 7 \quad \text{Distribute the } \frac{7}{3}.$$

$$y = \frac{7}{3}x - 3$$

In slope-intercept form, the equation is written as  $y = \frac{7}{3}x - 3$ .

To prove that either point can be used, let us use the second point  $(0, -3)$  and see if we get the same equation.

$$y - (-3) = \frac{7}{3}(x - 0)$$

$$y + 3 = \frac{7}{3}x$$

$$y = \frac{7}{3}x - 3$$

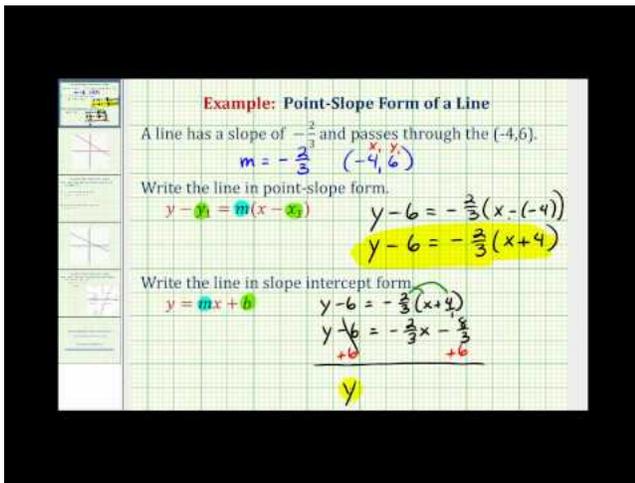
We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

$$y - (-2) = 0(x - 3)$$

$$y + 2 = 0$$

$$y = -2$$

The graph is a horizontal line through  $y = -2$ . Notice that all of the  $y$ -coordinates are the same.



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## EXAMPLE

Find the equation of the line with  $m = -6$  that passes through the point  $\left(\frac{1}{4}, -2\right)$ . Write the equation in standard form.

### Answer

We begin by using point-slope form.

$$y - (-2) = -6 \left( x - \frac{1}{4} \right)$$

$$y + 2 = -6x + \frac{3}{2}$$

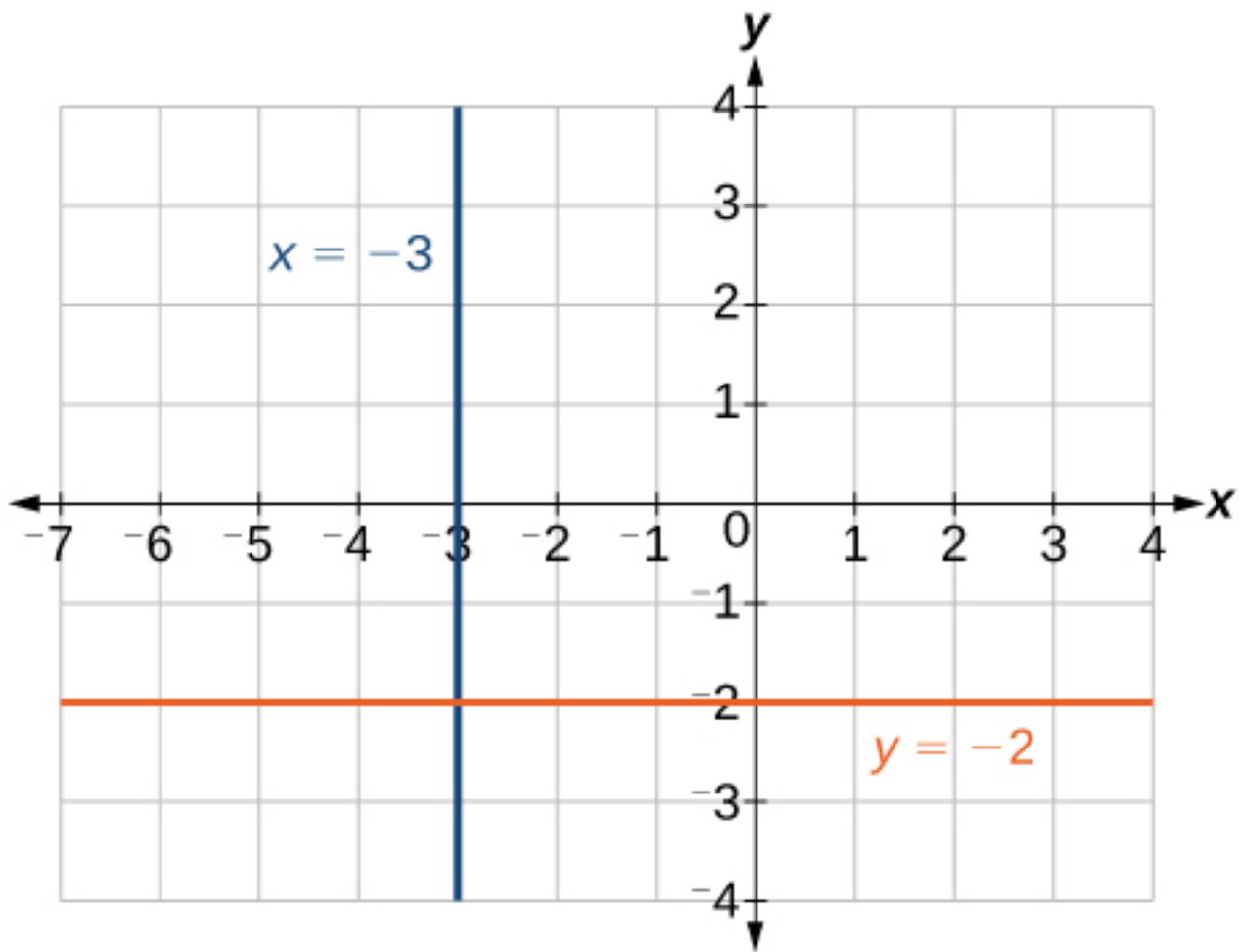
From here, we multiply through by 2, as no fractions are permitted in standard form. Then, move both variables to the left side of the equal sign and constants to the right side of the equal sign.

$$2(y + 2) = \left( -6x + \frac{3}{2} \right) 2$$

$$2y + 4 = -12x + 3$$

$$12x + 2y = -1$$

This equation is now written in standard form.



The line  $x = -3$  is a vertical line. The line  $y = -2$  is a horizontal line.

## EXAMPLE

Find the equation of the line passing through the given points:  $(1, -3)$  and  $(1, 4)$ .

### Answer

The x-coordinate of both points is 1. Therefore, we have a vertical line,  $x = 1$ .

Our last video shows another example of writing the equation of a line given two points on the line.

**Example: Point-Slope Form of a Line**  
 Determine the equation of the line passing through  $(1, 0)$  and  $(5, 4)$ .

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 0}{5 - 1} = \frac{4}{4} = 1$$

Write the line in point-slope form.

$$y - y_1 = m(x - x_1)$$

$$y - 0 = 1(x - 1) \quad \Bigg/ \quad y + 1 = -1$$

Write the line in slope intercept form.

$$y = mx + b$$

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<https://courses.lumenlearning.com/intermediatealgebra/?p=2006>

## Summary

- The **slope** of a line indicates the direction which it slants as well as its steepness. Slope is defined algebraically as:  $m = \frac{y_2 - y_1}{x_2 - x_1}$
- Given the slope and one point on a line, we can find the equation of the line using point-slope form.  
 $y - y_1 = m(x - x_1)$
- Standard form of a line is given as  $Ax + By = C$
- The equation of a **vertical line** is given as  $x = c$
- The equation of a **horizontal line** is given as  $y = c$

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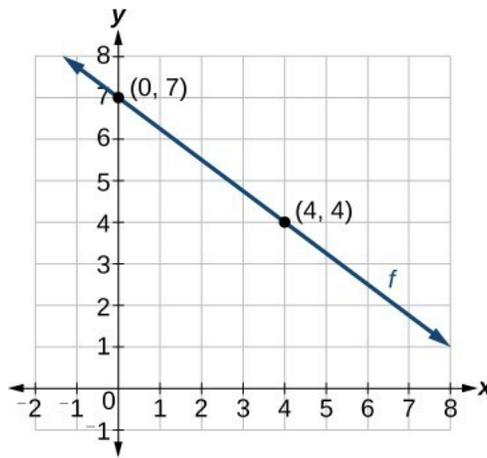
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# WRITE THE EQUATION OF A LINEAR FUNCTION PART II

## LEARNING OUTCOMES

- Write the equation of a linear function from a graph
- Write the equation for a linear function from slope given two points

Now that we have written equations for linear functions using both slope-intercept form and point-slope form, we can choose which method to use based on the information we are given. That information may be provided in the form of a graph, a point and a slope, two points, and so on. Look at the graph of the function  $f$  below.



We are not given the slope of the line, but we can choose any two points on the line to find the slope. Choose  $(0, 7)$  and  $(4, 4)$ . We can use these points to calculate the slope.

$$\begin{aligned}
 m &= \frac{y_2 - y_1}{x_2 - x_1} \\
 &= \frac{4 - 7}{4 - 0} \\
 &= -\frac{3}{4}
 \end{aligned}$$

Now we can substitute the slope and the coordinates of one of the points into point-slope form.

$$\begin{aligned}
 y - y_1 &= m(x - x_1) \\
 y - 4 &= -\frac{3}{4}(x - 4)
 \end{aligned}$$

If we want to rewrite the equation in slope-intercept form, we would do the following:

$$\begin{aligned}
 y - 4 &= -\frac{3}{4}(x - 4) \\
 y - 4 &= -\frac{3}{4}x + 3 \\
 y &= -\frac{3}{4}x + 7
 \end{aligned}$$

$$\begin{aligned}
 f(x) &= mx + b \\
 &\quad \uparrow \quad \uparrow \\
 &\quad -\frac{3}{4} \quad 7 \\
 f(x) &= -\frac{3}{4}x + 7
 \end{aligned}$$

If we wanted to find the equation of the line in slope-intercept form without first using point-slope form, we could have recognized that the line crosses the  $y$ -axis when the output value is 7. Therefore,  $b = 7$ . We now have the initial value  $b$  and the slope  $m$  so we can substitute  $m$  and  $b$  into slope-intercept form of a line.

The function is  $f(x) = -\frac{3}{4}x + 7$ , and the linear equation would be  $y = -\frac{3}{4}x + 7$ .

## HOW TO: GIVEN THE GRAPH OF A LINEAR FUNCTION, WRITE AN EQUATION TO REPRESENT THE FUNCTION

1. Identify two points on the line.
2. Use the two points to calculate the slope.
3. Determine where the line crosses the  $y$ -axis to identify the  $y$ -intercept by visual inspection.
4. Substitute the slope and  $y$ -intercept into slope-intercept form of a linear equation.

In the following video, we show an example of how to write the equation of a line given its graph.

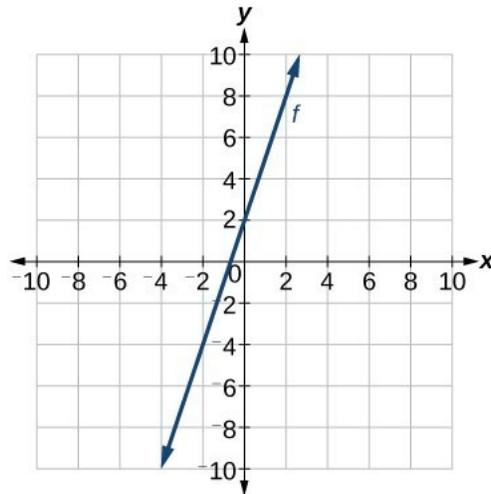
In the video we show another example of how to write a linear function given two points written with function notation.

## Summary

- Given a graph, you can write the equation of the line it represents.
- This method only works for graphs that have points that are easy to verify by visual inspection.

## EXAMPLE

Write an equation for a linear function  $f$  given its graph shown below.



### Answer

Identify two points on the line such as  $(0, 2)$  and  $(-2, -4)$ . Use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-4 - 2}{-2 - 0} \\ &= \frac{-6}{-2} \\ &= 3 \end{aligned}$$

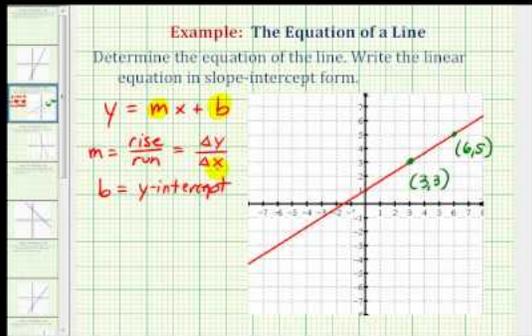
Substitute the slope and the coordinates of one of the points into point-slope form.

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-4) &= 3(x - (-2)) \\ y + 4 &= 3(x + 2) \end{aligned}$$

We can use algebra to rewrite the equation in slope-intercept form.

$$\begin{aligned} y + 4 &= 3(x + 2) \\ y + 4 &= 3x + 6 \\ y &= 3x + 2 \end{aligned}$$

This makes sense because we can see from the graph above that the line crosses the  $y$ -axis at the point  $(0, 2)$ , which is the  $y$ -intercept, so  $b = 2$



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## EXAMPLE

If  $f$  is a linear function, with  $f(3) = -2$ , and  $f(8) = 1$ , find an equation for the function in slope-intercept form.

### Answer

We can write the given points using coordinates.

$$f(3) = -2 \rightarrow (3, -2)$$

$$f(8) = 1 \rightarrow (8, 1)$$

We can then use the points to calculate the slope.

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1 - (-2)}{8 - 3} \\ &= \frac{3}{5} \end{aligned}$$

Substitute the slope and the coordinates of one of the points into point-slope form.

$$y - y_1 = m(x - x_1)$$

$$y - (-2) = \frac{3}{5}(x - 3)$$

We can use algebra to rewrite the equation in slope-intercept form.

$$y + 2 = \frac{3}{5}(x - 3)$$

$$y + 2 = \frac{3}{5}x - \frac{9}{5}$$

$$y = \frac{3}{5}x - \frac{19}{5}$$

The image shows a handwritten solution on a grid background. At the top, it says "Ex: Linear Functions" and "Determine the equation of the linear function given  $f(-1) = -7$  and  $f(2) = -2$ ". Two points are identified:  $(-1, -7)$  and  $(2, -2)$ . The slope  $m$  is calculated as  $m = \frac{-2 - (-7)}{2 - (-1)} = \frac{5}{3}$ . The point-slope form is written as  $y + 2 = \frac{5}{3}(x - 2)$ . This is then simplified to  $y + 2 = \frac{5}{3}x - \frac{10}{3}$ , and finally to the slope-intercept form  $y = \frac{5}{3}x - \frac{16}{3}$ . At the bottom, two formulas are highlighted in yellow:  $y = mx + b$  and  $m = \frac{y_2 - y_1}{x_2 - x_1}$ .

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# MODEL AN APPLICATION WITH A LINEAR FUNCTION

## LEARNING OUTCOMES

- Write the equation for a linear function given an application

In the real world, problems are not always explicitly stated in terms of a function or represented with a graph. Fortunately, we can analyze the problem by first representing it as a linear function and then interpreting the components of the function. As long as we know or can figure out the initial value and the rate of change of a linear function, we can solve many different kinds of real-world problems.

## HOW TO: GIVEN A LINEAR FUNCTION $f$ AND THE INITIAL VALUE AND RATE OF CHANGE, EVALUATE $f(c)$

1. Determine the initial value and the rate of change (slope).
2. Substitute the values into  $f(x) = mx + b$ .
3. Evaluate the function at  $x = c$ .

Initial value is a term that is typically used in applications of functions. It can be represented as the starting point of the relationship we are describing with a function. In the case of linear functions, the initial value is typically the  $y$ -intercept. Here are some characteristics of the initial value:

- The point  $(0, y)$  is often the initial value of a linear function
- The  $y$ -value of the initial value comes from  $b$  in slope-intercept form of a linear function,  $f(x) = mx + b$
- The initial value can be found by solving for  $b$  or substituting 0 in for  $x$  in a linear function.

In our first example, we are given a scenario where Marcus wants to increase the number of songs in his music collection by a fixed amount each month. This is a perfect candidate for a linear function because the increase in the number of songs stays the same each month. We will identify the initial value for the music collection and write an equation that represents the number of songs in the collection for any number of months  $t$ .

In the example we just completed, notice that  $N$  is an increasing linear function. As the input (the number of months) increases, the output (number of songs) increases as well.

In our next example, we will show that you can write the equation for a linear function given two data points.

## EXAMPLE

Marcus currently has 200 songs in his music collection. Every month he adds 15 new songs. Write a formula for the number of songs,  $N$ , in his collection as a function of time,  $t$ , the number of months. How many songs will he own in a year?

### Answer

The initial value for this function is 200 because he currently owns 200 songs, so  $N(0) = 200$ . This means that  $b = 200$ .

$$f(x) = mx + b$$
$$N(t) = 15t + 200$$

The number of songs increases by 15 songs per month, so the rate of change is 15 songs per month. Therefore, we know that  $m = 15$ . We can substitute the initial value and the rate of change into slope-intercept form.

We can write the formula  $N(t) = 15t + 200$ .

With this formula, we can then predict how many songs Marcus will have in 1 year (12 months). In other words, we can evaluate the function at  $t = 12$ .

$$\begin{aligned} N(12) &= 15(12) + 200 \\ &= 180 + 200 \\ &= 380 \end{aligned}$$

Marcus will have 380 songs in 12 months.

In this case, Ilya's weekly income depends on the number of insurance policies he sells. We are given his income for two different weeks and the number of policies sold. We first find the rate of change and then solve for the initial value.

In the following video example, we show how to identify the initial value, slope, and equation for a linear function.

We will show one more example of how to write a linear function that represents the monthly cost to run a company given monthly fixed costs and production costs per item.

The following video example shows how to write a linear function that represents how many miles you can travel in a rental car given a fixed amount of money.

In the next example, we will take data that is in tabular (table) form to write an equation that describes the rate of change of a rat population.

## Summary

- Sometimes we are given an initial value and sometimes we have to solve for it.
- Using units can help you verify that you have calculated slope correctly.
- We can write the equation for a line given a slope and a data point or from a table of data.

## EXAMPLE

Working as an insurance salesperson, Ilya earns a base salary plus a commission on each new policy. Therefore, Ilya's weekly income,  $I$ , depends on the number of new policies,  $n$ , he sells during the week. Last week, he sold 3 new policies and earned \$760 for the week. The week before, he sold 5 new policies and earned \$920. Find an equation for  $I(n)$  and interpret the meaning of the components of the equation.

### Answer

The given information gives us two input-output pairs:  $(3, 760)$  and  $(5, 920)$ . We start by finding the rate of change.

$$\begin{aligned} m &= \frac{920 - 760}{5 - 3} \\ &= \frac{\$160}{2 \text{ policies}} \\ &= \$80 \text{ per policy} \end{aligned}$$

Keeping track of units can help us interpret this quantity. Income increased by \$160 when the number of policies increased by 2, so the rate of change is \$80 per policy. Therefore, Ilya earns a commission of \$80 for each policy sold during the week.

We can then solve for the initial value.

$$\begin{aligned} I(n) &= 80n + b \\ 760 &= 80(3) + b \quad \text{when } n = 3, I(3) = 760 \\ 760 - 80(3) &= b \\ 520 &= b \end{aligned}$$

The value of  $b$  is the starting value for the function and represents Ilya's income when  $n = 0$ , or when no new policies are sold. We can interpret this as Ilya's base salary for the week which does not depend upon the number of policies sold.

We can now write the final equation.

$$I(n) = 80n + 520$$

Our final interpretation is that Ilya's base salary is \$520 per week and he earns an additional \$80 commission for each policy sold.

Notice that we used units to help us verify that we were calculating the rate correctly. It makes sense to speak in terms of the price per policy. To calculate the initial value, we solved for  $b$  by substituting values from one of the points we were given for  $n$  and  $I$ .

**Linear Function Application**

Tim found a piggy bank in the back of his closet that he hadn't seen in years. He decided to use it to save up for a summer vacation by depositing \$57 in the piggy bank every month. After three months, he counted the amount of money in the piggy bank and found he had 193 dollars.

How much money did Tim have in the piggy bank before he started making monthly deposits?

Tim had \$ 22 in the piggy bank before he started making monthly deposits.

Write a Linear Function that represents this situation. Write your function in the form  $S(t) = mt + b$  where  $S(t)$  represents the amount of money in the piggy bank after  $t$  months of saving.

Linear Function:  $S(t) =$   Preview

$m = 57$     $(3, 193)$     $(0, 22)$   
 $t$     $S(t)$

$S(t) = 57t + b$

$(x, y) \rightarrow (t, S(t))$   
 $S(t) = mt + b$

$y = f(x) = mx + b$   
 $m = \text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$   
 $b = \text{vertical intercept} \rightarrow (0, b)$

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## EXAMPLE

Suppose Ben starts a company in which he incurs a fixed cost of \$1,250 per month for the overhead which includes his office rent. His production costs are \$37.50 per item. Write a linear function  $C$  where  $C(x)$  is the cost for  $x$  items produced in a given month.

### Answer

The fixed cost is present every month, \$1,250. The costs that can vary include the cost to produce each item, which is \$37.50 for Ben. The variable cost, called the marginal cost, is represented by 37.5. The cost Ben incurs is the sum of these two costs represented by  $C(x) = 1250 + 37.5x$ .

It is important to note that we are writing a function based on monthly costs, so the initial cost will be \$1,250, because Ben has to pay that amount monthly for rent. If Ben produces 100 items in a month, his monthly cost is represented by

$$C(100) = 1250 + 37.5(100) = 5000$$

So his monthly cost would be \$5,000.

**Example: Problem Solving Using Linear Equations.**

A car rental company charges \$210 per week plus \$0.15 per mile to rent a car. How many miles can you travel in one week for \$336?

$C =$  the total cost in \$  
 $w =$  the number of weeks  
 $m =$  the number of miles

$C = 210w + 0.15m$

$w = 1$     $C = 336$

$$336 = 210(1) + 0.15m$$

$$336 = 210 + 0.15m$$

$$\begin{array}{r} 336 \\ -210 \\ \hline 126 \end{array} = \frac{0.15m}{0.15}$$

$$\frac{126}{0.15} = \frac{0.15m}{0.15}$$

**12**

A YouTube element has been excluded from this version of the text. You can view it online here:  
<https://courses.lumenlearning.com/intermediatealgebra/?p=2010>

## EXAMPLE

The table below relates the number of rats in a population to time (in weeks). Use the table to write a linear equation.

<b><math>w</math>, number of weeks</b>	0	2	4	6
<b><math>P(w)</math>, number of rats</b>	1000	1080	1160	1240

### Answer

We can see from the table that the initial value for the number of rats is 1000, so  $b = 1000$ .

Rather than solving for  $m$ , we can tell from looking at the table that the population increases by 80 for every 2 weeks that pass. This means that the rate of change is 80 rats per 2 weeks, which can be simplified to 40 rats per week.

$$P(w) = 40w + 1000$$

If we did not notice the rate of change from the table, we could still solve for the slope using any two points from the table. For example, using (2, 1080) and (6, 1240),

$$\begin{aligned} m &= \frac{1240 - 1080}{6 - 2} \\ &= \frac{160}{4} \\ &= 40 \end{aligned}$$

## THINK ABOUT IT

Is the initial value always provided in a table of values like the table in the previous example? Write your ideas in the textbox below before you look at the answer.

If your answer is no, give a description of how you would find the initial value.

### Answer

No. Sometimes the initial value is provided in a table of values, but sometimes it is not. If you see an input of 0, then the initial value would be the corresponding output. If the initial value is not provided because there is no value of input on the table equal to 0, find the slope, substitute one coordinate pair and the slope into  $f(x) = mx + b$ , and solve for  $b$ .

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# INTRODUCTION TO PIECEWISE-DEFINED LINEAR FUNCTIONS

Some functions come in pieces. In this section, we will learn about how to define and graph functions that are essentially collections of discrete pieces. Examples of something defined this way include designing the profile of a car, figuring out your mobile phone plan, and calculating income tax rates. For example, your tax rate depends on your income and is the same for a range of incomes as shown in the table below:

Marginal Tax Rate	Single Taxable Income	Married Filing Jointly or Qualified Widow(er) Taxable Income	Married Filing Separately Taxable Income	Head of Household Taxable Income
10%	\$0–\$9,275	\$0–\$18,550	\$0–\$9,275	\$0–\$13,250
15%	\$9,276–\$37,650	\$18,551–\$75,300	\$9,276–\$37,650	\$13,251–\$50,400
25%	\$37,651–\$91,150	\$75,301–\$151,900	\$37,651–\$75,950	\$50,401–\$130,150
28%	\$91,151–\$190,150	\$151,901–\$231,450	\$75,951–\$115,725	\$130,151–\$210,800
33%	\$190,151–\$413,350	\$231,451–\$413,350	\$115,726–\$206,675	\$210,801–\$413,350
35%	\$413,351–\$415,050	\$413,351–\$466,950	\$206,676–\$233,475	\$413,351–\$441,000
39.6%	\$415,051+	\$466,951+	\$233,476+	\$441,001+

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## DEFINE AND WRITE PIECEWISE FUNCTIONS

### LEARNING OUTCOMES

- Define piecewise function
- Evaluate a piecewise function
- Write a piecewise function given an application

A **piecewise function** is a function where more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations where a rule or relationship changes as the input value crosses certain “boundaries.” For example, we often encounter situations in business where the cost per

piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system where incomes up to \$10,000 are taxed at 10% and any additional income is taxed at 20%. The tax on a total income,  $S$ , would be  $0.1S$  if  $S \leq \$10,000$  and  $1000 + 0.2(S - \$10,000)$  if  $S > \$10,000$ .

## PIECEWISE FUNCTION

A piecewise function is a function where more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all of these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1 if } x \text{ is in domain 1} \\ \text{formula 2 if } x \text{ is in domain 2} \\ \text{formula 3 if } x \text{ is in domain 3} \end{cases}$$

In piecewise notation, the absolute value function is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

## Evaluate a Piecewise-Defined Function

In the first example, we will show how to evaluate a piecewise defined function. Note how it is important to pay attention to the domain to determine which expression to use to evaluate the input.

### EXAMPLE

Given the function  $f(x) = \begin{cases} 7x + 3 & \text{if } x < 0 \\ 7x + 6 & \text{if } x \geq 0 \end{cases}$ , evaluate:

1.  $f(-1)$
2.  $f(0)$
3.  $f(2)$

#### Answer

1.  $f(x)$  is defined as  $7x + 3$  for  $x = -1$  because  $-1 < 0$ .

Evaluate:  $f(-1) = 7(-1) + 3 = -7 + 3 = -4$

2.  $f(x)$  is defined as  $7x + 6$  for  $x = 0$  because  $0 \geq 0$ .

Evaluate:  $f(0) = 7(0) + 6 = 0 + 6 = 6$

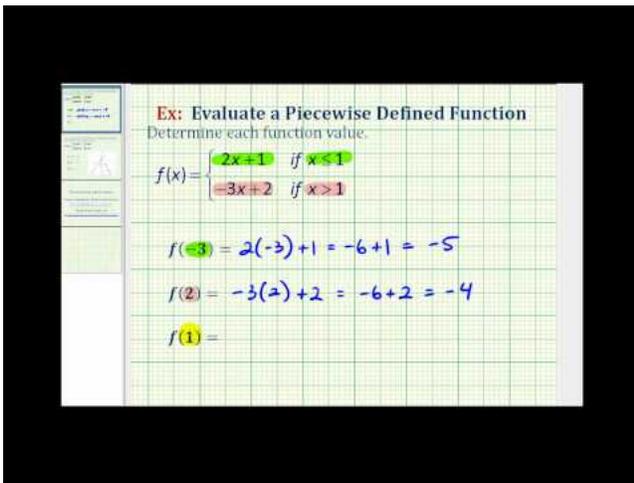
3.  $f(x)$  is defined as  $7x + 6$  for  $x = 2$  because  $2 \geq 0$ .

Evaluate:  $f(2) = 7(2) + 6 = 14 + 6 = 20$

In the following video, we show how to evaluate several values given a piecewise-defined function.

In the next example, we show how to evaluate a function that models the cost of data transfer for a phone company.

The function from the previous example is represented in the graph below. We can see where the function changes from a constant to a line with a positive slope at  $g = 2$ . We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.



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## EXAMPLE

A cell phone company uses the function below to determine the cost,  $C$ , in dollars for  $g$  gigabytes of data transfer.

$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 10g + 5 & \text{if } g \geq 2 \end{cases}$$

Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

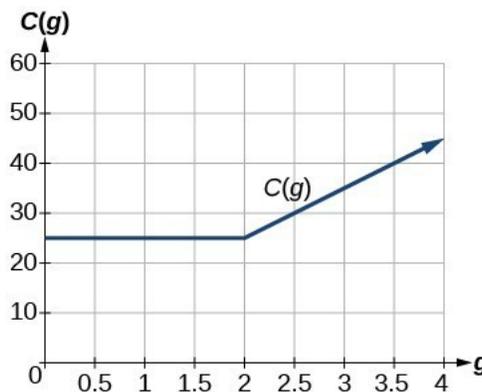
### Answer

To find the cost of using 1.5 gigabytes of data,  $C(1.5)$ , we first look to see which part of the domain our input falls. Because 1.5 is less than 2, we use the first formula.

$$C(1.5) = \$25$$

To find the cost of using 4 gigabytes of data,  $C(4)$ , we see that our input of 4 is greater than 2, so we use the second formula.

$$C(4) = 10(4) + 5 = \$45$$



## Write a Piecewise-Defined Function

In the last example, we will show how to write a piecewise-defined function that models the price of a guided museum tour.

A graph of the function for the previous example is shown below. The graph is a diagonal line from  $n = 0$  to

## EXAMPLE

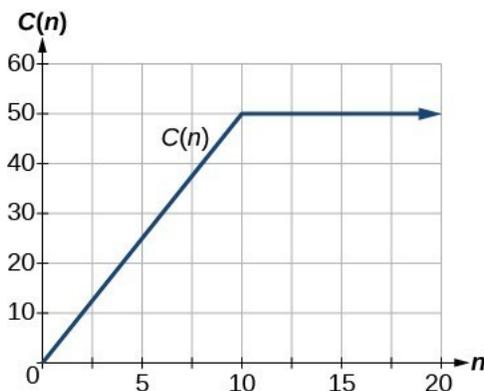
A museum charges \$5 per person for a guided tour for a group of 1 to 9 people or a fixed \$50 fee for a group of 10 or more people. Write a **function** relating the number of people,  $n$ , to the cost,  $C$ .

### Answer

Two different formulas will be needed. For  $n$ -values under 10,  $C = 5n$ . For values of  $n$  that are 10 or greater,  $C = 50$ .

$$C(n) = \begin{cases} 5n & \text{if } 0 < n < 10 \\ 50 & \text{if } n \geq 10 \end{cases}$$

$n = 10$  and a constant after that. In this example, the two formulas agree at the meeting point where  $n = 10$ , but not all piecewise functions have this property.



In the following video, we show an example of how to write a piecewise-defined function given a scenario.

**Determine a Piecewise Defined Function**

**PIECEWISE FUNCTION**

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:

$$f(x) = \begin{cases} \text{formula 1 if } x \text{ is in domain 1} \\ \text{formula 2 if } x \text{ is in domain 2} \\ \text{formula 3 if } x \text{ is in domain 3} \end{cases}$$

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## HOW TO: GIVEN A PIECEWISE FUNCTION, WRITE THE FORMULA AND IDENTIFY THE DOMAIN FOR EACH INTERVAL

1. Identify the intervals where different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use a bracket and “if” statements to write the function.

# Summary

- A **piecewise function** is a function where more than one formula is used to define the output over different pieces of the domain.
- Evaluating a piecewise function means you need to pay close attention to the correct expression used for the given input.

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# GRAPH PIECEWISE FUNCTIONS

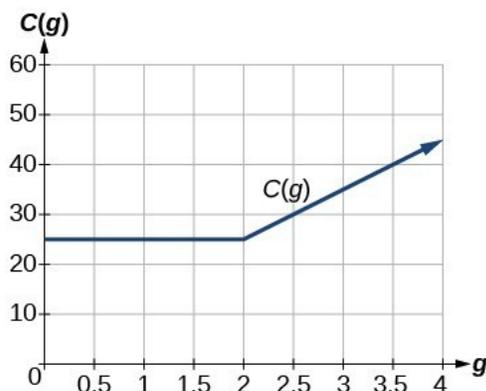
## LEARNING OUTCOMES

- Write a piecewise function that represents an application
- Graph a piecewise function using domain

In this section, we will plot piecewise functions. The function plotted below represents the cost to transfer data for a given cell phone company. We can see where the function changes from a constant to a line with a positive slope at  $g = 2$ . When we plot piecewise functions, it is important to make sure each formula is

applied on its proper domain. 
$$C(g) = \begin{cases} 25 & \text{if } 0 < g < 2 \\ 10g + 5 & \text{if } g \geq 2 \end{cases}$$

In this case, the output is 25 for any input between 0 and 2. For values equal to or greater than 2, the output is defined as  $10g + 5$ .



## HOW TO: GIVEN A PIECEWISE FUNCTION, SKETCH A GRAPH

1. Indicate on the  $x$ -axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.

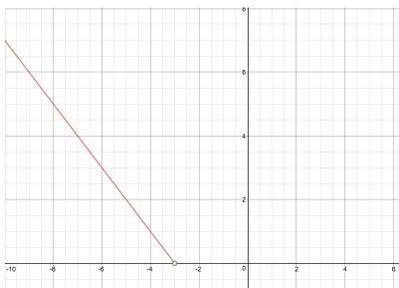
## EXAMPLE

Sketch a graph of the function and state its domain and range.

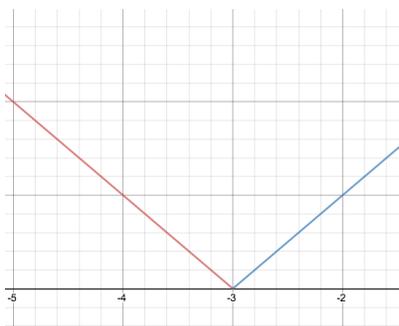
Given the piecewise definition  $f(x) = \begin{cases} -x - 3 & \text{if } x < -3 \\ x + 3 & \text{if } x \geq -3 \end{cases}$

### Answer

First, graph the line  $f(x) = -x - 3$  erasing the part where  $x$  is greater than  $-3$ . Place an open circle at  $(-3, 0)$ .



Now place the line  $f(x) = x + 3$  on the graph starting at the point  $(-3, 0)$ . Note that for this portion of the graph the point  $(-3, 0)$  is included so you can remove the open circle.



Notice the two pieces of the graph meet at the point  $(-3, 0)$ .

The domain of this function is all real numbers because  $(-3, 0)$  is not included as the endpoint of  $f(x) = -x - 3$ , but it is included as the endpoint for  $f(x) = x + 3$ .

The range of this function starts at  $f(x) = 0$  and includes 0, and goes to infinity, so we would write this as  $y \geq 0$ .

In the next example, we will graph a piecewise-defined function that models the cost of shipping for an online comic book retailer.

In the following video, we show how to graph a piecewise-defined function which is linear over both domains.

## Summary

To graph piecewise functions, first identify where the domain is partitioned by boundary values. Graph functions on the domain using tools such as plotting points or transformations. Be sure to use open or

## EXAMPLE

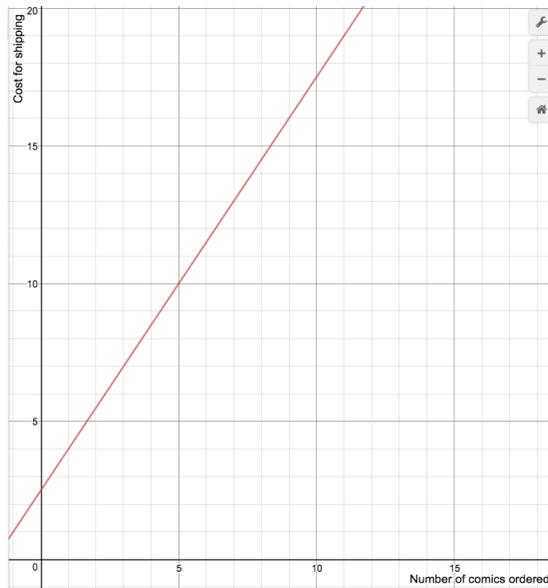
An on-line comic book retailer charges shipping costs according to the following formula, where  $n$  is the number of comic books:

$$S(n) = \begin{cases} 1.5n + 2.5 & \text{if } 1 \leq n \leq 14 \\ 0 & \text{if } n \geq 15 \end{cases}$$

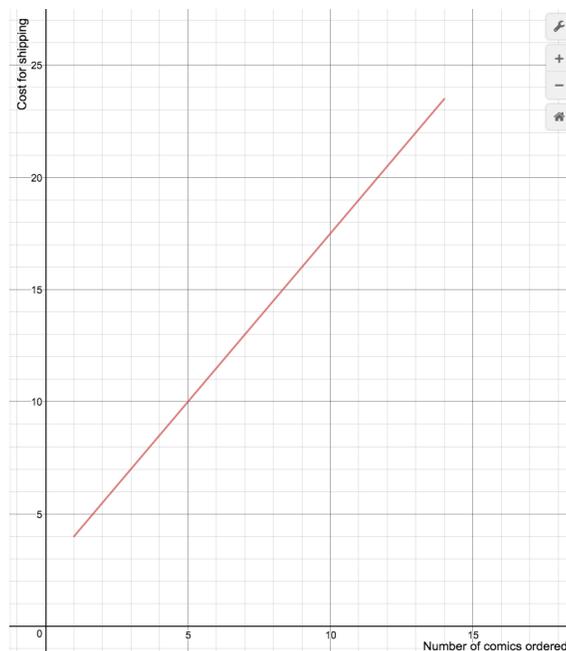
Draw a graph of the cost function.

### Answer

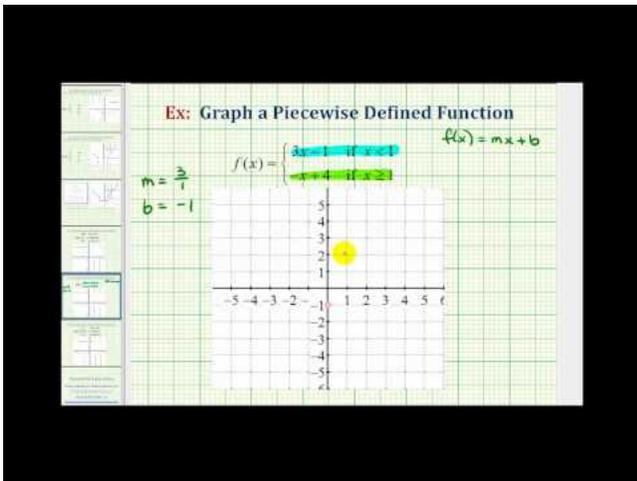
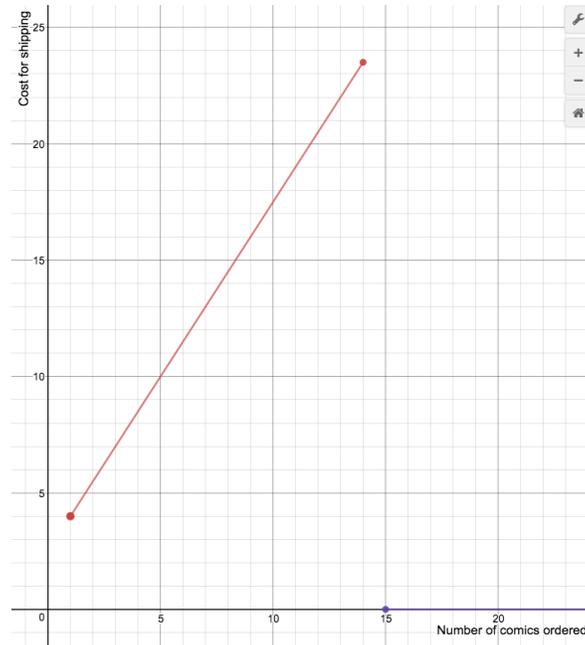
First, draw the line  $S(n) = 1.5n + 2.5$ . We can use transformations: this is a vertical stretch by a factor of 1.5 and a vertical shift by 2.5.



Now we can eliminate the portions of the graph that are not in the domain. This leaves us with the portion of the graph at  $1 \leq n \leq 14$ .



Last, add the constant function  $S(n) = 0$  for inputs greater than or equal to 15. Place closed dots on the ends of the graph to indicate the inclusion of the endpoints.



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closed circles on the endpoints of each domain based on whether the endpoint is included.

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# PUTTING IT TOGETHER: LINEAR FUNCTIONS

At the beginning of the module, Joan had applied for a credit card with airmiles so she could pay her regular bills with it. Her hope was to build up enough miles to take a trip. Joan has planned to use her air miles card for the following things each month:

- gas = \$30
- car insurance = \$40
- mobile phone bill = \$65
- groceries = \$150
- entertainment – restaurants, movies, and live music = \$50

The total monthly expenses she plans to put on her card is \$335.

Each dollar she spends gives her 1.25 air miles, and she expects to need about 1200 air miles to buy the ticket she wants. For signing up, the card gave Joan 100 free miles.

Help Joan plan for her trip by figuring out how many months she needs to pay her bills with her card before she has enough air miles. Because she earns a fixed amount of air miles for each dollar spent, this is a good candidate for a linear function.

Since we know the amount of air miles earned depends on the number of dollars spent, we can write a function with an input of dollars:

$A(d)$  = Number of air miles earned as a function of dollars spent

The initial amount of air miles Joan has is 100, so our function will look like:

$$A(d) = 100 + 1.25d$$

Since Joan needs to earn 1200 air miles to buy the ticket she wants, set the function  $A(d) = 1200$ .

$$1200 = 100 + 1.25d$$

$$1100 = 1.25d$$

$$880 = d$$

Joan needs to spend \$880 before she has enough air miles to buy the ticket she wants. Divide \$880 by Joan's monthly expenses of \$335 to determine the number of months Joan needs to pay her bills with her card before having enough air miles.

$$\frac{880}{335} \approx 2.63$$

Since the answer has to be in months and two months would not earn Joan enough air miles, Joan would need to pay her bills for three months with her card before having enough air miles to purchase her ticket.

# MODULE 6: LINEAR SYSTEMS

## WHY IT MATTERS: SYSTEMS OF EQUATIONS AND INEQUALITIES

### Why Learn About Systems of Equations and Inequalities?

Joan's dad loves to build and fly model RC airplanes. One hot Saturday afternoon, Joan goes with him to fly his favorite plane. She packs them a picnic lunch and they hit the road for their favorite spot. On the way there, Joan's dad expresses his concern that there may be a lot of turbulence because of the hot weather and the location of the field where they will fly the plane.

Joan was surprised by what her dad said. She thought turbulence only happened at high altitudes – like when you were flying in an airplane. Later, after she and her dad got home from flying his RC plane, Joan decided to Google turbulence. She was surprised to find that turbulence can happen on scales as small as gas or water passing through pipes.



*An Alliance 2x2 Pattern RC Plane ready for take off.*

She also learned that despite its occurrence in most fluid flows, such as wind, rivers, oceans, and fluids moving through pipes, scientists and mathematicians have yet to find a reliable way to model or describe its behavior with mathematics or physics. Joan was shocked; she thought math and science had answers for everything! She read further and found that the mathematical equations that are typically used to model fluid flows have not yet been proven to have solutions that always exist. Joan further found that the equations are called the Navier-Stokes equations after Claude-Louis Navier and George Gabriel Stokes. The Navier-Stokes equations are a system of three equations that contain many variables used to describe features of the fluid including temperature, speed, and density.

Joan found that physicists and mathematicians use the Navier–Stokes equations to model the weather, ocean currents, flow of water through a pipe, and air flow around a wing. Engineers and scientists use them to design aircraft and cars, study blood flow, design power stations, and even understand how pollution spreads.

The following week, in her algebra class, Joan learned about systems of linear equations and how to graph them and find solutions for them. Even though her classmates were surprised to find that some systems do not have solutions, Joan already knew about a famous system of equations that does not have any known solutions.

Her teacher was very impressed when she shared what she had learned by spending an afternoon with her father flying RC airplanes.

In this section, we will learn how to graph systems of equations in two and three dimensions and find whether solutions exist. We will also see how systems of equations can be used to solve problems where we have two or three unknown variables.

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# INTRODUCTION TO SOLVING SYSTEMS IN TWO VARIABLES

Systems of equations help us quantify and understand the relationship between different phenomena. Systems are used to understand and describe physical, chemical, fiscal, and even economic principles. We will use all of the skills we gained from solving and graphing linear equations to be able to solve and graph systems of equations.

## SOLUTIONS FOR SYSTEMS OF LINEAR EQUATIONS

### LEARNING OUTCOMES

- Identify solutions of systems of equations
- Solve systems of equations by graphing

A **system of linear equations** consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables which consist of two equations that each contain two different variables. For example, consider the following system of linear equations in two variables.

$$2x + y = 15$$

$$3x - y = 5$$

The *solution* to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair  $(4, 7)$  is the solution to the system of linear equations. We

can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly, we will investigate methods of finding such a solution if it exists.

$$2(4) + (7) = 15 \text{ True}$$

$$3(4) - (7) = 5 \text{ True}$$

In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A **consistent system** of equations has at least one solution. A consistent system is considered to be an **independent system** if it has a single solution like the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a **dependent system** if the equations have the same slope and the same  $y$ -intercepts. In other words, the lines coincide, so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

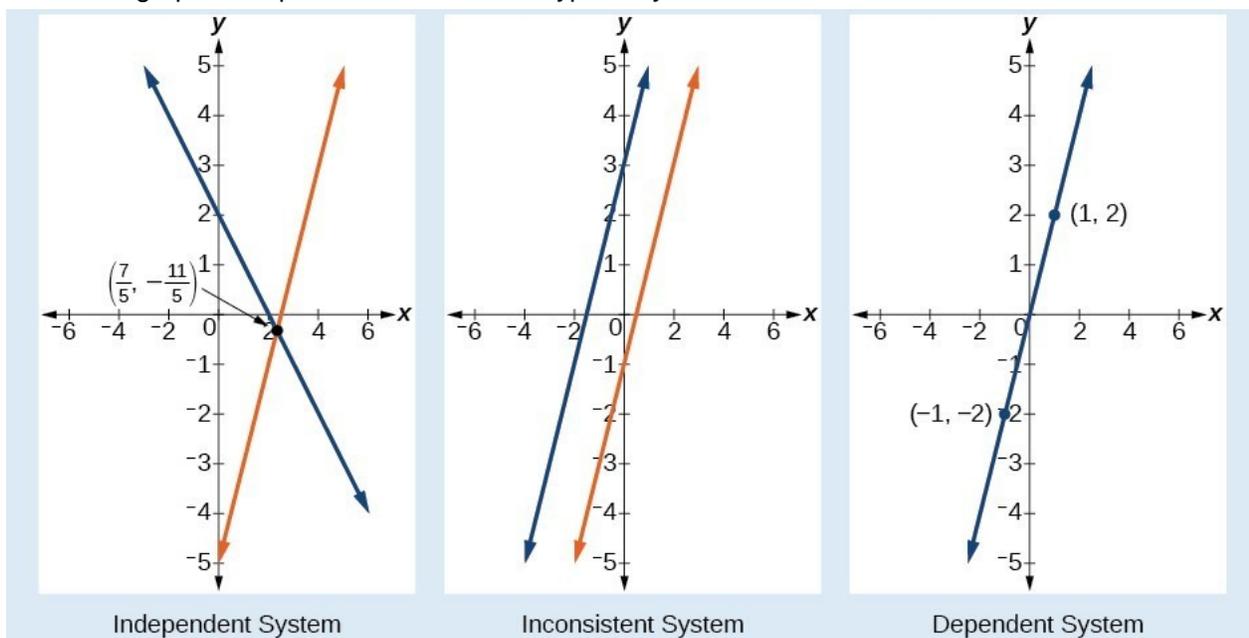
Another type of system of linear equations is an **inconsistent system** which is one where the equations represent two parallel lines. The lines have the same slope and different  $y$ -intercepts. There are no points common to both lines; hence, there is no solution to the system.

## A GENERAL NOTE: TYPES OF LINEAR SYSTEMS

There are three types of systems of linear equations in two variables and three types of solutions.

- An **independent system** has exactly one solution pair  $(x, y)$ . The point where the two lines intersect is the only solution.
- An **inconsistent system** has no solution. The two lines are parallel and will never intersect.
- A **dependent system** has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Below are graphical representations of each type of system.



The independent and dependent systems are also consistent because they both have at least one solution.

**HOW TO: GIVEN A SYSTEM OF LINEAR EQUATIONS AND AN ORDERED PAIR, DETERMINE WHETHER THE ORDERED PAIR IS A SOLUTION**

1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

## EXAMPLE

Determine whether the ordered pair  $(5, 1)$  is a solution to the given system of equations.

$$x + 3y = 8$$

$$2x - 9 = y$$

### Answer

Substitute the ordered pair  $(5, 1)$  into both equations.

$$(5) + 3(1) = 8$$

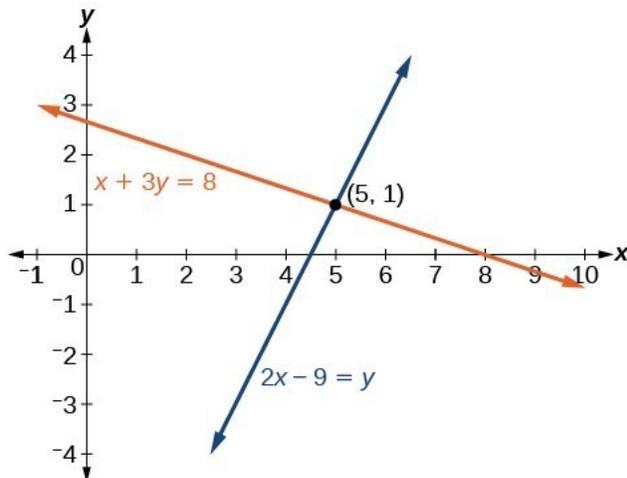
$$8 = 8 \quad \text{True}$$

$$2(5) - 9 = (1)$$

$$1 = 1 \quad \text{True}$$

The ordered pair  $(5, 1)$  satisfies both equations, so it is the solution to the system.

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines.



In the following video, we will show another example of how to verify whether an ordered pair is a solution to a system of equations.

## Solving Systems of Equations by Graphing

There are multiple methods of solving systems of linear equations. For a **system of linear equations** in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

Graphing can be used if the system is inconsistent or dependent. In both cases, we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.

**Ordered Pairs as Solutions to Systems of Equations**

**EXAMPLE**

Determine if the ordered pairs  $(9, -2)$  and  $(3, 2)$  are solutions to the system

$$\begin{cases} y = -\frac{2}{3}x + 4 \\ x + 4y = 1 \end{cases}$$

*(9, -2) is a solution to the system.*

$x$	$y$	$x$	$y$
9	-2	3	2
$y = -\frac{2}{3}x + 4$		$y = -\frac{2}{3}x + 4$	
$-2 = -\frac{2}{3}(9) + 4$		$2 = -\frac{2}{3}(3) + 4$	
$-2 = -6 + 4$		$2 = -2 + 4$	
$-2 = -2$ <i>True</i>		$2 = 2$ <i>True</i>	
$x + 4y = 1$		$x + 4y = 1$	
$9 + 4(-2) = 1$			
$9 + (-8) = 1$			
$1 = 1$ <i>True</i>			

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In the following video, we show another example of how to identify whether a graphed system has a solution and identify what type of solution is represented.

In our last video, we show how to solve a system of equations by first graphing the lines and then identifying the solution the system has.

## EXAMPLE

Solve the following system of equations by graphing. Identify the type of system.

$$2x + y = -8$$

$$x - y = -1$$

### Answer

Solve the first equation for  $y$ .

$$2x + y = -8$$

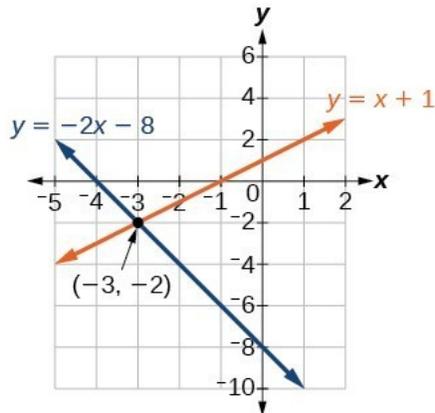
$$y = -2x - 8$$

Solve the second equation for  $y$ .

$$x - y = -1$$

$$y = x + 1$$

Graph both equations on the same set of axes as seen below.



The lines appear to intersect at the point  $(-3, -2)$ . We can check to make sure that this is the solution to the system by substituting the ordered pair into both equations

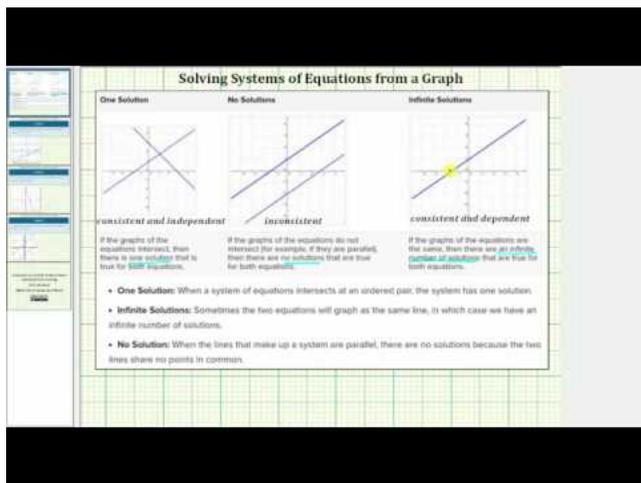
$$2(-3) + (-2) = -8$$

$$-8 = -8 \quad \text{True}$$

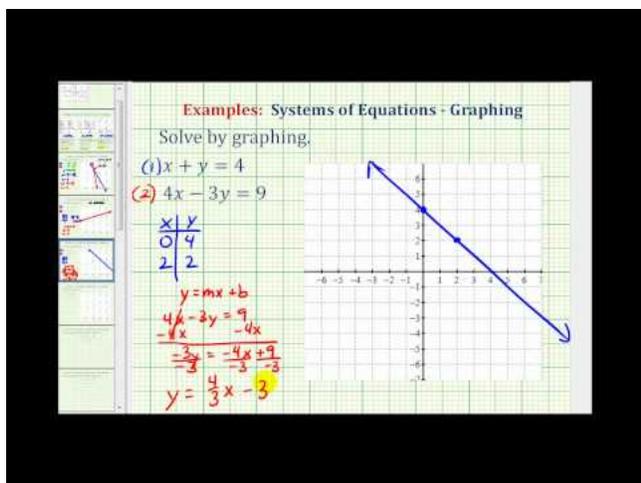
$$(-3) - (-2) = -1$$

$$-1 = -1 \quad \text{True}$$

The solution to the system is the ordered pair  $(-3, -2)$ , so the system is independent.



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# SOLVE SYSTEMS BY SUBSTITUTION

## LEARNING OUTCOMES

- Solve systems of equations by substitution
- Identify inconsistent systems of equations containing two variables

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a **system of linear equations** that are more precise than graphing. One such method is solving a system of equations by the **substitution method** where we solve one of the equations for one variable and then substitute the result into the other equation to solve for the second variable. Recall that we can solve for only one variable at a time which is the reason the substitution method is both valuable and practical.

## HOW TO: GIVEN A SYSTEM OF TWO EQUATIONS IN TWO VARIABLES, SOLVE USING THE SUBSTITUTION METHOD

1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, and then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the other variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

In the following video, you will be given an example of solving a system of two equations using the substitution method.

If you had chosen the other equation to start with in the previous example, you would still be able to find the same solution. It is really a matter of preference, because sometimes solving for a variable will result in having to work with fractions. As you become more experienced with algebra, you will be able to anticipate what choices will lead to more desirable outcomes.

Recall that an **inconsistent system** consists of parallel lines that have the same slope but different  $y$ -intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement such as  $12 = 0$ .

In the next video, we show another example of using substitution to solve a system that has no solution.

In our next video, we show that a system can have an infinite number of solutions.

Consider a skateboard manufacturer's **revenue function**; this is the function used to calculate the amount of money that comes into the business. It can be represented by the equation  $R = xp$ , where  $x$  = quantity and  $p$  = price. The revenue function is shown in orange in the graph below.

The **cost function** is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in the graph below. The  $x$ -axis represents the quantity of skateboards produced and sold in hundreds of units. The  $y$ -axis represents either cost or revenue in hundreds of dollars.

## EXAMPLE

Solve the following system of equations by substitution.

$$-x + y = -5$$

$$2x - 5y = 1$$

### Answer

First, we will solve the first equation for  $y$ .

$$-x + y = -5$$

$$y = x - 5$$

Now, we can substitute the expression  $x - 5$  for  $y$  in the second equation.

$$2x - 5y = 1$$

$$2x - 5(x - 5) = 1$$

$$2x - 5x + 25 = 1$$

$$-3x = -24$$

$$x = 8$$

Now, we substitute  $x = 8$  into the first equation and solve for  $y$ .

$$-(8) + y = -5$$

$$y = 3$$

Our solution is  $(8, 3)$ .

Check the solution by substituting  $(8, 3)$  into both equations.

$$-x + y = -5$$

$$-(8) + (3) = -5 \quad \text{True}$$

$$2x - 5y = 1$$

$$2(8) - 5(3) = 1 \quad \text{True}$$

The substitution method can be used to solve any linear system in two variables, but the method works best if one of the equations contains a coefficient of 1 or  $-1$  so that we do not have to deal with fractions.

**Examples: Systems of Equations - Substitution**  
Solve using the substitution method.

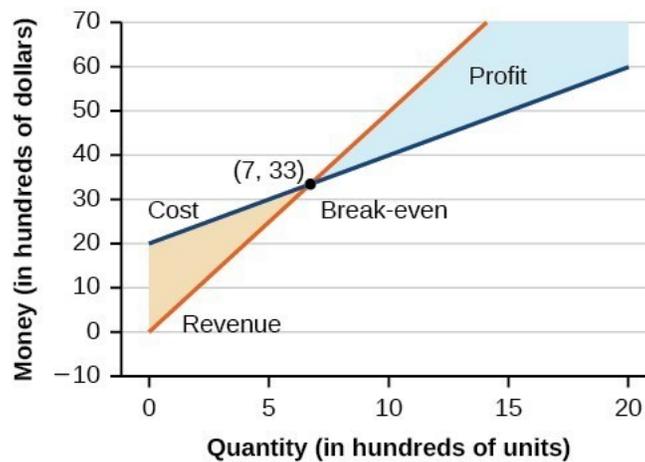
$$(1) \quad x - 3y = -24$$

$$(2) \quad y = -x + 4$$

$$(1) \quad x - 3(-x + 4) = -24$$

$$x + 3x - 12 = -24$$

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The point at which the two lines intersect is called the **break-even point**. We can see from the graph that if 700 units are produced, the cost is \$3,300 and the revenue is also \$3,300. In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

The shaded region to the right of the break-even point represents quantities for which the company makes a profit. The shaded region to the left represents quantities for which the company suffers a loss.

In the next example, we will show how to write a system of linear equations given attendance and ticket cost data. We will then find the number of tickets purchased based on our system.

In our last video example, we show how to set up a system of linear equations that represents the total cost for admission to a museum.

In the next section, we will introduce more methods for solving systems of equations that cannot be easily solved by substitution.

## EXAMPLE

Solve the following system of equations.

$$x = 9 - 2y$$

$$x + 2y = 13$$

### Answer

We can approach this problem in two ways. Because one equation is already solved for  $x$ , the most obvious step is to use substitution.

We can substitute the expression  $9 - 2y$  for  $x$  in the second equation.

$$x + 2y = 13$$

$$(9 - 2y) + 2y = 13$$

$$9 + 0y = 13$$

$$9 = 13$$

Clearly, this statement is a contradiction because  $9 \neq 13$ . Therefore, the system has no solution.

The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.

$$x = 9 - 2y$$

$$2y = -x + 9$$

$$y = -\frac{1}{2}x + \frac{9}{2}$$

We then convert the second equation to slope-intercept form.

$$x + 2y = 13$$

$$2y = -x + 13$$

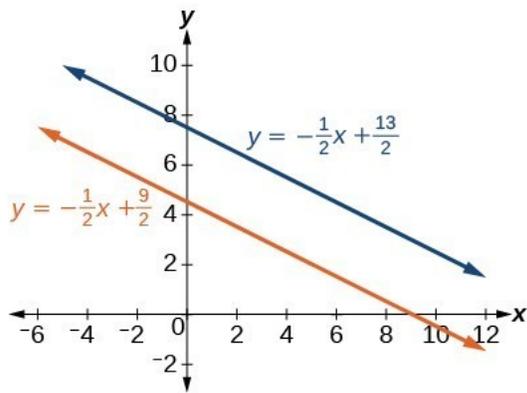
$$y = -\frac{1}{2}x + \frac{13}{2}$$

Comparing the equations, we see that they have the same slope but different  $y$ -intercepts. Therefore, the lines are parallel and do not intersect.

$$y = -\frac{1}{2}x + \frac{9}{2}$$

$$y = -\frac{1}{2}x + \frac{13}{2}$$

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown below.



**Examples: Systems of Equations - Substitution**  
Solve using the substitution method.

(1)  $2x - 5y = -24$

(2)  $\frac{5}{4}y = \frac{2x}{5} \rightarrow y = \frac{2}{5}x$

(1)  $2x - 5\left(\frac{2}{5}x\right) = -24$

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**Examples: Systems of Equations - Substitution**  
Solve using the substitution method.

(1)  $5x - 2y = 10$

(2)  $y = \frac{5}{2}x - 5$

(1)  $5x - 2\left(\frac{5}{2}x - 5\right) = 10$

$5x - 5x + 10 = 10$

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## EXAMPLE

Given the cost function  $C(x) = 0.85x + 35,000$  and the revenue function  $R(x) = 1.55x$ , find the break-even point.

### Answer

Write the system of equations using  $y$  to replace function notation.

$$y = 0.85x + 35,000$$

$$y = 1.55x$$

Substitute the expression  $0.85x + 35,000$  from the first equation into the second equation and solve for  $x$ .

$$0.85x + 35,000 = 1.55x$$

$$35,000 = 0.7x$$

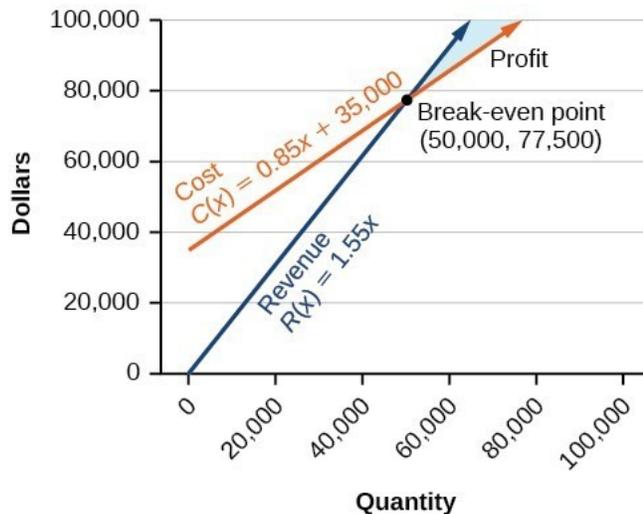
$$50,000 = x$$

Then, substitute  $x = 50,000$  into either the cost function or the revenue function.

$$1.55(50,000) = 77,500$$

The break-even point is  $(50,000, 77,500)$ .

The cost to produce 50,000 units is \$77,500, and the revenue from the sales of 50,000 units is also \$77,500. To make a profit, the business must produce and sell more than 50,000 units.



The company will make a profit after 50,000 units are produced.

## EXAMPLE

The cost of a ticket to the circus is \$25.00 for children and \$50.00 for adults. On a certain day, attendance at the circus is 2,000 and the total gate revenue is \$70,000. How many children and how many adults bought tickets?

### Answer

Let  $c$  = the number of children and  $a$  = the number of adults in attendance.

The total number of people is 2,000. We can use this to write an equation for the number of people at the circus that day.

$$c + a = 2,000$$

The revenue from all children can be found by multiplying \$25.00 by the number of children,  $25c$ . The revenue from all adults can be found by multiplying \$50.00 by the number of adults,  $50a$ . The total revenue is \$70,000. We can use this to write an equation for the revenue.

$$25c + 50a = 70,000$$

We now have a system of linear equations in two variables.

$$c + a = 2,000$$

$$25c + 50a = 70,000$$

In the first equation, the coefficient of both variables is 1. We can quickly solve the first equation for either  $c$  or  $a$ . We will solve for  $a$ .

$$c + a = 2,000$$

$$a = 2,000 - c$$

Substitute the expression  $2,000 - c$  in the second equation for  $a$  and solve for  $c$ .

$$25c + 50(2,000 - c) = 70,000$$

$$25c + 100,000 - 50c = 70,000$$

$$-25c = -30,000$$

$$c = 1,200$$

Substitute  $c = 1,200$  into the first equation to solve for  $a$ .

$$1,200 + a = 2,000$$

$$a = 800$$

We find that 1,200 children and 800 adults bought tickets to the circus that day.

**System of Linear Equations Application**

The Science Museum charges \$14 for adult admission and \$11 for each child. The total bill for 68 people from a school field trip was \$784. How many children went to the museum?

$a =$  the number of adults  
 $c =$  the number of children

$$a + c = 68$$

$$a = 68 - c$$

$$14(68 - c) + 11c = 784$$

$$a + c = 68$$

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# SOLVE SYSTEMS BY ELIMINATION

## LEARNING OUTCOMES

- Solve systems of equations by addition
- Express the solution of a system of dependent equations containing two variables

A third method of **solving systems of linear equations** is the **elimination method**. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by elimination.

In the following video, you will see another example of how to use the method of elimination to solve a system of linear equations.

Sometimes we have to do a couple of steps of algebra before we can eliminate a variable from a system and solve it. In the next example, you will see a technique where we multiply one of the equations in the system by a number that will allow us to eliminate one of the variables.

Below is another video example of using the elimination method to solve a system of linear equations.

In the next example, we will see that sometimes both equations need to be multiplied by different numbers in order for one variable to be eliminated.

## EXAMPLE

Solve the given system of equations by elimination.

$$x + 2y = -1$$

$$-x + y = 3$$

### Answer

Both equations are already set equal to a constant. Notice that the coefficient of  $x$  in the second equation,  $-1$ , is the opposite of the coefficient of  $x$  in the first equation,  $1$ . We can add the two equations to eliminate  $x$  without needing to multiply by a constant.

$$\begin{array}{r} x + 2y = -1 \\ -x + y = 3 \\ \hline 3y = 2 \end{array}$$

Now that we have eliminated  $x$ , we can solve the resulting equation for  $y$ .

$$3y = 2$$

$$y = \frac{2}{3}$$

Then, we substitute this value for  $y$  into one of the original equations and solve for  $x$

$$-x + y = 3$$

$$-x + \frac{2}{3} = 3$$

$$-x = 3 - \frac{2}{3}$$

$$-x = \frac{7}{3}$$

$$x = -\frac{7}{3}$$

The solution to this system is  $\left(-\frac{7}{3}, \frac{2}{3}\right)$ .

Check the solution in the first equation.

$$x + 2y = -1$$

$$\left(-\frac{7}{3}\right) + 2\left(\frac{2}{3}\right) = -1$$

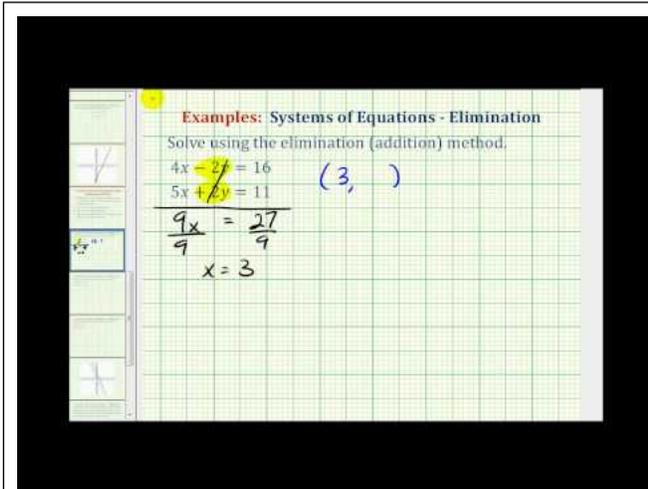
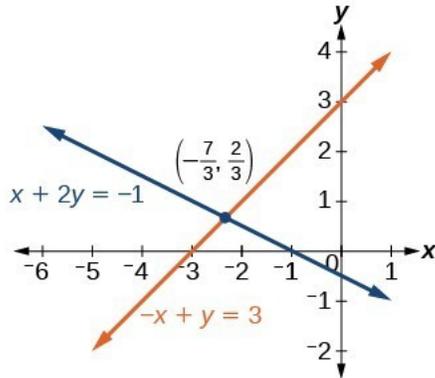
$$-\frac{7}{3} + \frac{4}{3} = -1$$

$$-\frac{3}{3} = -1$$

$$-1 = -1$$

True

We gain an important perspective on systems of equations by looking at the graphical representation. In the graph below, you will see that the equations intersect at the solution. We do not need to ask whether there may be a second solution, because observing the graph confirms that the system has exactly one solution.



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Below is a summary of the general steps for using the elimination method to solve a system of equations.

In the next example, we will show how to solve a system with fractions. As with single linear equations, the easiest way to solve is to clear the fractions first with the least common denominator.

In the following video, you will find one more example of using the elimination method to solve a system; this one has coefficients that are fractions.

Recall that a **dependent system** of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or elimination, the resulting equation will be an identity such as  $0 = 0$ . The last example includes two equations that represent the same line and are therefore dependent.

In the following video, we show another example of solving a system that is dependent using elimination.

In our last video example, we present a system that is inconsistent; it has no solutions which means the lines the equations represent are parallel to each other.

## EXAMPLE

Solve the given system of equations by the **elimination method**.

$$3x + 5y = -11$$

$$x - 2y = 11$$

### Answer

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has  $3x$  in it and the second equation has  $x$ . So if we multiply the second equation by  $-3$ , the  $x$ -terms will add to zero.

$$x - 2y = 11$$

$$-3(x - 2y) = -3(11) \quad \text{Multiply both sides by } -3.$$

$$-3x + 6y = -33 \quad \text{Use the distributive property.}$$

Now, let us add them.

$$3x + 5y = -11$$

$$-3x + 6y = -33$$

---

$$11y = -44$$

$$y = -4$$

For the last step, we substitute  $y = -4$  into one of the original equations and solve for  $x$ .

$$3x + 5y = -11$$

$$3x + 5(-4) = -11$$

$$3x - 20 = -11$$

$$3x = 9$$

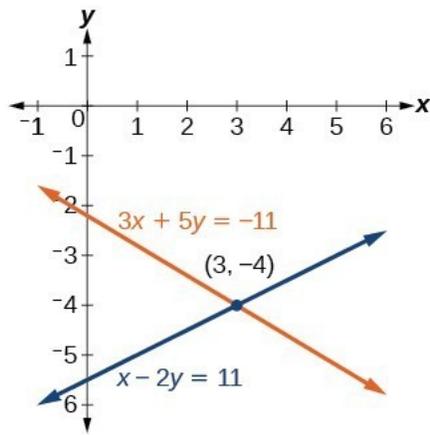
$$x = 3$$

Our solution is the ordered pair  $(3, -4)$ . Check the solution in the original second equation.

$$x - 2y = 11$$

$$(3) - 2(-4) = 11$$

$$11 = 11 \quad \text{True}$$



**Examples: Systems of Equations - Elimination**  
 Solve using the elimination (addition) method.

$$\begin{array}{r} 3x + y = -10 \quad (-5) \rightarrow -15x - 5y = 50 \\ 7x + 5y = -18 \quad \rightarrow \quad \quad \quad 7x + 5y = -18 \\ \hline -8x = 32 \end{array}$$

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## EXAMPLE

Solve the given system of equations in two variables by elimination.

$$2x + 3y = -16$$

$$5x - 10y = 30$$

### Answer

One equation has  $2x$  and the other has  $5x$ . The least common multiple is  $10x$ , so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate  $x$  by multiplying the first equation by  $-5$  and the second equation by  $2$ .

$$-5(2x + 3y) = -5(-16)$$

$$-10x - 15y = 80$$

$$2(5x - 10y) = 2(30)$$

$$10x - 20y = 60$$

Then, we add the two equations together.

$$-10x - 15y = 80$$

$$10x - 20y = 60$$

---

$$-35y = 140$$

$$y = -4$$

Substitute  $y = -4$  into the original first equation.

$$2x + 3(-4) = -16$$

$$2x - 12 = -16$$

$$2x = -4$$

$$x = -2$$

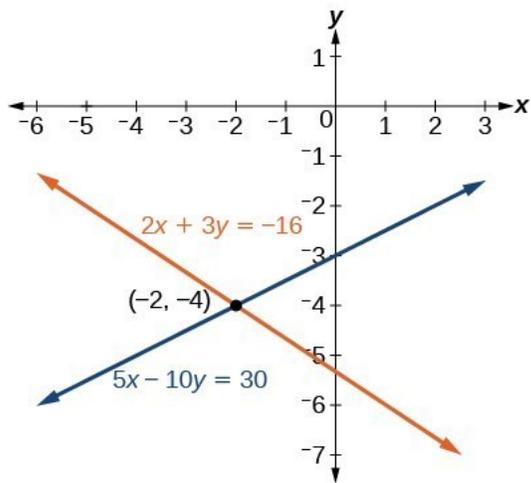
The solution is  $(-2, -4)$ . Check it in the second original equation.

$$5x - 10y = 30$$

$$5(-2) - 10(-4) = 30$$

$$-10 + 40 = 30$$

$$30 = 30$$



## HOW TO: GIVEN A SYSTEM OF EQUATIONS, SOLVE USING THE ELIMINATION METHOD

1. Write both equations with  $x$  and  $y$ -variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

## EXAMPLE

Solve the given system of equations in two variables by elimination.

$$\frac{x}{3} + \frac{y}{6} = 3$$

$$\frac{x}{2} - \frac{y}{4} = 1$$

### Answer

First clear each equation of fractions by multiplying both sides of the equation by the least common denominator

$$6 \left( \frac{x}{3} + \frac{y}{6} \right) = 6(3)$$

$$2x + y = 18$$

$$4 \left( \frac{x}{2} - \frac{y}{4} \right) = 4(1)$$

$$2x - y = 4$$

Now multiply the second equation by  $-1$  so that we can eliminate the  $x$ -variable.

$$-1(2x - y) = -1(4)$$

$$-2x + y = -4$$

Add the two equations to eliminate the  $x$ -variable and solve the resulting equation.

$$2x + y = 18$$

$$-2x + y = -4$$

---

$$2y = 14$$

$$y = 7$$

Substitute  $y = 7$  into the first equation.

$$2x + (7) = 18$$

$$2x = 11$$

$$x = \frac{11}{2}$$

The solution is  $\left(\frac{11}{2}, 7\right)$ . Check it in the other equation.

$$2x - y = 4$$

$$2\left(\frac{11}{2}\right) - 7 = 4$$

$$11 - 7 = 4$$

$$4 = 4$$

**Solve a System of Equations Using Elimination (Fractions)**

Solve the system of equations. Give your solution as an ordered pair.

$$4x - 2y = 6 \rightarrow 4x - 2y = 6$$
$$\frac{2}{3}x + \frac{2}{3}y = 2 \rightarrow 2x + 2y = 6$$

Preview: Solution:  $(x, y) = (2, \quad)$

$$\begin{array}{r} 4x - 2y = 6 \\ 2x + 2y = 6 \\ \hline 6x = 12 \\ \hline x = 2 \end{array}$$

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## EXAMPLE

Find a solution to the system of equations using the **elimination method**.

$$x + 3y = 2$$

$$3x + 9y = 6$$

### Answer

With the elimination method, we want to eliminate one of the variables by adding the equations. In this case, focus on eliminating  $x$ . If we multiply both sides of the first equation by  $-3$ , then we will be able to eliminate the  $x$ -variable.

$$x + 3y = 2$$

$$(-3)(x + 3y) = (-3)(2)$$

$$-3x - 9y = -6$$

Now add the equations.

$$-3x - 9y = -6$$

$$3x + 9y = 6$$

---

$$0 = 0$$

We can see that there will be an infinite number of solutions that satisfy both equations.

If we rewrote both equations in slope-intercept form, we might know what the solution would look like before adding. Look at what happens when we convert the system to slope-intercept form.

$$x + 3y = 2$$

$$3y = -x + 2$$

$$y = -\frac{1}{3}x + \frac{2}{3}$$

$$3x + 9y = 6$$

$$9y = -3x + 6$$

$$y = -\frac{3}{9}x + \frac{6}{9}$$

$$y = -\frac{1}{3}x + \frac{2}{3}$$

See the graph below. Notice the results are the same. The general solution to the system is

$$\left(x, -\frac{1}{3}x + \frac{2}{3}\right).$$



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# INTRODUCTION TO SYSTEMS OF EQUATIONS IN THREE VARIABLES

We will solve problems involving three equations and three variables in this lesson. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

## SYSTEMS OF THREE EQUATIONS IN THREE VARIABLES

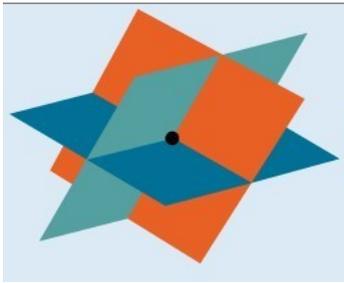
### LEARNING OUTCOMES

- Solve systems of three equations in three variables

The **solution set** to a system of three equations in three variables is an ordered triple  $(x, y, z)$ . Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

### SOLUTION SET, ONE SOLUTION

The figure below illustrates how a system with three variables can have one solution. Systems that have a single solution are those which result in a **solution set** consisting of an ordered triple  $(x, y, z)$ . Graphically, the ordered triple defines a point that is the intersection of three planes in space.



In the first example, we will determine whether an ordered triple is a solution for a system of three linear equations in three variables.

## EXAMPLE

Determine whether the ordered triple  $(3, -2, 1)$  is a solution to the system.

$$x + y + z = 2$$

$$6x - 4y + 5z = 31$$

$$5x + 2y + 2z = 13$$

### Answer

We will check each equation by substituting in the values of the ordered triple for  $x$ ,  $y$ , and  $z$ .

$$x + y + z = 2$$

$$(3) + (-2) + (1) = 2$$

True

$$6x - 4y + 5z = 31$$

$$6(3) - 4(-2) + 5(1) = 31$$

$$18 + 8 + 5 = 31$$

True

$$5x + 2y + 2z = 13$$

$$5(3) + 2(-2) + 2(1) = 13$$

$$15 - 4 + 2 = 13$$

True

The ordered triple  $(3, -2, 1)$  is indeed a solution to the system.

## HOW TO: GIVEN A LINEAR SYSTEM OF THREE EQUATIONS, SOLVE FOR THREE UNKNOWNNS

1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

Solving a system with three variables is very similar to solving one with two variables. It is important to keep track of your work as the addition of one more equation can create more steps in the solution process.

In the example that follows, we will solve the system by first using the elimination method to solve for  $x$  and then using back-substitution.

## EXAMPLE

Solve the given system.

$$\begin{cases} x - \frac{1}{3}y + \frac{1}{2}z = 1 \\ y - \frac{1}{2}z = 4 \\ z = -1 \end{cases}$$

**Answer**

The third equation states that  $z = -1$ , so we substitute this into the second equation to obtain a solution for  $y$ .

$$y - \frac{1}{2}(-1) = 4$$

$$y + \frac{1}{2} = 4$$

$$y = 4 - \frac{1}{2}$$

$$y = \frac{8}{2} - \frac{1}{2}$$

$$y = \frac{7}{2}$$

Now we have two of our solutions, and we can substitute them both into the first equation to solve for  $x$ .

$$x - \frac{1}{3}\left(\frac{7}{2}\right) + \frac{1}{2}(-1) = 1$$

$$x - \frac{7}{6} - \frac{1}{2} = 1$$

$$x - \frac{7}{6} - \frac{3}{6} = 1$$

$$x - \frac{10}{6} = 1$$

$$x = 1 + \frac{10}{6}$$

$$x = \frac{6}{6} + \frac{10}{6}$$

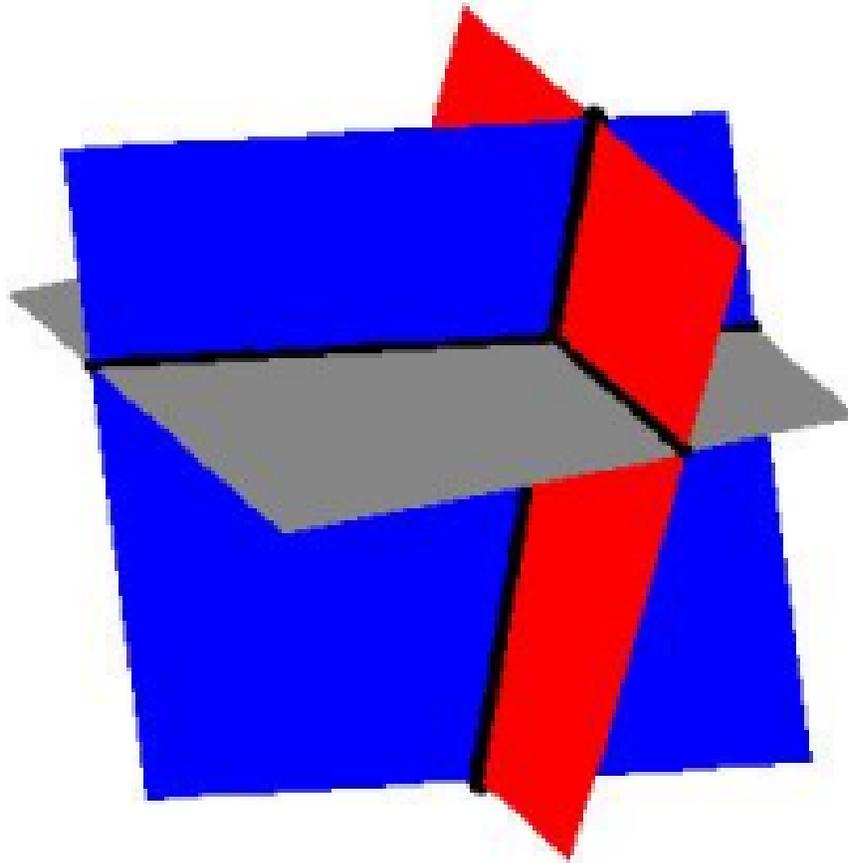
$$x = \frac{16}{6} = \frac{8}{3}$$

Now we have our ordered triple; remember that where you place the solutions matters!

$$(x, y, z) = \left(\frac{8}{3}, \frac{7}{2}, -1\right)$$

Analysis of the Solution:

Each of the lines in the system above represents a plane (think about a sheet of paper). If you imagine three sheets of notebook paper each representing a portion of these planes, you will start to see the complexities involved in how three such planes can intersect. Below is a sketch of the three planes. It turns out that any two of these planes intersect in a line, so our intersection point is where all three of these lines meet.



Three planes intersecting.

In the following video, we show another example of using back-substitution to solve a system in three variables.

**System of Three Equations - Back Substitution**

Solve the system by back substitution

$$\begin{cases} 2x - 4y + 5z = 0 \\ 3y - 2z = -5 \\ 2x = -4 \end{cases}$$

$x =$

$y =$

$z =$

$x = -2$

$\frac{2x}{2} = \frac{-4}{2}$        $3y - 2z = -5$

$x = -2$        $3y - 2(-2) = -5$

$3y + 4 = -5$

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<https://courses.lumenlearning.com/intermediatealgebra/?p=2094>

In the next example, we will not start with a solution, but will need to use the method of elimination to find our first solution.

## EXAMPLE

Find a solution to the following system:

$$x - y + z = 5 \quad (1)$$

$$-2y + z = 6 \quad (2)$$

$$2y - 2z = -12 \quad (3)$$

### Answer

We labeled the equations this time to be able to keep track of things a little more easily. The most obvious first step here is to eliminate  $y$  by adding equations (2) and (3).

$$-2y + z = 6 \quad (2)$$

$$\underline{2y - 2z = -12} \quad (3)$$

$$-z = -6$$

$$z = 6$$

Now we can substitute the value for  $z$  that we obtained into equation (2).

$$-2y + (6) = 6$$

$$-2y = 6 - 6$$

$$-2y = 0$$

$$y = 0$$

Be careful here not to get confused with a solution of  $y = 0$  and an inconsistent solution. It is ok for variables to equal 0.

Now we can substitute  $z = 6$  and  $y = 0$  back into the first equation.

$$x - y + z = 5$$

$$x - 0 + 6 = 5$$

$$x + 6 = 5$$

$$x = 5 - 6$$

$$x = -1$$

$$(x, y, z) = (-1, 0, 6)$$

In the following videos, we show more examples of the algebra you may encounter when solving systems with three variables.

## Summary

**Ex: Solve the System of Equations Using Elimination**

(a)  $3x - 8y + z = -14$   
 (b)  $x - y + z = -5$   
 (c)  $x - 3y = -4$   
 (d)  $-2x + 7y = 9$

$2(c) + (d)$   
 $2x - 6y = -8$   
 $-2x + 7y = 9$   
 $y = 1$

Using (b)  
 $-1 - 1 + z = -5$   
 $-2 + z = -5$   
 $z = -3$

Using (c)  
 $x - 3(1) = -4$   
 $x - 3 = -4$   
 $x = -1$

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**Ex: Solve the System of Equations Using Elimination**

(a)  $3x + y - 4z = -6$   
 (b)  $2x + 3y - 4z = 4$   
 (c)  $x - y + 2z = -4$

$2(c) + (b)$   
 $2x - 2y + 4z = -8$   
 $2x + 3y - 4z = 4$   
 $4x + y = -4$

$2(c) + (a)$   
 $2x - 2y + 4z = -8$   
 $3x + y - 4z = -6$   
 $5x - y = -14$

We can also say the system is consistent and independent.

$x = -2$

Using (d)  $y = -4x - 4$   
 $y = -4(-2) - 4$   
 $y = 4$

Using (c)  $-2 - 4 + 2z = -4$   
 $-6 + 2z = -4$   
 $2z = 2$   
 $z = 1$

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- The solution to a system of linear equations in three variables is an ordered triple of the form  $(x, y, z)$ .
- Solutions can be verified using substitution and the order of operations.
- Systems of three variables can be solved using the same techniques as we used to solve systems with two variables, including elimination and substitution.

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# INCONSISTENT AND DEPENDENT SYSTEMS IN THREE VARIABLES

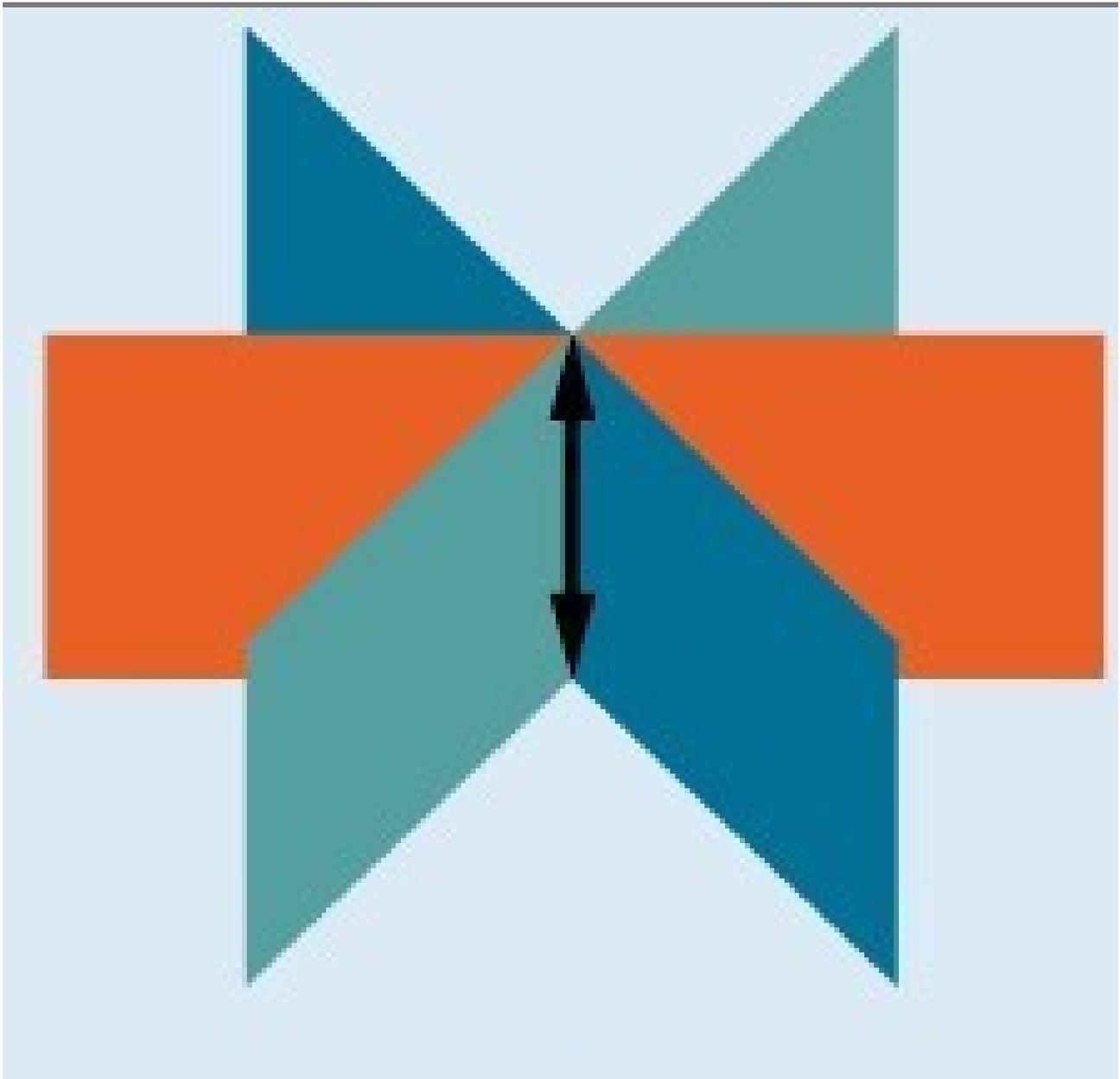
## LEARNING OUTCOMES

- Identify inconsistent systems of equations containing three variables
- Express the solution of a system of dependent equations containing three variables

Just as with systems of equations in two variables, we may come across an **inconsistent system** of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as  $3 = 7$  or some other contradiction.

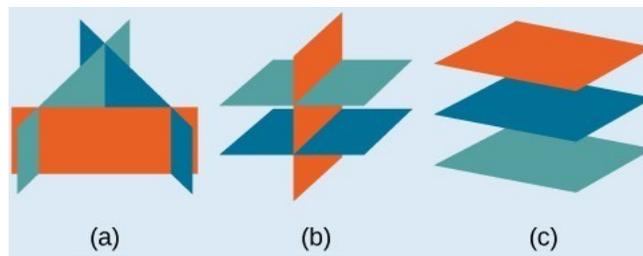
## INFINITELY MANY OR NO SOLUTIONS

- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as  $0 = 0$ . Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space. The graphic below shows how three planes can intersect to form a line giving the system infinitely many solutions.



*Infinitely many solutions.*

- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as  $3 = 0$ . Graphically, a system with no solution is represented by three planes with no point in common. Three parallel planes (c), two parallel planes and one intersecting plane (b), three planes that intersect the other two but not at the same location (a).



In the first example, we will see how it is possible to have a system with three variables and no solutions.

## EXAMPLE

Solve the following system.

$$x - 3y + z = 4 \quad (1)$$

$$-y - 4z = 7 \quad (2)$$

$$2y + 8z = -12 \quad (3)$$

### Answer

If you multiply equation (2) by 2 and add (2) and (3) together, you can eliminate  $y$  and solve for  $z$ .

First, multiply both sides of equation (2) by 2.

$$2(-y - 4z) = 2(7)$$

$$-2y - 8z = 14$$

Next, add equations (2) and (3) together to eliminate  $y$  and solve for  $z$ .

$$-2y - 8z = 14$$

$$\underline{2y + 8z = -12}$$

$$0 + 0 = 2$$

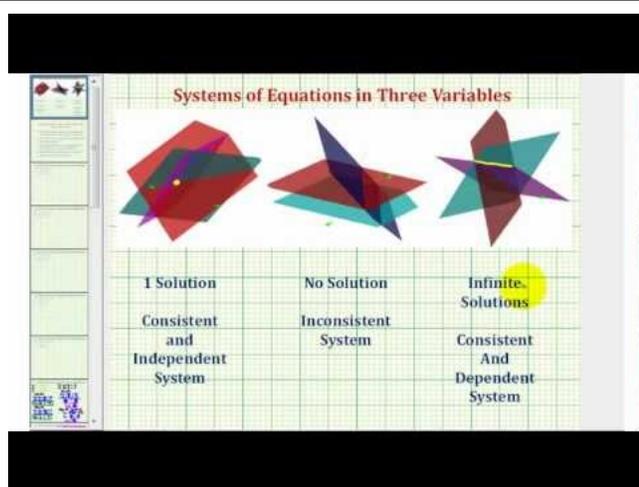
$$0 = 2$$

Recall that when we were solving systems with two variables, a non-true solution such as  $0 = 2$  implied that there was no solution to the system.

This can happen with a system of three variables as well.

This system has no solution.

We will show another example of using elimination to solve a system in three variables that ends up having no solution in the following video.



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<https://courses.lumenlearning.com/intermediatealgebra/?p=2096>

We know from working with systems of equations in two variables that a **dependent system** of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. The other possibility is that two of the equations could be the same and intersect the third on a line.

In our last video example, we show a system that has an infinite number of solutions.

## EXAMPLE

Find the solution to the given system of three equations in three variables.

$$2x + y - 3z = 0 \quad (1)$$

$$4x + 2y - 6z = 0 \quad (2)$$

$$x - y + z = 0 \quad (3)$$

### Answer

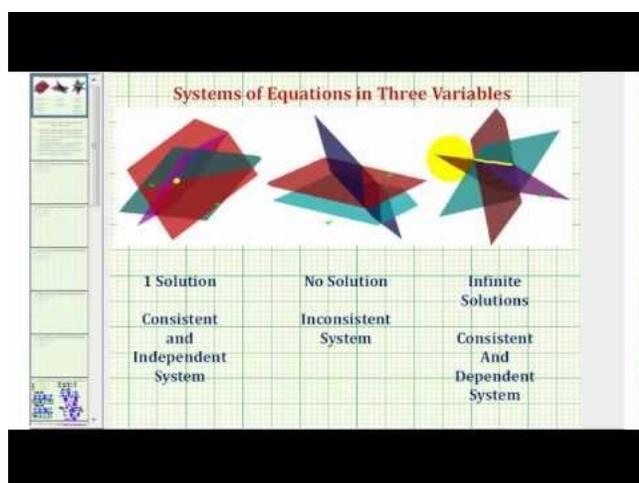
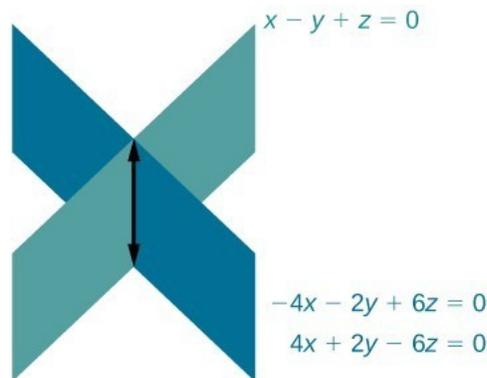
First, we can multiply equation (1) by  $-2$  and add it to equation (2).

$$4x - 2y + 6z = 0 \quad \text{equation (1) multiplied by } -2$$

$$4x + 2y - 6z = 0 \quad (2)$$

We do not need to proceed any further. The result we get when adding the two equations is an identity,  $0 = 0$  which tells us that this system has an infinite number of solutions.

As shown in the figure below, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.



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## Summary

- A system with three variables can have one, none, or many solutions.

- A system with no solutions will have a non-true result when solving.
- A system with many solutions will have an identity result when solving.

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# APPLICATIONS OF LINEAR EQUATIONS IN THREE VARIABLES

## LEARNING OUTCOMES

- Solve applications using systems of equations in three variables

Systems can be helpful for solving real-world problems. In this example, we will write three equations that model sales at an art fair to learn how many prints should be sold to break even for the cost of the booth rental.

In the following video example, we show how to define a system of three equations in three variables that represents a mixture needed by a chemist.

Our last example shows you how to write a system of three equations that represents ticket sales for a theater that has three different prices for tickets.

## EXAMPLE

Andrea sells photographs at art fairs. She prices the photos according to size: small photos cost \$10, medium photos cost \$15, and large photos cost \$40. She usually sells as many small photos as medium and large photos combined. She also sells twice as many medium photos as large. A booth at the art fair costs \$300.

If her sales go as usual, how many of each size photo must she sell to pay for the booth?

### Answer

To set up the system, first choose the variables. In this case the unknown values are the number of small, medium, and large photos.

$S$  = number of small photos sold

$M$  = number of medium photos sold

$L$  = number of large photos sold

The total of her sales must be \$300 to pay for the booth. We can find the total by multiplying the cost for each size by the number of that size sold.

$10S$  = money received for small photos

$15M$  = money received for medium photos

$40L$  = money received for large photos

Total Sales:  $10S + 15M + 40L = 300$

The number of small photos is the same as the total of medium and large photos combined.

$S = M + L$

She sells twice as many medium photos as large photos.

$M = 2L$

To make things easier, rewrite the equations to be in the same format, with all variables on the left side of the equal sign and only a constant number on the right.

$$\begin{cases} 10S + 15M + 40L = 300 \\ S - M - L = 0 \\ M - 2L = 0 \end{cases}$$

Now solve the system.

**Step 1:** First choose two equations and eliminate a variable. Since one equation has no  $S$  variable, it may be helpful to use the other two equations and eliminate the  $S$  variable from them. Multiply both sides of the second equation by  $-10$ .

$$-10(S - M - L) = -10(0)$$

$$-10s + 10M + 10L = 0$$

Now add this modified equation with the first equation in the original list of equation.

$$10S + 15M + 40L = 300$$

$$+(-10s + 10M + 10L = 0)$$

$$25M + 50L = 300$$

**Step 2:** The other equation for our two-variable system will be the remaining equation (that has no  $S$  variable). Eliminate a second variable using the equation from step 1. While you could multiply the third of the original equations by 25 to eliminate  $L$ , the numbers will stay nicer if you divide the resulting equation from step 1 by 25. Do not forget to be careful of the signs!

Divide first:

$$\frac{25}{25}M + \frac{50}{25}L = \frac{300}{25}$$

$$M + 2L = 12$$

Now eliminate L by adding  $M - 2L = 0$  to this new equation.

$$M + 2L = 12$$

$$\underline{M - 2L = 0}$$

$$2M = 12$$

$$M = \frac{12}{2} = 6$$

**Step 3:** Use  $M=6$  and one of the equations containing just two variables to solve for the second variable. It is best to use one of the original equation in case an error was made in multiplication.

$$M - 2L = 0$$

$$6 - 2L = 0$$

$$-2L = -6$$

$$L = 3$$

**Step 4:** Use the two found values and one of the original equations to solve for the third variable.

$$S - M - L = 0$$

$$S - 6 - 3 = 0$$

$$S - 9 = 0$$

$$S = 9$$

**Step 5: Check your answer.** With application problems, it is sometimes easier (and better) to use the original wording of the problem rather than the equations you write.

*She usually sells as many small photos as medium and large photos combined.*

- Medium and large photos combined:  $6 + 3 = 9$ , which is the number of small photos.

*She also sells twice as many medium photos as large.*

- Medium photos is 6, which is twice the number of large photos (3).

*A booth at the art fair costs \$300.*

- Andrea receives  $\$10(9)$  or  $\$90$  for the 9 small photos,  $\$15(6)$  or  $\$90$  for the 6 medium photos, and  $\$40(3)$  or  $\$120$  for the large photos.  $\$90 + \$90 + \$120 = \$300$ .

If Andrea sells 9 small photos, 6 medium photos, and 3 large photos, she will receive exactly the amount of money needed to pay for the booth.

**Systems of Three Equations with Three Unknowns**  
 A chemist has three different acid solutions. The first acid solution contains 10% acid, the second contains 40% acid and the third 60% acid. The chemist wants to use all three solutions to obtain a mixture of 100 liters containing 45% acid. The supply of 40% acid is low so twice as much 10% solution as 40% solution must be used. How many liters of each solution should be used?

$x = \text{amount of 10\% solution}$   
 $y = \text{amount of 40\% solution}$   
 $z = \text{amount of 60\% solution}$

$$x + y + z = 100$$

$$0.1x + 0.4y + 0.6z = 45$$

$$2x = y$$

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**Systems of Three Equations with Three Unknowns**  
 A movie theater is filled to capacity of 350. The theater charges \$4.50 for children, \$7.50 for students, and \$12.50 of adults. There are half as many adults as there are students. If the total ticket sales was \$2415, How many children, students, and adults attended?

$x = \text{the number of children}$   
 $y = \text{the number of students}$   
 $z = \text{the number of adults}$

$$x + y + z = 350$$

$$4.5x + 7.5y + 12.5z = 2415$$

$$y = 2z$$

60 adults

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# INTRODUCTION TO SYSTEMS OF LINEAR INEQUALITIES

In this section, we will define solutions to systems of linear inequalities and graph systems of linear inequalities. We will also see how systems of linear inequalities can be used to solve real life problems.

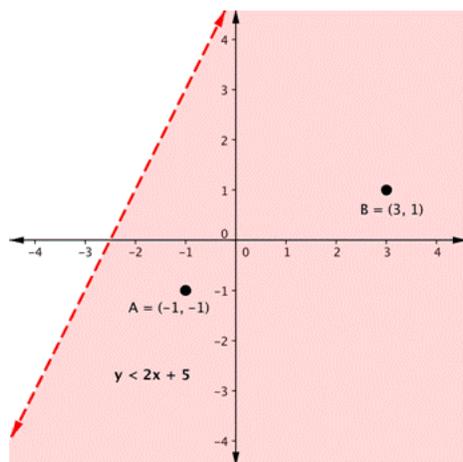
# DEFINE SOLUTIONS TO SYSTEMS OF LINEAR INEQUALITIES

## LEARNING OUTCOMES

- Identify solutions to systems of linear inequalities

## Graph a System of Two Inequalities

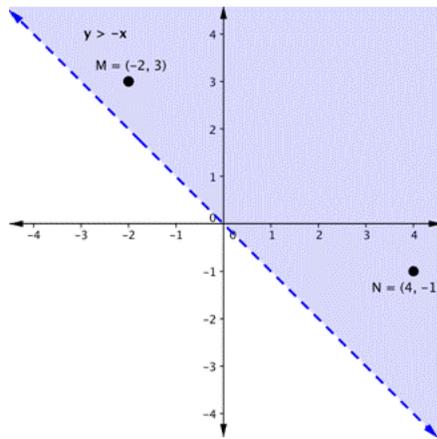
Remember from the module on graphing that the graph of a single linear inequality splits the **coordinate plane** into two regions. On one side lie all the solutions to the inequality. On the other side, there are no solutions. Consider the graph of the inequality  $y < 2x + 5$ .



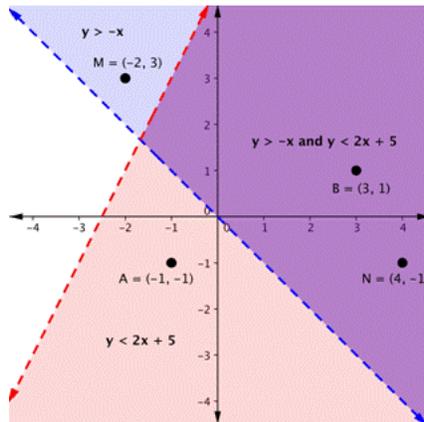
The dashed line is  $y = 2x + 5$ . Every ordered pair in the shaded area below the line is a solution to  $y < 2x + 5$ , as all of the points below the line will make the inequality true. If you doubt that, try substituting the  $x$  and  $y$  coordinates of Points A and B into the inequality; you will see that they work. So, the shaded area shows all of the solutions for this inequality.

The boundary line divides the coordinate plane in half. In this case, it is shown as a dashed line as the points on the line do not satisfy the inequality. If the inequality had been  $y \leq 2x + 5$ , then the boundary line would have been solid.

Now graph another inequality:  $y > -x$ . You can check a couple of points to determine which side of the boundary line to shade. Checking points M and N yield true statements. So, we shade the area above the line. The line is dashed as points on the line are not true.



To create a system of inequalities, you need to graph two or more inequalities together. Let us use  $y < 2x + 5$  and  $y > -x$  since we have already graphed each of them.



The purple area shows where the solutions of the two inequalities overlap. This area is the solution to the system of inequalities. Any point within this purple region will be true for both  $y > -x$  and  $y < 2x + 5$ .

In the following video examples, we show how to graph a system of linear inequalities and define the solution region.

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In the next section, we will see that points can be solutions to systems of equations and inequalities. We will verify algebraically whether a point is a solution to a linear equation or inequality.

**Examples: Systems of Equations - Graphing**

Solve by graphing.

(1)  $x - 3y < 6$

(2)  $x + 4y \leq 4$

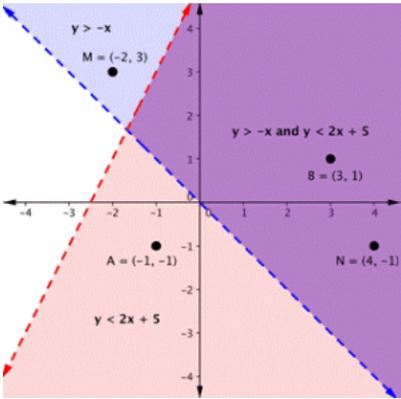
(1) dashed  
 $x - 3y = 6$

$x$	$y$	
0	-2	$-3y = 6$
6	0	

(2) solid  
 $x + 4y = 4$

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# Determine Whether an Ordered Pair is a Solution to a System of Linear Inequalities



On the graph above, you can see that the points B and N are solutions for the system because their coordinates will make both inequalities true statements.

In contrast, points M and A both lie outside the solution region (purple). While point M is a solution for the inequality  $y > -x$  and point A is a solution for the inequality  $y < 2x + 5$ , neither point is a solution for the system. The following example shows how to test a point to see whether it is a solution to a system of inequalities.

**Example**

Is the point  $(2, 1)$  a solution of the system  $x + y > 1$  and  $2x + y < 8$ ?

**Answer**

Check the point with each of the inequalities. Substitute 2 for  $x$  and 1 for  $y$ . Is the point a solution of both inequalities?

- $x + y > 1$
- $2 + 1 > 1$
- $3 > 1$
- TRUE

$(2, 1)$  is a solution for  $x + y > 1$ .

$$2x + y < 8$$

$$2(2) + 1 < 8$$

$$4 + 1 < 8$$

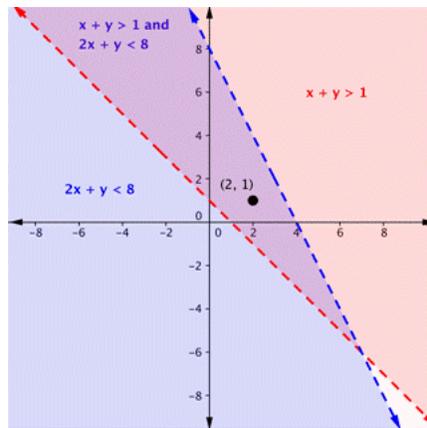
$$5 < 8$$

TRUE

$(2, 1)$  is a solution for  $2x + y < 8$ .

Since  $(2, 1)$  is a solution of each inequality, it is also a solution of the system.

Here is a graph of the system in the example above. Notice that  $(2, 1)$  lies in the purple area which is the overlapping area for the two inequalities.



### Example

Is the point  $(2, 1)$  a solution of the system  $x + y > 1$  and  $3x + y < 4$ ?

**Answer**

Check the point with each of the inequalities. Substitute 2 for  $x$  and 1 for  $y$ . Is the point a solution of both inequalities?

$$x + y > 1$$

$$2 + 1 > 1$$

$$3 > 1$$

TRUE

$(2, 1)$  is a solution for  $x + y > 1$ .

$$3x + y < 4$$

$$3(2) + 1 < 4$$

$$6 + 1 < 4$$

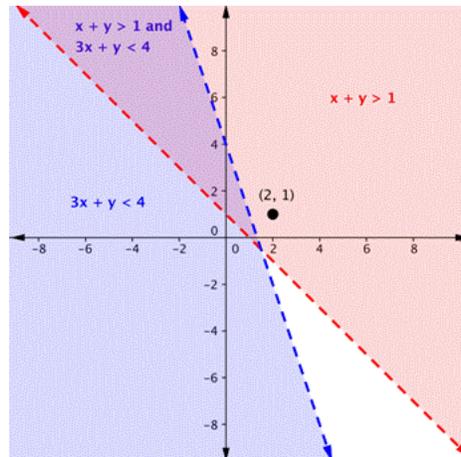
$$7 < 4$$

FALSE

$(2, 1)$  is *not* a solution for  $3x + y < 4$ .

Since  $(2, 1)$  is *not* a solution of one of the inequalities, it is not a solution of the system.

Here is a graph of the system above. Notice that  $(2, 1)$  is not in the purple area which is the overlapping area; it is a solution for one inequality (the red region), but it is not a solution for the second inequality (the blue region).



In the following video, we show another example of determining whether a point is in the solution of a system of linear inequalities.

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As shown above, finding the solutions of a system of inequalities can be done by graphing each inequality and identifying the region they share. Below, you are given more examples that show the entire process of defining the region of solutions on a graph for a system of two linear inequalities. The general steps are outlined below:

- Graph each inequality as a line and determine whether it will be solid or dashed.
- Determine which side of each boundary line represents solutions to the inequality by testing a point on each side.
- Shade the region that represents solutions for both inequalities.

## Systems with No Solutions

In the next example, we will show the solution to a system of two inequalities whose boundary lines are parallel to each other. When the graphs of a system of two linear equations are parallel to each other, we found that there was no solution to the system. We will get a similar result for the following system of linear inequalities.

## Summary

- Solutions to systems of linear inequalities are entire regions of points.
- You can verify whether a point is a solution to a system of linear inequalities in the same way you verify whether a point is a solution to a system of equations.
- Systems of inequalities can have no solutions when boundary lines are parallel.

## EXAMPLES

Graph the system

$$y \geq 2x + 1$$

$$y < 2x - 3$$

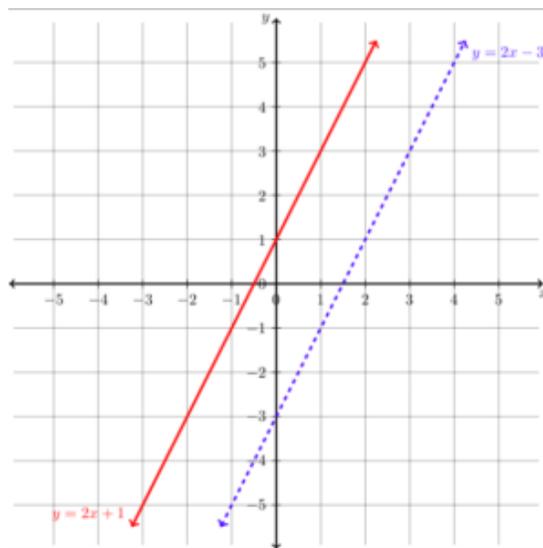
### Answer

The boundary lines for this system are parallel to each other. Note how they have the same slopes.

$$y = 2x + 1$$

$$y = 2x - 3$$

Plotting the boundary lines will give the graph below. Note that the inequality  $y < 2x - 3$  requires that we draw a dashed line, while the inequality  $y \geq 2x + 1$  requires a solid line.



Now we need to shade the regions that represent the inequalities. For the inequality  $y \geq 2x + 1$ , we can test a point on either side of the line to see which region to shade. Test  $(0, 0)$  to make it easy.

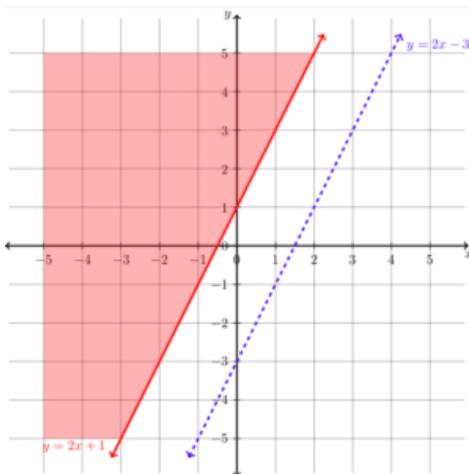
Substitute  $(0, 0)$  into  $y \geq 2x + 1$

$$y \geq 2x + 1$$

$$0 \geq 2(0) + 1$$

$$0 \geq 1$$

This is not true, so we know that we need to shade the other side of the boundary line for the inequality  $y \geq 2x + 1$ . The graph will now look like this:



Now shade the region that shows the solutions to the inequality  $y < 2x - 3$ . Again, we can pick  $(0, 0)$  to test, because it makes easy algebra.

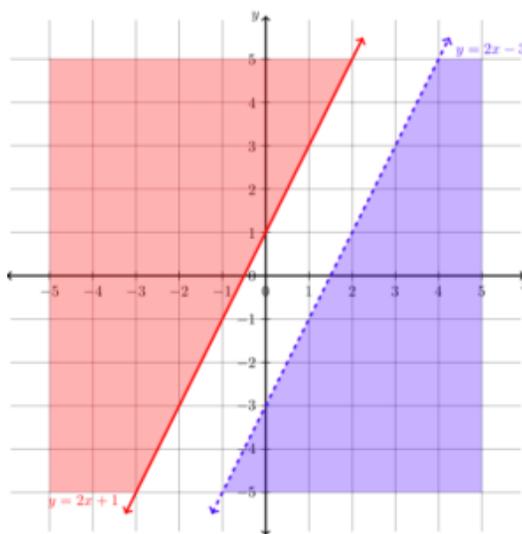
Substitute  $(0, 0)$  into  $y < 2x - 3$

$$y < 2x - 3$$

$$0 < 2(0) - 3$$

$$0 < -3$$

This is not true, so we know that we need to shade the other side of the boundary line for the inequality  $y < 2x - 3$ . The graph will now look like this:



This system of inequalities has no points in common so has no solution.

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# GRAPH SOLUTIONS TO SYSTEMS OF LINEAR INEQUALITIES

## LEARNING OUTCOMES

- Graph systems of linear inequalities

We will continue to practice graphing the solution region for systems of linear inequalities. We will also graph the solutions to a system that includes a compound inequality.

The videos that follow show more examples of graphing the solution set of a system of linear inequalities.

The system in our last example includes a compound inequality. We will see that you can treat a compound inequality like two lines when you are graphing them.

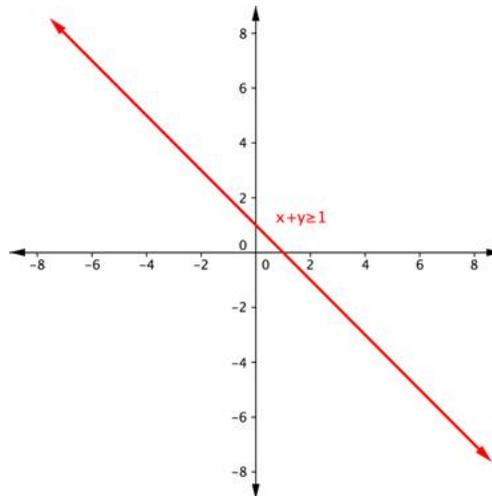
In the video that follows, we show how to solve another system of inequalities that contains a compound inequality.

## EXAMPLE

Shade the region of the graph that represents solutions for both inequalities.  $x + y \geq 1$  and  $y - x \geq 5$ .

### Answer

Graph one inequality. First graph the boundary line using a table of values, intercepts, or any other method you prefer. The boundary line for  $x + y \geq 1$  is  $x + y = 1$ , or  $y = -x + 1$ . Since the equal sign is included with the greater than sign, the boundary line is solid.



Find an ordered pair on either side of the boundary line. Insert the  $x$  and  $y$ -values into the inequality  $x + y \geq 1$  and see which ordered pair results in a true statement.

Test 1 :  $(-3, 0)$

$$x + y \geq 1$$

$$-3 + 0 \geq 1$$

$$-3 \geq 1$$

FALSE

Test 2 :  $(4, 1)$

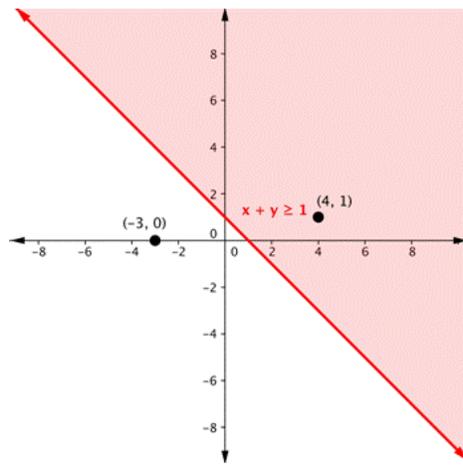
$$x + y \geq 1$$

$$4 + 1 \geq 1$$

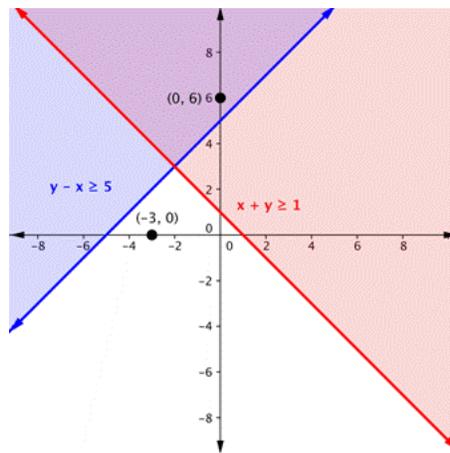
$$5 \geq 1$$

TRUE

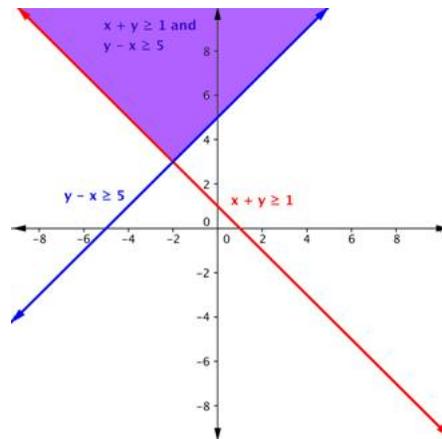
Since  $(4, 1)$  results in a true statement, the region that includes  $(4, 1)$  should be shaded.

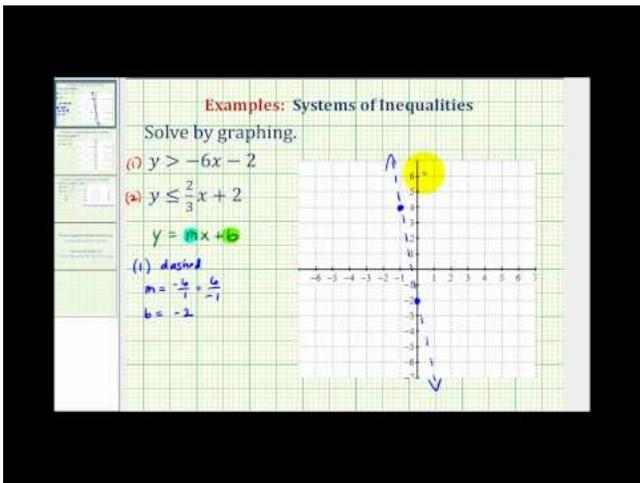


Do the same with the second inequality. Graph the boundary line, then test points to find which region is the solution to the inequality. In this case, the boundary line is  $y - x = 5$  (or  $y = x + 5$ ) and is solid. Test point  $(-3, 0)$  is not a solution of  $y - x \geq 5$  and test point  $(0, 6)$  is a solution.

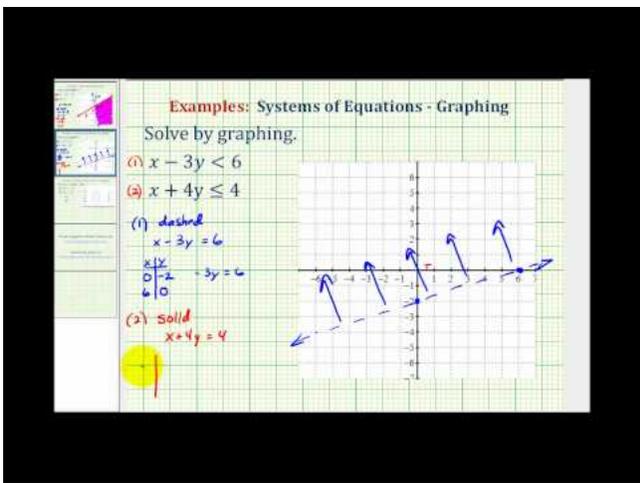


The purple region in this graph shows the set of all solutions of the system.





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## EXAMPLE

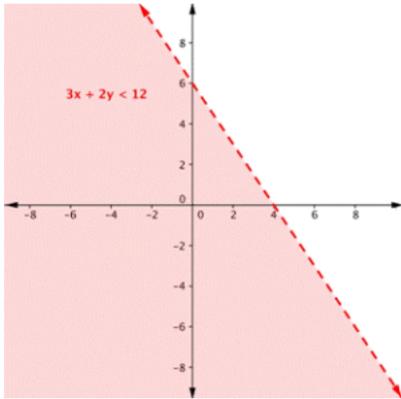
Find the solution to the system  $3x + 2y < 12$  and  $-1 \leq y \leq 5$ .

### Answer

Graph one inequality. First graph the boundary line, then test points.

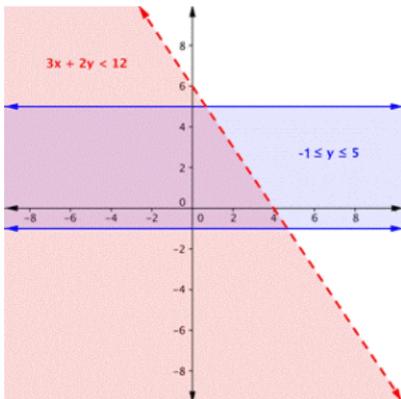
Remember, because the inequality  $3x + 2y < 12$  does not include the equal sign, draw a dashed border line.

Testing a point like  $(0, 0)$  will show that the area below the line is the solution to this inequality.

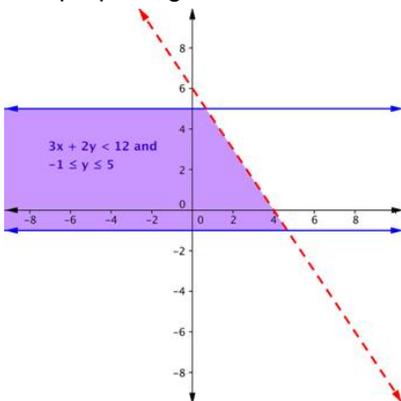


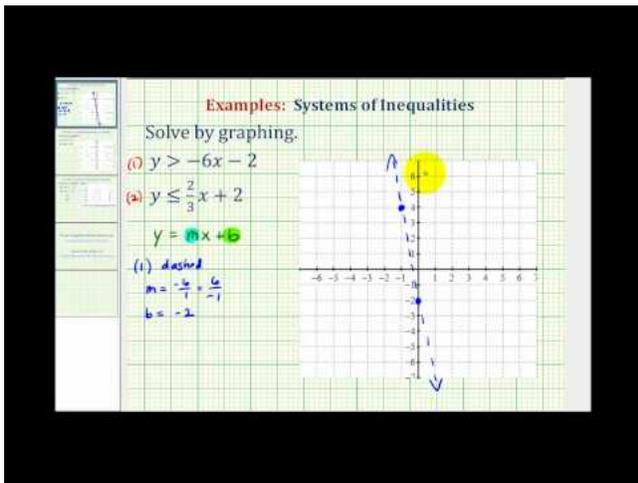
The inequality  $-1 \leq y \leq 5$  is actually two inequalities:  $-1 \leq y$ , and  $y \leq 5$ . Another way to think of this is  $y$  must be between  $-1$  and  $5$ . The border lines for both are horizontal. The region between those two lines contains the solutions of  $-1 \leq y \leq 5$ . We make the lines solid because we also want to include  $y = -1$  and  $y = 5$ .

Graph this region on the same axes as the other inequality.



The purple region shows the set of all solutions of the system.





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# APPLICATIONS OF SYSTEMS OF LINEAR INEQUALITIES

## Learning Outcomes

- Interpret graphs and solutions to systems of linear inequalities

In our first example we will show how to write and graph a system of linear inequalities that models the amount of sales needed to obtain a specific amount of money.

In a previous example for finding a solution to a system of linear equations, we introduced a manufacturer’s cost and revenue equations:

Cost:  $y = 0.85x + 35,000$

Revenue:  $y = 1.55x$

$x \geq 0, y \geq 0$

The cost equation is shown in blue in the graph below, and the revenue equation is graphed in orange. The point at which the two lines intersect is called the break-even point. We learned that this is the solution to the system of linear equations that cause the cost and revenue equations to equal each other. Note how the lines shown only represent where  $x \geq 0, y \geq 0$ . It is easy to forget to include this part in the graph.

## EXAMPLE

Cathy is selling ice cream cones at a school fundraiser. She is selling two sizes: small (which has 1 scoop) and large (which has 2 scoops). She knows that she can get a maximum of 70 scoops of ice cream out of her supply. She charges \$3 for a small cone and \$5 for a large cone.

Cathy wants to earn at least \$120 to give back to the school. Write and graph a system of inequalities that models this situation.

### Answer

First, identify the variables. There are two variables: the number of small cones and the number of large cones.

$s$  = small cone

$l$  = large cone

Write the first equation: the maximum number of scoops she can give out. The scoops she has available (70) must be greater than or equal to the number of scoops for the small cones ( $s$ ) and the large cones ( $2l$ ) she sells.

$$s + 2l \leq 70$$

Write the second equation: the amount of money she raises. She wants the total amount of money earned from small cones [ $3s$ ] and large cones [ $5l$ ] to be at least \$120.

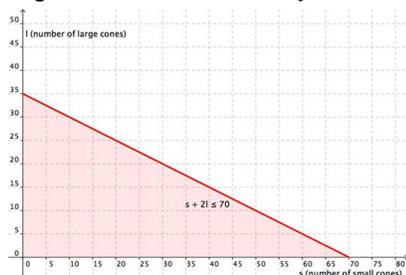
$$3s + 5l \geq 120$$

Write the system.

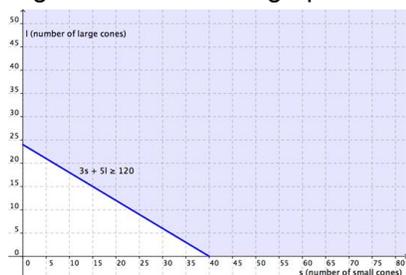
$$\begin{cases} s + 2l \leq 70 \\ 3s + 5l \geq 120 \\ s \geq 0 \\ l \geq 0 \end{cases}$$

Now graph the system. The variables  $x$  and  $y$  have been replaced by  $s$  and  $l$ ; graph  $s$  along the  $x$ -axis and  $l$  along the  $y$ -axis.

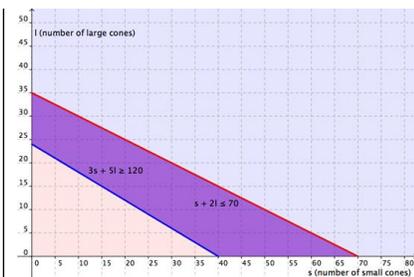
First graph the region  $s + 2l \leq 70$ . Graph the boundary line and then test individual points to see which region to shade. We only shade the regions that also satisfy  $x \geq 0$ ,  $y \geq 0$ . The graph is shown below.



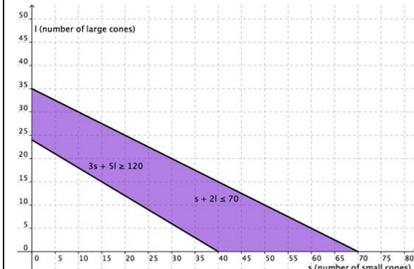
Now graph the region  $3s + 5l \geq 120$ . Graph the boundary line and then test individual points to see which region to shade. The graph is shown below.



Graphing the regions together, you find the following:



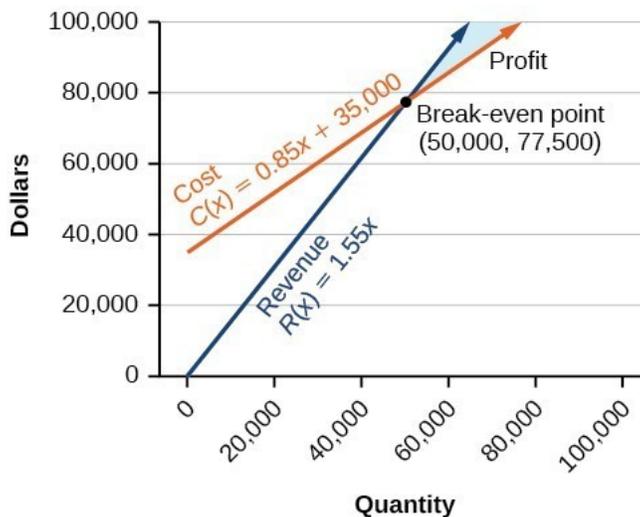
Looking at just the overlapping region, you have:



The region in purple is the solution. As long as the combination of small cones and large cones that Cathy sells can be mapped in the purple region, she will have earned at least \$120 and not used more than 70 scoops of ice cream.

The shaded region to the right of the break-even point represents quantities where the company makes a profit. The region to the left represents quantities where the company suffers a loss.

In the next example, you will see how the information you learned about systems of linear inequalities can be applied to answering questions about cost and revenue.



Note how the blue shaded region between the cost and revenue equations is labeled profit. This is the “sweet spot” that the company wants to achieve where they produce enough bike frames at a minimal enough cost to make money. They do not want more money going out than coming in!

In the following video, you will see an example of how to find the break-even point for a small sno-cone business.

Below is one more video example about solving an application using a system of linear inequalities.

We have seen that systems of linear equations and inequalities can help to define market behaviors that are very helpful to businesses. The intersection of cost and revenue equations gives the break-even point and also helps define the region where a company will make a profit.

## EXAMPLE

Define the profit region for the skateboard manufacturing business using inequalities given the system of linear equations:

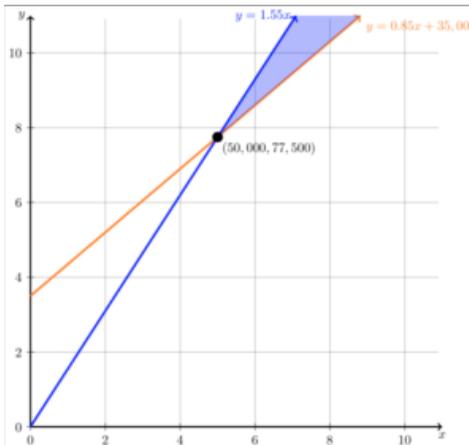
$$\text{Cost: } y = 0.85x + 35,000$$

$$\text{Revenue: } y = 1.55x$$

$$x \geq 0, y \geq 0$$

### Answer

We know that graphically, solutions to linear inequalities are entire regions, and we learned how to graph systems of linear inequalities earlier in this module. Based on the graph below and the equations that define cost and revenue, we can use inequalities to define the region where the skateboard manufacturer will make a profit. Again, note how only the region for  $x \geq 0, y \geq 0$  is included.



Start with the revenue equation. We know that the break-even point is at  $(50,000, 77,500)$ , and the profit region is the blue area. If we choose a point in the region and test it like we did for finding solution regions for inequalities, we will know which kind of inequality sign to use.

Test the point  $(65,000, 100,000)$  in both equations to determine which inequality sign to use.

Cost:

$$y = 0.85x + 35,000$$

$$100,000 ? 0.85(65,000) + 35,000$$

$$100,000 ? 90,250$$

We need to use  $>$  because 100,000 is greater than 90,250

The cost inequality that will ensure the company makes profit, not just break-even, is  $y > 0.85x + 35,000$

Now test the point in the revenue equation:

Revenue:

$$y = 1.55x$$

$$100,000 ? 1.55(65,000)$$

$$100,000 ? 100,750$$

We need to use  $<$  because 100,000 is less than 100,750

The revenue inequality that will ensure the company makes profit, not just break-even, is  $y < 1.55x$

The system of inequalities that defines the profit region for the bike manufacturer is:

$$y > 0.85x + 35,000$$

$$y < 1.55x$$

**Systems of Equations Application - Breakeven Point**

**EXAMPLE**

You decide to sell snow cones at a local park as a summer job. The snow cone machine costs \$120 per month to rent. The cost to make each snow cone is \$0.25. If you sell each snow cone for \$1.50, determine the break-even point each month.

Cost:  $y = 120 + 0.25x$   
 Revenue:  $y = 1.5x$        $(x, y)$

Got:  $y = 120 + 0.25(96)$        $1.5x = 120 + 0.25x$   
 $1.5x - 0.25x = 120$        $-0.25x$   
 $1.25x = 120$   
 $\frac{1.25x}{1.25} = \frac{120}{1.25}$   
 $x = 96$

The break-even point is the point at which cost and revenue are equal. There is no loss or profit or the company doesn't lose money or make money.

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**Example: Inequalities in Two Variables**

A phone company charges 20 cents for daytime minutes and 5 cents for night minutes. How many day and night minutes would you have to use to be charged more than \$10 in 24 hours?

$d$  = number of day min.  
 $n$  = number of night min.  
 $\$1 = 100¢ \rightarrow 10 = 1000¢$

$20d + 5n > 1000$   
 $20d + 5n = 1000$   
 $\frac{d}{50} \quad \frac{n}{200}$        $S_n = 1000$   
 $0 \quad 0$        $\leq, < \geq, >$  - solid line  
 $<, >$  - dashed line

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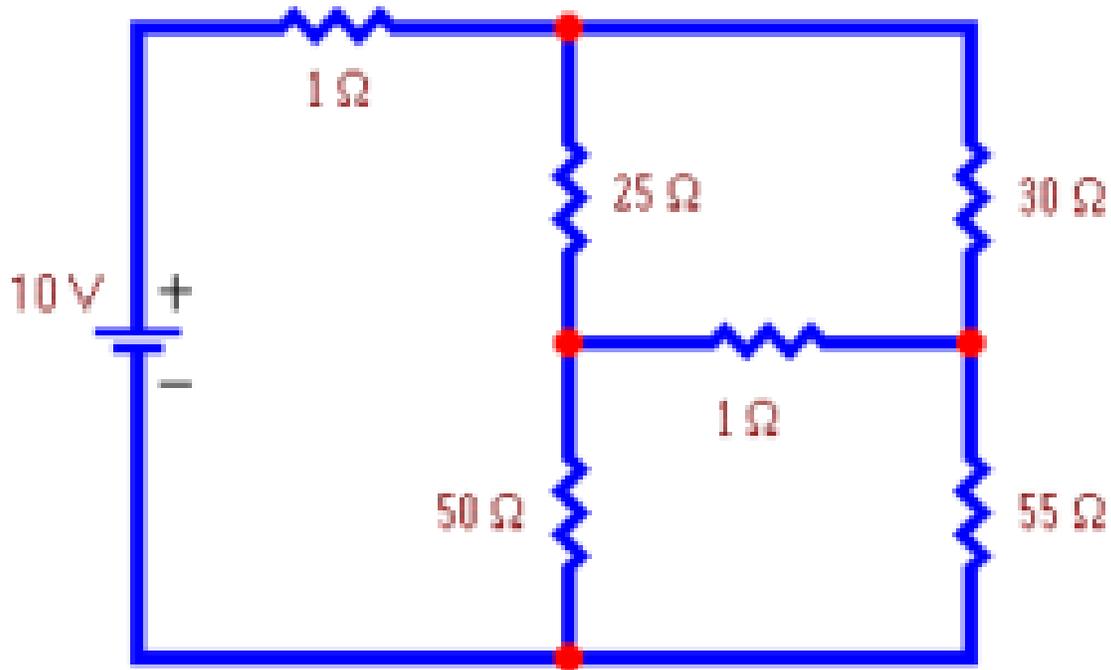
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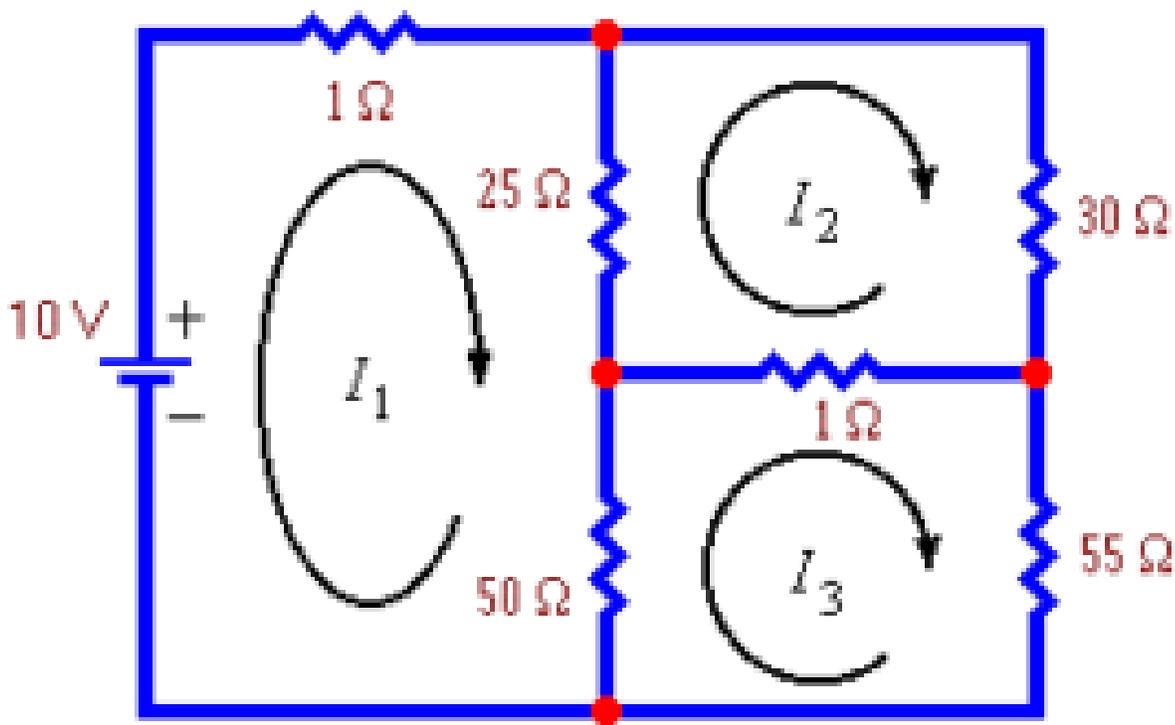
# PUTTING IT TOGETHER: LINEAR SYSTEMS

In this module, we came a few steps closer to solving the \$1,000,000 question by learning some of the basic principles of systems of equations. We learned that using systems of linear equations with two and three variables allows us to describe relationships that are affected by a number of variables. We saw that graphing systems is the same as graphing single equations, and that the solution to a system of two linear equations is the point where the two lines intersect. We also saw that the solution to a system of three linear equations in three variables is an ordered triple, and the geometric solution is the intersection of three planes in space. The number of practical applications for systems of equations spans nearly all disciplines from geology to business to economics to engineering and beyond. For example, electrical engineers may be interested in defining the strength of a current flowing through a circuit as shown in the diagram below.



*An electrical circuit.*

A system of equations based on Kirchoff's Voltage Law can be constructed and solved to find the current in amperes that flows through each branch of the circuit. Kirchoff's Voltage Law is based on the principle that voltage changes in intensity as it moves around a circuit. Each equation in the system represents one branch of the circuit, so the number of equations needed to find the current depends on the number of branches in the circuit. For example, the figure below depicts the circuit from above with its three loop currents identified:



Electrical circuit with 3 loop currents.

Because there are three loop currents, we would need to write three equations with three variables based on Kirchoff's Law to find the three unknown currents that comprise the system. If you are curious to learn more about Kirchoff's Law and loop currents, [visit this site](#).

If you continue to take more math courses, you may get to take an entire course in systems called Linear Algebra. From there you can discover more about the Navier-Stokes equations and, perhaps, solve one of the seven most important math problems in the world!

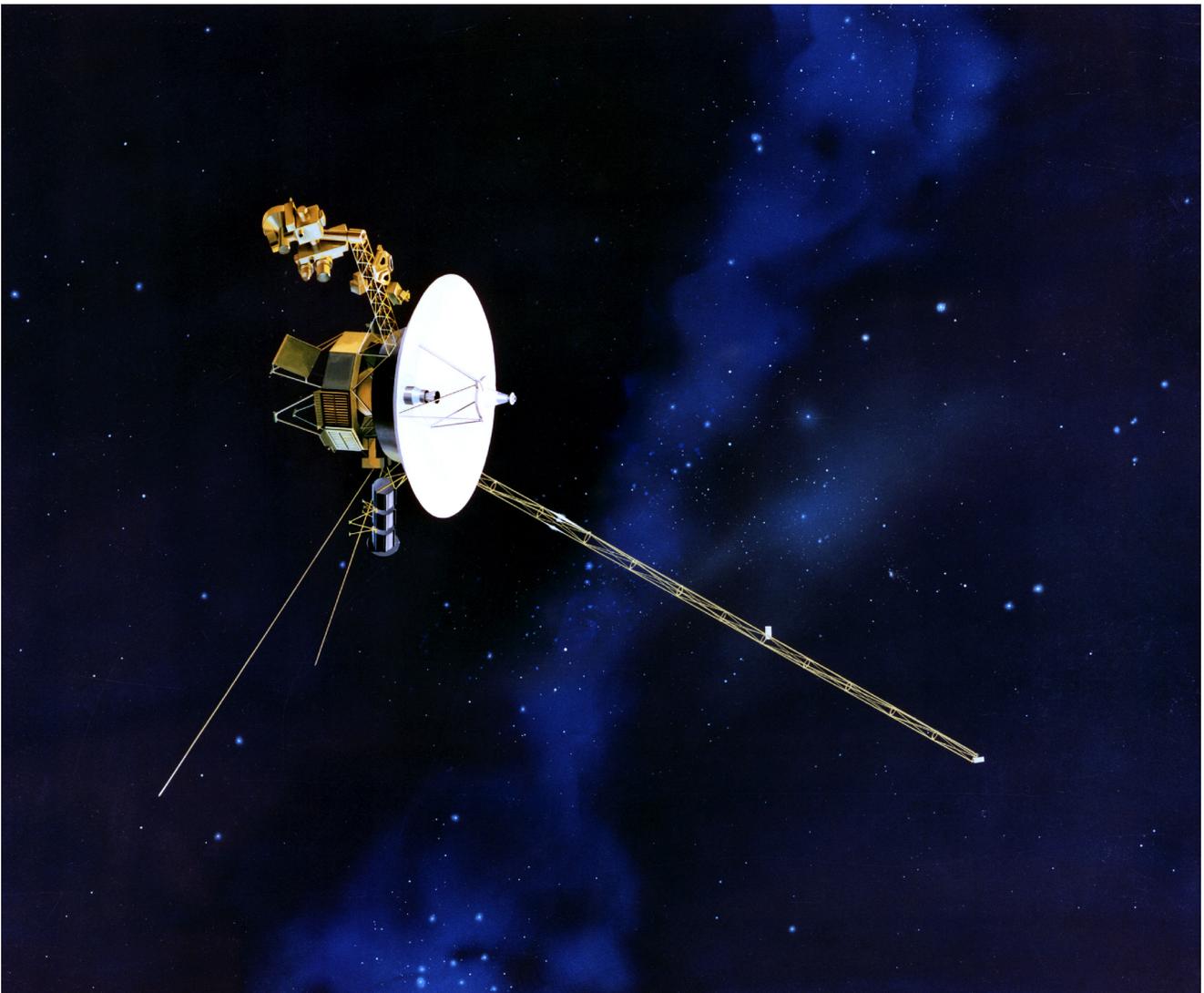
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# MODULE 7: EXPONENTS

## WHY IT MATTERS: EXPONENTS



*An artist's concept of Voyager in flight.*

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. For example, Star Wars fans may remember Han Solo bragging about the *Millennium Falcon's* ability to make the Kessel Run in less than 12 parsecs in Episode IV. He was referring to a smuggler's route with sections that were flown in hyperspace, making length an important factor in how quickly a ship could make the run.

In reality, a parsec is a unit of length used to measure large distances to objects outside the solar system. A parsec is equal to about 31 trillion kilometers or 19 trillion miles in length. Rather than writing all the zeros associated with the number 1 trillion (1, 000, 000, 000, 000), we commonly use the written words or scientific notation. Scientific notation uses exponents to represent the number of zeros that come before or after the important digits of a very small or large number. Using scientific notation, 19 trillion miles would be written as  $1.9 \times 10^{13}$  miles.

The most distant space probe, Voyager 1, was 0.0006 parsecs from Earth as of March 2015. It took Voyager 37 years to cover that distance. Voyager 1 was launched by NASA on September 5, 1977. As of 2013, the probe was moving with a relative velocity to the sun of about 17030 m/s. With the velocity the probe is currently maintaining, Voyager 1 is traveling about 325 million miles per year, or 520 million kilometers per year. Here are some more distances to well-known astronomical objects in parsecs:

- The distance to the open cluster Pleiades is 130 parsecs from Earth. That's  $1.7 \times 10^{15}$  miles.
- The center of the Milky Way is more than 8 kiloparsecs (a kiloparsec is 1000 parsecs) from Earth, and the Milky Way is roughly 34 kiloparsecs across.
- The nearest star to Earth, Proxima Centauri, is about 1.3 parsecs from the sun.
- Most of the stars visible to the unaided eye in the nighttime sky are within 500 parsecs of the sun.

In this module, you will learn the rules for algebraic operations on terms with exponents then apply them to calculations involving very large or small numbers.

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## INTRODUCTION TO APPLYING EXPONENT RULES

A common language is needed in order to communicate mathematical ideas clearly and efficiently. **Exponential notation** was developed to write repeated multiplication more efficiently. For example, growth occurs in living organisms by the division of cells. One type of cell divides 2 times in an hour. So in 12 hours, the cell will divide  $2 \cdot 2 \cdot 2$  times. This can be written more efficiently as  $2^{12}$ . In this section we will learn how to simplify and perform mathematical operations such as multiplication and division on terms that have exponents. We will also learn how to use scientific notation to represent very large or very small numbers, and perform mathematical operations on them.

## TERMS AND EXPRESSIONS WITH EXPONENTS

#### Learning Outcomes

- Evaluate exponential expressions



*An image that repeats itself.*

## Anatomy of Exponential Terms

We use exponential notation to write repeated multiplication. For example,  $10 \cdot 10 \cdot 10$  can be written more succinctly as  $10^3$ . The 10 in  $10^3$  is called the **base**. The 3 in  $10^3$  is called the **exponent**. The expression  $10^3$  is called the exponential expression. Knowing the names for the parts of an exponential expression or term will help you learn how to perform mathematical operations on them.

base  $\rightarrow 10^{3 \leftarrow \text{exponent}}$

$10^3$  is read as “10 to the third power” or “10 cubed.” It means  $10 \cdot 10 \cdot 10$ , or 1,000.

$8^2$  is read as “8 to the second power” or “8 squared.” It means  $8 \cdot 8$ , or 64.

$5^4$  is read as “5 to the fourth power.” It means  $5 \cdot 5 \cdot 5 \cdot 5$ , or 625.

$b^5$  is read as “ $b$  to the fifth power.” It means  $b \cdot b \cdot b \cdot b \cdot b$ . Its value will depend on the value of  $b$ .

The exponent applies only to the number that it is next to. Therefore, in the expression  $xy^4$ , only the  $y$  is affected by the 4.  $xy^4$  means  $x \cdot y \cdot y \cdot y \cdot y$ . The  $x$  in this term is a **coefficient** of  $y$ .

If the exponential expression is negative, such as  $-3^4$ , it means  $-(3 \cdot 3 \cdot 3 \cdot 3)$ , or  $-81$ .

If  $-3$  is to be the base, it must be written as  $(-3)^4$ , which means  $-3 \cdot -3 \cdot -3 \cdot -3$ , or  $81$ .

Likewise,  $(-x)^4 = (-x) \cdot (-x) \cdot (-x) \cdot (-x) = x^4$ , while  $-x^4 = -(x \cdot x \cdot x \cdot x)$ .

You can see that there is quite a difference, so you have to be very careful! The following examples show how to identify the base and the exponent as well as how to identify the expanded and exponential format of writing repeated multiplication.

### EXAMPLE

Identify the exponent and the base in the following terms, then simplify:

1.  $7^2$
2.  $\left(\frac{1}{2}\right)^3$
3.  $2x^3$
4.  $(-5)^2$

#### Answer

1)  $7^2$

The exponent in this term is 2, and the base is 7. To simplify, expand the term:  $7^2 = 7 \cdot 7 = 49$

2)  $\left(\frac{1}{2}\right)^3$

The exponent on this term is 3, and the base is  $\frac{1}{2}$ . To simplify, expand the multiplication and remember

how to multiply fractions:  $\left(\frac{1}{2}\right)^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$

3)  $2x^3$

The exponent on this term is 3, and the base is  $x$ . The 2 is not getting the exponent, because there are no parentheses that tell us it is. This term is in its most simplified form.

4)  $(-5)^2$

The exponent on this term is 2, and the base is  $-5$ . To simplify, expand the multiplication:

$$(-5)^2 = -5 \cdot -5 = 25$$

In the following video, you are provided more examples of applying exponents to various bases.

## Evaluate Expressions

Evaluating expressions containing exponents is the same as evaluating the linear expressions from earlier in the course. You substitute the value of the variable into the expression and simplify.

You can use the order of operations to evaluate the expressions containing exponents. First, evaluate anything in parentheses or grouping symbols. Next, look for exponents, followed by multiplication and division (reading from left to right), and lastly, addition and subtraction (again, reading from left to right).

So, when you evaluate the expression  $5x^3$  if  $x = 4$ , first substitute the value 4 for the variable  $x$ , then

**Simplifying Exponential Expressions**

**EXAMPLE**

Simplify.

1.  $5^2 = 25$
2.  $\left(\frac{2}{3}\right)^3 = \frac{8}{27}$
3.  $(-2)^4$
4.  $-2^4$
5.  $(2x^2)^3$

base  $\rightarrow$   $10^{\text{base}}$  - exponent

Handwritten solutions on a grid background:

1.  $5^2 = 5 \cdot 5 = 25$
2.  $\left(\frac{2}{3}\right)^3 = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$
3.  $(-2)^4 = -2 \cdot -2 \cdot -2 \cdot -2 = 16$

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evaluate using order of operations.

### Example

Evaluate the following expressions for the given value.

1.  $5x^3$  if  $x = 4$
2.  $(5x)^3$  if  $x = 4$
3.  $x^3$  if  $x = -4$
4.  $3^x$  if  $x = 4$

#### Answer

1) Substitute 4 for the variable  $x$ .

$$5 \cdot 4^3$$

Evaluate  $4^3$ . Multiply.

$$5(4 \cdot 4 \cdot 4) = 5 \cdot 64 = 320$$

2)  $(5x)^3$  if  $x = 4$

Substitute 4 for the variable  $x$ . Notice how including parentheses can change the outcome when you are simplifying terms with exponents.

$$(5 \cdot 4)^3$$

Multiply inside the parentheses, then apply the exponent—follow the order of operations.

$$20^3$$

Evaluate  $20^3$ .

$$20 \cdot 20 \cdot 20 = 8,000$$

3)  $x^3$  if  $x = -4$ .

Substitute  $-4$  for the variable  $x$ .

$$(-4)^3$$

Evaluate. Note how placing parentheses around the  $-4$  means the negative sign is included as part of the base.

$$-4 \cdot -4 \cdot -4$$

Multiply.

$$-4 \cdot -4 \cdot -4 = -64$$

4)  $3^x$  if  $x = 4$

Substitute  $x = 4$  into the exponent.  $3^x = 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$



Caution! Whether to include a negative sign as part of a base or not often leads to confusion. To clarify whether a negative sign is applied before or after the exponent, here is an example. What is the difference in the way you would evaluate these two terms?

1.  $-3^2$

2.  $(-3)^2$

To evaluate 1), you would apply the exponent to the three first, then apply the negative sign last, like this:

$$-(3^2) = -(9) = -9$$

To evaluate 2), you would apply the exponent to the 3 and the negative sign:

$$(-3)^2 = (-3) \cdot (-3) = 9$$

The key to remembering this is to follow the order of operations. The first expression does not include parentheses so you would apply the exponent to the integer 3 first, then apply the negative sign. The second expression includes parentheses, so hopefully you will remember that the negative sign also gets squared.

Coming up, you will learn how to simplify expressions that contain exponents. Come back to this page if you forget how to apply the order of operations to a term with exponents or forget which is the base and which is the exponent!

In the following video, you are provided with examples of evaluating exponential expressions for a given number.

Evaluate Expressions with Exponents

EXAMPLE

Evaluate each expression.

- $x^3$  when  $x = -4$
- $2x^3$  when  $x = 3$
- $(2x)^2$  when  $x = 3$

base  $\rightarrow 10^{\text{th}}$  exponent

1.  $(-4)^3 = -4 \cdot -4 \cdot -4 = -64$

2.

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# PRODUCT AND QUOTIENT RULES

## Learning Outcomes

- Simplify exponential expressions with like bases using the product, quotient, and power rules

## Use the Product Rule to Multiply Exponential Expressions

**Exponential notation** was developed to write repeated multiplication more efficiently. There are times when it is easier or faster to leave the expressions in exponential notation when multiplying or dividing. Let us look at rules that will allow you to do this.

For example, the notation  $5^4$  can be expanded and written as  $5 \cdot 5 \cdot 5 \cdot 5$ , or 625. Do not forget, the exponent only applies to the number immediately to its left unless there are parentheses.

What happens if you multiply two numbers in exponential form with the same base? Consider the expression  $2^3 2^4$ . Expanding each exponent, this can be rewritten as  $(2 \cdot 2 \cdot 2)(2 \cdot 2 \cdot 2 \cdot 2)$  or  $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ . In exponential form, you would write the product as  $2^7$ . Notice that 7 is the sum of the original two exponents, 3 and 4.

What about  $x^2 x^6$ ? This can be written as  $(x \cdot x)(x \cdot x \cdot x \cdot x \cdot x \cdot x) = x \cdot x$  or  $x^8$ . And, once again, 8 is the sum of the original two exponents. This concept can be generalized in the following way:

## THE PRODUCT RULE FOR EXPONENTS

For any number  $x$  and any integers  $a$  and  $b$ ,  $(x^a)(x^b) = x^{a+b}$ .

To multiply exponential terms with the same base, add the exponents.

### Example

Write each of the following products with a single base. Do not simplify further.

1.  $t^5 \cdot t^3$
2.  $(-3)^5 \cdot (-3)$
3.  $x^2 \cdot x^5 \cdot x^3$

### Answer

Use the product rule to simplify each expression.

- $t^5 \cdot t^3 = t^{5+3} = t^8$
- $(-3)^5 \cdot (-3) = (-3)^5 \cdot (-3)^1 = (-3)^{5+1} = (-3)^6$
- $x^2 \cdot x^5 \cdot x^3$

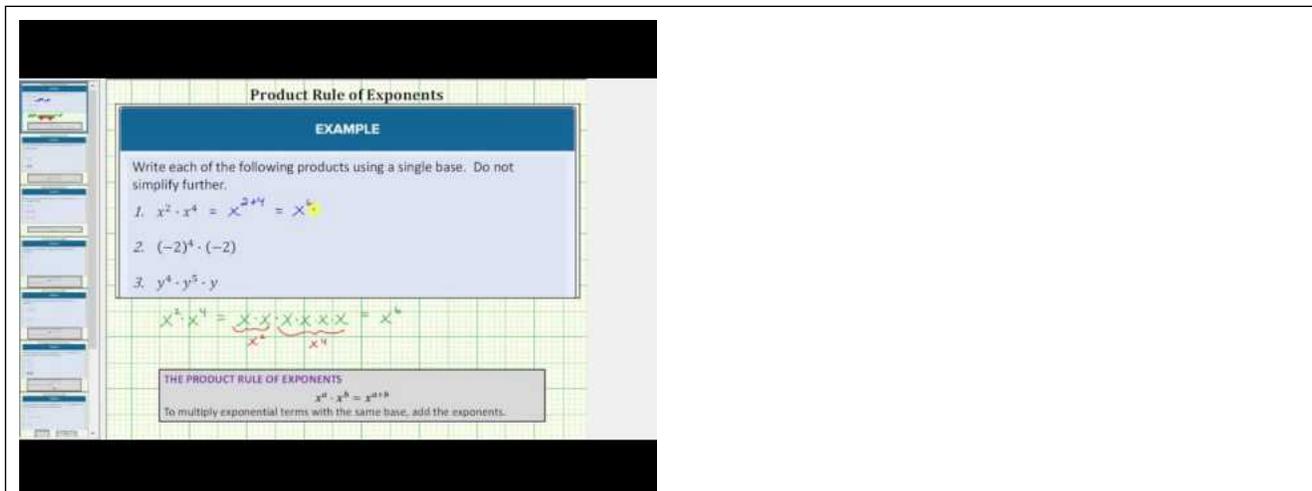
At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$x^2 \cdot x^5 \cdot x^3 = (x^2 \cdot x^5) \cdot x^3 = (x^{2+5}) \cdot x^3 = x^7 \cdot x^3 = x^{7+3} = x^{10}$$

Notice we get the same result by adding the three exponents in one step.

$$x^2 \cdot x^5 \cdot x^3 = x^{2+5+3} = x^{10}$$

In the following video, you will see more examples of using the product rule for exponents to simplify expressions.



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In our last product rule example, we will show that an exponent can be an algebraic expression. We can use the product rule for exponents no matter what the exponent looks like, as long as the base is the same.

## EXAMPLE

Multiply.  $x^{a+2} \cdot x^{3a-9}$

**Answer**

We have two exponential terms with the same base. The product rule for exponents says that we can add the exponents.

$$x^{a+2} \cdot x^{3a-9} = x^{(a+2)+(3a-9)} = x^{4a-7}$$

The expression cannot be simplified any further.

## Use the Quotient Rule to Divide Exponential Expressions

Let us look at dividing terms containing exponential expressions. What happens if you divide two numbers in exponential form with the same base? Consider the following expression.

$$\frac{4^5}{4^2}$$

You can rewrite the expression as:  $\frac{4 \cdot 4 \cdot 4 \cdot 4 \cdot 4}{4 \cdot 4}$ . Then you can cancel the common factors of 4 in the numerator and denominator:

Finally, this expression can be rewritten as  $4^3$  using exponential notation. Notice that the exponent, 3, is the difference between the two exponents in the original expression, 5 and 2.

$$\text{So, } \frac{4^5}{4^2} = 4^{5-2} = 4^3.$$

Be careful that you subtract the exponent in the denominator from the exponent in the numerator.

So, to divide two exponential terms with the same base, subtract the exponents.

## THE QUOTIENT (DIVISION) RULE FOR EXPONENTS

For any non-zero number  $x$  and any integers  $a$  and  $b$ :  $\frac{x^a}{x^b} = x^{a-b}$

### Example

Write each of the following products with a single base. Do not simplify further.

1.  $\frac{(-2)^{14}}{(-2)^9}$

2.  $\frac{t^{23}}{t^{15}}$

3.  $\frac{(z\sqrt{2})^5}{z\sqrt{2}}$

### Answer

Use the quotient rule to simplify each expression.

1.  $\frac{(-2)^{14}}{(-2)^9} = (-2)^{14-9} = (-2)^5$

2.  $\frac{t^{23}}{t^{15}} = t^{23-15} = t^8$

3.  $\frac{(z\sqrt{2})^5}{z\sqrt{2}} = (z\sqrt{2})^{5-1} = (z\sqrt{2})^4$

As we showed with the product rule, you may be given a quotient with an exponent that is an algebraic expression to simplify. As long as the bases agree, you may use the quotient rule for exponents.

## EXAMPLE

Simplify.  $\frac{y^{x-3}}{y^{9-x}}$

### Answer

We have a quotient whose terms have the same base so we can use the quotient rule for exponents.

$$\frac{y^{x-3}}{y^{9-x}} = y^{(x-3)-(9-x)} = y^{2x-12}$$

In the following video, you will see more examples of using the quotient rule for exponents.

**Quotient Rule of Exponents**

**EXAMPLE**

Write each of the following quotients using a single base. Do not simplify further.

1.  $\frac{t^7}{t^3} = t^{7-3} = t^4$        $\frac{t^7}{t^3} = \frac{t \cdot t \cdot t \cdot t \cdot t \cdot t \cdot t}{t \cdot t \cdot t} = t^4$
2.  $\frac{(-2)^{15}}{(-2)^{12}} = (-2)^{15-12} = (-2)^3$
3.  $\frac{(2\sqrt{x})^7}{2\sqrt{x}}$

**THE QUOTIENT (DIVISION) RULE OF EXPONENTS**

$\frac{x^a}{x^b} = x^{a-b} \quad x \neq 0$

To divide exponential terms with the same base, subtract the exponents.

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# THE POWER RULE FOR EXPONENTS

## Learning Outcome

- Simplify exponential expressions with like bases using the product, quotient, and power rules

## Raise Powers to Powers

Another word for exponent is power. You have likely seen or heard an example such as  $3^5$  can be described as 3 raised to the 5th power. In this section we will further expand our capabilities with exponents. We will learn what to do when a term with a power is raised to another power and what to do when two numbers or variables are multiplied and both are raised to a power. We will also learn what to do when numbers or variables that are divided are raised to a power. We will begin by raising powers to powers.

Let us simplify  $(5^2)^4$ . In this case, the base is  $5^2$  and the exponent is 4, so you multiply  $5^2$  four times:  
 $(5^2)^4 = 5^2 \cdot 5^2 \cdot 5^2 \cdot 5^2 = 5^8$  (using the Product Rule—add the exponents).

$(5^2)^4$  is a power of a power. It is the fourth power of 5 to the second power. We saw above that the answer is  $5^8$ . Notice that the new exponent is the same as the product of the original exponents:  $2 \cdot 4 = 8$ .

So,  $(5^2)^4 = 5^{2 \cdot 4} = 5^8$  (which equals 390,625 if you do the multiplication).

Likewise,  $(x^4)^3 = x^{4 \cdot 3} = x^{12}$

This leads to another rule for exponents—the **Power Rule for Exponents**. To simplify a power of a power, you multiply the exponents, keeping the base the same. For example,  $(2^3)^5 = 2^{15}$ .

## THE POWER RULE FOR EXPONENTS

For any positive number  $x$  and integers  $a$  and  $b$ :  $(x^a)^b = x^{a \cdot b}$ .

Take a moment to contrast how this is different from the product rule for exponents found on the previous page.

### Example

Write each of the following products with a single base. Do not simplify further.

1.  $(x^2)^7$
2.  $((2t)^5)^3$
3.  $((-3)^5)^{11}$

### Answer

Use the power rule to simplify each expression.

1.  $(x^2)^7 = x^{2 \cdot 7} = x^{14}$
2.  $((2t)^5)^3 = (2t)^{5 \cdot 3} = (2t)^{15}$
3.  $((-3)^5)^{11} = (-3)^{5 \cdot 11} = (-3)^{55}$

In the following video, you will see more examples of using the power rule to simplify expressions with exponents.

The screenshot shows a video player with a title 'Power Rule of Exponents'. Below the title is a blue box labeled 'EXAMPLE' containing the text: 'Write each of the following expressions using a single base. Do not simplify further.' followed by three problems: 1.  $(x^3)^2 = x^{3 \cdot 2} = x^6$ , 2.  $((2y)^4)^3$ , and 3.  $((-3)^4)^6$ . Below the problems, a handwritten-style equation shows  $(x^3)^4 = x^3 \cdot x^3 \cdot x^3 \cdot x^3 = x^{3+3+3+3} = x^{12}$ . At the bottom of the video frame, a box contains the text 'THE POWER RULE OF EXPONENTS' and the formula  $(x^a)^b = x^{a \cdot b}$ .

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Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power and typically contained within parentheses. In this case, you multiply the exponents.

Product Rule			Power Rule							
$5^3 \cdot 5^4$	=	$5^{3+4}$	=	$5^7$	but	$(5^3)^4$	=	$5^{3 \cdot 4}$	=	$5^{12}$
$x^5 \cdot x^2$	=	$x^{5+2}$	=	$x^7$	but	$(x^5)^2$	=	$x^{5 \cdot 2}$	=	$x^{10}$
$(3a)^7 \cdot (3a)^{10}$	=	$(3a)^{7+10}$	=	$(3a)^{17}$	but	$((3a)^7)^{10}$	=	$(3a)^{7 \cdot 10}$	=	$(3a)^{70}$

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## NEGATIVE AND ZERO EXPONENT RULES

### Learning Outcomes

- Simplify exponential expressions containing negative exponents and exponents of 0 and 1

## Define and Use the Zero Exponent Rule

Return to the quotient rule. We worked with expressions for which  $a > b$  so that the difference  $a - b$  would never be zero or negative.

### THE QUOTIENT (DIVISION) RULE FOR EXPONENTS

For any non-zero number  $x$  and any integers  $a$  and  $b$ :  $\frac{x^a}{x^b} = x^{a-b}$

What would happen if  $a = b$ ? In this case, we would use the *zero exponent rule of exponents* to simplify the expression to 1. To see how this is done, let us begin with an example.

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

If we were to simplify the original expression using the quotient rule, we would have:

$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

If we equate the two answers, the result is  $t^0 = 1$ . This is true for any nonzero real number, or any variable representing a real number.

$$a^0 = 1$$

The sole exception is the expression  $0^0$ . This appears later in more advanced courses, but for now, we will consider the value to be undefined.

## THE ZERO EXPONENT RULE OF EXPONENTS

For any nonzero real number  $a$ , the zero exponent rule of exponents states that

$$a^0 = 1$$

In the following video, you will see more examples of simplifying expressions whose exponents may be zero.

## Define and Use the Negative Exponent Rule

Another useful result occurs if we relax the condition that  $a > b$  in the quotient rule even further. For example, can we simplify  $\frac{h^3}{h^5}$ ? When  $a < b$ —that is, where the difference  $a - b$  is negative—we can use the *negative rule of exponents* to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example,  $\frac{h^3}{h^5}$ .

$$\begin{aligned}\frac{h^3}{h^5} &= \frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} \\ &= \frac{\cancel{h} \cdot \cancel{h} \cdot \cancel{h}}{\cancel{h} \cdot \cancel{h} \cdot \cancel{h} \cdot h \cdot h} \\ &= \frac{1}{h \cdot h} \\ &= \frac{1}{h^2}\end{aligned}$$

If we were to simplify the original expression using the quotient rule, we would have

$$\begin{aligned}\frac{h^3}{h^5} &= h^{3-5} \\ &= h^{-2}\end{aligned}$$

Putting the answers together, we have  $h^{-2} = \frac{1}{h^2}$ . This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar—from numerator to denominator or vice versa.

$$a^{-n} = \frac{1}{a^n} \quad \text{and} \quad a^n = \frac{1}{a^{-n}}$$

We have shown that the exponential expression  $a^n$  is defined when  $n$  is a natural number, 0, or the negative of a natural number. That means that  $a^n$  is defined for any integer  $n$ . Also, the product and quotient rules and all of the rules we will look at soon hold for any integer  $n$ .

In the following video, you will see examples of simplifying expressions with negative exponents.

## Combine Exponent Rules to Simplify Expressions

Now we will combine the use of the product and quotient rules to simplify expressions whose terms may have negative or zero exponents.

The following video shows more examples of how to combine the use of the product and quotient rules to simplify expressions whose terms may have negative or zero exponents.

## EXAMPLE

Simplify each expression using the zero exponent rule of exponents.

1.  $\frac{c^3}{c^3}$

2.  $\frac{-3x^5}{x^5}$

3.  $\frac{(j^2k)^4}{(j^2k) \cdot (j^2k)^3}$

4.  $\frac{5(rs^2)^2}{(rs^2)^2}$

### Answer

Use the zero exponent and other rules to simplify each expression.

1.

$$\begin{aligned}\frac{c^3}{c^3} &= c^{3-3} \\ &= c^0 \\ &= 1\end{aligned}$$

2.

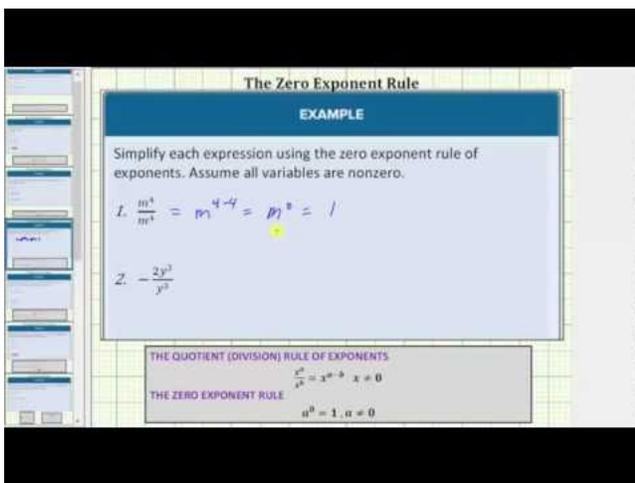
$$\begin{aligned}\frac{-3x^5}{x^5} &= -3 \cdot \frac{x^5}{x^5} \\ &= -3 \cdot x^{5-5} \\ &= -3 \cdot x^0 \\ &= -3 \cdot 1 \\ &= -3\end{aligned}$$

3.

$$\begin{aligned}\frac{(j^2k)^4}{(j^2k) \cdot (j^2k)^3} &= \frac{(j^2k)^4}{(j^2k)^{1+3}} && \text{Use the product rule in the denominator.} \\ &= \frac{(j^2k)^4}{(j^2k)^4} && \text{Simplify.} \\ &= (j^2k)^{4-4} && \text{Use the quotient rule.} \\ &= (j^2k)^0 && \text{Simplify.} \\ &= 1\end{aligned}$$

4.

$$\begin{aligned}\frac{5(rs^2)^2}{(rs^2)^2} &= 5(rs^2)^{2-2} && \text{Use the quotient rule.} \\ &= 5(rs^2)^0 && \text{Simplify.} \\ &= 5 \cdot 1 && \text{Use the zero exponent rule.} \\ &= 5 && \text{Simplify.}\end{aligned}$$



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## THE NEGATIVE RULE OF EXPONENTS

For any nonzero real number  $a$  and natural number  $n$ , the negative rule of exponents states that

$$a^{-n} = \frac{1}{a^n}$$

### EXAMPLE

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

$$1. \frac{(2b)^3}{(2b)^{10}}$$

$$2. \frac{z^2 \cdot z}{z^4}$$

$$3. \frac{(-5t^3)^4}{(-5t^3)^8}$$

**Answer**

$$1. \frac{(2b)^3}{(2b)^{10}} = (2b)^{3-10} = (2b)^{-7} = \frac{1}{(2b)^7}$$

$$2. \frac{z^2 \cdot z}{z^4} = \frac{z^{2+1}}{z^4} = \frac{z^3}{z^4} = z^{3-4} = z^{-1} = \frac{1}{z}$$

$$3. \frac{(-5t^3)^4}{(-5t^3)^8} = (-5t^3)^{4-8} = (-5t^3)^{-4} = \frac{1}{(-5t^3)^4}$$

**The Negative Rule of Exponents**

**EXAMPLE**

Write each quotient using a single base. Do not simplify further.  
Write all answers with positive exponents.

1.  $\frac{y^2}{y^5} = y^{2-5} = y^{-3} = \frac{1}{y^3}$       $\frac{y^4}{y^7} = \frac{\overbrace{y \cdot y \cdot y \cdot y}^4}{\overbrace{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}^7} = \frac{1}{y^3}$

2.  $\frac{(3x)^4}{(3x)^6} = (3x)^{4-6} = (3x)^{-2} = \frac{1}{(3x)^2}$

**THE QUOTIENT (DIVISION) RULE OF EXPONENTS**  
 $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$

**THE NEGATIVE RULE OF EXPONENTS**  
 $a^{-n} = \frac{1}{a^n}, a \neq 0$

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## EXAMPLE

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

1.  $b^2 \cdot b^{-8}$
2.  $(-x)^5 \cdot (-x)^{-5}$
3.  $\frac{-7z}{(-7z)^5}$

### Answer

1.  $b^2 \cdot b^{-8} = b^{2-8} = b^{-6} = \frac{1}{b^6}$
2.  $(-x)^5 \cdot (-x)^{-5} = (-x)^{5-5} = (-x)^0 = 1$
3.  $\frac{-7z}{(-7z)^5} = \frac{(-7z)^1}{(-7z)^5} = (-7z)^{1-5} = (-7z)^{-4} = \frac{1}{(-7z)^4}$

**The Rule of Exponents**

**EXAMPLE**

Write each expression using a single base. Do not simplify further.  
Write all answers with positive exponents.

1.  $p^3 \cdot p^{-6} = p^{3+(-6)} = p^{-3} = \frac{1}{p^3}$

2.  $(-2)^8 \cdot (-2)^{-8} = (-2)^{8+(-8)}$

3.  $\frac{-5r}{(-5r)^4}$

$x^a \cdot x^b = x^{a+b}$   
 $\frac{x^a}{x^b} = x^{a-b}, x \neq 0$   
 $(x^a)^b = x^{a \cdot b}$

$a^0 = 1, a \neq 0$   
 $a^{-n} = \frac{1}{a^n}, a \neq 0$

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# FIND THE POWER OF A PRODUCT AND A QUOTIENT

## LEARNING OUTCOMES

- Simplify compound expressions using the exponent rules

## Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the *power of a product rule of exponents* which breaks up the power of a product of factors into the product of the powers of the factors. For instance, consider  $(pq)^3$ . We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$\begin{aligned}(pq)^3 &= \overset{3 \text{ factors}}{(pq) \cdot (pq) \cdot (pq)} \\ &= p \cdot q \cdot p \cdot q \cdot p \cdot q \\ &= \overset{3 \text{ factors}}{p \cdot p \cdot p} \cdot \overset{3 \text{ factors}}{q \cdot q \cdot q} \\ &= p^3 \cdot q^3\end{aligned}$$

In other words,  $(pq)^3 = p^3 \cdot q^3$ .

## THE POWER OF A PRODUCT RULE OF EXPONENTS

For any real numbers  $a$  and  $b$  and any integer  $n$ , the power of a product rule of exponents states that

$$(ab)^n = a^n b^n$$

In the following video, we provide more examples of how to find the power of a product.

## Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the *power of a quotient rule* which states

## EXAMPLE

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

1.  $(ab^2)^3$
2.  $(2^a t)^{15}$
3.  $(-2w^3)^3$
4.  $\frac{1}{(-7z)^4}$
5.  $(e^{-2} f^2)^7$

### Answer

Use the product and quotient rules and the new definitions to simplify each expression.

1.  $(ab^2)^3 = (a)^3 \cdot (b^2)^3 = a^{1 \cdot 3} \cdot b^{2 \cdot 3} = a^3 b^6$
2.  $(2^a t)^{15} = (2^a)^{15} \cdot (t)^{15} = 2^{a \cdot 15} \cdot t^{15} = 2^{15a} \cdot t^{15}$
3.  $(-2w^3)^3 = (-2)^3 \cdot (w^3)^3 = -8 \cdot w^{3 \cdot 3} = -8w^9$
4.  $\frac{1}{(-7z)^4} = \frac{1}{(-7)^4 \cdot (z)^4} = \frac{1}{2,401z^4}$
5.  $(e^{-2} f^2)^7 = (e^{-2})^7 \cdot (f^2)^7 = e^{-2 \cdot 7} \cdot f^{2 \cdot 7} = e^{-14} f^{14} = \frac{f^{14}}{e^{14}}$



Caution! Do not try to apply this rule to sums.

Think about the expression  $(2 + 3)^2$

Does  $(2 + 3)^2$  equal  $2^2 + 3^2$ ?

No, it does not because of the order of operations!

$$(2 + 3)^2 = 5^2 = 25$$

and

$$2^2 + 3^2 = 4 + 9 = 13$$

Therefore, you can only use this rule when the numbers inside the parentheses are being multiplied or divided.

Power of a Product

EXAMPLE

Simplify each expression.

1.  $(xy^3)^2 = (x^2) \cdot (y^3)^2 = x^{1 \cdot 2} \cdot y^{3 \cdot 2} = x^2 y^6$
2.  $(3^a z)^6 = (3^a)^6 \cdot (z)^6 = 3^{a \cdot 6}$
3.  $(-4b^5)^2$

THE POWER OF A PRODUCT RULE OF EXPONENTS  
 $(ab)^n = a^n b^n$

THE POWER RULE OF EXPONENTS  
 $(a^m)^n = a^{m \cdot n}$

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that the power of a quotient of factors is the quotient of the powers of the factors. For example, look at the following example:

$$(e^{-2} f^2)^7 = \frac{f^{14}}{e^{14}}$$

Rewrite the original problem differently and look at the result:

$$\begin{aligned}(e^{-2} f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

$$\begin{aligned}(e^{-2} f^2)^7 &= \left(\frac{f^2}{e^2}\right)^7 \\ &= \frac{(f^2)^7}{(e^2)^7} \\ &= \frac{f^{2 \cdot 7}}{e^{2 \cdot 7}} \\ &= \frac{f^{14}}{e^{14}}\end{aligned}$$

## THE POWER OF A QUOTIENT RULE OF EXPONENTS

For any real numbers  $a$  and  $b$  and any integer  $n$ , the power of a quotient rule of exponents states that

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

The following video provides more examples of simplifying expressions using the power of a quotient and other exponent rules.

## Summary

- Evaluating expressions containing exponents is the same as evaluating any expression. You substitute the value of the variable into the expression and simplify.
- The product rule for exponents: For any number  $x$  and any integers  $a$  and  $b$ ,  $(x^a)(x^b) = x^{a+b}$ .
- The quotient rule for exponents: For any non-zero number  $x$  and any integers  $a$  and  $b$ ,  $\frac{x^a}{x^b} = x^{a-b}$
- The power rule for exponents:
  - For any nonzero numbers  $a$  and  $b$  and any integer  $n$ ,  $(ab)^n = a^n \cdot b^n$ .
  - For any number  $a$ , any non-zero number  $b$ , and any integer  $n$ ,  $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

## EXAMPLE

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

1.  $\left(\frac{4}{z^{11}}\right)^3$
2.  $\left(\frac{p}{q^3}\right)^6$
3.  $\left(\frac{-1}{t^2}\right)^{27}$
4.  $(j^3k^{-2})^4$
5.  $(m^{-2}n^{-2})^3$

**Answer**

1.  $\left(\frac{4}{z^{11}}\right)^3 = \frac{(4)^3}{(z^{11})^3} = \frac{64}{z^{11 \cdot 3}} = \frac{64}{z^{33}}$
2.  $\left(\frac{p}{q^3}\right)^6 = \frac{(p)^6}{(q^3)^6} = \frac{p^{1 \cdot 6}}{q^{3 \cdot 6}} = \frac{p^6}{q^{18}}$
3.  $\left(\frac{-1}{t^2}\right)^{27} = \frac{(-1)^{27}}{(t^2)^{27}} = \frac{-1}{t^{2 \cdot 27}} = \frac{-1}{t^{54}} = -\frac{1}{t^{54}}$
4.  $(j^3k^{-2})^4 = \left(\frac{j^3}{k^2}\right)^4 = \frac{(j^3)^4}{(k^2)^4} = \frac{j^{3 \cdot 4}}{k^{2 \cdot 4}} = \frac{j^{12}}{k^8}$
5.  $(m^{-2}n^{-2})^3 = \left(\frac{1}{m^2n^2}\right)^3 = \frac{(1)^3}{(m^2n^2)^3} = \frac{1}{(m^2)^3(n^2)^3} = \frac{1}{m^{2 \cdot 3} \cdot n^{2 \cdot 3}} = \frac{1}{m^6n^6}$

Power of a Quotient

EXAMPLE

Simplify each expression.

1.  $\left(\frac{2}{x^4}\right)^3 = \frac{(2)^3}{(x^4)^3} = \frac{8}{x^{4 \cdot 3}} = \frac{8}{x^{12}}$
2.  $\left(\frac{p^3}{q}\right)^4 = \frac{(p^3)^4}{(q)^4} = \frac{p^{3 \cdot 4}}{q^4} = \frac{p^{12}}{q^4}$
3.  $\left(\frac{-1}{y^2}\right)^{12} = \frac{(-1)^{12}}{(y^2)^{12}} = \frac{1}{y^{2 \cdot 12}}$

THE POWER OF A QUOTIENT RULE OF EXPONENTS  
 $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

THE POWER RULE OF EXPONENTS  
 $(a^m)^n = a^{m \cdot n}$

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## INTRODUCTION TO SCIENTIFIC NOTATION

In this section you will define and use scientific notation to write very large and very small numbers and solve problems with them.

In the same way that exponents help us to be able to write repeated multiplication with little effort, they are also used to express large and small numbers without a lot of zeros and confusion. Scientists and engineers make use of exponents regularly to keep track of the place value of numbers that they are working with to make calculations.

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## WRITING SCIENTIFIC NOTATION

### Learning Outcomes

- Define decimal and scientific notation
- Convert between scientific and decimal notation

## Convert Between Scientific and Decimal Notation

Before we can convert between scientific and decimal notation, we need to know the difference between the two. **Scientific notation** is used by scientists, mathematicians, and engineers when they are working with very large or very small numbers. Using exponential notation, large and small numbers can be written in a way that is easier to read.

When a number is written in scientific notation, the **exponent** tells you if the term is a large or a small number. A positive exponent indicates a large number and a negative exponent indicates a small number that is between 0 and 1. It is difficult to understand just how big a billion or a trillion is. Here is a way to help you think about it.

Word	How many thousands	Number	Scientific Notation
million	$1000 \times 1000 =$ a thousand thousands	1, 000, 000	$10^6$
billion	$(1000 \times 1000) \times 1000 =$ a thousand millions	1, 000, 000, 000	$10^9$
trillion	$(1000 \times 1000 \times 1000) \times 1000 =$ a thousand billions	1, 000, 000, 000, 000	$10^{12}$

1 billion can be written as 1, 000, 000, 000 or represented as  $10^9$ . How would 2 billion be represented? Since 2 billion is 2 times 1 billion, then 2 billion can be written as  $2 \times 10^9$ .

A light year is the number of miles light travels in one year, about 5, 880, 000, 000, 000. That is a lot of zeros, and it is easy to lose count when trying to figure out the place value of the number. Using scientific notation, the distance is  $5.88 \times 10^{12}$  miles. The exponent of 12 tells us how many places to count to the left of the decimal. Another example of how scientific notation can make numbers easier to read is the diameter of a hydrogen atom, which is about 0.00000005 mm, and in scientific notation is  $5 \times 10^{-8}$  mm. In this case, the  $-8$  tells us how many places to count to the right of the decimal.

Outlined in the box below are some important conventions of scientific notation format.

## SCIENTIFIC NOTATION

A positive number is written in scientific notation if it is written as  $a \times 10^n$  where the coefficient  $a$  is  $1 \leq a < 10$ , and  $n$  is an integer.

Look at the numbers below. Which of the numbers is written in scientific notation?

Number	Scientific Notation?	Explanation
$1.85 \times 10^{-2}$	yes	$1 \leq 1.85 < 10$ $-2$ is an integer
$1.083 \times 10^{\frac{1}{2}}$	no	$\frac{1}{2}$ is not an integer
$0.82 \times 10^{14}$	no	$0.82$ is not $\geq 1$
$10 \times 10^3$	no	$10$ is not $< 10$

Now compare some numbers expressed in both scientific notation and standard decimal notation in order to understand how to convert from one form to the other. Take a look at the tables below. Pay close attention to the exponent in the scientific notation and the position of the decimal point in the decimal notation.

Large Numbers		Small Numbers	
Decimal Notation	Scientific Notation	Decimal Notation	Scientific Notation
500.0	$5 \times 10^2$	0.05	$5 \times 10^{-2}$
80,000.0	$8 \times 10^4$	0.0008	$8 \times 10^{-4}$
43,000,000.0	$4.3 \times 10^7$	0.00000043	$4.3 \times 10^{-7}$
62,500,000,000.0	$6.25 \times 10^{10}$	0.000000000625	$6.25 \times 10^{-10}$

## Convert from Decimal Notation to Scientific Notation

To write a *large* number in scientific notation, move the decimal point to the left to obtain a number between 1 and 10. Since moving the decimal point changes the value, you have to multiply the decimal by a power of 10 so that the expression has the same value.

Let us look at an example:

$$180,000. = 18,000.0 \times 10^1$$

←

$$1,800.00 \times 10^2$$

$$180.000 \times 10^3$$

$$18.0000 \times 10^4$$

$$1.80000 \times 10^5$$

$$180,000 = 1.8 \times 10^5$$

Notice that the decimal point was moved 5 places to the left, and the exponent is 5.

To write a small number (between 0 and 1) in scientific notation, you move the decimal to the *right* and the exponent will have to be *negative*, as in the following example.

$$0.00004 = 00.0004 \times 10^{-1}$$

→

$$000.004 \times 10^{-2}$$

$$0000.04 \times 10^{-3}$$

$$00000.4 \times 10^{-4}$$

$$000004. \times 10^{-5}$$

$$0.00004 = 4 \times 10^{-5}$$

You may notice that the decimal point was moved five places to the *right* until you got to the number 4 which is between 1 and 10. The exponent is  $-5$ .

In the following video, you are provided with examples of how to convert both a large and a small number in decimal notation to scientific notation.

## EXAMPLE

Write the following numbers in scientific notation.

1. 920,000,000
2. 10,200,000
3. 100,000,000,000

### Answer

1. 920,000,000 We will move the decimal point to the left. It helps to place it at the end of the number  
←  
and then count how many times you move it to get one number before it that is between 1 and 10. Move the decimal point 8 times to the left and you will have 9.20000000. Now we can replace the zeros with an exponent of 8 giving us  $9.2 \times 10^8$ .
2. 10,200,000 = 10,200,000.0 =  $1.02 \times 10^7$ . Note here how we included the 0 and the 2 after the  
←  
decimal point. In some disciplines, you may learn about when to include both of these. Follow instructions from your teacher on rounding rules.
3. 100,000,000,000 = 100,000,000,000.0 =  $1.0 \times 10^{11}$   
←

## EXAMPLE

Write the following numbers in scientific notation.

1. 0.00000000000035
2. 0.0000000102
3. 0.00000000000000793

### Answer

1. 0.00000000000035 =  $3.5 \times 10^{-12}$ . We moved the decimal 12 times to get to a number between 1 and  
→  
10
2. 0.0000000102 =  $1.02 \times 10^{-8}$   
→
3. 0.00000000000000793 =  $7.93 \times 10^{-15}$   
→

The image shows a handwritten example on a grid background. At the top, it says "Example: Rewrite in scientific notation." Below this, the number "15,600,000" is written with a yellow highlight and a red bracket underneath it labeled "7 decimal places". To its right, the number "0.0000872" is written. Below "15,600,000", a green arrow points down to the result "1.56 x 10^7". To the right of the arrow, the general form "a x 10^b" is written, with "1 ≤ a < 10" written below it.

## Convert from Scientific Notation to Decimal Notation

You can also write scientific notation as decimal notation. Recall the number of miles that light travels in a year is  $5.88 \times 10^{12}$ , and a hydrogen atom has a diameter of  $5 \times 10^{-8}$  mm. To write each of these numbers in decimal notation, you move the decimal point the same number of places as the exponent. If the exponent is *positive*, move the decimal point to the *right*. If the exponent is *negative*, move the decimal point to the *left*.

$$5.88 \times 10^{12} = 5.880000000000. \xrightarrow{\hspace{1.5cm}} = 5,880,000,000,000$$

$$5 \times 10^{-8} = 0.00000005. \xleftarrow{\hspace{1.5cm}} = 0.00000005$$

For each power of 10, you move the decimal point one place. Be careful here and do not get carried away with the zeros—the number of zeros after the decimal point will always be 1 *less* than the exponent because it takes one power of 10 to shift that first number to the left of the decimal.

### EXAMPLE

Write the following in decimal notation.

1.  $4.8 \times 10^{-4}$
2.  $3.08 \times 10^6$

#### Answer

1.  $4.8 \times 10^{-4}$ , the exponent is negative, so we need to move the decimal to the left.  
 $4.8 \times 10^{-4} = .00048$   
 $\xleftarrow{\hspace{1.5cm}} \quad \xleftarrow{\hspace{1.5cm}}$
2.  $3.08 \times 10^6$ , the exponent is positive, so we need to move the decimal to the right.  
 $3.08 \times 10^6 = 3080000$   
 $\xrightarrow{\hspace{1.5cm}} \quad \xrightarrow{\hspace{1.5cm}}$

### THINK ABOUT IT

To help you get a sense of the relationship between the sign of the exponent and the relative size of a number written in scientific notation, answer the following questions. You can use the textbox to write your ideas before you reveal the solution.

1. You are writing a number whose absolute value is greater than 1 in scientific notation. Will your exponent be positive or negative?
2. You are writing a number whose absolute value is between 0 and 1 in scientific notation. Will your exponent be positive or negative?
3. What power do you need to put on 10 to get a result of 1?

#### Answer

1. *You are writing a number whose absolute value is greater than 1 in scientific notation. Will your exponent be positive or negative?* For numbers greater than 1, the exponent on 10 will be positive when you are using scientific notation. Refer to the table presented above:

Word	How many thousands	Number	Scientific Notation
million	$1000 \times 1000 =$ a thousand thousands	1,000,000	$10^6$
billion	$(1000 \times 1000) \times 1000 =$ a thousand millions	1,000,000,000	$10^9$
trillion	$(1000 \times 1000 \times 1000) \times 1000 =$ a thousand billions	1,000,000,000,000	$10^{12}$

2. You are writing a number whose absolute value is between 0 and 1 in scientific notation. Will your exponent be positive or negative? We can reason that since numbers greater than 1 will have a positive exponent, numbers between 0 and 1 will have a negative exponent. Why are we specifying numbers between 0 and 1? The numbers between 0 and 1 represent amounts that are fractional. Recall that we defined numbers with a negative exponent as  $a^{-n} = \frac{1}{a^n}$ , so if we have  $10^{-2}$  we have  $\frac{1}{10 \times 10} = \frac{1}{100}$  which is a number between 0 and 1.

3. What power do you need to put on 10 to get a result of 1? Recall that any number or variable with an exponent of 0 is equal to 1, as in this example:

$$\frac{t^8}{t^8} = \frac{\cancel{t^8}}{\cancel{t^8}} = 1$$

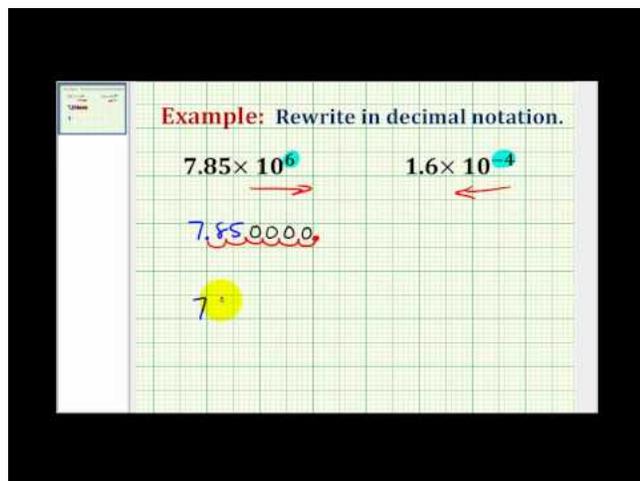
$$\frac{t^8}{t^8} = t^{8-8} = t^0$$

therefore

$$t^0 = 1$$

We now have described the notation necessary to write all possible numbers on the number line in scientific notation.

In the next video, you will see how to convert a number written in scientific notation into decimal notation.



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## Summary

Large and small numbers can be written in scientific notation to make them easier to understand. In the next section, you will see that performing mathematical operations such as multiplication and division on large and small numbers is made easier by scientific notation and the rules of exponents.

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# MULTIPLY AND DIVIDE NUMBERS IN SCIENTIFIC NOTATION

## Learning Outcome

- Multiply and divide numbers expressed in scientific notation

## Multiplying and Dividing Numbers Expressed in Scientific Notation

Numbers that are written in scientific notation can be multiplied and divided rather simply by taking advantage of the properties of numbers and the rules of exponents that you may recall. To multiply numbers in scientific notation, first multiply the numbers that are not powers of 10 (the  $a$  in  $a \times 10^n$ ). Then multiply the powers of ten by adding the exponents.

This will produce a new number times a different power of 10. All you have to do is check to make sure this new value is in scientific notation. If it is not, you convert it.

Let us look at some examples.

### Example

Multiply  $(3 \times 10^8)(6.8 \times 10^{-13})$

**Answer**

Regroup using the commutative and associative properties.

$$(3 \times 6.8)(10^8 \times 10^{-13})$$

Multiply the coefficients.

$$(20.4)(10^8 \times 10^{-13})$$

Multiply the powers of 10 using the Product Rule, so add the exponents.

$$20.4 \times 10^{-5}$$

Convert 20.4 into scientific notation by moving the decimal point one place to the left and multiplying by  $10^1$ .

$$(2.04 \times 10^1) \times 10^{-5}$$

Group the powers of 10 using the associative property of multiplication.

$$2.04 \times (10^1 \times 10^{-5})$$

Multiply using the Product Rule, so add the exponents.

$$2.04 \times 10^{1+(-5)}$$

The answer is  $2.04 \times 10^{-4}$

### Example

Multiply  $(8.2 \times 10^6) (1.5 \times 10^{-3}) (1.9 \times 10^{-7})$

**Answer**

Regroup using the commutative and associative properties.

$$(8.2 \times 1.5 \times 1.9) (10^6 \times 10^{-3} \times 10^{-7})$$

Multiply the numbers.

$$(23.37) (10^6 \times 10^{-3} \times 10^{-7})$$

Multiply the powers of 10 using the Product Rule, so add the exponents.

$$23.37 \times 10^{-4}$$

Convert 23.37 into scientific notation by moving the decimal point one place to the left and multiplying by  $10^1$ .

$$(2.337 \times 10^1) \times 10^{-4}$$

Group the powers of 10 using the associative property of multiplication.

$$2.337 \times (10^1 \times 10^{-4})$$

Multiply using the Product Rule, so add the exponents.

$$2.337 \times 10^{1+(-4)}$$

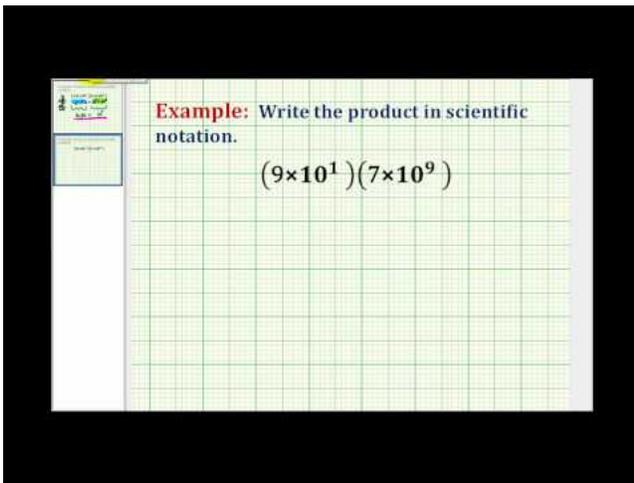
The answer is  $2.337 \times 10^{-3}$

In the following video, you will see an example of how to multiply two numbers that are written in scientific notation.

In order to divide numbers in scientific notation, you once again apply the properties of numbers and the rules of exponents. You begin by dividing the numbers that are not powers of 10 (the  $a$  in  $a \times 10^n$ ). Then you divide the powers of ten by subtracting the exponents.

This will produce a new number times a different power of 10. If it is not already in scientific notation, you convert it, and then you are done.

Let us look at some examples.



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### Example

Divide  $\frac{2.829 \times 10^{-9}}{3.45 \times 10^{-3}}$

**Answer**

Regroup using the associative property.

$$\left(\frac{2.829}{3.45}\right) \left(\frac{10^{-9}}{10^{-3}}\right)$$

Divide the coefficients.

$$(0.82) \left(\frac{10^{-9}}{10^{-3}}\right)$$

Divide the powers of 10 using the Quotient Rule, so subtract the exponents.

$$0.82 \times 10^{-9-(-3)}$$

$$0.82 \times 10^{-6}$$

Convert 0.82 into scientific notation by moving the decimal point one place to the right and multiplying by  $10^{-1}$ .

$$(8.2 \times 10^{-1}) \times 10^{-6}$$

Group the powers of 10 together using the associative property.

$$8.2 \times (10^{-1} \times 10^{-6})$$

Multiply the powers of 10 using the Product Rule, so add the exponents.

$$8.2 \times 10^{-1+(-6)}$$

The answer is  $8.2 \times 10^{-7}$

### Example

Calculate  $\frac{(1.37 \times 10^4)(9.85 \times 10^6)}{5.0 \times 10^{12}}$

**Answer**

Regroup the terms in the numerator according to the associative and commutative properties.

$$\frac{(1.37 \times 9.85)(10^6 \times 10^4)}{5.0 \times 10^{12}}$$

Multiply.

$$\frac{13.4945 \times 10^{10}}{5.0 \times 10^{12}}$$

Regroup using the associative property.

$$\left(\frac{13.4945}{5.0}\right)\left(\frac{10^{10}}{10^{12}}\right)$$

Divide the numbers.

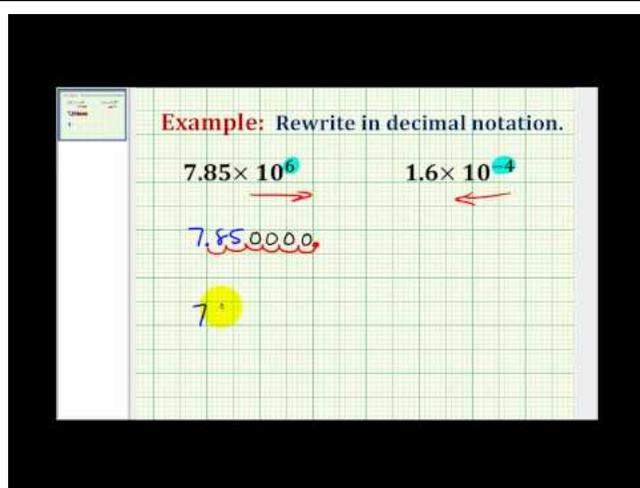
$$(2.6989)\left(\frac{10^{10}}{10^{12}}\right)$$

Divide the powers of 10 using the Quotient Rule, so subtract the exponents.

$$(2.6989)(10^{10-12})$$
$$2.6989 \times 10^{-2}$$

The answer is  $2.6989 \times 10^{-2}$

Notice that when you divide exponential terms, you subtract the exponent in the denominator from the exponent in the numerator. You will see another example of dividing numbers written in scientific notation in the following video.



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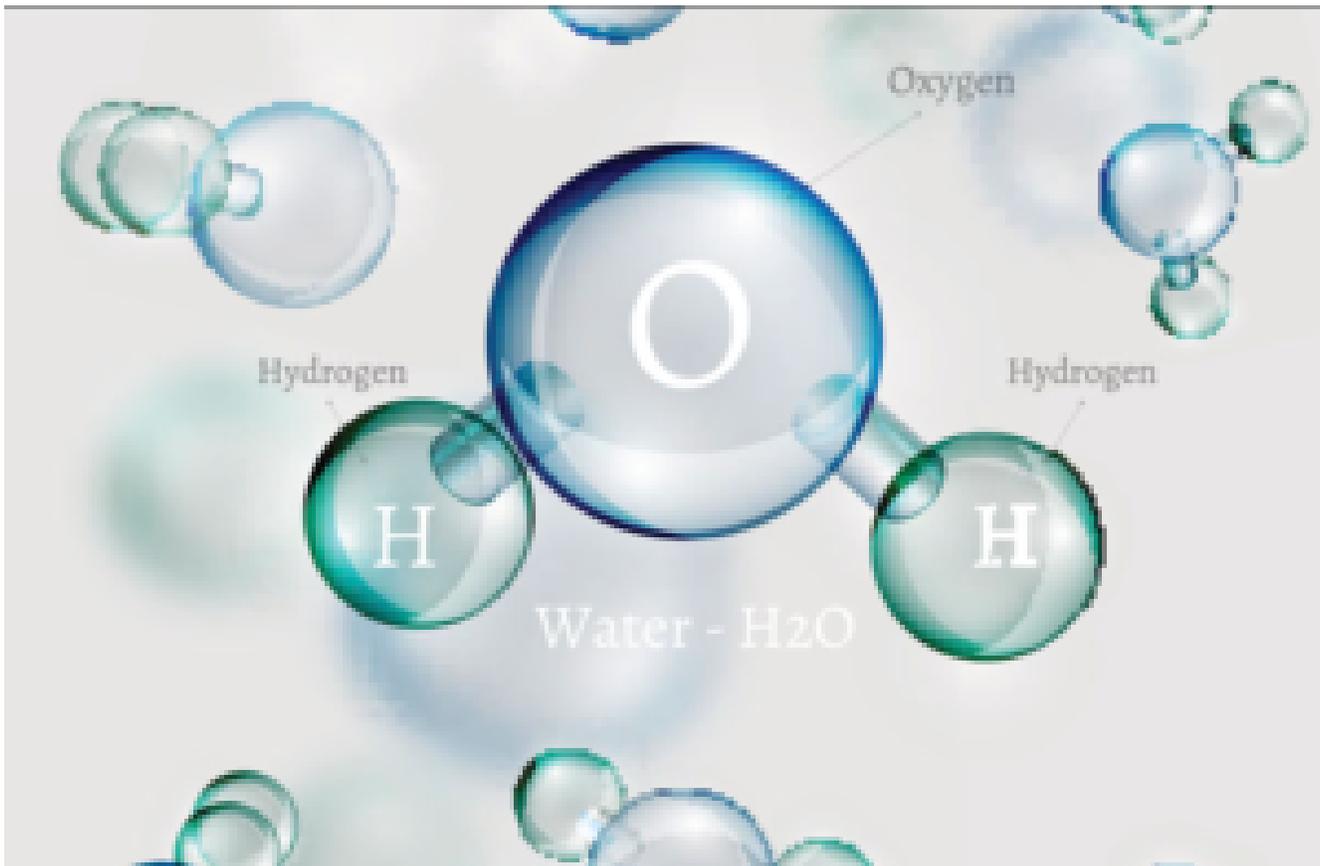
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# PROBLEM SOLVING WITH SCIENTIFIC NOTATION

## Learning Outcome

- Solve application problems involving scientific notation



*A water molecule.*

## Solve Application Problems

Learning rules for exponents seems pointless without context, so let us explore some examples of using scientific notation that involve real problems. First, let us look at an example of how scientific notation can be used to describe real measurements.

## THINK ABOUT IT

Match each length in the table with the appropriate number of meters described in scientific notation below. Write your ideas in the textboxes provided before you look at the solution.

The height of a desk	Diameter of a water molecule	Diameter of the sun at its equator
Distance from Earth to Neptune	Diameter of Earth at the equator	Height of Mt. Everest (rounded)
Diameter of an average human cell	Diameter of a large grain of sand	Distance a bullet travels in one second

Power of 10, units in meters	Length from table above
$10^{12}$	
$10^9$	
$10^6$	
$10^4$	
$10^2$	
$10^0$	
$10^{-3}$	
$10^{-5}$	
$10^{-10}$	

### Answer

Power of 10, units in meters	Length from table above
$10^{12}$	Distance from Earth to Neptune
$10^9$	Diameter of the sun at its equator
$10^6$	Diameter of Earth at the equator
$10^4$	Height of Mt. Everest (rounded)
$10^2$	Distance a bullet travels in one second
$10^0$	The height of a desk
$10^{-3}$	Diameter of a large grain of sand
$10^{-5}$	Diameter of an average human cell
$10^{-10}$	Diameter of a water molecule

One of the most important parts of solving a “real-world” problem is translating the words into appropriate mathematical terms and recognizing when a well known formula may help. Here is an example that requires you to find the density of a cell given its mass and volume. Cells are not visible to the naked eye, so their measurements, as described with scientific notation, involve negative exponents.



Several red blood cells.

### Example

Human cells come in a wide variety of shapes and sizes. The mass of an average human cell is about  $2 \times 10^{-11}$  grams. (Note: Orders of magnitude (mass). (n.d.). Retrieved May 26, 2016, from [https://en.wikipedia.org/wiki/Orders\\_of\\_magnitude\\_\(mass\)](https://en.wikipedia.org/wiki/Orders_of_magnitude_(mass))) Red blood cells are one of the smallest types of cells (Note: How Big is a Human Cell?), clocking in at a volume of approximately  $10^{-6}$  meters<sup>3</sup>. (Note: How big is a human cell? - Weizmann Institute of Science. (n.d.). Retrieved May 26, 2016, from <http://www.weizmann.ac.il/plants/Milo/images/humanCellSize120116Clean.pdf>) Biologists have recently discovered how to use the density of some types of cells to indicate the presence of disorders such as sickle cell anemia or leukemia. (Note: Grover, W. H., Bryan, A. K., Diez-Silva, M., Suresh, S., Higgins, J. M., & Manalis, S. R. (2011). Measuring single-cell density. *Proceedings of the National Academy of Sciences*, 108(27), 10992-10996. doi:10.1073/pnas.1104651108) Density is calculated as  $\frac{\text{mass}}{\text{volume}}$ . Calculate the density of an average human cell.

#### Answer

**Read and Understand:** We are given an average cellular mass and volume as well as the formula for density. We are looking for the density of an average human cell.

**Define and Translate:**  $m = \text{mass} = 2 \times 10^{-11}$ ,  $v = \text{volume} = 10^{-6}$  meters<sup>3</sup>, density =  $\frac{\text{mass}}{\text{volume}}$

**Write and Solve:** Use the quotient rule to simplify the ratio.

$$\begin{aligned}\text{density} &= \frac{2 \times 10^{-11} \text{ grams}}{10^{-6} \text{ meter}^3} \\ &= 2 \times 10^{-11 - (-6)} \frac{\text{grams}}{\text{meter}^3} \\ &= 2 \times 10^{-5} \frac{\text{grams}}{\text{meter}^3}\end{aligned}$$

If scientists know the density of healthy cells, they can compare the density of a sick person's cells to that to rule out or test for disorders or diseases that may affect cellular density.

The average density of a human cell is  $2 \times 10^{-5} \frac{\text{grams}}{\text{meter}^3}$

The following video provides an example of how to find the number of operations a computer can perform in a very short amount of time.

In the next example, you will use another well known formula,  $d = r \cdot t$ , to find how long it takes light to travel from the sun to Earth. Unlike the previous example, the distance between the earth and the sun is massive, so the numbers you will work with have positive exponents.

## Summary

**Scientific Notation Application**

**EXAMPLE**

If a computer can conduct 400 trillion operations per second, how long would it take the computer to perform 500 million operations?

400 trillion = 400,000,000,000,000 =  $4 \times 10^{14}$   
 500 million = 500,000,000 =  $5 \times 10^8$

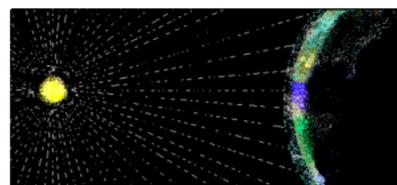
Number of Operations:  $5 \times 10^8$  operations  
 Rate of Operations:  $4 \times 10^{14}$  operations per second

$Time = \frac{5 \times 10^8}{4 \times 10^{14}} = \frac{5}{4} \times \frac{10^8}{10^{14}} = 1.25 \times 10^{-6}$

Number of Operations = Rate of Operations  $\times$  Time  
 $Time = \frac{\text{Number of Operations}}{\text{Rate of Operations}}$

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Scientific notation was developed to assist mathematicians, scientists, and others when expressing and working with very large and very small numbers. Scientific notation follows a very specific format in which a number is expressed as the product of a number greater than or equal to one and less than ten times a power of 10. The format is written  $a \times 10^n$ , where  $1 \leq a < 10$  and  $n$  is an integer. To multiply or divide numbers in scientific notation, you can use the commutative and associative properties to group the exponential terms together and apply the rules of exponents.



*Light traveling from the sun to the earth.*

## EXAMPLE

The speed of light is  $3 \times 10^8 \frac{\text{meters}}{\text{second}}$ . If the sun is  $1.5 \times 10^{11}$  meters from earth, how many seconds does it take for sunlight to reach the earth? Write your answer in scientific notation.

### Answer

**Read and Understand:** We are looking for how long—an amount of time. We are given a rate which has units of meters per second and a distance in meters. This is a  $d = r \cdot t$  problem.

### Define and Translate:

$$d = 1.5 \times 10^{11}$$

$$r = 3 \times 10^8 \frac{\text{meters}}{\text{second}}$$

$$t = ?$$

**Write and Solve:** Substitute the values we are given into the  $d = r \cdot t$  equation. We will work without units to make it easier. Often, scientists will work with units to make sure they have made correct calculations.

$$d = r \cdot t$$

$$1.5 \times 10^{11} = 3 \times 10^8 \cdot t$$

Divide both sides of the equation by  $3 \times 10^8$  to isolate  $t$ .

$$\frac{1.5 \times 10^{11}}{3 \times 10^8} = \frac{3 \times 10^8}{3 \times 10^8} \cdot t$$

On the left side, you will need to use the quotient rule of exponents to simplify, and on the right, you are left with  $t$ .

$$\left(\frac{1.5}{3}\right) \times \left(\frac{10^{11}}{10^8}\right) = t$$

$$(0.5) \times (10^{11-8}) = t$$

$$0.5 \times 10^3 = t$$

This answer is not in scientific notation, so we will move the decimal to the right, which means we need to subtract one factor of 10.

$$0.5 \times 10^3 = 5.0 \times 10^2 = t$$

The time it takes light to travel from the sun to Earth is  $5.0 \times 10^2$  seconds, or in standard notation, 500 seconds. That is not bad considering how far it has to travel!

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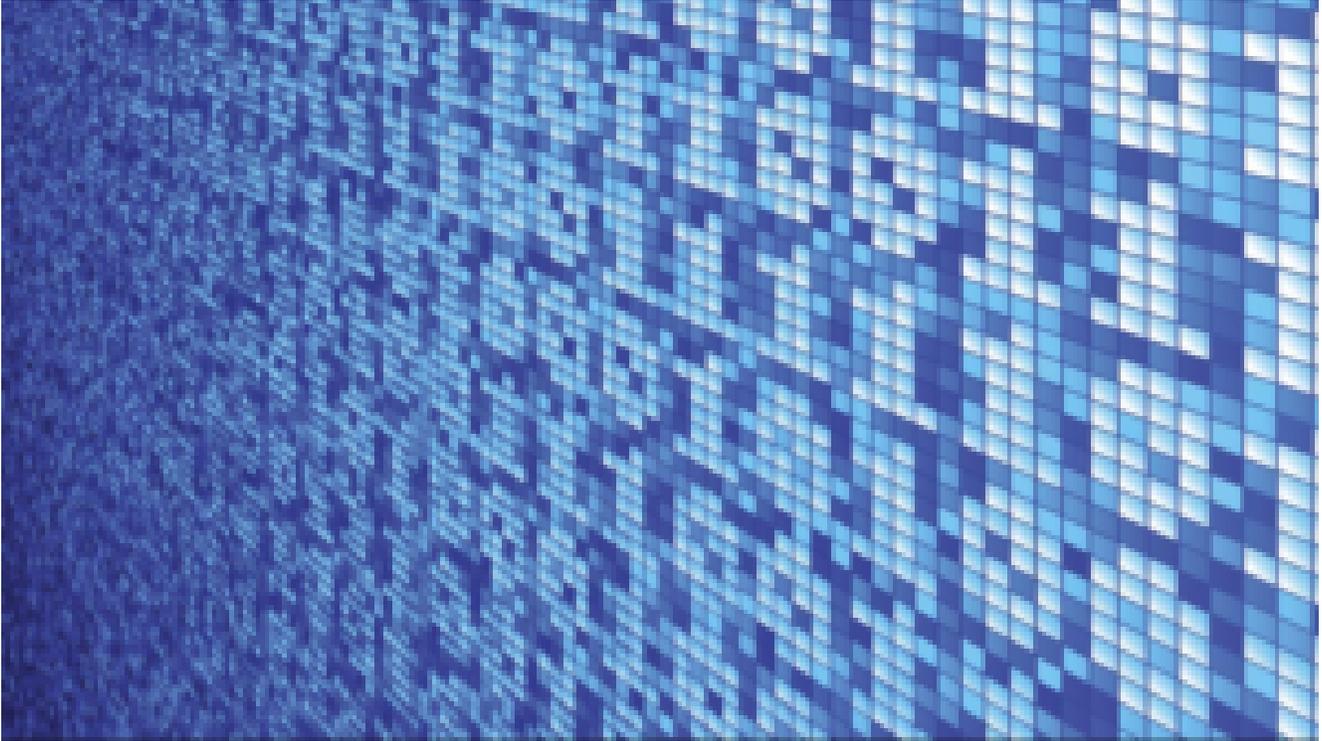
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# PUTTING IT TOGETHER: EXPONENTS



*Bits and Bytes.*

Exponents allow us to perform repeated multiplication quickly, and write very large or small numbers in a way that is easy to understand. There are many and varied uses of exponents that you may be surprised to find are all around us. You may have heard a computer described as having a 32-bit processor with 64 megabytes of RAM and 2.1 gigabytes of hard disk space. What does that even mean? This terminology is used to describe how fast a computer can process information which results in how quickly it responds after you click on a link or an app or open a file. It also describes how much memory the computer has to store the files and apps you want to use with it.

Computers use binary digits (0 and 1) instead of the decimal digits (0 to 9) we are used to working with; therefore, the descriptions 32-bit and 64 megabytes do not make much sense at first glance.

Each binary digit is equivalent to one bit, and a byte contains 8 bits, often called a binary octet. This is what a byte represents relative to decimal numbers:

Type of Number									Decimal Equivalent
Decimal Powers of 2	$2^7$	$2^6$	$2^5$	$2^4$	$2^3$	$2^2$	$2^1$	$2^0$	
Decimal Number	128	64	32	16	8	4	2	1	$128 + 64 + 32 + 16 + 8 + 4 + 2 + 1 = 255$
Binary Octet	0	0	0	0	0	0	0	1	1

	0	0	0	0	0	0	1	1	$2 + 1 = 3$
	0	0	0	0	1	1	1	0	$8 + 4 + 2 = 14$
	1	0	1	0	1	1	0	0	$128 + 32 + 8 + 4 = 172$

The 1's in the binary octet act as light switches within a computer. The computer uses the binary octets to represent characters such as numbers, letters, and other information. You can think of the number of bits assigned to a computer processor as the amount of "scratch pad" it has available to store calculations and information while it works. Having more "scratch pad" available means the computer can do more without needing to access memory which takes time. There are many more factors that influence the speed and agility of a computer, but the basics can be described with exponents!

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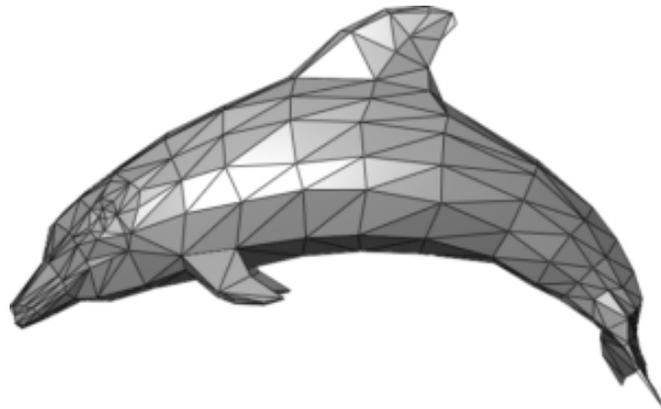
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# MODULE 8: POLYNOMIALS AND POLYNOMIAL FUNCTIONS

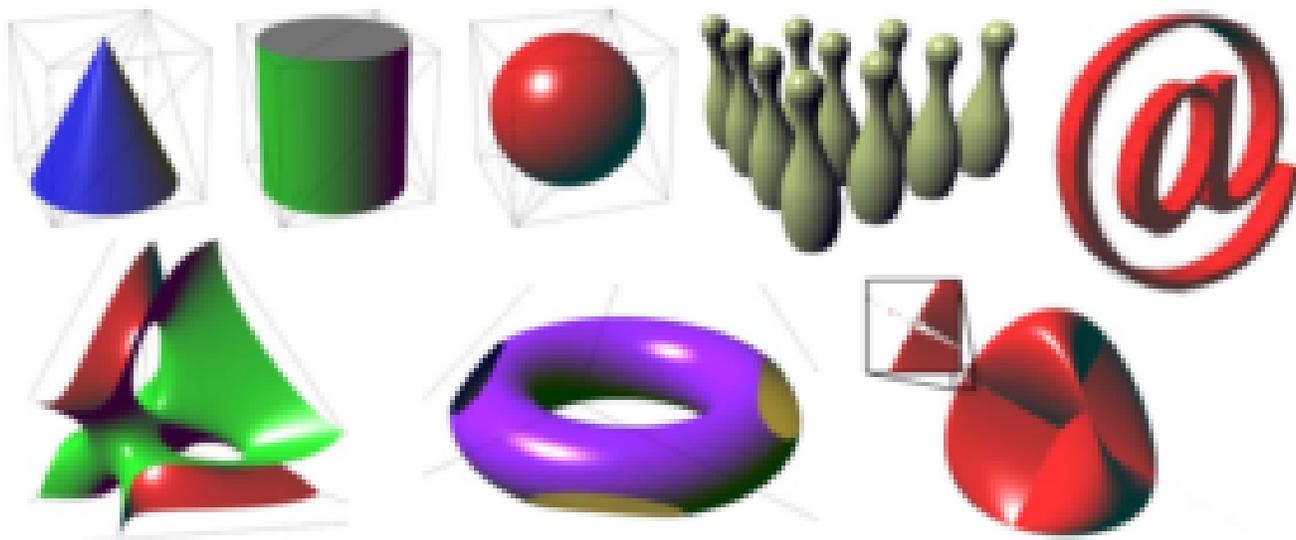
## WHY IT MATTERS: POLYNOMIALS

### Why Learn About Polynomials?

If you have ever watched a Pixar movie, you have seen computer generated images. A very common method for generating graphics with a computer is to use what is called a wire mesh. You can think of a wire mesh as a grid – like the ones we have used to graph lines – that has been bent and stretched to make a shape we want as in the image of a dolphin below.



The dolphin in the image was created by plotting points in space to create connected triangles. This method of rendering graphics works well and is in wide use, but it takes a lot of computer memory. Recently, researchers have been investigating the use of polynomials for rendering graphics in part because it demands less memory in the process. (Note: Loop, Charles, and Jim Blinn. Real-time GPU Rendering of Piecewise Algebraic Surfaces. *ACM SIGGRAPH 2006 Papers on - SIGGRAPH '06* (2006): n. pag. Web.). In this process, the surfaces that are rendered are made from solutions to algebraic polynomials. Below are images of smooth-surfaced objects that were rendered using polynomials by researchers Charles Loop and Jim Blinn from Microsoft.



Surfaces rendered using polynomials.

In this module, you will learn how to identify a polynomial and how to perform algebraic operations on them. Like the linear equations and inequalities you learned about earlier, polynomials are useful in many applications of mathematics as well as in other disciplines like biology, economics, and even cryptology. Gaining a basic understanding of their qualities and how the rules of algebra we have learned so far apply to them will help you learn how to use polynomials both inside and out of your math class.

In your next math class you will likely learn how to solve certain kinds of polynomials and how to graph them as well.

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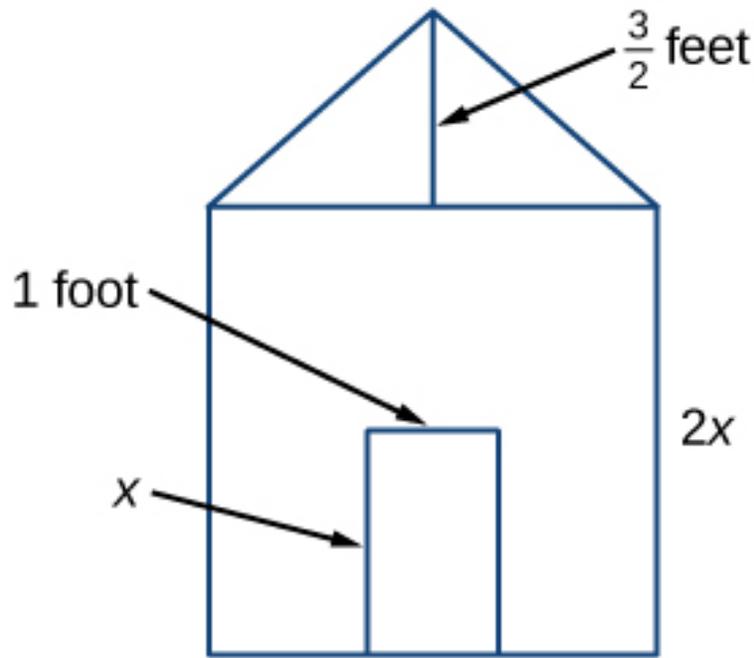
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## INTRODUCTION TO ALGEBRAIC OPERATIONS ON POLYNOMIALS

Earl is building a doghouse, the front of which is in the shape of a square topped with a triangle. There will be a rectangular door so that Felix, Earl's dog, can enter and exit the house. In order to save money for treats, Earl wants to buy exactly as much paint as he needs but not any extra. In order to do that, Earl needs to find the area of the front of the doghouse. Using the measurements of the front of the house shown below, we can create a mathematical expression that combines several terms, allowing us to solve this problem and others like it.



The front of the doghouse.

First find the area of the square in square feet.

$$A = s^2 = (2x)^2 = 4x^2$$

Then find the area of the triangle in square feet.

$$\begin{aligned} A &= \frac{1}{2}bh \\ &= \frac{1}{2}(2x)\left(\frac{3}{2}\right) \\ &= \frac{3}{2}x \end{aligned}$$

Next find the area of the rectangular door in square feet.

$$A = lw = x \cdot 1 = x$$

The area of the front of the doghouse can be found by adding the areas of the square and the triangle and then subtracting the area of the rectangle. When we do this, we get

$$4x^2 + \frac{3}{2}x - x \text{ ft}^2 = 4x^2 + \frac{1}{2}x \text{ ft}^2.$$

In this section, we will examine expressions such as this one, which combine several terms both with and without variables.

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# DEFINE AND EVALUATE POLYNOMIALS

## Learning Outcomes

- Define, evaluate, and simplify single variable polynomials

In the example on the previous page, we saw how combining the formulas for different shapes provides a way to accurately predict the amount of paint needed for a construction project. The result was a **polynomial**.

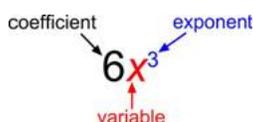
A polynomial function is a function consisting of the sum or difference of terms in which each term is a real number, a variable, or the product of a real number and variable(s) with a non-negative integer exponents. Non negative integers are 0, 1, 2, 3, 4, ...

You may see a resemblance between expressions and polynomials which we have been studying in this course. Polynomials are a special sub-group of mathematical expressions and equations.

The following table is intended to help you tell the difference between what is a polynomial and what is not.

IS a Polynomial	Is NOT a Polynomial	Reason
$2x^2 - \frac{1}{2}x - 9$	$\frac{2}{x^2} + x$	Polynomials only have variables in the numerator
$\frac{y}{4} - y^3$	$\frac{2}{y} + 4$	Polynomials only have variables in the numerator
$\sqrt{12}(a) + 9$	$\sqrt{a} + 7$	Roots are equivalent to rational exponents, and polynomials only have integer exponents on variables

The basic building block of a polynomial is a **monomial**. A monomial is one term and can be a number, a variable, or the product of a number and variable(s) with an exponent. The number part of the term is called the **coefficient**.



A polynomial containing two terms, such as  $2x - 9$ , is called a **binomial**. A polynomial containing three terms, such as  $-3x^2 + 8x - 7$ , is called a **trinomial**.

We can find the **degree** of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the **leading term** because it is usually written first. The coefficient of the leading term is called the **leading coefficient**. When a polynomial is written so that the powers on a specified variable are descending, we say that it is in standard form. It is important to note that polynomials only have integer exponents.

Leading Coefficient = 4 →  $4x^3 - 9x^2 + 6x$ 
Degree = 3 ↖

}  
**Polynomial**

## HOW TO: GIVEN A POLYNOMIAL EXPRESSION, IDENTIFY THE DEGREE AND LEADING COEFFICIENT

1. Find the highest power of  $x$  to determine the degree.
2. Identify the term containing the highest power of  $x$  to find the leading term.
3. Identify the coefficient of the leading term.

### EXAMPLE

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

1.  $3 + 2x^2 - 4x^3$
2.  $5t^5 - 2t^3 + 7t$
3.  $6p - p^3 - 2$

#### Answer

1. The highest power of  $x$  is 3, so the degree is 3. The leading term is the term containing that degree,  $-4x^3$ . The leading coefficient is the coefficient of that term,  $-4$ .
2. The highest power of  $t$  is 5, so the degree is 5. The leading term is the term containing that degree,  $5t^5$ . The leading coefficient is the coefficient of that term, 5.
3. The highest power of  $p$  is 3, so the degree is 3. The leading term is the term containing that degree,  $-p^3$ , The leading coefficient is the coefficient of that term,  $-1$ .

In the following video, we will identify the terms, leading coefficient, and degree of a polynomial.

**Examples: Intro to Polynomials**

A polynomial is an expression involving a sum of the product of constants and variables with non-negative integer exponents.

Given:  $9y + 7y^3 - 5 - 4y^2$   $-5y^0 = -5$

1 <sup>st</sup> Term:	$9y$	Degree:	1	Coefficient:	9
2 <sup>nd</sup> Term:	$7y^3$	Degree:	3	Coefficient:	7
3 <sup>rd</sup> Term:	$-5$	Degree:	0	Coefficient:	$-5$
4 <sup>th</sup> Term:	$-4y^2$	Degree:	2	Coefficient:	$-4$

Leading coefficient:  
 Degree of leading term:  
 Degree of polynomial:  
 Write the polynomial in descending order:

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The table below illustrates some examples of monomials, binomials, trinomials, and other polynomials. They are all written in standard form.

Monomials	Binomials	Trinomials	Other Polynomials
15	$3y + 13$	$x^3 - x^2 + 1$	$5x^4 + 3x^3 - 6x^2 + 2x$
$\frac{1}{2}x$	$4p - 7$	$3x^2 + 2x - 9$	$\frac{1}{3}x^5 - 2x^4 + \frac{2}{9}x^3 - x^2 + 4x - \frac{5}{6}$
$-4y^3$	$3x^2 + \frac{5}{8}x$	$3y^3 + y^2 - 2$	$3t^3 - 3t^2 - 3t - 3$
$16n^4$	$14y^3 + 3y$	$a^7 + 2a^5 - 3a^3$	$q^7 + 2q^5 - 3q^3 + q$

When the coefficient of a polynomial term is 0, you usually do not write the term at all (because 0 times anything is 0, and adding 0 does not change the value). The last binomial above could be written as a trinomial,  $14y^3 + 0y^2 + 3y$ .

A term without a variable is called a **constant** term, and the degree of that term is 0. For example, 13 is the constant term in  $3y + 13$ . You would usually say that  $14y^3 + 3y$  has no constant term or that the constant term is 0.

## Evaluate a Polynomial

You can evaluate polynomials just as you have been evaluating expressions all along. To evaluate an expression for a value of the variable, you substitute the value for the variable *every time* it appears. Then use the order of operations to find the resulting value for the expression.

### Example

Evaluate  $3x^2 - 2x + 1$  for  $x = -1$ .

**Answer**

Substitute  $-1$  for each  $x$  in the polynomial.

$$3(-1)^2 - 2(-1) + 1$$

Following the order of operations, evaluate exponents first.

$$3(1) - 2(-1) + 1$$

Multiply 3 times 1, and then multiply  $-2$  times  $-1$ .

$$3 + 2 + 1$$

Find the sum:  $3 + 2 + 1 = 6$

### Example

Evaluate  $-\frac{2}{3}p^4 + 2^3 - p$  for  $p = 3$ .

**Answer**

Substitute 3 for each  $p$  in the polynomial.

$$-\frac{2}{3}(3)^4 + 2(3)^3 - 3$$

Following the order of operations, evaluate exponents first and then multiply.

$$-\frac{2}{3}(81) + 2(27) - 3$$

Add and then subtract to get  $-3$ .

$$-54 + 54 - 3 = -3$$

In the following video, we show more examples of evaluating polynomials for given values of the variable.

**Evaluate a Polynomial in One Variable**

**EXAMPLE**

Evaluate each polynomial for the given value of the variable.

1.  $4x^3 - 5x + 7$  for  $x = -2$

2.  $-\frac{2}{3}y^4 + 2y^2$  for  $y = 4$

1.  $4(-2)^3 - 5(-2) + 7$       2.  $-\frac{2}{3}$

$= 4(-8) - 5(-2) + 7$

$= -32 - (-10) + 7$

$= -32 + 10 + 7$

$= -22 + 7$

$= -15$

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## SUMS AND PRODUCTS OF POLYNOMIALS

### Learning Outcomes

- Add and subtract polynomials
- Multiply and divide polynomials

## Add and Subtract Polynomials

We can add and subtract polynomials by combining like terms which are terms that contain the same variables raised to the same exponents. For example,  $5x^2$  and  $-2x^2$  are like terms and can be added to get  $3x^2$ , but  $3x$  and  $3x^2$  are not like terms; therefore, they cannot be added.

### EXAMPLE

Find the sum.

$$(12x^2 + 9x - 21) + (4x^3 + 8x^2 - 5x + 20)$$

**Answer**

$$4x^3 + (12x^2 + 8x^2) + (9x - 5x) + (-21 + 20) \quad \text{Combine like terms.}$$

$$4x^3 + 20x^2 + 4x - 1 \quad \text{Simplify.}$$

Here is a summary of some helpful steps for adding and subtracting polynomials.

### HOW TO: GIVEN MULTIPLE POLYNOMIALS, ADD OR SUBTRACT THEM TO SIMPLIFY THE EXPRESSIONS

1. Combine like terms.
2. Simplify and write in standard form.

When you subtract polynomials, you will still be looking for like terms to combine, but you will need to pay attention to the sign of the terms you are combining. In the following example we will show how to distribute the negative sign to each term of a polynomial that is being subtracted from another.

### EXAMPLE

Find the difference.

$$(7x^4 - x^2 + 6x + 1) - (5x^3 - 2x^2 + 3x + 2)$$

**Answer**

$$7x^4 - x^2 + 6x + 1 - 5x^3 + 2x^2 - 3x - 2 \quad \text{Distribute.}$$

$$7x^4 - 5x^3 + (-x^2 + 2x^2) + (6x - 3x) + (1 - 2) \quad \text{Combine like terms.}$$

$$7x^4 - 5x^3 + x^2 + 3x - 1 \quad \text{Simplify.}$$

Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.

In the following video, we show more examples of adding and subtracting polynomials.

## Multiplying Polynomials

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms.

You may have used the distributive property to help you solve linear equations such as  $2(x + 7) = 21$ . We can distribute the 2 in  $2(x + 7)$  to obtain the equivalent expression  $2x + 14$ . When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second.

**Examples: Adding and Subtracting Polynomials**

$$1(2x^3 - 6x^2 - 12x^2 - 4) + (-11x^5 + 8x + 2x^2 + 6)$$

$$= -9x^5 - 6x^3$$
  

$$(-9y^3 - 6y^2 - 11x + 2) - (-9y^4 - 8y^3 + 4x^2 + 2x)$$

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We then add the products together and combine like terms to simplify.

The following video will provide you with examples of using the distributive property to find the product of monomials and polynomials.

**Examples: Multiplying Polynomials**

Multiply and simplify.  $a^m \cdot a^n = a^{m+n}$

$$2(x + 5) = 2 \cdot x + 2 \cdot 5 = 2x + 10$$

$$-5n(n - 2) = -5n^2$$
  

$$5y(3y^3 - 5y + 9)$$
  

$$-2a^2b(7ab^3 - 4a^2)$$

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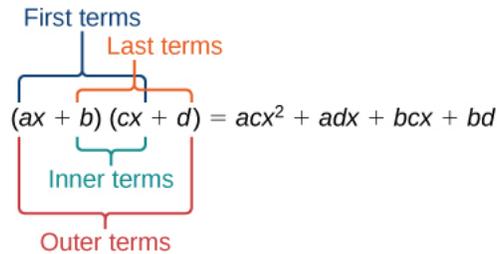
Below is a summary of the steps we used to find the product of two polynomials using the distributive property.

## HOW TO: GIVEN THE MULTIPLICATION OF TWO POLYNOMIALS, USE THE DISTRIBUTIVE PROPERTY TO SIMPLIFY THE EXPRESSION

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.
3. Simplify.

## Using FOIL to Multiply Binomials

We can also use a shortcut called the FOIL method when multiplying two binomials. It is called FOIL because we multiply the **f**irst terms, the **o**uter terms, the **i**nner terms, and then the **l**ast terms of each binomial.



The FOIL method arises out of the distributive property. We are simply multiplying each term of the first binomial by each term of the second binomial and then combining like terms.

### EXAMPLE

Use FOIL to find the product.  $(2x - 18)(3x + 3)$

**Answer**

Find the product of the first terms.

$$\begin{array}{ccc} \downarrow & & \downarrow \\ 2x - 18 & & 3x + 3 \\ \hline & & \end{array} \qquad 2x \cdot 3x = 6x^2$$

Find the product of the outer terms.

$$\begin{array}{ccc} \downarrow & & \downarrow \\ 2x - 18 & & 3x + 3 \\ \hline & & \end{array} \qquad 2x \cdot 3 = 6x$$

Find the product of the inner terms.

$$\begin{array}{ccc} & \downarrow & \downarrow \\ 2x - 18 & & 3x + 3 \\ \hline & & \end{array} \qquad -18 \cdot 3x = -54x$$

Find the product of the last terms.

$$\begin{array}{ccc} & \downarrow & \downarrow \\ 2x - 18 & & 3x + 3 \\ \hline & & \end{array} \qquad -18 \cdot 3 = -54$$

$$6x^2 + 6x - 54x - 54 \quad \text{Add the products.}$$

$$6x^2 + (6x - 54x) - 54 \quad \text{Combine like terms.}$$

$$6x^2 - 48x - 54 \quad \text{Simplify.}$$

In the following video, we show an example of how to use the FOIL method to multiply two binomials.

The following steps summarize the process for using FOIL to multiply two binomials. It is very important to note that this process only works for the product of two binomials. If you are multiplying a binomial and a trinomial, it is better to use a table to keep track of your terms.

**Multiplying Binomials Using FOIL**

**EXAMPLE**

Find the product using the FOIL acronym.

$$(3x - 7)(4x + 5)$$

**FIRST:**  $3x(4x) = 12x^2$   
**OUTER:**  $3x(5) = 15x$   
**INNER:**  $-7(4x) = -28x$   
**LAST:**  $-7(5) = -35$

Multiplying Two Binomials:  
Using FOIL  
FIRST  
OUTER  
INNER  
LAST

$$(3x - 7)(4x + 5) = 12x^2 + 15x - 28x - 35$$

$$= 12x^2 - 13x - 35$$

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## HOW TO: GIVEN TWO BINOMIALS, USE FOIL TO SIMPLIFY THE EXPRESSION

1. Multiply the first terms of each binomial.
2. Multiply the outer terms of the binomials.
3. Multiply the inner terms of the binomials.
4. Multiply the last terms of each binomial.
5. Add the products.
6. Combine like terms and simplify.

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# MULTIPLY A TRINOMIAL AND A BINOMIAL

## LEARNING OUTCOME

- Multiply and divide polynomials

Another type of polynomial multiplication problem is the product of a binomial and trinomial. Although the FOIL method cannot be used since there are more than two terms in a trinomial, you still use the distributive

property to organize the individual products. Using the distributive property, each term in the binomial must be multiplied by each of the terms in the trinomial.

For our first examples, we will show you two ways to organize all of the terms that result from multiplying polynomials with more than two terms. The most important part of the process is finding a way to organize terms.

#### Example

Find the product.  $(3x + 6)(5x^2 + 3x + 10)$

**Answer**

Distribute the trinomial to each term in the binomial.

$$3x(5x^2 + 3x + 10) + 6(5x^2 + 3x + 10)$$

Use the distributive property to distribute the monomials to each term in the trinomials.

$$3x(5x^2) + 3x(3x) + 3x(10) + 6(5x^2) + 6(3x) + 6(10)$$

Multiply.

$$15x^3 + 9x^2 + 30x + 30x^2 + 18x + 60$$

Group like terms.

$$15x^3 + (9x^2 + 30x^2) + (30x + 18x) + 60$$

Combine like terms.

$$(3x + 6)(5x^2 + 3x + 10) = 15x^3 + 39x^2 + 48x + 60$$

As you can see, multiplying a binomial by a trinomial leads to a lot of individual terms! Using the same problem as above, we will show another way to organize all the terms produced by multiplying two polynomials with more than two terms.

#### Example

Multiply.  $(3x + 6)(5x^2 + 3x + 10)$

**Answer**

Set up the problem in a vertical form, and begin by multiplying  $3x + 6$  by  $+10$ . Place the products underneath, as shown.

$$\begin{array}{r} 3x + 6 \\ \times \quad 5x^2 + 3x + 10 \\ \hline +30x + 60 \end{array}$$

Now multiply  $3x + 6$  by  $+3x$ . Notice that  $(6)(3x) = 18x$ ; since this term is like  $30x$ , place it directly beneath it.

$$\begin{array}{r} 3x + 6 \\ \times \quad 5x^2 + 3x + 10 \\ \hline +30x + 60 \\ +9x^2 + 18x \end{array}$$

Finally, multiply  $3x + 6$  by  $5x^2$ . Notice that  $30x^2$  is placed underneath  $9x^2$ .

$$\begin{array}{r}
 3x + 6 \\
 \times \quad 5x^2 + 3x + 10 \\
 \hline
 +30x + 60 \\
 +9x^2 + 18x \\
 +15x^3 + 30x^2 \\
 \hline
 \end{array}$$

Now add like terms.

$$\begin{array}{r}
 3x + 6 \\
 \times \quad 5x^2 + 3x + 10 \\
 \hline
 +30x + 60 \\
 +9x^2 + 18x \\
 +15x^3 + 30x^2 \\
 \hline
 +15x^3 + 39x^2 + 48x + 60
 \end{array}$$

The answer is  $15x^3 + 39x^2 + 48x + 60$ .

Notice that although the two problems were solved using different strategies, the product is the same. Both the horizontal and vertical methods apply the distributive property to multiply a binomial by a trinomial.

In our next example, we will multiply a binomial and a trinomial that contains subtraction. Pay attention to the signs on the terms. Forgetting a negative sign is the easiest mistake to make in this case.

### EXAMPLE

Find the product.

$$(2x + 1)(3x^2 - x + 4)$$

**Answer**

$$2x(3x^2 - x + 4) + 1(3x^2 - x + 4) \quad \text{Use the distributive property.}$$

$$(6x^3 - 2x^2 + 8x) + (3x^2 - x + 4) \quad \text{Multiply.}$$

$$6x^3 + (-2x^2 + 3x^2) + (8x - x) + 4 \quad \text{Combine like terms.}$$

$$6x^3 + x^2 + 7x + 4 \quad \text{Simplify.}$$

Another way to keep track of all the terms involved in the above product is to use a table, as shown below. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify. Notice how we kept the sign on each term; for example, we are subtracting  $x$  from  $3x^2$ , so we place  $-x$  in the table.

	$3x^2$	$-x$	$+4$
$2x$	$6x^3$	$-2x^2$	$8x$
$+1$	$3x^2$	$-x$	$4$

## EXAMPLE

Multiply.  $(2p - 1)(3p^2 - 3p + 1)$

**Answer**

Distribute  $2p$  and  $-1$  to each term in the trinomial.

$$2p(3p^2 - 3p + 1) - 1(3p^2 - 3p + 1)$$

$$2p(3p^2) + 2p(-3p) + 2p(1) - 1(3p^2) - 1(-3p) - 1(1)$$

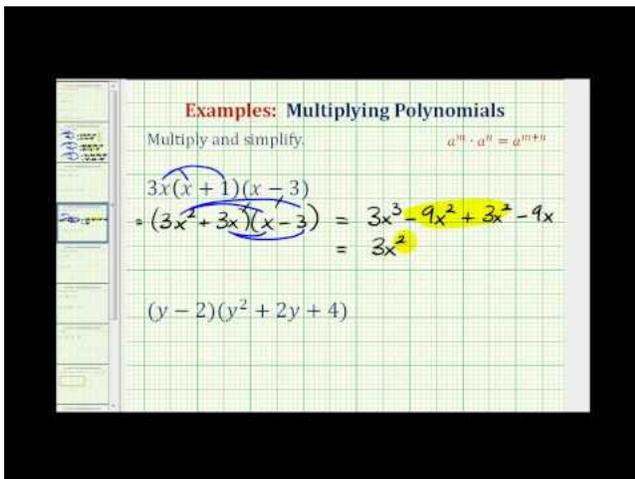
Multiply. Notice that the subtracted 1 and the subtracted  $3p$  have a positive product that is added.

$$6p^3 - 6p^2 + 2p - 3p^2 + 3p - 1$$

Combine like terms.

$$6p^3 - 9p^2 + 5p - 1$$

In the following video, we show more examples of multiplying polynomials.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2841>

## Summary

Multiplication of binomials and polynomials requires use of the distributive property as well as the commutative and associative properties of multiplication. Whether the polynomials are monomials, binomials, or trinomials, carefully multiply each term in one polynomial by each term in the other polynomial. Be careful to watch the addition and subtraction signs and negative coefficients. A product is written in simplified form if all of its like terms have been combined.

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# DIVIDE POLYNOMIALS PART I

## Learning Outcome

- Multiply and divide polynomials

## Divide a Polynomial by a Binomial

Dividing a **polynomial** by a monomial can be handled by dividing each term in the polynomial separately. This cannot be done when the divisor has more than one term. However, the process of long division can be very helpful with polynomials.

Recall how you can use long division to divide two whole numbers, say 900 divided by 37.

$$37 \overline{)900}$$

First, you would think about how many 37s are in 90, as 9 is too small. (*Note:* you could also think, how many 40s are there in 90.)

There are two 37s in 90, so write 2 above the last digit of 90. Two 37s is 74; write that product below the 90.

$$\begin{array}{r} 2 \\ 37 \overline{)900} \\ \underline{74} \phantom{0} \end{array}$$

Subtract: 90–74 is 16. (If the result is larger than the divisor, 37, then you need to use a larger number for the quotient.)

$$\begin{array}{r} 2 \\ 37 \overline{)900} \\ \underline{-74} \phantom{0} \\ 16 \phantom{0} \end{array}$$

Bring down the next digit (0) and consider how many 37s are in 160.

$$\begin{array}{r} 2 \\ 37 \overline{)900} \\ \underline{-74} \phantom{0} \\ 160 \phantom{0} \end{array}$$

There are four 37s in 160, so write the 4 next to the two in the quotient. Four 37s is 148; write that product below the 160.

$$\begin{array}{r} 24 \\ 37 \overline{)900} \\ \underline{-74} \phantom{0} \\ 160 \phantom{0} \\ \underline{-148} \phantom{0} \end{array}$$

Subtract: 160–148 is 12. This is less than 37 so the 4 is correct. Since there are no more digits in the dividend to bring down, you are done.

$$\begin{array}{r} 24 \\ 37 \overline{)900} \\ \underline{-74} \phantom{0} \\ 160 \\ \underline{-148} \\ 12 \end{array}$$

The final answer is 24 R12, or  $24\frac{12}{37}$ . You can check this by multiplying the quotient (without the remainder) by the divisor, and then adding in the remainder. The result should be the dividend:

$$24 \cdot 37 + 12 = 888 + 12 = 900$$

To divide polynomials, use the same process. This example shows how to do this when dividing by a **binomial**.

### Example

Divide:  $\frac{(x^2-4x-12)}{(x+2)}$

**Answer**

How many  $x$ 's are there in  $x^2$ ? That is, what is  $\frac{x^2}{x}$ ?

$$x + 2 \overline{)x^2 - 4x - 12}$$

$\frac{x^2}{x} = x$ . Put  $x$  in the quotient above the  $-4x$  term. These are like terms, which helps to organize the problem.

Write the product of the divisor and the part of the quotient you just found under the dividend. Since  $x(x + 2) = x^2 + 2x$ , write this underneath, and get ready to subtract.

$$\begin{array}{r} x \\ x + 2 \overline{)x^2 - 4x - 12} \\ \underline{-(x^2 + 2x)} \phantom{-12} \end{array}$$

Rewrite  $-(x^2 + 2x)$  as its opposite  $-x^2 - 2x$  so that you can add the opposite. Adding the opposite is the same as subtracting, and it is easier to do.

$$\begin{array}{r} x \\ x + 2 \overline{)x^2 - 4x - 12} \\ \underline{-x^2 - 2x} \phantom{-12} \end{array}$$

Add  $-x^2$  to  $x^2$  and  $-2x$  to  $-4x$ .

$$\begin{array}{r} x \\ x + 2 \overline{)x^2 - 4x - 12} \\ \underline{-x^2 - 2x} \phantom{-12} \\ -6x \phantom{-12} \end{array}$$

Bring down  $-12$ .

$$\begin{array}{r} x \\ x + 2 \overline{)x^2 - 4x - 12} \\ \underline{-x^2 - 2x} \phantom{-12} \\ -6x - 12 \end{array}$$

Repeat the process. How many times does  $x$  go into  $-6x$ ? In other words, what is  $\frac{-6x}{x}$ ?

$$\begin{array}{r} x \\ x + 2 \overline{)x^2 - 4x - 12} \\ \underline{-x^2 - 2x} \phantom{-12} \\ -6x - 12 \end{array}$$

Since  $\frac{-6x}{x} = -6$ , write  $-6$  in the quotient. Multiply  $-6$  and  $x + 2$  and prepare to subtract the product.

$$\begin{array}{r}
 x - 6 \\
 x + 2 \overline{) x^2 - 4x - 12} \\
 \underline{-x^2 - 2x} \phantom{-12} \\
 -6x - 12 \\
 \underline{-(-6x - 12)} \\
 0
 \end{array}$$

Rewrite  $-(-6x - 12)$  as  $6x + 12$ , so that you can add the opposite.

$$\begin{array}{r}
 x - 6 \\
 x + 2 \overline{) x^2 - 4x - 12} \\
 \underline{-x^2 - 2x} \phantom{-12} \\
 -6x - 12 \\
 \underline{6x + 12} \\
 0
 \end{array}$$

Add. In this case, there is no remainder, so you are done.

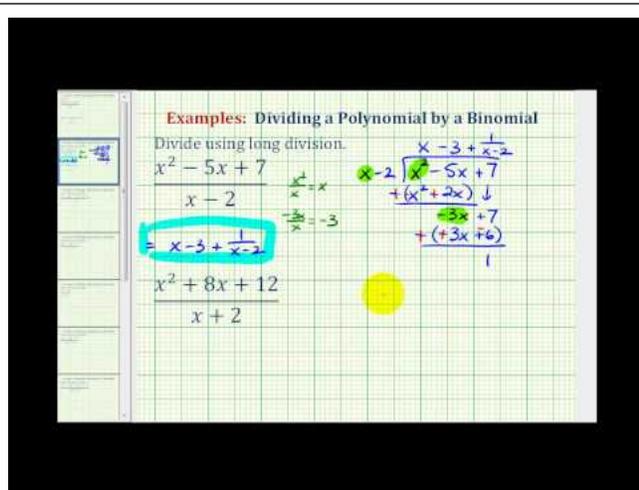
$$\begin{array}{r}
 x - 6 \\
 x + 2 \overline{) x^2 - 4x - 12} \\
 \underline{-x^2 - 2x} \phantom{-12} \\
 -6x - 12 \\
 \underline{6x + 12} \\
 0
 \end{array}$$

The answer is  $x - 6$ .

Check this by multiplying:

$$(x - 6)(x + 2) = x^2 + 2x - 6x - 12 = x^2 - 4x - 12$$

In the following video, we show another example of dividing a degree two trinomial by a degree one binomial.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2590>

Let us try another example. In this example, a term is “missing” from the dividend.

### Example

Divide:  $\frac{(x^3 - 6x - 10)}{(x - 3)}$

**Answer**

In setting up this problem, notice that there is an  $x^3$  term but no  $x^2$  term. Add  $0x^2$  as a “place holder” for this term. Since 0 times anything is 0, you are not changing the value of the dividend.

$$x - 3 \overline{) x^3 + 0x^2 - 6x - 10}$$

Focus on the first terms again: how many  $x$ 's are there in  $x^3$ ? Since  $\frac{x^3}{x} = x^2$ , put  $x^2$  in the quotient.

Multiply:  $x^2(x-3) = x^3 - 3x^2$ ; write this underneath the dividend, and prepare to subtract.

$$\begin{array}{r} x-3 \overline{) x^3 + 0x^2 - 6x - 10} \\ \underline{-(x^3 - 3x^2)} \phantom{- 10} \end{array}$$

Rewrite the subtraction using the opposite of the expression  $x^3 - 3x^2$ . Then add.

$$\begin{array}{r} x-3 \overline{) x^3 + 0x^2 - 6x - 10} \\ \underline{-x^3 + 3x^2} \phantom{- 10} \\ 3x^2 \phantom{- 6x - 10} \end{array}$$

Bring down the rest of the expression in the dividend. It is helpful to bring down *all* of the remaining terms.

$$\begin{array}{r} x-3 \overline{) x^3 + 0x^2 - 6x - 10} \\ \underline{-x^3 + 3x^2} \phantom{- 10} \\ 3x^2 - 6x - 10 \end{array}$$

Now, repeat the process with the remaining expression,  $3x^2 - 6x - 10$ , as the dividend.

$$\begin{array}{r} x-3 \overline{) x^3 + 0x^2 - 6x - 10} \\ \underline{-x^3 + 3x^2} \phantom{- 10} \\ 3x^2 - 6x - 10 \\ \underline{-(3x^2 - 9x)} \phantom{- 10} \end{array}$$

Remember to watch the signs!

$$\begin{array}{r} x-3 \overline{) x^3 + 0x^2 - 6x - 10} \\ \underline{-x^3 + 3x^2} \phantom{- 10} \\ 3x^2 - 6x - 10 \\ \underline{-3x^2 + 9x} \phantom{- 10} \\ 3x - 10 \end{array}$$

How many  $x$ 's are there in  $3x$ ? Since there are 3, multiply  $3(x-3) = 3x-9$ , write this underneath the dividend, and prepare to subtract.

$$\begin{array}{r} x-3 \overline{) x^3 + 0x^2 - 6x - 10} \\ \underline{-x^3 + 3x^2} \phantom{- 10} \\ 3x^2 - 6x - 10 \\ \underline{-3x^2 + 9x} \phantom{- 10} \\ 3x - 10 \\ \underline{-(3x - 9)} \phantom{- 10} \end{array}$$

Continue until the **degree** of the remainder is *less* than the degree of the divisor. In this case the degree of the remainder,  $-1$ , is 0, which is less than the degree of  $x - 3$ , which is 1.

Also notice that you have brought down all the terms in the dividend and that the quotient extends to the right edge of the dividend. These are other ways to check whether you have completed the problem.

$$\begin{array}{r} x-3 \overline{) x^3 + 0x^2 - 6x - 10} \\ \underline{-x^3 + 3x^2} \phantom{- 10} \\ 3x^2 - 6x - 10 \\ \underline{-3x^2 + 9x} \phantom{- 10} \\ 3x - 10 \\ \underline{-3x + 9} \phantom{- 10} \\ -1 \end{array}$$

You can write the remainder using the symbol  $R$  or as a fraction added to the rest of the quotient with the remainder in the numerator and the divisor in the denominator. In this case, since the remainder is negative, you can also subtract the opposite.

Possible forms of the solution are:

$$x^2 + 3x + 3 + R - 1,$$

$$x^2 + 3x + 3 + \frac{-1}{x-3}, \text{ or}$$

$$x^2 + 3x + 3 - \frac{1}{x-3}$$

Check the result:

$$\begin{aligned} (x-3)(x^2 + 3x + 3) &= x(x^2 + 3x + 3) - 3(x^2 + 3x + 3) \\ &= x^3 + 3x^2 + 3x - 3x^2 - 9x - 9 \\ &= x^3 - 6x - 9 \end{aligned}$$

$$x^3 - 6x - 9 + (-1) = x^3 - 6x - 10$$

In the video that follows, we show another example of dividing a degree three trinomial by a binomial. Note the “missing” term and how we work with it.

Polynomial Division - Long Division

EXAMPLE

Divide:  $\frac{x^3 - 3x + 5}{x + 2} = (x^3 - 3x + 5) \div (x + 2)$

Handwritten work on grid paper:

$$\begin{array}{r} x^2 - 2x - 2 \\ x + 2 \overline{) x^3 + 0x^2 - 3x + 5} \\ \underline{+(x^3 + 2x^2)} \phantom{+ 5} \\ -2x^2 - 3x \phantom{+ 5} \\ \underline{+2x^2 + 4x} \phantom{+ 5} \\ -2x + 5 \\ \underline{+2x + 4} \\ -2 \end{array}$$

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The process above works for dividing any polynomials no matter how many terms are in the divisor or the dividend. The main things to remember are:

- When subtracting, be sure to subtract the whole expression, not just the first term. *This is very easy to forget, so be careful!*
- Stop when the degree of the remainder is less than the degree of the divisor or when you have brought down all the terms in the dividend and your quotient extends to the right edge of the dividend.

In the next video, we present one more example of polynomial long division.

Examples: Dividing a Polynomial by a Binomial

Divide using long division.

$2x^3 - 8x^2 + 7x - 13$

$2x^2 + 5$

Handwritten work on grid paper:

$$\begin{array}{r} x - 4 \\ 2x^2 + 5 \overline{) 2x^3 - 8x^2 + 7x - 13} \\ \underline{+(2x^3 + 10x^2)} \phantom{+ 7x - 13} \\ -18x^2 + 7x - 13 \\ \underline{+18x^2 + 25x} \phantom{- 13} \\ -18x - 13 \\ \underline{+18x + 45} \\ 32 \end{array}$$

A YouTube element has been excluded from this version of the text. You can view it online here:  
<https://courses.lumenlearning.com/intermediatealgebra/?p=2590>

## Summary

Dividing polynomials by polynomials of more than one term can be done using a process very much like long division of whole numbers. You must be careful to subtract entire expressions, not just the first term. Stop when the degree of the remainder is less than the degree of the divisor. The remainder can be written using R notation or as a fraction added to the quotient with the remainder in the numerator and the divisor in the denominator.

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# DIVIDE POLYNOMIALS PART II

## Learning Outcome

- Multiply and divide polynomials

As we have seen, long division of polynomials can involve many steps and be quite cumbersome. **Synthetic division** is a shorthand method of dividing polynomials for the special case of dividing by a polynomial whose leading coefficient is 1.

## SYNTHETIC DIVISION

Synthetic division is a shortcut that can be used when the divisor is a binomial in the form  $x - k$ , for a real number  $k$ . In **synthetic division**, only the coefficients are used in the division process.

To illustrate the process, divide  $2x^3 - 3x^2 + 4x + 5$  by  $x + 2$  using the long division algorithm.

$$\begin{array}{r} 2x^2 - 7x + 18 \\ x+2 \overline{) 2x^3 - 3x^2 + 4x + 5} \\ \underline{-2x^3 - 4x^2} \phantom{+ 5} \\ -7x^2 + 4x \phantom{+ 5} \\ \underline{+7x^2 + 14x} \phantom{+ 5} \\ +18x + 5 \\ \underline{-18x - 36} \\ -31 \end{array}$$

There is a lot of repetition in this process. If we do not write the variables but, instead, line up their coefficients in columns under the division sign, we already have a simpler version of the entire problem.

$$\begin{array}{r} 2 \overline{) 2 \quad -3 \quad 4 \quad 5} \\ \underline{-2 \quad -4} \phantom{0} \\ -7 \quad 14 \phantom{0} \\ \underline{18 \quad -36} \\ -31 \end{array}$$

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2, as we would in division of whole numbers, then multiplying and subtracting the middle product, we change the sign of the “divisor” to  $-2$ , multiply and add. The process starts by bringing down the leading coefficient.

$$\begin{array}{r|rrrr} -2 & 2 & -3 & 4 & 5 \\ & & -4 & 14 & -36 \\ \hline & 2 & -7 & 18 & -31 \end{array}$$

We then multiply it by the “divisor” and add, repeating this process column by column, until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is  $2x^2 - 7x + 18$  and the remainder is  $-31$ . The process will be made more clear in the following example.

#### Analysis of the solution

It is important to note that the result,  $5x + 12$ , of  $5x^2 - 3x - 36 \div x - 3$  is one degree less than  $5x^2 - 3x - 36$ . Why is that? Think about how you would have solved this using long division. The first thing you would ask yourself is how many  $x$ 's are in  $5x^2$ ?

$$x - 3 \overline{) 5x^2 - 3x - 36}$$

To get a result of  $5x^2$ , you need to multiply  $x$  by  $5x$ . The next step in long division is to subtract this result from  $5x^2$ . This leaves us with no  $x^2$  term in the result.

In the next video example, you will see another example of using synthetic division for division of a degree two polynomial by a degree one binomial.

In the next example, we will use synthetic division to divide a third-degree polynomial.

In the next example, we will show division of a fourth degree polynomial by a binomial. Note how there is no  $x$  term in the fourth degree polynomial, so we need to use a placeholder of 0 to ensure proper alignment of terms.

In our last video example, we show another example of how to use synthetic division to divide a degree three polynomial by a degree one binomial.

## EXAMPLE

Use synthetic division to divide  $5x^2 - 3x - 36$  by  $x - 3$ .

### Answer

Begin by setting up the synthetic division. Write 3 and the coefficients of the polynomial.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & & \end{array}$$

Bring down the lead coefficient. Multiply the lead coefficient by 3 and place the result in the second column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & \\ \hline & 5 & & \end{array}$$

Continue by adding  $-3 + 15$  in the second column. Multiply the resulting number, 12, by 3. Write the result, 36, in the next column. Then add the numbers in the third column.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

The result is  $5x + 12$ .

We can check our work by multiplying the result by the original divisor  $x - 3$ . If we get  $5x^2 - 3x - 36$ , then we have used the method correctly.

Check:

$$\begin{aligned} & (5x + 12)(x - 3) \\ &= 5x^2 - 15x + 12x - 36 \\ &= 5x^2 - 3x - 36 \end{aligned}$$

Because we got a result of  $5x^2 - 3x - 36$  when we multiplied the divisor and our answer, we can be sure that we have used synthetic division correctly.

## THINK ABOUT IT

Reflect on this idea – if you multiply two polynomials and get a result whose degree is 2, what are the possible degrees of the two polynomials that were multiplied? Write your ideas in the box below before looking at the discussion.

### Answer

A degree two polynomial will have a leading term with  $x^2$ . Let us use  $2x^2 - 2x - 24$  as an example. We can write two products that will give this as a result of multiplication:

$$\begin{aligned} 2(x^2 - x - 12) &= 2x^2 - 2x - 24 \\ (2x + 6)(x - 4) &= 2x^2 - 2x - 24 \end{aligned}$$

If we work backward, starting from  $2x^2 - 2x - 24$  and we divide by a binomial with degree one, such as  $(x - 4)$ , our result will also have degree one.

**Example: Dividing a Polynomial by a Binomial**  
Divide using synthetic division.

$$\begin{array}{r} x^2 - 5x + 7 \\ x - 2 \end{array}$$

2 | 1   -5   7  
   ↓   2   -6  
 1   -3   1

Use synthetic division to divide  $ax^2 + bx^2 + cx + d$  by  $(x - k)$ , as follows.

k	a	b	c	d
	↓	↓	↓	↓
	a	ka	b + ka	c + k(b + ka)

**Vertical Pattern:** Add terms.  
**Diagonal Pattern:** Multiply by k.  
 The bottom row represent the coefficients of the quotient, which is 1 degree less than the dividend.

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## HOW TO: GIVEN TWO POLYNOMIALS, USE SYNTHETIC DIVISION TO DIVIDE

1. Write  $k$  for the divisor.
2. Write the coefficients of the dividend.
3. Bring the lead coefficient down.
4. Multiply the lead coefficient by  $k$ . Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by  $k$ . Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0. The next number from the right has degree 1, and the next number from the right has degree 2, and so on.

## EXAMPLE

Use synthetic division to divide  $4x^3 + 10x^2 - 6x - 20$  by  $x + 2$ .

**Answer**

The binomial divisor is  $x + 2$  so  $k = -2$ . Add each column, multiply the result by  $-2$ , and repeat until the last column is reached.

$$\begin{array}{r} -2 \quad | \quad 4 \quad 10 \quad -6 \quad -20 \\ \quad \quad | \quad -8 \quad -4 \quad 20 \\ \hline \quad \quad | \quad 4 \quad 2 \quad -10 \quad 0 \end{array}$$

The result is  $4x^2 + 2x - 10$ . Again notice the degree of the result is less than the degree of the quotient,  $4x^3 + 10x^2 - 6x - 20$ .

We can check that we are correct by multiplying the result with the divisor:

$$(x + 2)(4x^2 + 2x - 10) = 4x^3 + 2x^2 - 10x + 8x^2 + 4x - 20 = 4x^3 + 10x^2 - 6x - 20$$

We just verified that the answer is  $4x^2 + 2x - 10$

## EXAMPLE

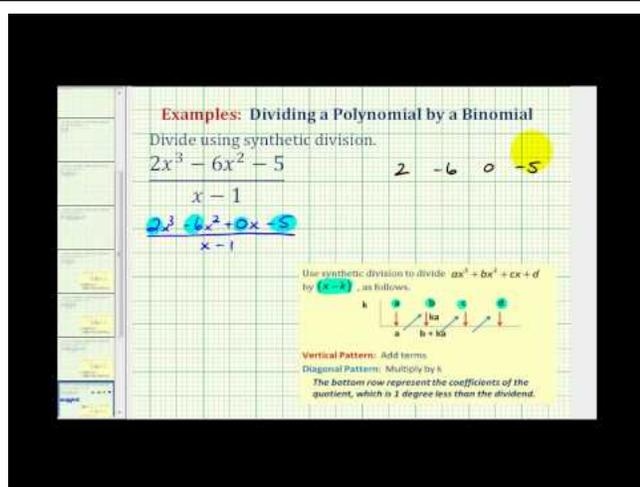
Use synthetic division to divide  $-9x^4 + 10x^3 + 7x^2 - 6$  by  $x - 1$ .

### Answer

Notice there is no  $x$ -term. We will use a zero as the coefficient for that term.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

The result is  $-9x^3 + x^2 + 8x + 8 + \frac{2}{x-1}$ .



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# INTRODUCTION TO POLYNOMIAL FUNCTIONS

Polynomial functions can be used to model a wide variety of things that either grow or shrink, like populations and the area of oil spills. Remember that a polynomial is a mathematical expression of two or more algebraic terms that contain variables and coefficients and only uses addition, subtraction, and multiplication of non-negative integer exponents. A polynomial function consists of either zero or the sum of a finite number of these non-zero terms, each of which is a product of a number called the coefficient of the term and a variable raised to a non-negative integer power.

In this section, we will combine what we know about functions and polynomials to gain a basic understanding of polynomial functions.

# DEFINE AND IDENTIFY POLYNOMIAL FUNCTIONS

## LEARNING OUTCOMES

- Define and identify polynomial functions

## Identify Polynomial Functions

We have introduced polynomials and functions, so now we will combine these ideas to describe polynomial functions. Polynomials are algebraic expressions that are created by adding or subtracting monomial terms, such as  $-3x^2$ , where the exponents are only integers. Functions are a specific type of relation in which each input value has one and only one output value. Polynomial functions have all of these characteristics as well as a domain and range, and corresponding graphs. In this section, we will identify and evaluate polynomial functions. Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the powers of the variables.

When we introduced polynomials, we presented the following:  $4x^3 - 9x^2 + 6x$ . We can turn this into a polynomial function by using function notation:

$$f(x) = 4x^3 - 9x^2 + 6x$$

Polynomial functions are written with the leading term first and all other terms in descending order as a matter of convention. In the first example, we will identify some basic characteristics of polynomial functions.

In the following video, you will see additional examples of how to identify a polynomial function using the definition.

## Define the Degree and Leading Coefficient of a Polynomial Function

Just as we identified the degree of a polynomial, we can identify the degree of a polynomial function. To review: the **degree** of the polynomial is the highest power of the variable that occurs in the polynomial; the **leading term** is the term containing the highest power of the variable or the term with the highest degree. The **leading coefficient** is the coefficient of the leading term.

In the next video, we will show more examples of how to identify the degree, leading term and leading coefficient of a polynomial function.

## Summary

## EXAMPLE

Which of the following are polynomial functions?

$$f(x) = 2x^3 \cdot 3x + 4$$

$$g(x) = -x(x^2 - 4)$$

$$h(x) = 5\sqrt{x} + 2$$

### Answer

The first two functions are examples of polynomial functions because they contain powers that are non-negative integers and the coefficients are real numbers.

- $f(x)$  can be written as  $f(x) = 6x^4 + 4$ .
- $g(x)$  can be written as  $g(x) = -x^3 + 4x$ .
- $h(x) = 5\sqrt{x} + 2$  is not a polynomial function because the variable is under a square root – therefore the exponent is not a positive integer.

The screenshot shows a video player interface. The video title is "Introduction to Polynomial Functions". The main content is a slide titled "EXAMPLE" with the question "Which of the following are polynomial functions?". The slide lists four functions: 1.  $f(x) = -2x(4x^2) + 4$ , 2.  $g(x) = x^2(5x^3 - 2)$ , 3.  $h(x) = 4\sqrt{x} + 9$ , and 4.  $f(x) = \frac{7}{x} + 3$ . A definition of a polynomial function is provided: "A polynomial function is a function consisting of the sum or difference of terms in which each term is a real number, a variable, or the product of a real number and variables with non-negative integer exponents or powers. Non-negative integers are 0, 1, 2, 3, 4, ...". Below the slide, handwritten notes on a grid background show: 1.  $f(x) = -8x^3 + 4$  and  $h(x)$  is a polynomial function. 2.  $g(x) = 5x^5 - 2x^2$ .

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Polynomial functions contain powers that are non-negative integers and the coefficients are real numbers. It is often helpful to know how to identify the degree and leading coefficient of a polynomial function. To do this, follow these suggestions:

1. Find the highest power of  $x$  to determine the degree of the function.
2. Identify the term containing the highest power of  $x$  to find the leading term.
3. Identify the coefficient of the leading term.

## EXAMPLE

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

$$f(x) = 3 + 2x^2 - 4x^3$$

$$g(t) = 5t^5 - 2t^3 + 7t$$

$$h(p) = 6p - p^3 - 2$$

### Answer

For the function  $f(x)$ , the highest power of  $x$  is 3, so the degree is 3. The leading term is the term containing that degree,  $-4x^3$ . The leading coefficient is the coefficient of that term,  $-4$ .

For the function  $g(t)$ , the highest power of  $t$  is 5, so the degree is 5. The leading term is the term containing that degree,  $5t^5$ . The leading coefficient is the coefficient of that term, 5.

For the function  $h(p)$ , the highest power of  $p$  is 3, so the degree is 3. The leading term is the term containing that degree,  $-p^3$ . The leading coefficient is the coefficient of that term,  $-1$ .

Introduction to Polynomial Functions

**EXAMPLE**

Identify the degree, leading term, and leading coefficient of the following polynomial functions.

1.  $f(x) = 4 - 3x^2 - 5x^3$
2.  $g(t) = \frac{1}{2}t^6 - t^4 + 3t^2 - 1$
3.  $h(s) = s - s^4 - 5$

**HOW TO: GIVEN A POLYNOMIAL FUNCTION, IDENTIFY THE DEGREE, LEADING TERM, AND LEADING COEFFICIENT**

1. Identify the highest power of the variable to find the degree.
2. Identify the term containing the highest power of variable to find the leading term.
3. Identify the coefficient of the leading term to find the leading coefficient.

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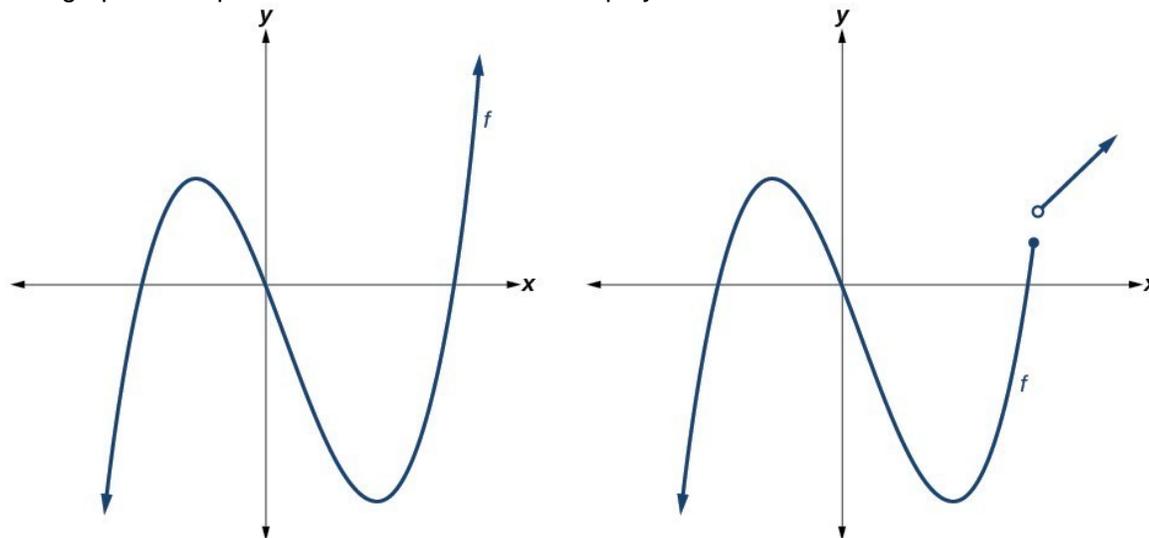
# GRAPHS OF POLYNOMIAL FUNCTIONS

## LEARNING OUTCOMES

- Identify general characteristics of a polynomial function from its graph

Plotting polynomial functions using tables of values can be misleading because of some of the inherent characteristics of polynomials. Additionally, the algebra of finding points like x-intercepts for higher degree polynomials can get very messy and oftentimes be impossible to find by hand. We have therefore developed some techniques for describing the general behavior of polynomial graphs.

Polynomial functions of degree 2 or more have graphs that do not have sharp corners. These types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. The figure below shows a graph that represents a **polynomial function** and a graph that represents a function that is not a polynomial.



## Identifying the Shape of the Graph of a Polynomial Function

Knowing the degree of a polynomial function is useful in helping us predict what its graph will look like. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as  $x$  gets very large or very small, so its behavior will dominate the graph. For any polynomial, the graph of the polynomial will match the end behavior of the term of highest degree.

As an example, we compare the outputs of a degree 2 polynomial and a degree 5 polynomial in the following table.

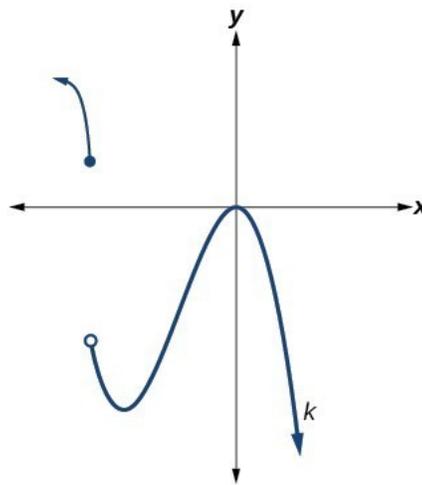
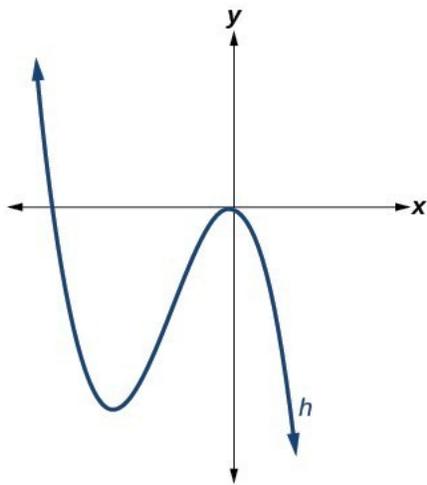
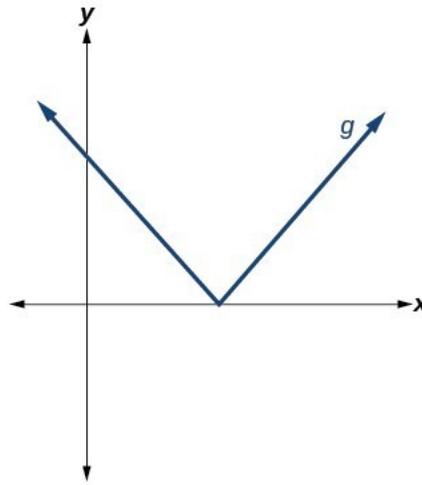
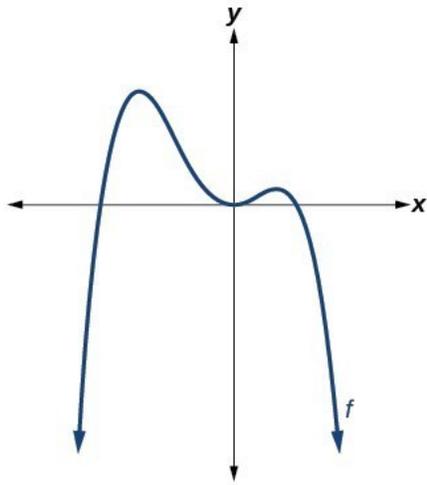
$x$	$f(x) = 2x^2 - 2x + 4$	$g(x) = x^5 + 2x^3 - 12x + 3$
1	4	8
10	184	98117
100	19804	9998001197
1000	1998004	9999980000000000

As the inputs for both functions get larger, the degree 5 polynomial outputs get much larger than the degree 2 polynomial outputs. This is why we use the leading term to get a rough idea of the behavior of polynomial graphs.

There are two other important features of polynomials that influence the shape of its graph. The first is whether the degree is even or odd, and the second is whether the leading term is negative.

## EXAMPLE

Which of the graphs below represents a polynomial function?



### Answer

The graphs of  $f$  and  $h$  are graphs of polynomial functions. They are smooth and **continuous**.

The graphs of  $g$  and  $k$  are graphs of functions that are not polynomials. The graph of function  $g$  has a sharp corner. The graph of function  $k$  is not continuous.

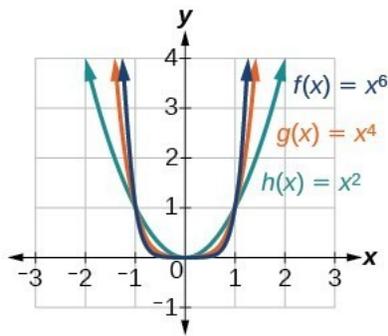
## Q & A

**Do all polynomial functions have “all real numbers” as their domain?**

*Yes. Any real number is a valid input for a polynomial function.*

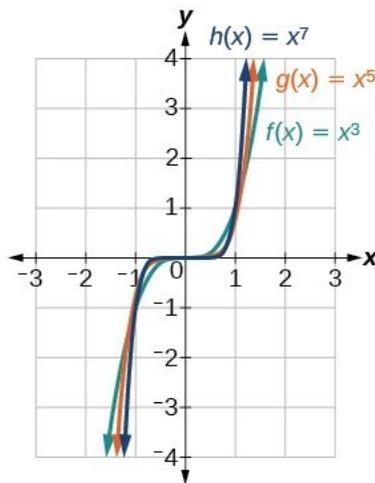
## Even Degree Polynomials

In the figure below, we show the graphs of  $f(x) = x^2$ ,  $g(x) = x^4$ , and  $h(x) = x^6$  which all have even degrees. Notice that these graphs have similar shapes, very much like that of a quadratic function. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.



## Odd Degree Polynomials

The next figure shows the graphs of  $f(x) = x^3$ ,  $g(x) = x^5$ , and  $h(x) = x^7$  which all have odd degrees.



Notice that one arm of the graph points down and the other points up. This is because when your input is negative, you will get a negative output if the degree is odd. The following table of values shows this.

x	$f(x) = x^4$	$h(x) = x^5$
-1	1	-1
-2	16	-32
-3	81	-243

Now you try it.

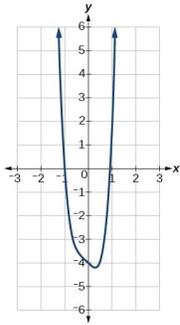
## The Sign of the Leading Term

What would happen if we change the sign of the leading term of an even degree polynomial? For example, let us say that the leading term of a polynomial is  $-3x^4$ . We will use a table of values to compare the outputs for a polynomial with leading term  $-3x^4$  and  $3x^4$ .

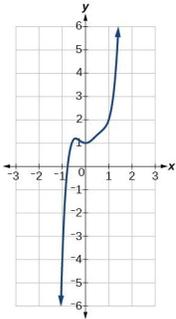
## EXAMPLE

Identify whether each graph represents a polynomial function that has a degree that is even or odd.

a)



b)



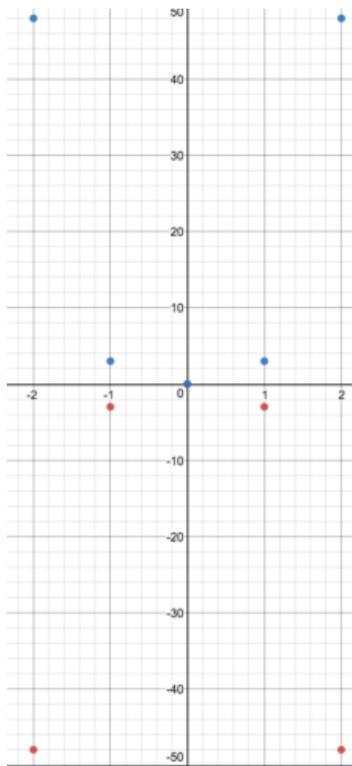
### Answer

a) Both arms of this polynomial point upward, similar to a quadratic polynomial, therefore the degree must be even. If you apply negative inputs to an even degree polynomial, you will get positive outputs back.

b) As the inputs of this polynomial become more negative the outputs also become negative. The only way this is possible is with an odd degree polynomial. Therefore, this polynomial must have an odd degree.

x	$-3x^4$	$3x^4$
-2	-48	48
-1	-3	3
0	0	0
1	-3	3
2	-48	48

The grid below shows a plot with these points. The red points indicate a negative leading coefficient, and the blue points indicate a positive leading coefficient:



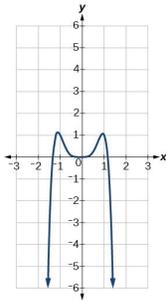
The negative sign creates a reflection of  $3x^4$  across the x-axis. The arms of a polynomial with a leading term of  $-3x^4$  will point down, whereas the arms of a polynomial with leading term  $3x^4$  will point up.

Now you try it.

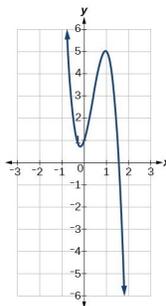
## EXAMPLE

Identify whether the leading term is positive or negative and whether the degree is even or odd for the following graphs of polynomial functions.

a)



b)



### Answer

a) Both arms of this polynomial point in the same direction so it must have an even degree. The leading term of the polynomial must be negative since the arms are pointing downward.

b) The arms of this polynomial point in different directions, so the degree must be odd. As the inputs get really big and positive, the outputs get really big and negative, so the leading coefficient must be negative.

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# ALGEBRA OF POLYNOMIAL FUNCTIONS

## LEARNING OUTCOMES

- Perform algebraic operations on polynomial functions

Just as we have performed algebraic operations on polynomials, we can do the same with polynomial functions. In this section, we will show you how to perform algebraic operations on polynomial functions and

introduce related notation.

The four basic operations on functions are adding, subtracting, multiplying, and dividing. The notation for these functions is as follows:

$$\text{Addition } (f + g)(x) = f(x) + g(x)$$

$$\text{Subtraction } (f - g)(x) = f(x) - g(x)$$

$$\text{Multiplication } (f \cdot g)(x) = f(x)g(x)$$

$$\text{Division } \frac{f}{g}(x) = \frac{f(x)}{g(x)}$$

We will focus on applying these operations to polynomial functions in this section.

## Add and Subtract Polynomial Functions

Adding and subtracting polynomial functions is the same as adding and subtracting polynomials. When you evaluate a sum or difference of functions, you can either evaluate first or perform the operation on the functions first as we will see. Our next examples describe the notation used to add and subtract polynomial functions.

### EXAMPLE

Let  $f(x) = 2x^3 - 5x + 3$  and  $h(x) = x - 5$ ,

Find the following:

$$(f + h)(x) \text{ and } (h - f)(x)$$

**Answer**

$$\begin{aligned}(f + h)(x) &= f(x) + h(x) = (2x^3 - 5x + 3) + (x - 5) \\ &= 2x^3 - 5x + 3 + x - 5 && \text{combine like terms} \\ &= 2x^3 - 4x - 2 && \text{simplify}\end{aligned}$$

$$\begin{aligned}(h - f)(x) &= h(x) - f(x) = (x - 5) - (2x^3 - 5x + 3) \\ &= x - 5 - 2x^3 + 5x - 3 && \text{combine like terms} \\ &= -2x^3 + 6x - 8 && \text{simplify}\end{aligned}$$

In our next example, we will evaluate a sum and difference of functions and show that you can get to the same result in one of two ways.

## Multiply and Divide Polynomial Functions

We saw that multiplying polynomials often required the use of the distributive property and that the algebra of dividing polynomials could get messy fast! The same techniques can be used to multiply and divide polynomial functions. Additionally, the same idea applies to evaluating a product or quotient of functions as we discovered in the previous example. We can either evaluate the function and then perform the indicated operation or vice-versa. You may already be thinking it will be a lot less work to evaluate the polynomials and then divide the results!

In the next example, we will divide polynomial functions and then evaluate the new function.

## EXAMPLE

Let  $f(x) = 2x^3 - 5x + 3$  and  $h(x) = x - 5$

Evaluate:  $(f + h)(2)$

Show that you get the same result by

- 1) Evaluating the functions first then performing the indicated operation on the result.
- 2) Performing the operation on the functions first then evaluating the result.

### Answer

1) First, we will evaluate the functions separately:

$$f(2) = 2(2)^3 - 5(2) + 3 = 16 - 10 + 3 = 9$$

$$h(2) = (2) - 5 = -3$$

Now we will perform the indicated operation using the results:

$$(f + h)(2) = f(2) + h(2) = 9 + (-3) = 6$$

2) We can get the same result by adding the functions first and then evaluating the result at  $x = 2$ .

$$(f + h)(x) = f(x) + h(x) = 2x^3 - 4x - 2 \text{ from the previous example.}$$

Now we can evaluate this result at  $x = 2$

$$(f + h)(2) = 2(2)^3 - 4(2) - 2 = 16 - 8 - 2 = 6$$

Both methods give the same result, and both require about the same amount of work.

## EXAMPLE

Let  $g(t) = 2t^3 - t^2 + 7$  and  $f(t) = 5t^2 - 3$

Find  $(g \cdot f)(t)$ , and evaluate  $(g \cdot f)(-1)$

### Answer

$$\begin{aligned}(g \cdot f)(t) &= (2t^3 - t^2 + 7)(5t^2 - 3) \\ &= (2t^3 \cdot (5t^2) - t^2 \cdot (5t^2) + 7 \cdot (5t^2)) \\ &\quad + (2t^3 \cdot (-3) - t^2 \cdot (-3) + 7 \cdot (-3)) \\ &\quad \text{apply the distributive property} \\ &= (10t^5 - 5t^4 + 35t^2) + (-6t^3 + 3t^2 - 21) \\ &\quad \text{simplify} \\ &= 10t^5 - 5t^4 - 6t^3 + 38t^2 - 21 \\ &\quad \text{combine like terms}\end{aligned}$$

Evaluate  $(g \cdot f)(-1)$

$$(g \cdot f)(t) = 10t^5 - 5t^4 - 6t^3 + 38t^2 - 21$$

$$(g \cdot f)(-1) = 10(-1)^5 - 5(-1)^4 - 6(-1)^3 + 38(-1)^2 - 21 = -10 - 5 + 6 + 38 - 21 = 8$$

# PUTTING IT TOGETHER: POLYNOMIALS AND POLYNOMIAL FUNCTIONS

## EXAMPLE

Given  $p(x) = 2x^2 + x - 15$  and  $r(x) = x + 3$

Find  $\frac{p}{r}(x)$  and evaluate  $\frac{p}{r}(2)$

### Answer

We can use synthetic division for this polynomial division since the coefficient on  $r(x) = x + 3$  is 1.

$$\begin{array}{r|rrr} -3 & 2 & 1 & -15 \\ & & -6 & 15 \\ \hline & 2 & -5 & 0 \end{array}$$

This result means that  $\frac{p}{r}(x) = \frac{2x^2+x-15}{x+3} = 2x - 5$

Now evaluate this quotient for  $x = -3$  both ways as we did in a previous example.

First, we will evaluate the result after polynomial division:

$$\frac{p}{r}(x) = 2x - 5$$

$$\frac{p}{r}(2) = 2(2) - 5 = 4 - 5 = -1$$

Next, we will evaluate each function for  $x = 2$ , then we will divide the results.

$$p(2) = 2(2)^2 + (2) - 15 = 8 + 2 - 15 = -5$$

$$r(2) = (2) + 3 = 5$$

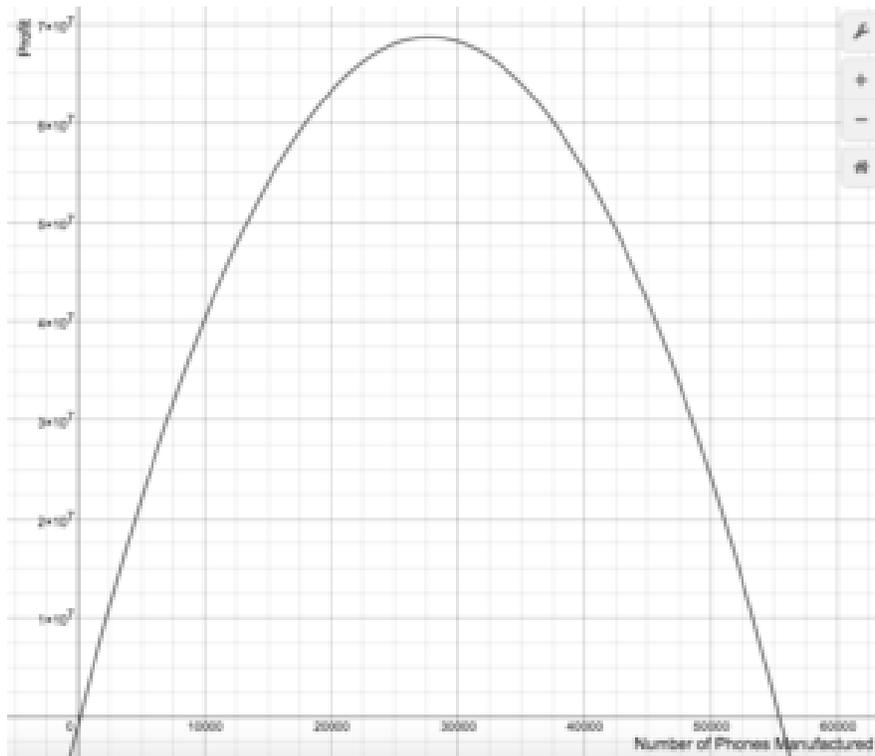
Divide the results:

$$\frac{p}{r}(2) = \frac{-5}{5} = -1$$

We began this module by considering how polynomials can be used by computers to create digital images (remember the dolphin?). Let us close with a different example of polynomials in real life, the profits of a cell phone company. In the following polynomial function, the variable  $x$  represents the number of phones manufactured.

$$P(x) = -0.09x^2 + 5000x - 750,000$$

Using an online graphing calculator, we generated the graph of the profit equation as seen below.



The  $x$ -axis represents the number of phones manufactured, and the  $y$ -axis represents the profit earned by the company. Because the company must invest money into machinery and workforce to manufacture phones, when no phones have been made, the profit for the company is negative. In other words, the point  $(0, -750,000)$  represents no phones made and \$750,000 invested.

The top of the curve represents the “sweet spot” that many business would like to achieve between making and spending money. Using some tools from later math classes, we can find that the maximum amount of profit that can be gained from producing 27,777 phones.

Polynomials have many practical uses and have been studied widely by mathematicians for many years. Our vast knowledge of how they behave mixed with high speed computing allows us to continue to find even more uses for them.

# MODULE 9: FACTORING

## WHY IT MATTERS: FACTORING

### Why Learn How to Factor?

There are hard ways to do things and there are easy ways to do things. For example, you can free climb the brick wall of your apartment building and crawl into the third floor window, or you could take the stairs. Easier still, you could use the elevator. Math also offers efficient ways to accomplish a task.

Remember Joan? Joan has a friend, Nate, who is never far from his phone or a computer. Nate posts numerous pictures, videos, memes, quips, and links on his various social media accounts daily. Joan follows Nate on Instagram and has lately been annoyed with the sheer number of his posts. She wants to let him know that he is probably annoying others, too, but she does not want to hurt his feelings.

Since Joan is studying polynomial functions in her math class, she comes up with a plan. She will appeal to Nate's data-savvy, technical side by proposing that you can fall out of favor with your Instagram followers by posting too many times in one day.

She sees the following polynomial function in her math homework:

$$L(x) = -x^2 + 4x$$

In this function, Joan decides that  $x$  represents the number of pictures posted on Instagram each day by her friend Nate.  $L(x)$ , then, represents the number of likes or comments his pictures on Instagram receive based on the number posted. If he does not post any, obviously no one will like his pictures.

Instagram

If Nate posts too many, people will get bored of his spam and ignore him. Joan wants to know how much is too much so Nate can stay popular on Instagram and not become an annoying spammer. By solving this polynomial function for zero, Joan hopes to find out exactly how many is too many posts for Nate. This translates to, when will he get zero likes or comments on his Instagram pictures?



Joan is not sure how to solve this kind of equation, so she starts guessing values for  $x$ .

Her first guess is 10 since Nate posts more than that.

$$L(10) = -10^2 + 4(10) = -100 + 40 = -60$$

Whoa, if Nate posts 10 pictures on Instagram each day, he will get -60 likes or comments. Joan assumes negative numbers are not a good thing.

Next, she guesses 5.

$$L(5) = -5^2 + 4(5) = -25 + 20 = -5$$

OK, Joan is getting closer.

Then Joan asks herself, why am I guessing the solution to a math problem when I can ask my math teacher how to solve it?

The point is that there are hard ways to do things, and there are easier ways to do things. The hard way to solve a quadratic equation such as  $0 = -x^2 + 4x$  is to guess. An easier way is to factor.

In this lesson, we will learn different techniques for factoring a wide range of polynomials. Keep in mind as you move through this module that it may seem meaningless to learn how to factor, but it is far more efficient than guessing!

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## INTRODUCTION TO FACTORING TRINOMIALS

In this section, we will introduce factoring methods that will allow you to factor trinomials which are algebraic expressions with, you guessed it, three terms.

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## THE GREATEST COMMON FACTOR

### LEARNING OUTCOMES

- Identify the greatest common factor of a polynomial

**Factors** are the building blocks of multiplication. They are the numbers that you can multiply together to produce another number: 2 and 10 are factors of 20, as are 4, 5, 1, 20. To factor a number is to rewrite it as a product.  $20 = 4 \cdot 5$  or  $20 = 1 \cdot 20$ . In algebra, we use the word factor as both a noun – something being multiplied – and as a verb – the action of rewriting a sum or difference as a product. **Factoring** is very helpful in simplifying expressions and solving equations involving polynomials.

The **greatest common factor** (GCF) of two numbers is the largest number that divides evenly into both numbers. For instance, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20. The GCF of polynomials works the same way:  $4x$  is the GCF of  $16x$  and  $20x^2$  because it is the largest polynomial that divides evenly into both  $16x$  and  $20x^2$ .

When factoring a polynomial expression, our first step should be to check for a GCF. Look for the GCF of the coefficients, and then look for the GCF of the variables.

## GREATEST COMMON FACTOR

The **greatest common factor** (GCF) of a group of given polynomials is the largest polynomial that divides evenly into the polynomials.

### Example

Find the greatest common factor of  $25b^3$  and  $10b^2$ .

**Answer**

$$25b^3 = 5 \cdot 5 \cdot b \cdot b \cdot b$$

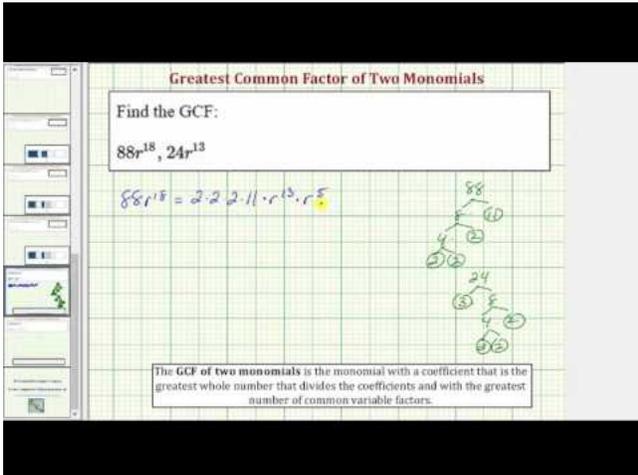
$$10b^2 = 5 \cdot 2 \cdot b \cdot b$$

$$\text{GCF} = 5 \cdot b \cdot b$$

$$\text{GCF} = 5b^2$$

In the example above, the monomials have the factors 5,  $b$ , and  $b$  in common, which means their greatest common factor is  $5 \cdot b \cdot b$ , or simply  $5b^2$ .

The video that follows gives an example of finding the greatest common factor of two monomials with only one variable.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2714>

Sometimes you may encounter a polynomial with more than one variable, so it is important to check whether both variables are part of the GCF. In the next example, we find the GCF of two terms which both contain two variables.

### Example

Find the greatest common factor of  $81c^3d$  and  $45c^2d^2$ .

**Answer**

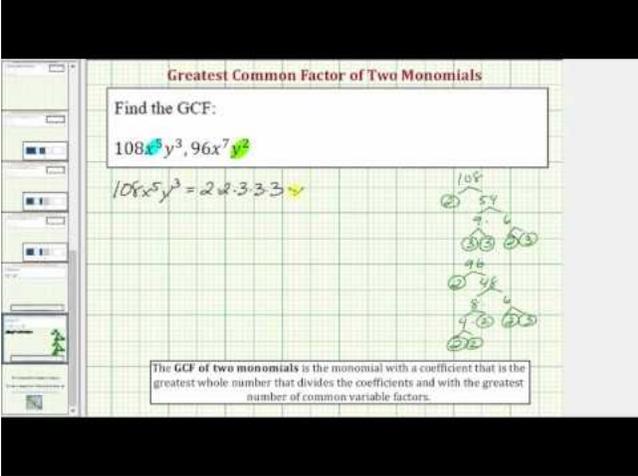
$$81c^3d = 3 \cdot 3 \cdot 3 \cdot 3 \cdot c \cdot c \cdot c \cdot d$$

$$45c^2d^2 = 3 \cdot 3 \cdot 5 \cdot c \cdot c \cdot d \cdot d$$

$$\text{GCF} = 3 \cdot 3 \cdot c \cdot c \cdot d$$

$$\text{GCF} = 9c^2d$$

The video that follows shows another example of finding the greatest common factor of two monomials with more than one variable.



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Now that you have practiced identifying the GCF of terms with one and two variables, we can apply this idea to factoring the GCF out of a polynomial. Notice that the instructions are now “Factor” instead of “Find the greatest common factor.”

To factor a polynomial, first identify the greatest common factor of the terms. You can then use the distributive property to rewrite the polynomial in factored form. Recall that the **distributive property of multiplication over addition** states that a product of a number and a sum is the same as the sum of the products.

## DISTRIBUTIVE PROPERTY FORWARD AND BACKWARD

Forward: Product of a number and a sum:  $a(b + c) = a \cdot b + a \cdot c$ . You can say that “ $a$  is being distributed over  $b + c$ .”

Backward: Sum of the products:  $a \cdot b + a \cdot c = a(b + c)$ . Here you can say that “ $a$  is being factored out.”

We first learned that we could distribute a factor over a sum or difference, now we are learning that we can “undo” the distributive property with factoring.

### Example

Factor  $25b^3 + 10b^2$ .

**Answer**

Find the GCF. From a previous example, you found the GCF of  $25b^3$  and  $10b^2$  to be  $5b^2$ .

$$25b^3 = 5 \cdot 5 \cdot b \cdot b \cdot b$$

$$10b^2 = 5 \cdot 2 \cdot b \cdot b$$

$$\text{GCF} = 5 \cdot b \cdot b = 5b^2$$

Rewrite each term with the GCF as one factor.

$$25b^3 = 5b^2 \cdot 5b$$

$$10b^2 = 5b^2 \cdot 2$$

Rewrite the polynomial using the factored terms in place of the original terms.

$$5b^2 (5b) + 5b^2 (2)$$

Factor out the  $5b^2$ .

$$5b^2 (5b + 2)$$

The factored form of the polynomial  $25b^3 + 10b^2$  is  $5b^2 (5b + 2)$ . You can check this by doing the multiplication.  $5b^2 (5b + 2) = 25b^3 + 10b^2$ .

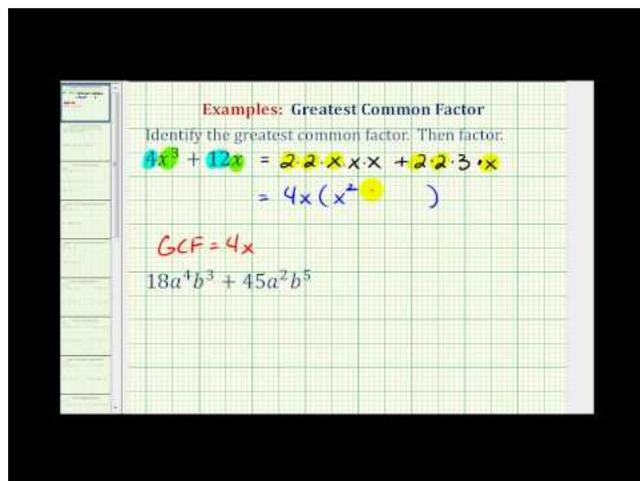
Note that if you do not factor the greatest common factor at first, you can continue factoring, rather than start all over.

For example:

$$\begin{aligned} 25b^3 + 10b^2 &= 5 (5b^3 + 2b^2) && \text{Factor out 5} \\ &= 5b^2 (5b + 2) && \text{Factor out } b^2 \end{aligned}$$

Notice that you arrive at the same simplified form whether you factor out the GCF immediately or if you pull out factors individually.

In the following video, we show two more examples of how to find and factor the GCF from binomials.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2714>

We will show one last example of finding the GCF of a polynomial with several terms and two variables. No matter how large the polynomial, you can use the same technique described below to factor out its GCF.

## HOW TO: GIVEN A POLYNOMIAL EXPRESSION, FACTOR OUT THE GREATEST COMMON FACTOR

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Write together to find the GCF of the expression.

- Determine what the GCF needs to be multiplied by to obtain each term in the expression.
- Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

## EXAMPLE

Factor  $6x^3y^3 + 45x^2y^2 + 21xy$ .

### Answer

First, find the GCF of the expression. The GCF of 6, 45, and 21 is 3. The GCF of  $x^3$ ,  $x^2$ , and  $x$  is  $x$ . (Note that the GCF of a set of expressions in the form  $x^n$  will always be the exponent of lowest degree.) And the GCF of  $y^3$ ,  $y^2$ , and  $y$  is  $y$ . Put these together to find the GCF of the polynomial,  $3xy$ .

Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that  $3xy(2x^2y^2) = 6x^3y^3$ ,  $3xy(15xy) = 45x^2y^2$ , and  $3xy(7) = 21xy$ .

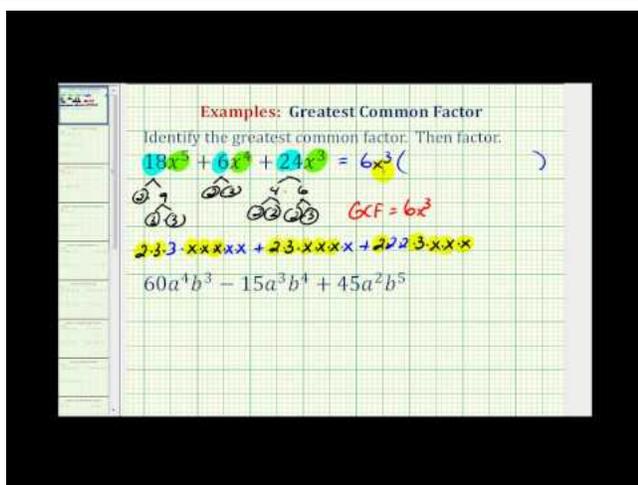
Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by

$$(3xy)(2x^2y^2 + 15xy + 7)$$

After factoring, we can check our work by multiplying. Use the distributive property to confirm that

$$(3xy)(2x^2y^2 + 15xy + 7) = 6x^3y^3 + 45x^2y^2 + 21xy.$$

In the following video, you will see two more examples of how to find and factor out the greatest common factor of a polynomial.



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# FACTOR A TRINOMIAL WITH LEADING COEFFICIENT = 1

## LEARNING OUTCOMES

- Factor a trinomial with leading coefficient = 1

Trinomials are polynomials with three terms. We are going to show you a method for factoring a trinomial whose leading coefficient is 1. Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that trinomials can be factored. The trinomial  $x^2 + 5x + 6$  has a GCF of 1, but it can be written as the product of the factors  $(x + 2)$  and  $(x + 3)$ .

Recall how to use the distributive property to multiply two binomials:

$$(x + 2)(x + 3) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6$$

We can reverse the distributive property and return  $x^2 + 5x + 6$  to  $(x + 2)(x + 3)$  by finding two numbers with a product of 6 and a sum of 5.

## FACTORING A TRINOMIAL WITH LEADING COEFFICIENT 1

In general, for a trinomial of the form  $x^2 + bx + c$ , you can factor a trinomial with leading coefficient 1 by finding two numbers,  $p$  and  $q$  whose product is  $c$  and whose sum is  $b$ .

Let us put this idea to practice with the following example.

## EXAMPLE

Factor  $x^2 + 2x - 15$ .

### Answer

We have a trinomial with leading coefficient 1,  $b = 2$ , and  $c = -15$ . We need to find two numbers with a product of  $-15$  and a sum of 2. In the table, we list factors until we find a pair with the desired sum.

Factors of $-15$	Sum of Factors
1, $-15$	$-14$
$-1$ , 15	14
3, $-5$	$-2$
$-3$ , 5	2

Now that we have identified  $p$  and  $q$  as  $-3$  and 5, write the factored form as  $(x - 3)(x + 5)$ .

In the following video, we present two more examples of factoring a trinomial with a leading coefficient of 1.

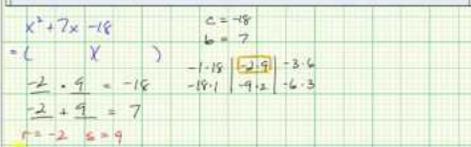
To summarize our process, consider the following steps:

**Shortcut Method for Factoring Trinomials:  $x^2 + bx + c$**

**EXAMPLE**

Factor each trinomial.

$x^2 + 7x - 18$                        $y^2 - 11y + 24$



Factoring Trinomials in the Form:  $x^2 + bx + c$   
 Find the two integers,  $r$  and  $s$ , whose product is  $c$  and whose sum is  $b$ .  
 $r \cdot s = c$   
 and  
 $r + s = b$   
 $x^2 + bx + c = (x + r)(x + s) = (x + s)(x + r)$

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## HOW TO: GIVEN A TRINOMIAL IN THE FORM $x^2 + bx + c$ , FACTOR IT

1. List factors of  $c$ .
2. Find  $p$  and  $q$ , a pair of factors of  $c$  with a sum of  $b$ .
3. Write the factored expression  $(x + p)(x + q)$ .

We will now show an example where the trinomial has a negative  $c$  term. Pay attention to the signs of the numbers that are considered for  $p$  and  $q$ .

In our next example, we show that when  $c$  is negative, either  $p$  or  $q$  will be negative.

In our last example, we will show how to factor a trinomial whose  $b$  term is negative.

In the last example, the  $b$  term was negative and the  $c$  term was positive. This will always mean that if it can be factored,  $p$  and  $q$  will both be negative.

## EXAMPLE

Factor  $x^2 + x - 12$ .

### Answer

Consider all the combinations of numbers whose product is  $-12$  and list their sum.

Factors whose product is $-12$	Sum of the factors
$1 \cdot -12 = -12$	$1 + -12 = -11$
$2 \cdot -6 = -12$	$2 + -6 = -4$
$3 \cdot -4 = -12$	$3 + -4 = -1$
$4 \cdot -3 = -12$	$4 + -3 = 1$
$6 \cdot -2 = -12$	$6 + -2 = 4$
$12 \cdot -1 = -12$	$12 + -1 = 11$

Choose the values whose sum is  $+1$ :  $p = 4$  and  $q = -3$ , and place them into a product of binomials.

$$(x + 4)(x - 3)$$

## THINK ABOUT IT

Which property of multiplication can be used to describe why  $(x + 4)(x - 3) = (x - 3)(x + 4)$ . Use the textbox below to write down your ideas before you look at the answer.

### Answer

The **commutative property of multiplication** states that factors may be multiplied in any order without affecting the product.

$$a \cdot b = b \cdot a$$

## EXAMPLE

Factor  $x^2 - 7x + 6$ .

### Answer

List the factors of 6. Note that the b term is negative, so we will need to consider negative numbers in our list.

Factors of 6	Sum of Factors
1, 6	7
2, 3	5
-1, -6	-7
-2, -3	-5

Choose the pair that sum to  $-7$ , which is  $-1, -6$

Write the pair as constant terms in a product of binomials.

$$(x - 1)(x - 6)$$

## THINK ABOUT IT

Can every trinomial be factored as a product of binomials?

Mathematicians often use a counterexample to prove or disprove a question. A counterexample means you provide an example where a proposed rule or definition is not true. Can you create a trinomial with leading coefficient 1 that *cannot* be factored as a product of binomials?

Use the textbox below to write your ideas.

### Answer

Can every trinomial be factored as a product of binomials? No. Some polynomials cannot be factored. These polynomials are said to be prime.

A counterexample would be:  $x^2 + 3x + 7$

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## FACTOR BY GROUPING

### LEARNING OUTCOME

- Factor a trinomial with leading coefficient  $\neq 1$

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can **factor by grouping** by dividing the  $x$  term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial  $2x^2 + 5x + 3$  can be rewritten as  $(2x + 3)(x + 1)$  using this process. We begin by rewriting the original expression as  $2x^2 + 2x + 3x + 3$  and then factor each portion of the expression to obtain  $2x(x + 1) + 3(x + 1)$ . We then pull out the GCF of  $(x + 1)$  to find the factored expression.

The first step in this process is to figure out what two numbers to use to re-write the  $x$  term as the sum of two new terms. Making a table to keep track of your work is helpful. We are looking for two numbers with a product of 6 and a sum of 5

Factors of $2 \cdot 3 = 6$	Sum of Factors
1, 6	7
-1, -6	-7
2, 3	5
-2, -3	-5

The pair  $p = 2$  and  $q = 3$  will give the correct  $x$  term, so we will rewrite it using the new factors:

$$2x^2 + 5x + 3 = 2x^2 + 2x + 3x + 3$$

Now we can group the polynomial into two binomials.

$$2x^2 + 2x + 3x + 3 = (2x^2 + 2x) + (3x + 3)$$

Identify the GCF of each binomial.

$2x$  is the GCF of  $(2x^2 + 2x)$  and 3 is the GCF of  $(3x + 3)$ . Use this to rewrite the polynomial:

$$(2x^2 + 2x) + (3x + 3) = 2x(x + 1) + 3(x + 1)$$

Note how we leave the signs in the binomials and the addition that joins them. Be careful with signs when you factor out the GCF. The GCF of our new polynomial is  $(x + 1)$ . We factor this out as well:

$$2x(x + 1) + 3(x + 1) = (x + 1)(2x + 3).$$

Sometimes it helps visually to write the polynomial as  $(x + 1)2x + (x + 1)3$  before you factor out the GCF. This is purely a matter of preference. Multiplication is commutative, so order does not matter.

### A GENERAL NOTE: FACTOR BY GROUPING

To factor a trinomial in the form  $ax^2 + bx + c$  by grouping, we find two numbers with a product of  $ac$  and a sum of  $b$ . We use these numbers to divide the  $x$  term into the sum of two terms and factor each portion of the expression separately. Then we factor out the GCF of the entire expression.

We can summarize our process in the following way:

In the following video, we present one more example of factoring a trinomial whose leading coefficient is not 1 using the grouping method.

Factoring trinomials whose leading coefficient is not 1 becomes quick and kind of fun once you get the idea. Give the next example a try on your own before you look at the solution.

## EXAMPLE

Factor  $5x^2 + 7x - 6$  by grouping.

### Answer

We have a trinomial with  $a = 5$ ,  $b = 7$ , and  $c = -6$ . First, determine  $ac = -30$ . We need to find two numbers with a product of  $-30$  and a sum of  $7$ . In the table, we list factors until we find a pair with the desired sum.

Factors of $-30$	Sum of Factors
1, $-30$	$-29$
$-1, 30$	$29$
2, $-15$	$-13$
$-2, 15$	$13$
3, $-10$	$-7$
$-3, 10$	$7$

So  $p = -3$  and  $q = 10$ .

$$5x^2 - 3x + 10x - 6 \quad \text{Rewrite the original expression as } ax^2 + px + qx + c$$

$$x(5x - 3) + 2(5x - 3) \quad \text{Factor out the GCF of each part}$$

$$(5x - 3)(x + 2) \quad \text{Factor out the GCF of the expression}$$

We can check our work by multiplying. Use FOIL to confirm that  $(5x - 3)(x + 2) = 5x^2 + 7x - 6$ .

## HOW TO: GIVEN A TRINOMIAL IN THE FORM $ax^2 + bx + c$ , FACTOR BY GROUPING

1. List factors of  $ac$ .
2. Find  $p$  and  $q$ , a pair of factors of  $ac$  with a sum of  $b$ .
3. Rewrite the original expression as  $ax^2 + px + qx + c$ .
4. Pull out the GCF of  $ax^2 + px$ .
5. Pull out the GCF of  $qx + c$ .
6. Factor out the GCF of the expression.

We will show two more examples so you can become acquainted with the variety of possible outcomes for factoring this type of trinomial.

Here is an example where the  $x$  term is positive and  $c$  is negative.

In the following video example, we will factor a trinomial whose leading term is negative.

For our last example, you will see that sometimes, you will encounter polynomials that, despite your best efforts, cannot be factored into the product of two binomials.

**Factoring Trinomials in the  $ax^2 + bx + c$  by Grouping**

**EXAMPLE**

Factor the trinomial.

$6y^2 - 13y + 5$

$6y^2 - 13y + 5$

$a = 6 \quad b = -13 \quad c = 5$

$-3 \cdot 10 = 30$

$-2 \cdot -15 = 30$

$-3 + -10 = -13$

$-2 + 15 = 13$

$r = -3 \quad s = 10$

Factoring Trinomials in the Form:  $ax^2 + bx + c$

Find the two integers,  $r$  and  $s$ , whose product is  $a \cdot c$  and whose sum is  $b$ .

$r \cdot s = a \cdot c$

and

$r + s = b$

Rewrite the trinomial in the form  $ax^2 + bx + c = ax^2 + rx + sx + c$  and factor by grouping.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2723>

## EXAMPLE

Factor  $2x^2 + 9x + 9$ .

**Answer**

Find two numbers  $p, q$  such that  $p \cdot q = 18$  and  $p + q = 9$ .

9 and 18 are both positive, so we will only consider positive factors.

Factors of $2 \cdot 9$ $= 18$	Sum of Factors
1, 18	19
3, 6	9

We can stop because we have found our factors.

Rewrite the original expression and group.

$$2x^2 + 3x + 6x + 9 = (2x^2 + 3x) + (6x + 9)$$

Factor out the GCF of each binomial and write as a product of two binomials:

$$(2x^2 + 3x) + (6x + 9) = x(2x + 3) + 3(2x + 3) = (x + 3)(2x + 3)$$

$$2x^2 + 9x + 9 = (x + 3)(2x + 3)$$

## EXAMPLE

Factor  $6x^2 + x - 1$ .

**Answer**

Factors of $6 \cdot -1 = -6$	Sum of Factors
-1, 6	5
1, -6	-5
-2, 3	1

We can stop because we have found our factors.

Rewrite the original expression and group.

$$6x^2 + x - 1 = 6x^2 - 2x + 3x - 1$$

Factor out the GCF of each binomial and write as a product of two binomials:

$$(6x^2 - 2x) + (3x - 1) = 2x(3x - 1) + 1(3x - 1) = (2x + 1)(3x - 1)$$

$$6x^2 + x - 1 = (2x + 1)(3x - 1)$$

**Factoring Trinomials in the  $ax^2 + bx + c$  by Grouping**

**EXAMPLE**

Factor the trinomial.

$$-4x^2 + 8x + 21$$

$= -1[4x^2 - 8x - 21]$

$a = 4$	$b = -8$	$c = -21$	$-1 \cdot 84$	$-3 \cdot 28$	$-6 \cdot 14$
$\cdot$	$= -84$		$-44 \cdot 1$	$-28 \cdot 3$	$-14 \cdot 6$
$-$	$= -8$		$-2 \cdot 42$	$-4 \cdot 21$	$-12 \cdot 7$
$-$	$= -8$		$-42 \cdot 2$	$-21 \cdot 4$	$-7 \cdot 12$
					$84 = 2 \cdot 2 \cdot 3 \cdot 7$

Factoring Trinomials in the Form:  $ax^2 + bx + c$   
 Find the two integers,  $r$  and  $s$ , whose product is  $a \cdot c$  and whose sum is  $b$ .  
 $r \cdot s = a \cdot c$   
 and  
 $r + s = b$   
 Rewrite the trinomial in the form  $ax^2 + bx + c = ax^2 + rx + sx + c$  and factor by grouping.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2723>

### Example

Factor  $7x^2 - 16x - 5$ .

**Answer**

Find  $p, q$  such that  $p \cdot q = -35$  and  $p + q = -16$

Factors of $7 \cdot -5 =$ $-35$	Sum of Factors
$-1, 35$	34
$1, -35$	-34
$-5, 7$	2
$-7, 5$	-2

The trinomial cannot be factored. None of the factors add up to  $-16$

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# INTRODUCTION TO SPECIAL CASES

## Why Learn How to Factor Special Cases?



Some people like to find patterns in the world around them, like a game. There are some polynomials that, when factored, follow a specific pattern.

These include:

Perfect square trinomials of the form:  $a^2 + 2ab + b^2$

A difference of squares:  $a^2 - b^2$

A sum of cubes:  $a^3 + b^3$

A difference of cubes:  $a^3 - b^3$

In this lesson you will see you can factor each of these types of polynomials following a specific pattern. You will also learn how to factor polynomials that have negative or fractional exponents.

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## SPECIAL CASES - SQUARES

### LEARNING OUTCOMES

- Factor special products



Some people find it helpful to know when they can take a shortcut to avoid doing extra work. There are some polynomials that will always factor a certain way, and for those, we offer a shortcut. Most people find it helpful to memorize the factored form of a perfect square trinomial or a difference of squares. The most important skill you will use in this section will be recognizing when you can use the shortcuts.

### Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and the square of the last term.

$$a^2 + 2ab + b^2 = (a + b)^2$$

and

$$a^2 - 2ab + b^2 = (a - b)^2$$

We can use these equations to factor any perfect square trinomial.

### A GENERAL NOTE: PERFECT SQUARE TRINOMIALS

A perfect square trinomial can be written as the square of a binomial:

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

In the following example, we will show you how to define a and b so you can use the shortcut.

## EXAMPLE

Factor  $25x^2 + 20x + 4$ .

### Answer

First, notice that  $25x^2$  and 4 are perfect squares because  $25x^2 = (5x)^2$  and  $4 = 2^2$ .

This means that  $a = 5x$  and  $b = 2$

Next, check to see if the middle term is equal to  $2ab$ , which it is:

$$2ab = 2(5x)(2) = 20x$$

Therefore, the trinomial is a perfect square trinomial and can be written as  $(a + b)^2 = (5x + 2)^2$ .

In the next example, we will show that we can use  $1 = 1^2$  to factor a polynomial with a term equal to 1.

## EXAMPLE

Factor  $49x^2 - 14x + 1$ .

### Answer

First, notice that  $49x^2$  and 1 are perfect squares because  $49x^2 = (7x)^2$  and  $1 = 1^2$ .

This means that  $a = 7x$  and  $b = 1$ .

Next, check to see if the middle term is equal to  $2ab$ , which it is:

$$2ab = 2(7x)(1) = 14x$$

Therefore, the trinomial is a perfect square trinomial and can be written as  $(a - b)^2 = (7x - 1)^2$ .

In the following video, we provide another short description of what a perfect square trinomial is and show how to factor them using a formula.

The screenshot shows a worksheet titled "Perfect Square Trinomials" with a section for "EXERCISES". The exercises are:

- Factor:  $36x^2 + 60x + 25$
- $49x^2 - 28x + 4$

Formulas provided are:

$$a^2 + 2ab + b^2 = (a + b)^2$$

and

$$a^2 - 2ab + b^2 = (a - b)^2$$

Handwritten solutions for exercise 1:

$$1. 36x^2 + 60x + 25 = (6x)^2 + 60x + (5)^2$$
$$a = 6x \quad b = 5$$
$$2ab = 2(6x)(5) = 60x$$
$$(6x + 5)^2 = (6x + 5)(6x + 5)$$
$$= 36x^2 + 30x + 30x + 25$$

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We can summarize our process in the following way:

## HOW TO: GIVEN A PERFECT SQUARE TRINOMIAL, FACTOR IT INTO THE SQUARE OF A BINOMIAL

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of  $ab$ .
3. Write the factored form as  $(a + b)^2$  or  $(a - b)^2$ .

## Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.

$$a^2 - b^2 = (a + b)(a - b)$$

We can use this equation to factor any differences of squares.

### A GENERAL NOTE: DIFFERENCES OF SQUARES

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$a^2 - b^2 = (a + b)(a - b)$$

### EXAMPLE

Factor  $9x^2 - 25$ .

**Answer**

Notice that  $9x^2$  and 25 are perfect squares because  $9x^2 = (3x)^2$  and  $25 = 5^2$ .

This means that  $a = 3x$ , and  $b = 5$

The polynomial represents a difference of squares and can be rewritten as  $(3x + 5)(3x - 5)$ .

Check that you are correct by multiplying.

$$(3x + 5)(3x - 5) = 9x^2 - 15x + 15x - 25 = 9x^2 - 25$$

The most helpful thing for recognizing a difference of squares that can be factored with the shortcut is knowing which numbers are perfect squares, as you will see in the next example.

### EXAMPLE

Factor  $81y^2 - 144$ .

**Answer**

Notice that  $81y^2$  and 144 are perfect squares because  $81y^2 = (9y)^2$  and  $144 = 12^2$ .

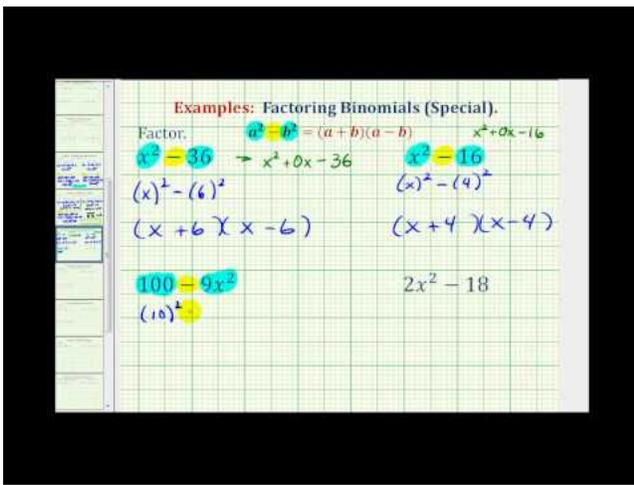
This means that  $a = 9y$ , and  $b = 12$

The polynomial represents a difference of squares and can be rewritten as  $(9y + 12)(9y - 12)$ .

Check that you are correct by multiplying.

$$(9y + 12)(9y - 12) = 81y^2 - 108y + 108y - 144 = 81y^2 - 144$$

In the following video, we show another example of how to use the formula for factoring a difference of squares.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2726>

We can summarize the process for factoring a difference of squares with the shortcut this way:

## HOW TO: GIVEN A DIFFERENCE OF SQUARES, FACTOR IT INTO BINOMIALS

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as  $(a + b)(a - b)$ .

## THINK ABOUT IT

Is there a formula to factor the sum of squares,  $a^2 + b^2$ , into a product of two binomials?

Write down some ideas for how you would answer this in the box below before you look at the answer.

### Answer

There is no way to factor a sum of squares into a product of two binomials. This is because of addition – the middle term needs to “disappear” and the only way to do that is with opposite signs. To get a positive result, you must multiply two numbers with the same signs.

The only time a sum of squares can be factored is if they share any common factors, as in the following case:

$$9x^2 + 36$$

The only way to factor this expression is by pulling out the GCF which is 9.

$$9x^2 + 36 = 9(x^2 + 4)$$

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# SPECIAL CASES - CUBES

## Learning Outcomes

- Factor special products

Some interesting patterns arise when you are working with cubed quantities within polynomials. Specifically, there are two more special cases to consider:  $a^3 + b^3$  and  $a^3 - b^3$ .

Let us take a look at how to factor sums and differences of cubes.

## Sum of Cubes

The term “cubed” is used to describe a number raised to the third power. In geometry, a cube is a six-sided shape with equal width, length, and height; since all these measures are equal, the volume of a cube with width  $x$  can be represented by  $x^3$ . (Notice the exponent!)

Cubed numbers get large very quickly:  $1^3 = 1$ ,  $2^3 = 8$ ,  $3^3 = 27$ ,  $4^3 = 64$ , and  $5^3 = 125$

Before looking at factoring a sum of two cubes, let us look at the possible factors.

It turns out that  $a^3 + b^3$  can actually be factored as  $(a + b)(a^2 - ab + b^2)$ . Check these factors by multiplying.

## Example

Does  $(a + b)(a^2 - ab + b^2) = a^3 + b^3$ ?

**Answer**

Apply the distributive property.

$$(a)(a^2 - ab + b^2) + (b)(a^2 - ab + b^2)$$

Multiply by  $a$ .

$$(a^3 - a^2b + ab^2) + (b)(a^2 - ab + b^2)$$

Multiply by  $b$ .

$$(a^3 - a^2b + ab^2) + (a^2b - ab^2 + b^3)$$

Rearrange terms in order to combine the like terms.

$$a^3 - a^2b + a^2b + ab^2 - ab^2 + b^3$$

Simplify.

$$a^3 + b^3$$

Did you see that? Four of the terms cancelled out, leaving us with the (seemingly) simple binomial  $a^3 + b^3$ . So, the factors are correct.

You can use this pattern to factor binomials in the form  $a^3 + b^3$ , otherwise known as “the sum of cubes.”

## THE SUM OF CUBES

A binomial in the form  $a^3 + b^3$  can be factored as  $(a + b)(a^2 - ab + b^2)$ .

### EXAMPLES

The factored form of  $x^3 + 64$  is  $(x + 4)(x^2 - 4x + 16)$ .

The factored form of  $8x^3 + y^3$  is  $(2x + y)(4x^2 - 2xy + y^2)$ .

#### Example

Factor  $x^3 + 8y^3$ .

**Answer**

Identify that this binomial fits the sum of cubes pattern  $a^3 + b^3$ .

$a = x$ , and  $b = 2y$  (since  $2y \cdot 2y \cdot 2y = 8y^3$ ).

$$x^3 + 8y^3$$

Factor the binomial as  $(a + b)(a^2 - ab + b^2)$ , substituting  $a = x$  and  $b = 2y$  into the expression.

$$(x + 2y)(x^2 - x(2y) + (2y)^2)$$

Square  $(2y)^2 = 4y^2$ .

$$(x + 2y)(x^2 - x(2y) + 4y^2)$$

Multiply  $-x(2y) = -2xy$  (writing the coefficient first).

The factored form is  $(x + 2y)(x^2 - 2xy + 4y^2)$ .

Now try another one.

You should always look for a common factor before you follow any of the patterns for factoring.

#### Example

Factor  $16m^3 + 54n^3$ .

**Answer**

Factor out the common factor 2.

$$16m^3 + 54n^3$$

$$2(8m^3 + 27n^3)$$

$8m^3$  and  $27n^3$  are cubes, so you can factor  $8m^3 + 27n^3$  as the sum of two cubes:  $a = 2m$  and  $b = 3n$ .

Factor the binomial  $8m^3 + 27n^3$  substituting  $a = 2m$  and  $b = 3n$  into the expression  $(a + b)(a^2 - ab + b^2)$ .

$$2(2m + 3n) \left[ (2m)^2 - (2m)(3n) + (3n)^2 \right]$$

Square:  $(2m)^2 = 4m^2$  and  $(3n)^2 = 9n^2$ .

$2(2m + 3n) [4m^2 - (2m)(3n) + 9n^2]$

Multiply  $-(2m)(3n) = -6mn$ .

The factored form is  $2(2m + 3n)(4m^2 - 6mn + 9n^2)$ .

## Difference of Cubes

Having seen how binomials in the form  $a^3 + b^3$  can be factored, it should not come as a surprise that binomials in the form  $a^3 - b^3$  can be factored in a similar way.

### THE DIFFERENCE OF CUBES

A binomial in the form  $a^3 - b^3$  can be factored as  $(a - b)(a^2 + ab + b^2)$ .

#### EXAMPLES

The factored form of  $x^3 - 64$  is  $(x - 4)(x^2 + 4x + 16)$ .

The factored form of  $27x^3 - 8y^3$  is  $(3x - 2y)(9x^2 + 6xy + 4y^2)$ .

Notice that the basic construction of the factorization is the same as it is for the sum of cubes; the difference is in the + and - signs. Take a moment to compare the factored form of  $a^3 + b^3$  with the factored form of  $a^3 - b^3$ .

Factored form of  $a^3 + b^3$ :  $(a + b)(a^2 - ab + b^2)$

Factored form of  $a^3 - b^3$ :  $(a - b)(a^2 + ab + b^2)$

This can be tricky to remember because of the different signs. The factored form of  $a^3 + b^3$  contains a negative, and the factored form of  $a^3 - b^3$  contains a positive! Some people remember the different forms like this:

“Remember one sequence of variables:  $a^3b^3 = (ab)(a^2abb^2)$ . There are 4 missing signs. Whatever the first sign is, it is also the second sign. The third sign is the opposite, and the fourth sign is always +.”

Try this for yourself. If the first sign is +, as in  $a^3 + b^3$ , according to this strategy, how do you fill in the rest:  $(ab)(a^2abb^2)$ ? Does this method help you remember the factored form of  $a^3 + b^3$  and  $a^3 - b^3$ ?

Let us go ahead and look at a couple of examples. Remember to factor out all common factors first.

#### Example

Factor  $8x^3 - 1,000$ .

**Answer**

Factor out 8.

$8(x^3 - 125)$

Identify that the binomial fits the pattern  $a^3 - b^3$ :  $a = x$ , and  $b = 5$  (since  $5^3 = 125$ ).

Factor  $x^3 - 125$  as  $(a - b)(a^2 + ab + b^2)$ , substituting  $a = x$  and  $b = 5$  into the expression.

$$8(x - 5) [x^2 + (x)(5) + 5^2]$$

Square the first and last terms, and rewrite  $(x)(5)$  as  $5x$ .

$$8(x - 5)(x^2 + 5x + 25)$$

Here is one more example. Note that  $r^9 = (r^3)^3$  and that  $8s^6 = (2s^2)^3$ .

### Example

Factor  $r^9 - 8s^6$ .

**Answer**

Identify this binomial as the difference of two cubes. As shown above, it is.

Rewrite  $r^9$  as  $(r^3)^3$  and rewrite  $8s^6$  as  $(2s^2)^3$ .

$$(r^3)^3 - (2s^2)^3$$

Now the binomial is written in terms of cubed quantities. Thinking of  $a^3 - b^3$ ,  $a = r^3$  and  $b = 2s^2$ .

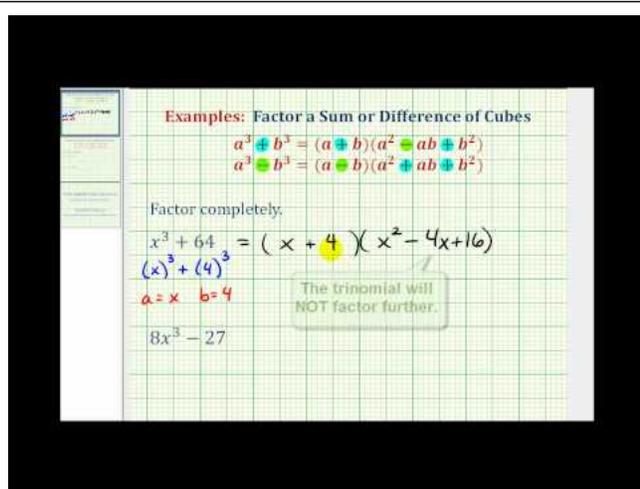
Factor the binomial as  $(a - b)(a^2 + ab + b^2)$ , substituting  $a = r^3$  and  $b = 2s^2$  into the expression.

$$(r^3 - 2s^2) [(r^3)^2 + (r^3)(2s^2) + (2s^2)^2]$$

Multiply and square the terms.

$$(r^3 - 2s^2)(r^6 + 2r^3s^2 + 4s^4)$$

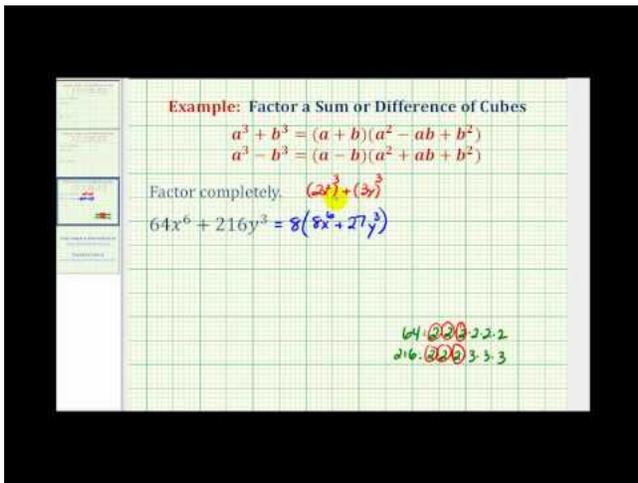
In the following two video examples, we show more binomials that can be factored as a sum or difference of cubes.



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You encounter some interesting patterns when factoring. Two special cases—the sum of cubes and the difference of cubes—can help you factor some binomials that have a degree of three (or higher, in some cases). The special cases are:

- A binomial in the form  $a^3 + b^3$  can be factored as  $(a + b)(a^2 - ab + b^2)$
- A binomial in the form  $a^3 - b^3$  can be factored as  $(a - b)(a^2 + ab + b^2)$



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Always remember to factor out any common factors first.

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# MORE FACTORING METHODS

**Learning Outcomes**

- Factor polynomials with negative or fractional exponents
- Factor by substitution

Expressions with fractional or negative exponents can be factored using the same factoring techniques as those with integer exponents. It is important to remember a couple of things first.

- When you multiply two exponentiated terms with the same base, you can add the exponents:  

$$x^{-1} \cdot x^{-1} = x^{-1+(-1)} = x^{-2}$$
- When you add fractions, you need a common denominator:  $\frac{1}{2} + \frac{1}{3} = \frac{3}{3} \cdot \frac{1}{2} + \frac{2}{2} \cdot \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$
- Polynomials have positive integer exponents – if it has a fractional or negative exponent it is an expression.

First, practice finding a GCF that is a negative exponent.

Now let us factor a trinomial that has negative exponents.

In the next example, we will see a difference of squares with negative exponents. We can use the same shortcut as we have before, but be careful with the exponent.

In the following video, you will see more examples that are similar to the previous three written examples.

## EXAMPLE

Factor  $12y^{-3} - 2y^{-2}$ .

### Answer

If the exponents in this expression were positive, we could determine that the GCF is  $2y^2$ , but since we have negative exponents, we will need to use  $2y^{-3}$ .

Therefore,  $12y^{-3} - 2y^{-2} = 2y^{-3}(6 - y)$

We can check that we are correct by multiplying:

$$2y^{-3}(6 - y) = 12y^{-3} - 2y^{-3+1} = 12y^{-3} - 2y^{-2}$$

## EXAMPLE

Factor  $x^{-2} + 5x^{-1} + 6$ .

### Answer

If the exponents on this trinomial were positive, we could factor this as  $(x + 2)(x + 3)$ . Note that the exponent on the x's in the factored form is 1, in other words  $(x + 2) = (x^1 + 2)$ .

Also note that  $-1 + (-1) = -2$ ; therefore, if we factor this trinomial as  $(x^{-1} + 2)(x^{-1} + 3)$ , we will get the correct result if we check by multiplying.

$$(x^{-1} + 2)(x^{-1} + 3) = x^{-1+(-1)} + 2x^{-1} + 3x^{-1} + 6 = x^{-2} + 5x^{-1} + 6$$

The factored form is  $(x^{-1} + 2)(x^{-1} + 3)$

## EXAMPLE

Factor  $25x^{-4} - 36$ .

### Answer

Recall that a difference of squares factors in this way:  $a^2 - b^2 = (a - b)(a + b)$ , and the first thing we did was identify a and b to see whether we could factor this as a difference of squares.

Given  $25x^{-4} - 36$ , we can define  $a = 5x^{-2}$  and  $b = 6$  because  $(5x^{-2})^2 = 25x^{-4}$  and  $6^2 = 36$

Therefore, the factored form is:  $(5x^{-2} - 6)(5x^{-2} + 6)$

Factoring Expressions with Negative Exponents

EXERCISES

Factor:  $x^a \cdot x^b = x^{a+b}$

- $15x^{-4} - 3x^{-3}$
- $y^{-2} + 6y^{-1} + 8$
- $4x^{-2} - 9$

1.  $15x^{-4} - 3x^{-3} = 3x^{-3}(5x^{-1} - 1)$

2.  $y^{-2} + 6y^{-1} + 8 = (y^{-1} + 4)(y^{-1} + 2)$

y

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# Fractional Exponents

Again, we will first practice finding a GCF that has a fractional exponent.

## EXAMPLE

Factor  $x^{\frac{2}{3}} + 3x^{\frac{1}{3}}$ .

### Answer

First, look for the term with the lowest value exponent. In this case, it is  $3x^{\frac{1}{3}}$ .

Recall that when you multiply terms with exponents, you add the exponents. To get  $\frac{2}{3}$  you would need to add  $\frac{1}{3}$  to  $\frac{1}{3}$ , so we will need a term whose exponent is  $\frac{1}{3}$ .

$x^{\frac{1}{3}} \cdot x^{\frac{1}{3}} = x^{\frac{2}{3}}$ , therefore:

$$x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = x^{\frac{1}{3}}(x^{\frac{1}{3}} + 3)$$

In our next example, we will factor a perfect square trinomial that has fractional exponents.

## EXAMPLE

Factor  $25y^{\frac{1}{2}} + 70x^{\frac{1}{4}} + 49$ .

### Answer

Recall that a perfect square trinomial of the form  $a^2 + 2ab + b^2$  factors as  $(a + b)^2$

The first step in factoring a perfect square trinomial was to identify a and b.

To find a, we ask:  $(?)^2 = 25y^{\frac{1}{2}}$ , and recall that  $(x^a)^b = x^{a \cdot b}$ , therefore we are looking for an exponent for x that when multiplied by 2, will give  $\frac{1}{2}$ . You can also think about the fact that the middle term is defined as  $2ab$  so a will probably have an exponent of  $\frac{1}{4}$ , therefore a choice for a may be  $5x^{\frac{1}{4}}$ .

We can check that this is right by squaring a:  $(5x^{\frac{1}{4}})^2 = 25x^{2 \cdot \frac{1}{4}} = 25x^{\frac{1}{2}}$

$$b = 7 \text{ and } b^2 = 49$$

Now we can check whether  $2ab = 70x^{\frac{1}{4}}$

$$2ab = 2 \cdot 5x^{\frac{1}{4}} \cdot 7 = 70x^{\frac{1}{4}}$$

Our terms work out, so we can use the shortcut to factor:

$$25y^{\frac{1}{2}} + 70x^{\frac{1}{4}} + 49 = (5x^{\frac{1}{4}} + 7)^2$$

In our next video, you will see more examples of how to factor expressions with fractional exponents.

## Factor Using Substitution

We are going to move back to factoring polynomials; our exponents will be positive integers. Sometimes we encounter a polynomial that looks similar to something we know how to factor but is not quite the same. Substitution is a useful tool that can be used to “mask” a term or expression to make algebraic operations easier.

You may recall that substitution can be used to solve systems of linear equations and to check whether a point is a solution to a system of linear equations.

For example, consider the following system of equations:

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$$x + 3y = 8$$

$$2x - 9 = y$$

To determine whether the ordered pair (5, 1) is a solution to the given system of equations, we can substitute the ordered pair (5, 1) into both equations.

$$(5) + 3(1) = 8$$

$$8 = 8 \quad \text{True}$$

$$2(5) - 9 = (1)$$

$$1 = 1 \quad \text{True}$$

We replaced the variable with a number and then performed the algebraic operations specified. In the next example, we will see how we can use a similar technique to factor a fourth degree polynomial.

## EXAMPLE

Factor  $x^4 + 3x^2 + 2$ .

### Answer

This looks a lot like a trinomial that we know how to factor:  $x^2 + 3x + 2 = (x + 2)(x + 1)$ . The only thing different is the exponents.

If we substitute  $u = x^2$  and recognize that  $u^2 = (x^2)^2 = x^4$ , we may be able to factor this beast!

Everywhere there is a  $x^2$  we will replace it with a  $u$  then factor.

$$u^2 + 3u + 2 = (u + 1)(u + 2)$$

We are not quite done yet. We want to factor the original polynomial which had  $x$  as its variable, so we need to replace  $x^2 = u$  now that we are done factoring.

$$(u + 1)(u + 2) = (x^2 + 1)(x^2 + 2)$$

We conclude that  $x^4 + 3x^2 + 2 = (x^2 + 1)(x^2 + 2)$ .

In the following video, we show two more examples of how to use substitution to factor a fourth degree polynomial and an expression with fractional exponents.

Factoring Using Substitution

**EXERCISES**

Factor:

1.  $x^4 - 8x^2 + 15$

2.  $2y^{1/2} - 13y^{1/4} + 21$

$$1. \quad x^4 - 8x^2 + 15 = (x^2)^2 - 8(x^2) + 15 \quad u = x^2$$

$$= u^2 - 8u + 15$$

$$\begin{array}{l} -5 \cdot (-3) = 15 \\ -5 + (-3) = -8 \end{array} \quad = (u - 5)(u - 3)$$

$$= (x^2 - 5)(x^2 - 3)$$

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## Factor Completely

Sometimes you may encounter a polynomial that takes an extra step to factor. In our next example, we will first find the GCF of a trinomial, and after factoring it out, we will be able to factor again so that we end up with a product of a monomial and two binomials.

In our last example, we show why it is important to factor out a GCF, if there is one, before you begin using the techniques shown in this module.

## Summary

In this section, we used factoring with special cases and factoring by grouping to factor expressions with negative and fractional exponents. We also returned to factoring polynomials and used the substitution method to factor a 4<sup>th</sup> degree polynomial. The last topic we covered was what it means to factor completely.

## EXAMPLE

Factor  $6m^2k - 3mk - 3k$  completely.

### Answer

Whenever you factor, first try the easy route and ask yourself if there is a GCF. In this case, there is one, and it is  $3k$ .

Factor  $3k$  from the trinomial:

$$6m^2k - 3mk - 3k = 3k(2m^2 - m - 1)$$

We are left with a trinomial that can be factored using your choice of factoring methods. We will create a table to find the factors of  $2 \cdot -1 = -2$  that sum to  $-1$

Factors of $2 \cdot -1 = -2$	Sum of Factors
$2, -1$	1
$-2, 1$	$-1$

Our factors are  $-2, 1$  which will allow us to factor by grouping:

Rewrite the middle term with the factors we found:

$$(2m^2 - m - 1) = 2m^2 - 2m + m - 1$$

Regroup and find the GCF of each group:

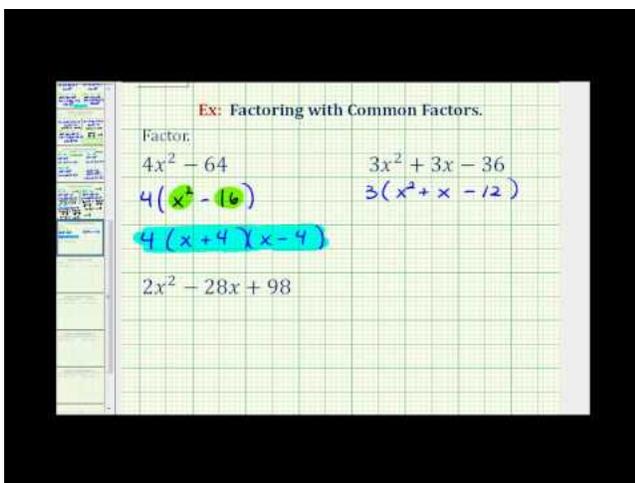
$$(2m^2 - 2m) + (m - 1) = 2m(m - 1) + 1(m - 1)$$

Now factor  $(m - 1)$  from each term:

$$2m^2 - m - 1 = (m - 1)(2m + 1)$$

Do not forget the original GCF that we factored out! Our final factored form is:

$$6m^2k - 3mk - 3k = 3k(m - 1)(2m + 1)$$



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# INTRODUCTION TO POLYNOMIAL EQUATIONS

## Why Learn About Polynomial Equations?

In this section, we will introduce the zero product principle as a tool for solving polynomial equations. We will also explore a polynomial function that models projectile motion and see how factoring can be used to find an unknown side length of a triangle.

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# SOLVE POLYNOMIAL EQUATIONS

## LEARNING OUTCOMES

- Use factoring methods to factor polynomial equations

Not all of the techniques we use for solving linear equations will apply to solving polynomial equations. In this section, we will introduce a method for solving polynomial equations that combines factoring and the zero product principle.

## The Principle of Zero Products



What if we told you that we multiplied two numbers together and got an answer of zero? What could you say about the two numbers? Could they be 2 and 5? Could they be 9 and 1? No! When the result (answer) from multiplying two numbers is zero, that means that one of them *had* to be zero. This idea is called the zero product principle, and it is useful for solving polynomial equations that can be factored.

## PRINCIPLE OF ZERO PRODUCTS

The Principle of Zero Products states that if the product of two numbers is 0, then at least one of the factors is 0. If  $ab = 0$ , then either  $a = 0$  or  $b = 0$ , or both  $a$  and  $b$  are 0.

Let us start with a simple example. We will factor a GCF from a binomial and apply the principle of zero products to solve a polynomial equation.

### EXAMPLE

Solve:

$$-t^2 + t = 0$$

**Answer**

Each term has a common factor of  $t$ , so we can factor and use the zero product principle. Rewrite each term as the product of the GCF and the remaining terms.

$$-t^2 = t(-t)$$

$$t = t(1)$$

Rewrite the polynomial equation using the factored terms in place of the original terms.

$$-t^2 + t = 0$$

$$t(-t) + t(1)$$

$$t(-t + 1) = 0$$

Now we have a product on one side and zero on the other, so we can set each factor equal to zero using the zero product principle.

$$t = 0 \quad \text{OR} \quad -t + 1 = 0$$

$$\frac{-1}{-1} \quad \frac{-1}{-1}$$

$$-t = -1$$

$$\frac{-t}{-1} = \frac{-1}{-1}$$

$$t = 1$$

Therefore,  $t = 0$  OR  $t = 1$ .

In the following video, we show two more examples of using both factoring and the principle of zero products to solve a polynomial equation.

In the next video, we show that you can use previously learned methods to factor a trinomial in order to solve a quadratic equation.

We all know that it is rare to be given an equation to solve that has zero on one side, so let us try another example.

We will work through one more example that is similar to the one above, except this example has fractions, yay!

In our last video, we show how to solve another quadratic equation that contains fractions.

**Examples: Solve an equation by factoring (GCF).**

Solve.

$$x^2 + 4x = 0 \qquad 14x^2 - 35x = 0$$

$$x(x+4) = 0 \qquad 7x \cdot x - 5 \cdot 7x$$

$$x = 0 \text{ or } x + 4 = 0$$

$$x = 0 \text{ or } x = -4$$

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**Examples: Solve by Factoring.**

Solve.  $ax^2 + bx + c = 0$

$$x^2 + 6x + 8 = 0 \qquad x^2 + 23x - 50 = 0$$

$$(x+2)(x+4) = 0 \qquad (x-2)(x+25) = 0$$

$$x+2 = 0 \text{ or } x+4 = 0$$

$$x = -2 \text{ or } x = -4$$

$$x^2 - 8x + 15 = 0 \qquad x^2 - 5x - 24 = 0$$

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## EXAMPLE

Solve:  $s^2 - 4s = 5$

### Answer

First, move all the terms to one side. The goal is to try and see if we can use the zero product principle since that is the only tool we know for solving polynomial equations.

$$s^2 - 4s = 5$$

$$s^2 - 4s - 5 = 0$$

We now have all the terms on the left side and zero on the other side. The polynomial  $s^2 - 4s - 5$  factors nicely which makes this equation a good candidate for the zero product principle.

$$s^2 - 4s - 5 = 0$$

$$(s + 1)(s - 5) = 0$$

We separate our factors into two linear equations using the principle of zero products.

$$(s - 5) = 0$$

$$s - 5 = 0$$

$$s = 5$$

OR

$$(s + 1) = 0$$

$$s + 1 = 0$$

$$s = -1$$

Therefore,  $s = -1$  OR  $s = 5$ .

## EXAMPLE

$$\text{Solve } y^2 - 5 = -\frac{7}{2}y + \frac{5}{2}$$

### Answer

We can solve this in one of two ways. One way is to eliminate the fractions like you may have done when solving linear equations, and the second is to find a common denominator and factor fractions. Eliminating fractions is easier, so we will show that way.

Start by multiplying the whole equation by 2 to eliminate fractions:

$$2(y^2 - 5 = -\frac{7}{2}y + \frac{5}{2})$$

$$2(y^2) + 2(-5) = 2(-\frac{7}{2}y) + 2(\frac{5}{2})$$

$$2y^2 - 10 = -7y + 5$$

Now we can move all the terms to one side and see if this will factor so we can use the principle of zero products.

$$2y^2 - 10 = -7y + 5$$

$$2y^2 - 10 + 7y - 5 = 0$$

$$2y^2 - 15 + 7y = 0$$

$$2y^2 + 7y - 15 = 0$$

We can now check whether this polynomial will factor. Using a table we can list factors until we find two numbers with a product of  $2 \cdot -15 = -30$  and a sum of 7.

Factors of $2 \cdot -15 = -30$	Sum of Factors
1, -30	-29
-1, 30	29
2, -15	-13
-2, 15	13
3, -10	-7
-3, 10	7

We have found the factors that will produce the middle term we want,  $-3, 10$ . We need to place the factors in a way that will lead to a middle term of  $7y$ :

$$(2y - 3)(y + 5) = 0$$

Now we can set each factor equal to zero and solve:

$$(2y - 3) = 0 \text{ OR } (y + 5) = 0$$

$$2y = 3 \text{ OR } y = -5$$

$$y = \frac{3}{2} \text{ OR } y = -5$$

You can always check your work to make sure your solutions are correct:

$$\text{Check } y = \frac{3}{2}$$

$$\left(\frac{3}{2}\right)^2 - 5 = -\frac{7}{2}\left(\frac{3}{2}\right) + \frac{5}{2}$$

$$\frac{9}{4} - 5 = -\frac{21}{4} + \frac{5}{2}$$

common denominator = 4

$$\frac{9}{4} - \frac{20}{4} = -\frac{21}{4} + \frac{10}{4}$$

$$-\frac{11}{4} = -\frac{11}{4}$$

$y = \frac{3}{2}$  is indeed a solution, now check  $y = -5$

$$(-5)^2 - 5 = -\frac{7}{2}(-5) + \frac{5}{2}$$

$$25 - 5 = \frac{35}{2} + \frac{5}{2}$$

$$20 = \frac{40}{2}$$

$$20 = 20$$

$y = -5$  is also a solution, so we must have done something right!

Therefore,  $y = \frac{3}{2}$  OR  $y = -5$ .

The image shows a handwritten solution on a grid background. The title is "Solving Equations by Factoring" and the example is "Solve.  $x^2 - 1 = \frac{11}{6}x + \frac{2}{3}$ ". The steps are:
   
1.  $x^2 - 1 = \frac{11}{6}x + \frac{2}{3}$ 
  
2.  $6(x^2 - 1) = 6\left(\frac{11}{6}x + \frac{2}{3}\right)$ 
  
3.  $6x^2 - 6 = \frac{11}{2}x + \frac{4}{3}$ 
  
4.  $6x^2 - 6 = \frac{11}{2}x + \frac{4}{3}$ 
  
5.  $11x + 4 = 11x + 4$ 
  
6.  $-11x - 4 = -11x - 4$ 
  
7.  $= 0$

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## PROJECTILES

## LEARNING OUTCOMES

- Model projectile motion

Projectile motion happens when you throw a ball into the air and it comes back down because of gravity. A projectile will follow a curved path that behaves in a predictable way. This predictable motion has been studied for centuries, and in simple cases, an object's height from the ground at a given time,  $t$ , can be modeled with a polynomial function of the form  $h(t) = at^2 + bt + c$ , where  $h(t)$  = height of an object at a given time,  $t$ . Projectile motion is also called a parabolic trajectory because of the shape of the path of a projectile's motion, as in the water in the image of the fountain below.



*Parabolic water trajectory in a fountain.*

Parabolic motion and its related functions allow us to launch satellites for telecommunications and rockets for space exploration. Recently, police departments have even begun using projectiles with GPS to track fleeing suspects in vehicles rather than pursuing them by high-speed chase (Note: "Cops' Latest Tool in High-speed Chases: GPS Projectiles." *CBSNews*. CBS Interactive, n.d. Web. 14 June 2016.).

In this section, we will use polynomial functions to answer questions about the parabolic motion of a projectile. The real mathematical model for the path of a rocket or a police GPS projectile may have different coefficients or more variables, but the concept remains the same. We will also learn to interpret the meaning of the variables in a polynomial function that models projectile motion.

### Example

A small toy rocket is launched from a 4-foot pedestal. The height ( $h$ , in feet) of the rocket  $t$  seconds after taking off is given by the function  $h(t) = -2t^2 + 7t + 4$ . How long will it take the rocket to hit the ground?

#### Answer

The rocket will be on the ground when  $h(t) = 0$ . We want to know how long,  $t$ , the rocket is in the air.

$$h(t) = -2t^2 + 7t + 4 = 0$$

$$0 = -2t^2 + 7t + 4$$

We can factor the polynomial  $-2t^2 + 7t + 4$  more easily by first factoring out a  $-1$

$$0 = -1(2t^2 - 7t - 4)$$

$$0 = -1(2t + 1)(t - 4)$$

Use the Zero Product Property. There is no need to set the constant factor  $-1$  to zero, because  $-1$  will never equal zero.

$$2t + 1 = 0 \quad \text{OR} \quad t - 4 = 0$$

Solve each equation.

$$t = -\frac{1}{2} \quad \text{OR} \quad t = 4$$

Interpret the answer. Since  $t$  represents time, it cannot be a negative number; only  $t = 4$  makes sense in this context.

We can check our answer when  $t = 4$ .

$$h(4) = -2(4)^2 + 7(4) + 4 = 0$$

$$h(4) = -2(16) + 28 + 4 = 0$$

$$h(4) - 32 + 32 = 0$$

$$h(4) = 0$$

Therefore, the rocket will hit the ground 4 seconds after being launched.

In the next example, we will solve for the time that the rocket reaches a given height other than zero.

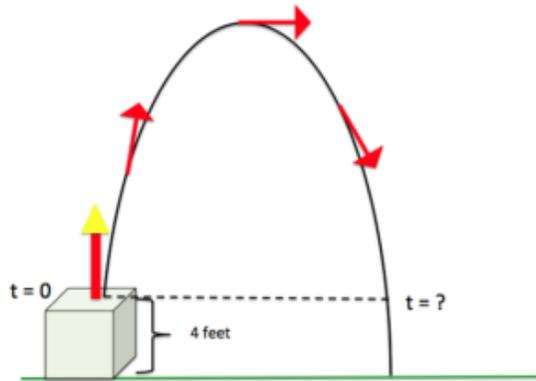
In the following video, we show another example of how to find the time when a object following a parabolic trajectory hits the ground.

In this section we introduced the concept of projectile motion and showed that it can be modeled with polynomial function. While the models used in these examples are simple, the concepts and interpretations

## EXAMPLE

Use the formula for the height of the rocket in the previous example to find the time when the rocket is 4 feet from hitting the ground on its way back down. Refer to the image.

$$h(t) = -2t^2 + 7t + 4$$



### Answer

We are given that the height of the rocket is 4 feet from the ground on its way back down. We want to know how long it will take for the rocket to get to that point in its path. We are going to solve for  $t$ .

Substitute  $h(t) = 4$  into the formula for height, and try to get zero on one side since we know we can use the zero product principle to solve polynomials.

### Write and Solve:

$$h(t) = 4 = -2t^2 + 7t + 4$$

$$4 = -2t^2 + 7t + 4$$

$$\underline{-4} \qquad \underline{-4}$$

$$0 = -2t^2 + 7t$$

Now we can factor out a  $t$  from each term:

$$0 = t(-2t + 7)$$

Solve each equation for  $t$  using the zero product principle:

$$t = 0 \text{ OR } -2t + 7 = 0$$

$$\underline{-7} \quad \underline{-7}$$

$$\underline{-2t} = \underline{-7}$$

$$t = \frac{7}{2} = 3.5$$

It does not make sense for us to choose  $t = 0$ , because we are interested in the amount of time that has passed when the projectile is 4 feet from hitting the ground on its way back down. We will choose  $t = 3.5$ .

Check the answer on your own for practice.

The rocket will be 4 feet from the ground at  $t = 3.5$  seconds.

are the same as what would happen in “real life.”

**Quadratic Equation Application - Projective**

You launch a toy rocket from a height of 5 feet. The height ( $h$ , in feet) of the rocket  $t$  seconds after taking off is given by the formula  $h = -3t^2 + 14t + 5$ .

a) How long will it take for the rocket to hit the ground?  
 b) Find the time when the rocket is 5 feet from hitting the ground.

a)  $0 = -3t^2 + 14t + 5$        $a = -3 \quad b = 14 \quad c = 5$   
 $0 = -1[3t^2 - 14t - 5]$

$\begin{array}{r} -15 \cdot 1 = -15 \\ - \quad - = -14 \end{array}$

$\begin{array}{r} -1 \cdot 15 = -15 \\ -15 \cdot 1 = -15 \\ - \quad - = -14 \end{array}$

**Factoring Trinomials in the Form:  $ax^2 + bx + c$**

Find the two integers,  $r$  and  $s$ , whose product is  $a \cdot c$  and whose sum is  $b$ .

$r \cdot s = a \cdot c$   
and  
 $r + s = b$

Rewrite the trinomial in the form  $ax^2 + bx + c = ax^2 + rx + sx + c$  and factor by grouping.

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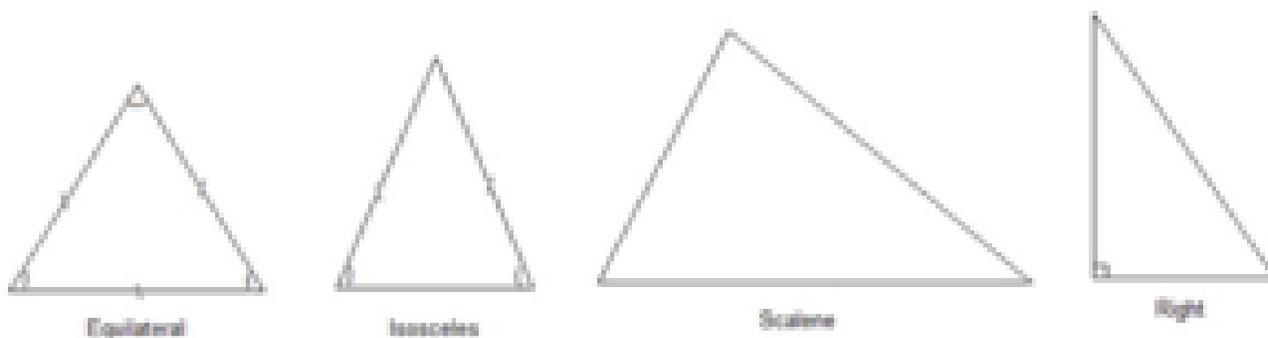
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# PYTHAGOREAN THEOREM

## LEARNING OUTCOMES

- Recognize a right triangle from other types of triangles

Types of Triangles



Some common types of triangles.

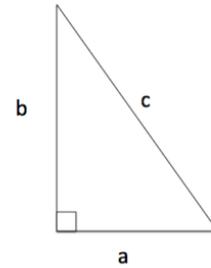
The **Pythagorean theorem** or **Pythagoras's theorem** is a statement about the sides of a right triangle. One of the angles of a right triangle is always equal to 90 degrees. This angle is the right angle. The two sides

next to the right angle are called the legs and the other side is called the hypotenuse. The hypotenuse is the side opposite the right angle, and it is always the longest side. The image above shows four common kinds of triangle, including a right triangle.

The Pythagorean theorem is often used to find unknown lengths of the sides of right triangles. If the longest leg of a right triangle is labeled  $c$ , and the other two  $a$ , and  $b$  as in the image on the left, The Pythagorean Theorem states that

$$a^2 + b^2 = c^2$$

Given enough information, we can solve for an unknown length. This relationship has been used for many, many years for things such as celestial navigation and early civil engineering projects. We now have digital GPS and survey equipment that have been programmed to do the calculations for us.



*A right triangle with sides labeled.*

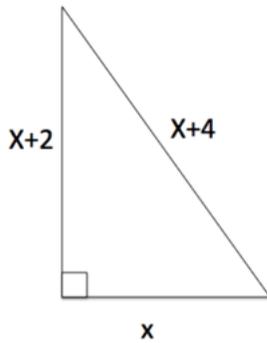
In the next example, we will combine the power of the Pythagorean theorem and what we know about solving quadratic equations to find unknown lengths of right triangles.

The following video example shows another way a quadratic equation can be used to find unknown side lengths of a right triangle.

If you are interested in celestial navigation and the mathematics behind it, watch the video below for fun.

## EXAMPLE

A right triangle has one leg with length  $x$ , another whose length is greater by two, and the length of the hypotenuse is greater by four. Find the lengths of the sides of the triangle. Use the image below.



### Answer

**Read and understand:** We know the lengths of all the sides of a triangle in terms of one side. We also know that the Pythagorean theorem will give us a relationship between the side lengths of a right triangle.

### Translate:

$$a^2 + b^2 = c^2$$

$$x^2 + (x + 2)^2 = (x + 4)^2$$

**Solve:** If we can move all the terms to one side and factor, we can use the zero product principle to solve. Since this is the only method we know, let us hope it works!

First, multiply the binomials and simplify so we can see what we are working with.

$$x^2 + (x + 2)^2 = (x + 4)^2$$

$$x^2 + x^2 + 4x + 4 = x^2 + 8x + 16$$

$$2x^2 + 4x + 4 = x^2 + 8x + 16$$

Now move all the terms to one side and see if we can factor.

$$2x^2 + 4x + 4 = x^2 + 8x + 16$$

$$\underline{-x^2} \quad \underline{-8x} \quad \underline{-16} \quad \underline{-x^2} \quad \underline{-8x} \quad \underline{-16}$$

$$x^2 - 4x - 12 = 0$$

This went from a messy looking problem to something promising. We can factor using the shortcut:

$$-6 \cdot 2 = -12 \text{ and } -6 + 2 = -4$$

So we can build our binomial factors with  $-6$  and  $2$ :

$$(x - 6)(x + 2) = 0$$

Set each factor equal to zero:

$$x - 6 = 0, x = 6$$

$$x + 2 = 0, x = -2$$

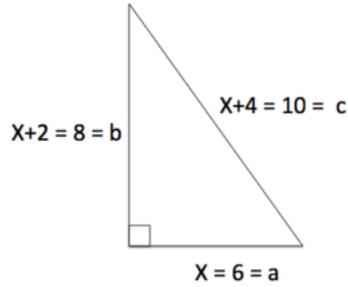
**Interpret:** Ok, it does not make sense to have a length equal to  $-2$ , so we can safely throw that solution out. The lengths of the sides are as follows:

$$x = 6$$

$$x + 2 = 6 + 2 = 8$$

$$x + 4 = 6 + 4 = 10$$

**Check:** Since we know the relationship between the sides of a right triangle, we can check that we are correct. Sometimes it helps to draw a picture



We know that  $a^2 + b^2 = c^2$ , so we can substitute the values we found:

$$6^2 + 8^2 = 10^2$$

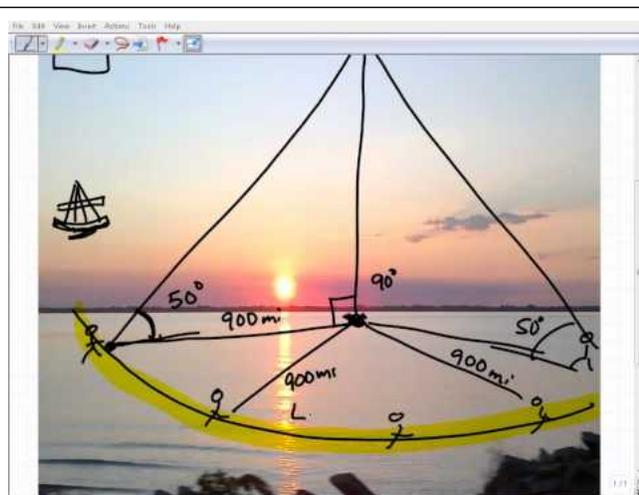
$$36 + 64 = 100$$

$$100 = 100$$

Our solution checks out.

The lengths of the sides of the right triangle are 6, 8, and 10

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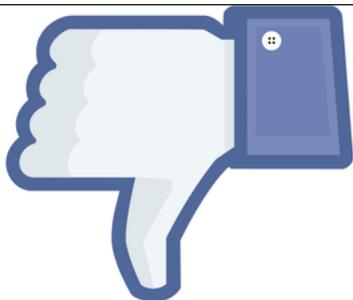
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## PUTTING IT TOGETHER: FACTORING

Factoring polynomials may seem like a tedious task that is put forth only to cause you frustration. Hopefully, now that you have worked through the exercises, text, and videos in this module, you are able to see that it can be a useful tool for solving polynomial equations.

Let us revisit the scenario we put forth in the beginning of the module. Recall that Joan proposed a polynomial function to represent the number of likes or comments received on Instagram based on the number of posts,  $x$ , in a day.

$$L(x) = -x^2 + 4x$$



Also recall that she tried to find the number of posts that would generate no likes or comments for her friend, Nate, by solving the function for zero:

$$0 = -x^2 + 4x$$

She tried guessing, but after attending office hours and asking her teacher how to solve an equations like this, she knows a more elegant and efficient way to solve the equation. Hopefully she can convince Nate with math that there is a limit to how much people enjoy his posts on Instagram.

First, Joan factors out  $-x$

$$0 = -x^2 + 4x$$

$$0 = -x(x - 4)$$

Next, she can use the principle of zero products to solve for  $x$ .

$$0 = -x(x - 4)$$

$$0 = -x \text{ OR } 0 = x - 4$$

$$0 = x \text{ OR } 4 = x$$

Factoring has led Joan to a solution much faster than guessing would have. The solution  $x = 0$  is obvious; if Nate does not post, nobody can like or comment on his posts. The solution  $x = 4$  can be interpreted to mean that once Nate makes four posts on Instagram in one day, people will stop paying attention.

Optimizing the amount of social media posting you do can influence how widely your ideas are shared. If you use social media for business, it is worth researching how often and when to post so that you will not only gain the audience you want, but also keep their attention.

Factoring may seem meaningless without the context of solving polynomial equations. We hope you will see that it can actually be kind of like solving a fun puzzle, too. Taking the hard route of guessing, much like free climbing a building, may seem fun at first. . . but, like using an elevator, once you discover how to factor you will be able to get where you are going with much less effort (less sweat, too).

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# MODULE 10: RATIONAL EXPRESSIONS AND EQUATIONS

## WHY IT MATTERS: RATIONAL EXPRESSIONS AND EQUATIONS

### Why Study Rational Expressions and Equations?

If you were around in 2011, you probably heard about Occupy Wall Street (OWS), the protest against social and economic inequality in the US. According to Wikipedia, the main issues raised by OWS were social and economic inequality, greed, corruption, and the perceived undue influence of corporations on government—particularly from the financial services sector. The OWS slogan, “We are the 99%”, refers to income inequality and wealth distribution in the U.S. between the wealthiest 1% and the rest of the population. To achieve their goals, protesters acted on consensus-based decisions made in general assemblies which emphasized direct action over petitioning authorities for redress.

**WHAT  
IS OUR  
ONE  
DEMAND?**



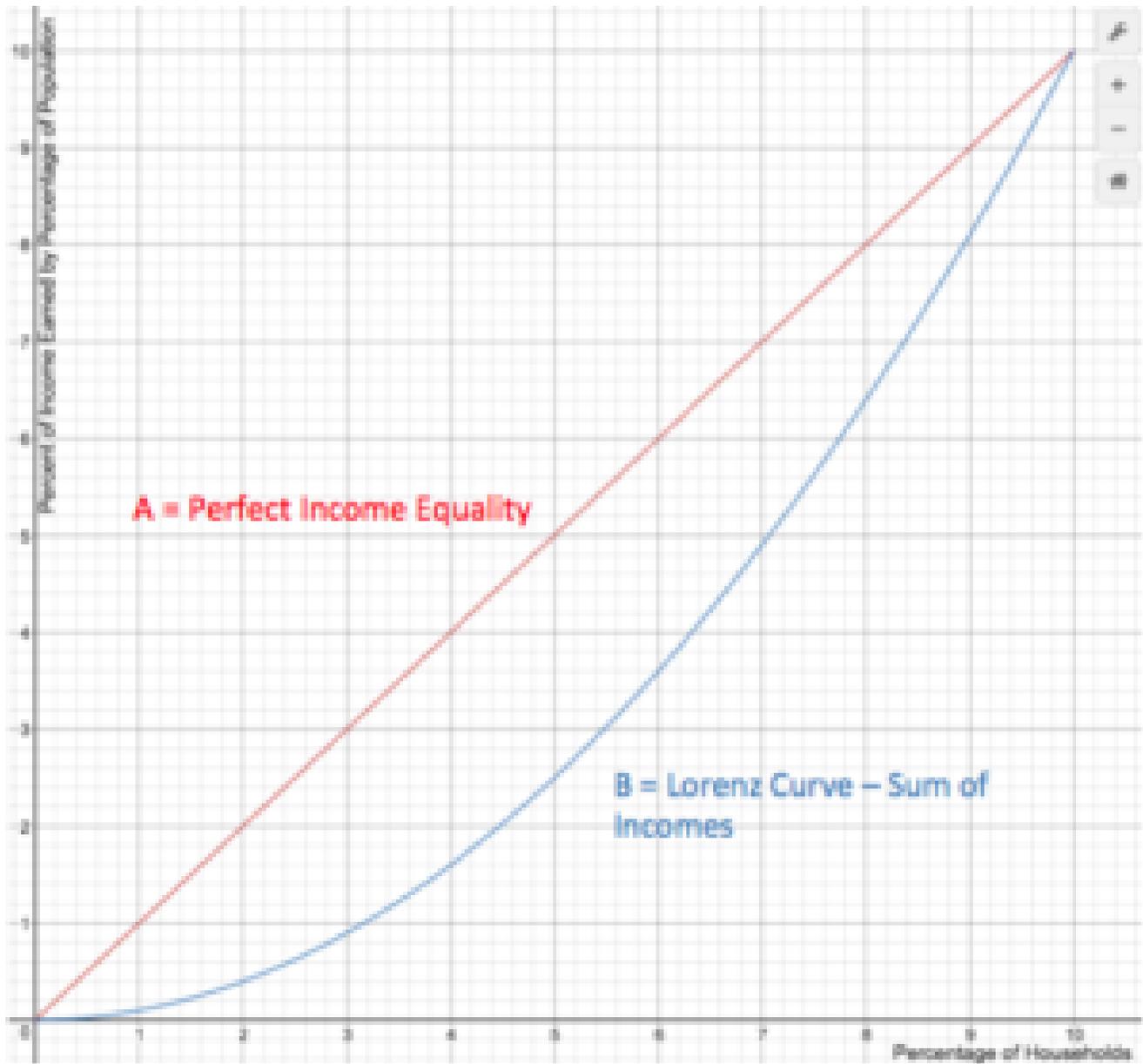
**#OCCUPY WALLSTREET**

SEPTEMBER 17TH,  
BRING TENT.

An Occupy Wall Street Ad.

Economists are also interested in how wealth is distributed in economies. The Lorenz curve is a mathematical representation of the distribution of income or of wealth. Max O. Lorenz developed it in 1905 to represent inequality of the wealth distribution.

The curve is a graph showing the proportion of overall income or wealth assumed by the bottom  $x\%$  of the people. It is often used to represent income distribution, where it shows for the bottom  $x\%$  of households, what percentage ( $y\%$ ) of the total income they have. The percentage of households is plotted on the  $x$ -axis, the percentage of income is on the  $y$ -axis. It can also be used to show distribution of assets. In such use, many economists consider it to be a measure of social inequality.



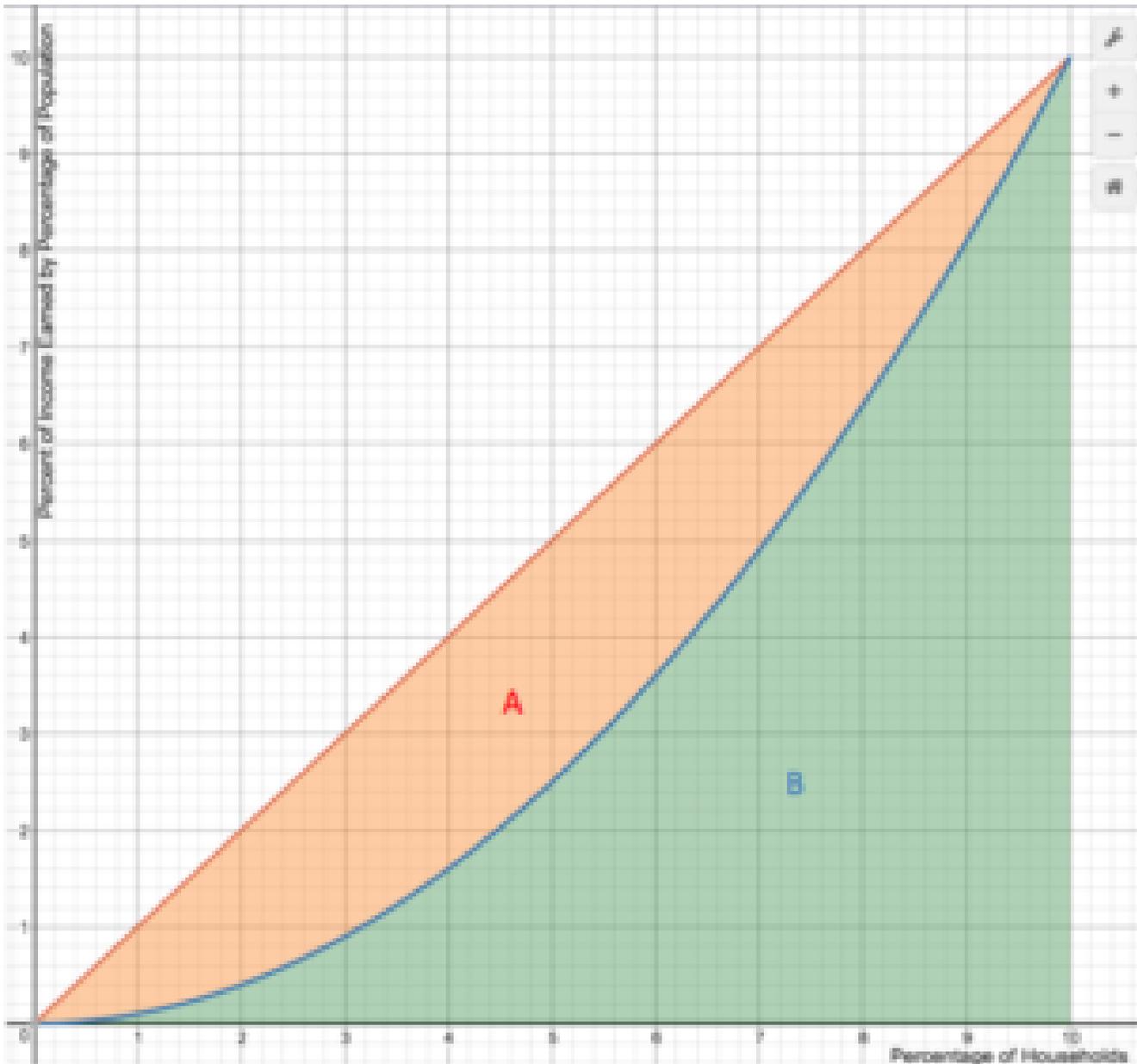
The Lorenz Curve.

Points on the Lorenz curve represent statements like “the bottom 20% of all households have 10% of the total income.” A perfectly equal income distribution would be one in which every person has the same

income. In this case, the bottom  $N\%$  of society would always have  $N\%$  of the income. This can be depicted by the straight line  $y = x$ , called the “line of perfect equality.”

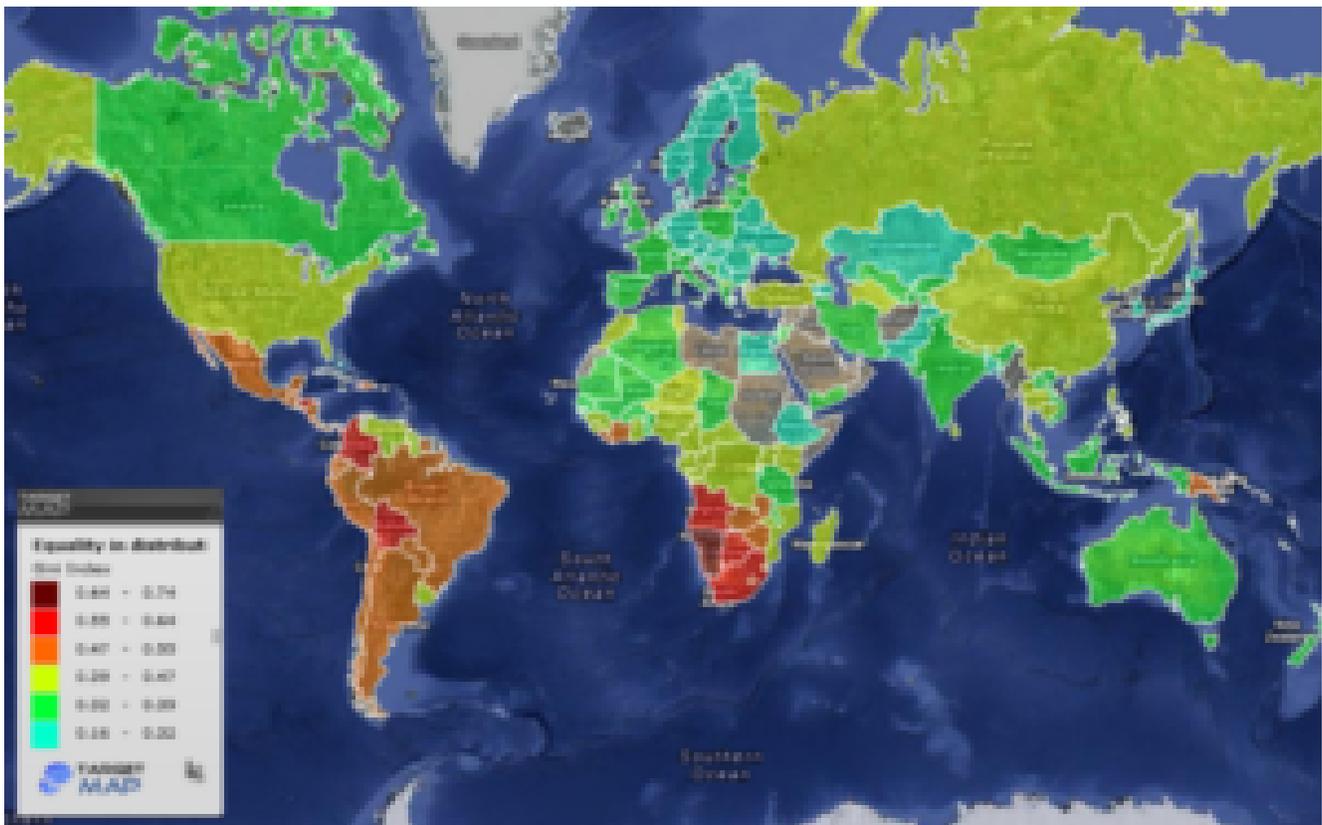
By contrast, a perfectly unequal distribution would be one in which one person has all the income and everyone else has none. In that case, the curve would be at  $y = 0\%$  for all  $x < 100\%$ , and  $y = 100\%$  when  $x = 100\%$ . This curve is called the “line of perfect inequality.”

The Gini coefficient is the ratio of the area between the line of perfect equality and the observed Lorenz curve to the area between the line of perfect equality and the line of perfect inequality. The higher the coefficient, the more unequal the distribution is. This is given by the ratio  $G = \frac{A}{A+B}$ , where  $A$  and  $B$  are the areas of regions as marked in the graph below.



A graph showing the regions which make up the Gini coefficient.

The closer the Gini coefficient is to 1, the less equally wealth is distributed among an economy. The closer it is to 0, the more equally wealth is distributed. Below is a map of the world with Gini coefficient color coded.



A world map color coded by Gini coefficient.

The Gini Index is a rational expression, or ratio. In this module, we will define and apply mathematical operations to rational expressions. We will also solve rational equations.

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# INTRODUCTION TO OPERATIONS WITH RATIONAL EXPRESSIONS

## Why Perform Mathematical Operations on Rational Expressions?

As we have moved through this course, we have been focusing on different aspects of the order of operations and how they apply to a wide range of mathematical expressions. We continue that here as we learn about rational expressions which have unique behaviors and are used to model many relationships. Because they have unique behaviors, it helps to isolate rational expressions and learn how mathematical operations apply to them.

# DEFINE AND SIMPLIFY RATIONAL EXPRESSIONS

## LEARNING OUTCOMES

- Define and simplify rational expressions
- Identify the domain of a rational expression

**Rational expressions** are fractions that have a polynomial in the numerator, denominator, or both. Although rational expressions can seem complicated because they contain variables, they can be simplified using the techniques used to simplify expressions such as  $\frac{4x^3}{12x^2}$  combined with techniques for factoring polynomials. There are a couple ways to get yourself into trouble when working with rational expressions, equations, and functions. One of them is dividing by zero, and the other is trying to divide across addition or subtraction.

## Determine the Domain of a Rational Expression

One sure way you can break math is to divide by zero. Consider the following rational expression evaluated at  $x = 2$ :

Evaluate  $\frac{x}{x-2}$  for  $x = 2$

Substitute  $x = 2$

$$\frac{2}{2-2} = \frac{2}{0}$$

This means that for the expression  $\frac{x}{x-2}$ ,  $x$  cannot be 2 because it will result in an undefined ratio. In general, finding values for a variable that will not result in division by zero is called finding the domain. Finding the domain of a rational expression or function will help you not break math.

## DOMAIN OF A RATIONAL EXPRESSION OR EQUATION

The domain of a rational expression or equation is a collection of the values for the variable that will not result in an undefined mathematical operation such as division by zero. For  $a =$  any real number, we can notate the domain in the following way:

$x$  is all real numbers where  $x \neq a$

The reason you cannot divide any number  $c$  by zero ( $\frac{c}{0} = ?$ ) is that you would have to find a number that when you multiply it by 0 you would get back  $c$  ( $? \cdot 0 = c$ ). There are no numbers that can do this, so we say "division by zero is undefined". When simplifying rational expressions, you need to pay attention to what values of the variable(s) in the expression would make the denominator equal zero. These values cannot be

included in the domain, so they are called excluded values. Discard them right at the start before you go any further.

(Note that although the *denominator* cannot be equal to 0, the *numerator* can—this is why you only look for excluded values in the denominator of a rational expression.)

For rational expressions, the domain will exclude values where the the denominator is 0. The following example illustrates finding the domain of an expression. Note that this is exactly the same algebra used to find the domain of a function.

### Example

Identify the domain of the expression.  $\frac{x+7}{x^2+8x-9}$

#### Answer

Find any values for  $x$  that would make the denominator equal to 0 by setting the denominator equal to 0 and solving the equation.

$$x^2 + 8x - 9 = 0$$

Solve the equation by factoring. The solutions are the values that are excluded from the domain.

$$(x + 9)(x - 1) = 0$$

$$x = -9 \text{ or } x = 1$$

The domain is all real numbers except  $-9$  and  $1$ .

## Simplify Rational Expressions

Before we dive in to simplifying rational expressions, let us review the difference between a factor, a term, and an expression. This will hopefully help you avoid another way to break math when you are simplifying rational expressions.

**Factors** are the building blocks of multiplication. They are the numbers that you can multiply together to produce another number: 2 and 10 are factors of 20, as are 4, 5, 1, 20.

**Terms** are single numbers, or variables and numbers connected by multiplication.  $-4$ ,  $6x$  and  $x^2$  are all terms.

**Expressions** are groups of terms connected by addition and subtraction.  $2x^2 - 5$  is an expression.

This distinction is important when you are required to divide. Let us use an example to show why this is important.

Simplify:  $\frac{2x^2}{12x}$

The numerator and denominator of this fraction consist of factors. To simplify it, we can divide without being impeded by addition or subtraction.

$$\begin{aligned} & \frac{2x^2}{12x} \\ &= \frac{2 \cdot x \cdot x}{2 \cdot 3 \cdot 2 \cdot x} \end{aligned}$$

$$= \frac{\cancel{x} \cdot \cancel{x} \cdot x}{\cancel{x} \cdot 3 \cdot 2 \cdot \cancel{x}}$$

We can do this because  $\frac{2}{2} = 1$  and  $\frac{x}{x} = 1$ , so our expression simplifies to  $\frac{x}{6}$ .

Compare that to the expression  $\frac{2x^2 + x}{12 - 2x}$ . Notice the denominator and numerator consist of two terms connected by addition and subtraction. We have to tip-toe around the addition and subtraction. When asked to simplify, it is tempting to want to cancel out like terms as we did when we just had factors. But you cannot do that, it will break math!



*Be careful not to break math when working with rational expressions.*

In the examples that follow, the numerator and the denominator are polynomials with more than one term, and we will show you how to properly simplify them by factoring. This turns expressions connected by addition and subtraction into terms connected by multiplication.

#### Example

Simplify and state the domain for the expression.  $\frac{x+3}{x^2+12x+27}$

#### Answer

To find the domain (and the excluded values), find the values where the denominator is equal to 0. Factor the quadratic and apply the zero product principle.

$$x + 3 = 0 \quad \text{or} \quad x + 9 = 0$$

$$x = 0 - 3 \quad \text{or} \quad x = 0 - 9$$

$$x = -3 \quad \text{or} \quad x = -9$$

The domain is all real numbers except  $x = -3$  or  $x = -9$ .

Factor the numerator and denominator. Identify the factors that are the same in the numerator and denominator then simplify.

$$\frac{x+3}{x^2+12x+27}$$
$$= \frac{x+3}{(x+3)(x+9)}$$
$$\frac{\cancel{x+3}}{(\cancel{x+3})(x+9)}$$
$$= 1 \cdot \frac{1}{x+9}$$
$$\frac{x+3}{x^2+12x+27} = \frac{1}{x+9}$$

The domain is all real numbers except  $-3$  and  $-9$ .

### Example

Simplify and state the domain for the expression.  $\frac{x^2+10x+24}{x^3-x^2-20x}$

#### Answer

To find the domain, determine the values where the denominator is equal to 0.

$$x^3 - x^2 - 20x = 0$$
$$x(x^2 - x - 20) = 0$$
$$x(x - 5)(x + 4) = 0$$

The domain is all real numbers except 0, 5, and  $-4$ .

To simplify, factor the numerator and denominator of the rational expression. Identify the factors that are the same in the numerator and denominator then simplify.

$$\frac{x^2+10x+24}{x^3-x^2-20x}$$
$$= \frac{(x+4)(x+6)}{x(x-5)(x+4)}$$
$$= \frac{\cancel{(x+4)}(x+6)}{x(x-5)\cancel{(x+4)}}$$

It is acceptable to either leave the denominator in factored form or to distribute/multiply.

$$\frac{x+6}{x(x-5)} \text{ or } \frac{x+6}{x^2-5x}$$

The domain is all real numbers except 0, 5, and  $-4$ .

We will show one last example of simplifying a rational expression. See if you can recognize the special product in the numerator.

## EXAMPLE

Simplify  $\frac{x^2-9}{x^2+4x+3}$  and state the domain.

### Answer

To find the domain, determine the values where the denominator is equal to 0. Be sure to factor the denominator first.

$$(x + 3)(x + 1) = 0$$

The domain is all real numbers except  $-3$  and  $-1$ .

Now factor and simplify the entire rational expression. Notice the numerator is a difference of squares.

$$\begin{aligned} & \frac{x^2-9}{x^2+4x+3} \\ &= \frac{(x+3)(x-3)}{(x+3)(x+1)} \\ &= \frac{\cancel{(x+3)}(x-3)}{\cancel{(x+3)}(x+1)} \\ & \frac{x^2-9}{x^2+4x+3} = \frac{x-3}{x+1} \end{aligned}$$

Domain:  $x \neq -3, -1$

In the following video, we present additional examples of simplifying and finding the domain of a rational expression.

The video shows three examples of simplifying rational expressions and finding their domains:

- $\frac{6x^4}{24x^2}$ : Simplified to  $\frac{x^2}{4}$ . Domain: all real numbers except 0.
- $\frac{2x-10}{x^2-2x-15}$ : Simplified to  $\frac{2(x-5)}{(x-5)(x+3)} = \frac{2}{x+3}$ . Domain: all real numbers except  $-3$  and  $5$ .
- $\frac{x^2-x-6}{x^2-7x^2+12x}$ : Simplified to  $\frac{(x-3)(x+2)}{x(x-3)}$ . Domain: all real numbers except  $-3$  and  $3$ .

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2922>

## STEPS FOR SIMPLIFYING A RATIONAL EXPRESSION

To simplify a rational expression, follow these steps:

- Determine the domain. The excluded values are those values for the variable that result in the expression having a denominator of 0.
- Factor the numerator and denominator.

- Cancel out common factors in the numerator and denominator and simplify.

## Summary

An additional consideration for rational expressions is to determine what values are excluded from the domain. Since division by 0 is undefined, any values of the variable that result in a denominator of 0 must be excluded from the domain. Excluded values must be identified in the original equation, not from its factored form. Rational expressions are fractions containing polynomials. They can be simplified much like numeric fractions. To simplify a rational expression, first determine common factors of the numerator and denominator then cancel them out.

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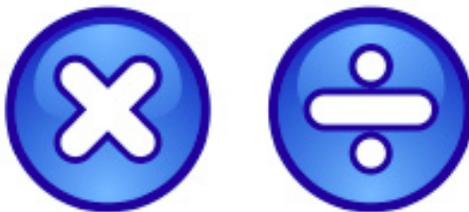
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# MULTIPLY AND DIVIDE RATIONAL EXPRESSIONS

## LEARNING OUTCOMES

- Multiply and divide rational expressions

Just as you can multiply and divide fractions, you can multiply and divide **rational expressions**. In fact, you use the same processes for multiplying and dividing rational expressions as you use for multiplying and dividing numeric fractions. The process is the same even though the expressions look different!



## Multiply Rational Expressions

Remember that there are two ways to multiply numeric fractions.

One way is to multiply the numerators and the denominators and then simplify the product, as shown here.

$$\frac{4}{5} \cdot \frac{9}{8} = \frac{36}{40} = \frac{3 \cdot 3 \cdot 2 \cdot 2}{5 \cdot 2 \cdot 2 \cdot 2} = \frac{3 \cdot 3 \cdot \cancel{2} \cdot \cancel{2}}{5 \cdot \cancel{2} \cdot \cancel{2} \cdot 2} = \frac{3 \cdot 3}{5 \cdot 2} \cdot 1 = \frac{9}{10}$$

A second way is to factor and simplify the fractions *before* performing the multiplication.

$$\frac{4}{5} \cdot \frac{9}{8} = \frac{2 \cdot 2}{5} \cdot \frac{3 \cdot 3}{2 \cdot 2 \cdot 2} = \frac{\cancel{2} \cdot \cancel{2} \cdot 3 \cdot 3}{\cancel{2} \cdot 5 \cdot \cancel{2} \cdot 2} = 1 \cdot \frac{3 \cdot 3}{5 \cdot 2} = \frac{9}{10}$$

Notice that both methods result in the same product. In some cases you may find it easier to multiply and then simplify, while in others it may make more sense to simplify fractions before multiplying.

The same two approaches can be applied to rational expressions. Our first two examples apply both techniques to one expression. After that we will let you decide which works best for you.

#### Example

Multiply.  $\frac{5a^2}{14} \cdot \frac{7}{10a^3}$

State the product in simplest form.

**Answer**

Multiply the numerators and then multiply the denominators.

$$\frac{5a^2}{14} \cdot \frac{7}{10a^3} = \frac{35a^2}{140a^3}$$

Simplify by finding common factors in the numerator and denominator. Cancel the common factors.

$$\begin{aligned} \frac{35a^2}{140a^3} &= \frac{5 \cdot 7 \cdot a^2}{5 \cdot 7 \cdot 2 \cdot 2 \cdot a^2 \cdot a} \\ &= \frac{\cancel{5} \cdot \cancel{7} \cdot \cancel{a^2}}{\cancel{5} \cdot \cancel{7} \cdot 2 \cdot 2 \cdot \cancel{a^2} \cdot a} \\ &= 1 \cdot \frac{1}{4a} \end{aligned}$$

$$\frac{5a^2}{14} \cdot \frac{7}{10a^3} = \frac{1}{4a}$$

Okay, that worked. This time let us simplify first then multiply. When using this method, it helps to look for the **greatest common factor**. You can factor out *any* common factors, but finding the greatest one will take fewer steps.

#### Example

Multiply.  $\frac{5a^2}{14} \cdot \frac{7}{10a^3}$

State the product in simplest form.

**Answer**

Factor the numerators and denominators. Look for the greatest common factors.

$$\frac{5 \cdot a^2}{7 \cdot 2} \cdot \frac{7}{5 \cdot 2 \cdot a^2 \cdot a}$$

Cancel common factors and then multiply.

$$\frac{5 \cdot a^2}{7 \cdot 2} \cdot \frac{7}{5 \cdot 2 \cdot a^2 \cdot a}$$

$$= \frac{\cancel{5} \cdot \cancel{a^2}}{\cancel{7} \cdot 2} \cdot \frac{\cancel{7}}{\cancel{5} \cdot 2 \cdot \cancel{a^2} \cdot a}$$

$$= \frac{1 \cdot 1 \cdot 1}{2 \cdot 2 \cdot a} = \frac{1}{4a}$$

$$\frac{5a^2}{14} \cdot \frac{7}{10a^3} = \frac{1}{4a}$$

Both methods produced the same answer.

Also, remember that when working with rational expressions, you should get into the habit of identifying any values for the variables that would result in division by 0. These excluded values must be eliminated from the domain, the set of all possible values of the variable. In the example above,  $\frac{5a^2}{14} \cdot \frac{7}{10a^3}$ , the domain is all real numbers where  $a$  is not equal to 0. When  $a = 0$ , the denominator of the fraction  $\frac{7}{10a^3}$  equals 0 which will make the fraction undefined.

Some rational expressions contain quadratic expressions and other multi-term polynomials. To multiply these rational expressions, the best approach is to first factor the polynomials and then look for common factors. Multiplying the terms before factoring will often create complicated polynomials...and then you will have to factor these polynomials anyway! For this reason, it is easier to factor, simplify, and then multiply. Just take it step by step like in the examples below.

#### Example

Multiply.  $\frac{a^2 - a - 2}{5a} \cdot \frac{10a}{a + 1}$ ,  $a \neq -1, 0$

State the product in simplest form.

**Answer**

Factor the numerators and denominators.

$$\frac{(a-2)(a+1)}{5 \cdot a} \cdot \frac{5 \cdot 2 \cdot a}{(a+1)}$$

Simplify common factors:

$$\frac{(a-2) \cancel{(a+1)}}{\cancel{5} \cdot \cancel{a}} \cdot \frac{\cancel{5} \cdot 2 \cdot \cancel{a}}{\cancel{(a+1)}}$$

$$= \frac{a-2}{1} \cdot \frac{2}{1}$$

Multiply simplified rational expressions. This expression can be left with the numerator in factored form or multiplied out.

$$\frac{(a-2)}{1} \cdot \frac{2}{1}$$

$$= 2(a-2)$$

$$\frac{a^2 - a - 2}{5a} \cdot \frac{10a}{a + 1} = 2a - 4$$

### Example

Multiply.  $\frac{a^2+4a+4}{2a^2-a-10} \cdot \frac{a+5}{a^2+2a}$ ,  $a \neq -2, 0, \frac{5}{2}$

State the product in simplest form.

**Answer**

Factor the numerators and denominators.

$$\frac{(a+2)(a+2)}{(2a-5)(a+2)} \cdot \frac{a+5}{a(a+2)}$$

Simplify common factors.

$$\frac{\cancel{(a+2)} \cancel{(a+2)}}{(2a-5) \cancel{(a+2)}} \cdot \frac{a+5}{a \cancel{(a+2)}}$$

Multiply simplified rational expressions. This expression can be left with the denominator in factored form or multiplied out.

$$\frac{1}{(2a-5)} \cdot \frac{a+5}{a} = \frac{a+5}{a(2a-5)}$$

$$\frac{a^2+4a+4}{2a^2-a-10} \cdot \frac{a+5}{a^2+2a} = \frac{a+5}{a(2a-5)}$$

Note that in the answer above, you cannot simplify the rational expression any further. It may be tempting to express the 5's in the numerator and denominator as the fraction  $\frac{5}{5}$ , but these 5's are terms because they are being added or subtracted. Remember that only common factors, not terms, can be regrouped to form factors of 1!

In the following video, we present another example of multiplying rational expressions.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2925>

## Divide Rational Expressions

You have seen that you multiply rational expressions as you multiply numeric fractions. It should come as no surprise that you also divide rational expressions the same way you divide numeric fractions. Specifically, to divide rational expressions, keep the first rational expression, change the division sign to multiplication, and then take the reciprocal of the second rational expression.

Let us begin by recalling division of numeric fractions.

$$\frac{2}{3} \div \frac{5}{9} = \frac{2}{3} \cdot \frac{9}{5} = \frac{18}{15} = \frac{6}{5}$$

Use the same process to divide rational expressions. You can think of division as multiplication by the reciprocal and then use what you know about multiplication to simplify.



*An example of reciprocal architecture.*

You still need to think about the domain, specifically, the variable values that would make either denominator equal zero. But there is a new consideration this time—because you divide by multiplying by the reciprocal of one of the rational expressions, you also need to find the values that would make the *numerator* of that expression equal zero. Have a look.

#### Example

State the domain and then divide.  $\frac{5x^2}{9} \div \frac{15x^3}{27}$

**Answer**

State the Domain:

Find excluded values. Notice the two denominators, 9 and 27, can never equal 0.

Because  $15x^3$  becomes the denominator in the reciprocal of  $\frac{15x^3}{27}$ , you must find the values of  $x$  that would make  $15x^3$  equal 0.

$$15x^3 = 0$$

$x = 0$  is an excluded value.

Divide:

State the quotient in simplest form. Rewrite division as multiplication by the reciprocal.

$$\frac{5x^2}{9} \cdot \frac{27}{15x^3}$$

Factor the numerators and denominators.

$$\frac{5 \cdot x \cdot x}{3 \cdot 3} \cdot \frac{3 \cdot 3 \cdot 3}{5 \cdot 3 \cdot x \cdot x \cdot x}$$

Cancel common factors.

$$\frac{\cancel{3} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{3} \cdot \cancel{3}} \cdot \frac{\cancel{3} \cdot \cancel{3} \cdot \cancel{3}}{\cancel{3} \cdot \cancel{3} \cdot \cancel{x} \cdot \cancel{x} \cdot x} = \frac{1}{x}$$

$$\frac{5x^2}{9} \div \frac{15x^3}{27} = \frac{1}{x}, x \neq 0$$

### Example

Divide.  $\frac{3x^2}{x+2} \div \frac{6x^4}{(x^2+5x+6)}$

State the quotient in simplest form and give the domain of the expression.

**Answer**

Determine the excluded values that make the denominators and the numerator of the divisor equal to 0.

$$(x + 2) = 0$$

$$x = -2$$

$$(x^2 + 5x + 6) = 0$$

$$(x + 3)(x + 2) = 0$$

$$x = -3 \text{ or } -2$$

$$6x^4 = 0$$

$$x = 0$$

The domain is all real numbers except 0,  $-2$ , and  $-3$ .

Rewrite division as multiplication by the reciprocal.

$$\frac{3x^2}{x+2} \cdot \frac{(x^2+5x+6)}{6x^4}$$

Factor the numerators and denominators.

$$\frac{3 \cdot x \cdot x}{x+2} \cdot \frac{(x+2)(x+3)}{2 \cdot 3 \cdot x \cdot x \cdot x \cdot x}$$

Cancel common factors.

$$\frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} + 2} \cdot \frac{\cancel{(x+2)}(x+3)}{2 \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x}$$

Simplify.

$$\frac{(x+3)}{2x^2}$$

$$\frac{3x^2}{x+2} \div \frac{6x^4}{(x^2+5x+6)} = \frac{x+3}{2x^2}$$

The domain is all real numbers except 0, -2, and -3.

Notice that once you rewrite the division as multiplication by a reciprocal, you follow the same process you used to multiply rational expressions.

In the video that follows, we present another example of dividing rational expressions.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2925>

## Summary

Rational expressions are multiplied and divided the same way as numeric fractions. To multiply, first find the greatest common factors of the numerator and denominator. Next, cancel common factors. Then, multiply any remaining factors. To divide, first rewrite the division as multiplication by the reciprocal. The steps are then the same as those for multiplication.

When expressing a product or quotient, it is important to state the excluded values. These are all values of a variable that would make a denominator equal zero at any step in the calculations.

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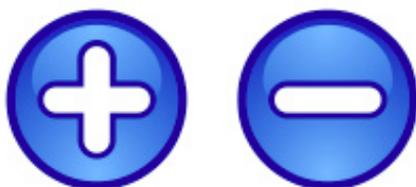
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# ADD AND SUBTRACT RATIONAL EXPRESSIONS

## LEARNING OUTCOMES

- Add and subtract rational expressions

In beginning math, students usually learn how to add and subtract whole numbers before they are taught multiplication and division. However, with fractions and rational expressions, multiplication and division are sometimes taught first because these operations are easier to perform than addition and subtraction. Addition and subtraction of rational expressions are not as easy to perform as multiplication because, as with numeric fractions, the process involves finding common denominators.



## Adding Rational Expressions

To find the least common denominator (LCD) of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, consider the following rational expressions:

$$\frac{6}{(x+3)(x+4)}, \text{ and } \frac{9x}{(x+4)(x+5)}$$

The LCD would be  $(x+3)(x+4)(x+5)$ .

To find the LCD, we count the greatest number of times a factor appears in each denominator and include it in the LCD that many times.

For example, in  $\frac{6}{(x+3)(x+4)}$ ,  $(x+3)$  is represented once and  $(x+4)$  is represented once, so they both appear exactly once in the LCD.

In  $\frac{9x}{(x+4)(x+5)}$ ,  $(x+4)$  appears once and  $(x+5)$  appears once.

We have already accounted for  $(x+4)$ , so the LCD just needs one factor of  $(x+5)$  to be complete.

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD.

What do we mean by "the form of 1"?

$\frac{x+5}{x+5} = 1$  so multiplying an expression by it will not change its value.

For example, we would need to multiply the expression  $\frac{6}{(x+3)(x+4)}$  by  $\frac{x+5}{x+5}$  and the expression  $\frac{9x}{(x+4)(x+5)}$  by  $\frac{x+3}{x+3}$ .

Hopefully this process will become clear after you practice it yourself. As you look through the examples on this page, try to identify the LCD before you look at the answers. Also, try figuring out which “form of 1” you will need to multiply each expression by so that it has the LCD.

### Example

Add the rational expressions  $\frac{5}{x} + \frac{6}{y}$  and define the domain.

State the sum in simplest form.

**Answer**

First, define the domain of each expression. Since we have  $x$  and  $y$  in the denominators, we can say  $x \neq 0$ , and  $y \neq 0$ .

Now we have to find the LCD. Since  $x$  appears once and  $y$  appears once, the LCD will be  $xy$ . We then multiply each expression by the appropriate form of 1 to obtain  $xy$  as the denominator for each fraction.

$$\frac{5}{x} \cdot \frac{y}{y} + \frac{6}{y} \cdot \frac{x}{x}$$

$$\frac{5y}{xy} + \frac{6x}{xy}$$

Now that the expressions have the same denominator, we simply add the numerators to find the sum.

$$\frac{6x+5y}{xy}$$

$$\frac{5}{x} + \frac{6}{y} = \frac{(6x+5y)}{xy}, \quad x \neq 0, \text{ and } y \neq 0$$

Here is one more example of adding rational expressions, but in this case, the expressions have denominators with multi-term polynomials. First, we will factor and then find the LCD. Note that  $x^2 - 4$  is a difference of squares and can be factored using special products.

In the video that follows, we present an example of adding two rational expression whose denominators are binomials with no common factors.

## Subtracting Rational Expressions

To subtract rational expressions, follow the same process you use to add rational expressions. You will need to be careful with signs though.

In the next example, we will give less instruction. See if you can find the LCD yourself before you look at the answer.

## EXAMPLE

Simplify  $\frac{2x^2}{x^2-4} + \frac{x}{x-2}$  and give the domain.

State the result in simplest form.

### Answer

Find the least common denominator by factoring each denominator. The least common denominator includes the maximum number of times it appears in a single factorization. Remember that  $x$  cannot be 2 or  $-2$  because the denominators would be 0.

$(x + 2)$  appears a maximum of one time and so does  $(x - 2)$ . This means the LCD is  $(x + 2)(x - 2)$ .

Multiply each expression by the equivalent of 1 that will give it the common denominator. Notice the first fraction already has the LCD and as a result remains unchanged.

$$\frac{2x^2}{x^2-4} = \frac{2x^2}{(x+2)(x-2)}$$

$$\frac{x}{x-2} \cdot \frac{x+2}{x+2} = \frac{x(x+2)}{(x+2)(x-2)}$$

Rewrite the original problem with the common denominator. It makes sense to keep the denominator in factored form in order to check for common factors.

$$\frac{2x^2}{(x+2)(x-2)} + \frac{x(x+2)}{(x+2)(x-2)}$$

Combine the numerators.

$$\frac{2x^2+x(x+2)}{(x+2)(x-2)}$$

$$\frac{2x^2+x^2+2x}{(x+2)(x-2)}$$

$$\frac{3x^2+2x}{(x+2)(x-2)}$$

Check for simplest form. Since neither  $(x + 2)$  nor  $(x - 2)$  is a factor of  $3x^2 + 2x$ , this expression is in simplest form.

$$\frac{2x^2}{x^2-4} + \frac{x}{x-2} = \frac{3x^2+2x}{(x+2)(x-2)} \quad x \neq 2, -2$$

The image shows a handwritten solution on a grid background. At the top, it says "Examples: Add / Subtract Rational Expressions - Unlike Binomial Denominators". The problem is  $\frac{x-3}{x+6} + \frac{x+7}{x-4}$ . The solution shows the fractions with their denominators factored:  $\frac{(x-3)(x-4)}{(x+6)(x-4)} + \frac{(x+7)(x+6)}{(x-4)(x+6)}$ . The numerators are then expanded:  $(x^2 - 7x + 12) + (x^2 + 13x + 42)$ . The combined numerator is  $2x^2 + 6x + 54$ , and the denominator is  $(x+6)(x-4)$ . The final simplified form is  $\frac{2(x^2 + 3x + 27)}{(x+6)(x-4)}$ , which is boxed in green.

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## EXAMPLE

Subtract  $\frac{2}{t+1} - \frac{t-2}{t^2-t-2}$  and define the domain.

State the difference in simplest form.

### Answer

Find the LCD of each expression.  $t + 1$  cannot be factored any further but  $t^2 - t - 2$  can be. Note that  $t$  cannot be  $-1$  or  $2$  because the denominators would be  $0$ .

Find the least common multiple.  $t + 1$  appears exactly once in both of the expressions, so it will appear once in the least common denominator.  $t - 2$  also appears once.

This means that  $(t - 2)(t + 1)$  is the LCD. In this case, it is easier to leave the common denominator in terms of the factors. You will not need to multiply it out.

Use the LCD as your new common denominator.

Compare each original denominator and the new common denominator. Now rewrite the rational expressions to each have the common denominator of  $(t + 1)(t - 2)$ .

In the first expression, you need to multiply  $t + 1$  by  $t - 2$  to get the LCD, so multiply the entire rational expression by  $\frac{t - 2}{t - 2}$ .

The second expression already has a denominator of  $(t + 1)(t - 2)$ , so you do not need to multiply it by anything.

$$\frac{2}{t+1} \cdot \frac{t-2}{t-2} = \frac{2(t-2)}{(t+1)(t-2)}$$

$$\frac{t-2}{t^2-t-2} = \frac{t-2}{(t+1)(t-2)}$$

Now rewrite the subtraction problem using the expressions with the common denominator.

$$\frac{2(t-2)}{(t+1)(t-2)} - \frac{t-2}{(t+1)(t-2)}$$

Subtract the numerators and simplify. Remember that parentheses need to be included around the second  $(t - 2)$  in the numerator because the whole quantity is subtracted. Otherwise, you would be subtracting just the  $t$ .

$$\frac{2(t-2)-(t-2)}{(t+1)(t-2)}$$

$$\frac{2t-4-t+2}{(t+1)(t-2)}$$

$$\frac{t-2}{(t+1)(t-2)}$$

The numerator and denominator have a common factor of  $t - 2$ , so the rational expression can be simplified.

$$\frac{\cancel{t-2}}{(t+1)\cancel{(t-2)}} = \frac{1}{t+1}$$

$$\frac{2}{t+1} - \frac{t-2}{t^2-t-2} = \frac{1}{t+1}, \quad t \neq -1, 2$$

### Example

Subtract the rational expressions:  $\frac{6}{x^2+4x+4} - \frac{2}{x^2-4}$ , and define the domain.

State the difference in simplest form.

### Answer

Note that the denominator of the first expression is a perfect square trinomial, and the denominator of the second expression is a difference of squares so they can be factored using special products.

$$\frac{6}{(x+2)^2} - \frac{2}{(x+2)(x-2)} \quad \text{Factor.}$$

$$\frac{6}{(x+2)^2} \cdot \frac{x-2}{x-2} - \frac{2}{(x+2)(x-2)} \cdot \frac{x+2}{x+2} \quad \text{Multiply each fraction to get LCD as denominator.}$$

$$\frac{6(x-2)}{(x+2)^2(x-2)} - \frac{2(x+2)}{(x+2)^2(x-2)} \quad \text{Multiply.}$$

$$\frac{6x-12-(2x+4)}{(x+2)^2(x-2)} \quad \text{Apply distributive property.}$$

$$\frac{4x-16}{(x+2)^2(x-2)} \quad \text{Subtract.}$$

$$\frac{4(x-4)}{(x+2)^2(x-2)} \quad \text{Simplify.}$$

The domain is  $x \neq -2, 2$

$$\frac{6}{x^2+4x+4} - \frac{2}{x^2-4} = \frac{4(x-4)}{(x+2)^2(x-2)}, \quad x \neq -2, 2$$

In the previous example, the LCD was  $(x+2)^2(x-2)$ . The reason we need to include  $(x+2)$  two times is because it appears two times in the expression  $\frac{6}{x^2+4x+4}$ .

The video that follows contains an example of subtracting rational expressions.

Add and Subtract Rational Expressions (Unlike Denominators)

EXAMPLE

Subtract. Then state the domain.

$$\frac{x^2 + 11x + 18}{x^2 - 2x - 8} - \frac{x + 3}{x - 4}$$

$$\frac{x^2 + 11x + 18}{(x - 4)(x + 2)} - \frac{(x + 3)(x + 2)}{(x - 4)(x + 2)}$$

$$= \frac{(x^2 + 11x + 18) - (x^2 + 5x + 6)}{(x - 4)(x + 2)}$$

$$= \frac{(x^2 + 11x + 18) - (x^2 + 5x + 6)}{(x - 4)(x + 2)}$$

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# COMPLEX RATIONAL EXPRESSIONS

## LEARNING OUTCOMES

- Simplify complex rational expressions

Fractions and rational expressions can be interpreted as quotients. When both the dividend (numerator) and divisor (denominator) include fractions or rational expressions, you have something more *complex* than usual. Do not fear—you have all the tools you need to simplify these quotients!

A **complex fraction** is the quotient of two fractions. These complex fractions are never considered to be in simplest form, but they can always be simplified using division of fractions. Remember, to divide fractions, you multiply by the reciprocal.

*Before* you multiply the numbers, it is often helpful to factor the fractions. You can then cancel factors.

### Example

Simplify.

$$\frac{\frac{12}{35}}{\frac{6}{7}}$$

**Answer**

Rewrite the complex fraction as a division problem.

$$\frac{\frac{12}{35}}{\frac{6}{7}} = \frac{12}{35} \div \frac{6}{7}$$

Rewrite the division as multiplication and take the reciprocal of the divisor.

$$= \frac{12}{35} \cdot \frac{7}{6}$$

Factor the numerator and denominator looking for common factors before multiplying numbers together.

$$= \frac{2 \cdot 6 \cdot 7}{5 \cdot 7 \cdot 6}$$

$$= \frac{2}{5} \cdot \frac{6 \cdot 7}{6 \cdot 7}$$

$$= \frac{2}{5} \cdot 1$$

$$\frac{\frac{12}{35}}{\frac{6}{7}} = \frac{2}{5}$$

If two fractions appear in the numerator or denominator (or both), first combine them. Then simplify the quotient as shown above.

## Example

Simplify.

$$\frac{\frac{3}{4} + \frac{1}{2}}{\frac{4}{5} - \frac{1}{10}}$$

**Answer**

First combine the numerator and denominator by adding or subtracting. You may need to find a common denominator first. Note that we do not show the steps for finding a common denominator, so please review that in the previous section if you are confused.

$$\frac{\frac{3}{4} + \frac{1}{2}}{\frac{4}{5} - \frac{1}{10}} = \frac{\frac{5}{4}}{\frac{7}{10}}$$

Rewrite the complex fraction as a division problem.

$$\frac{\frac{5}{4}}{\frac{7}{10}} = \frac{5}{4} \div \frac{7}{10}$$

Rewrite the division as multiplication and take the reciprocal of the divisor.

$$= \frac{5}{4} \cdot \frac{10}{7}$$

Multiply and simplify as needed.

$$\frac{5}{4} \cdot \frac{10}{7} = \frac{5 \cdot 5 \cdot 2}{2 \cdot 2 \cdot 7} = \frac{25}{14}$$

$$\frac{\frac{3}{4} + \frac{1}{2}}{\frac{4}{5} - \frac{1}{10}} = \frac{25}{14}$$

In the following video, we will show a couple more examples of how to simplify complex fractions.

**Complex Fractions**

Simplify.

$$\frac{\frac{5}{8}}{\frac{3}{4}} = \frac{5}{8} \div \frac{3}{4} = \frac{5}{8} \cdot \frac{4}{3} = \frac{5}{6}$$

$$\frac{\frac{2}{3} + \frac{1}{2}}{\frac{3}{4} - \frac{1}{12}} = \frac{\frac{4}{6} + \frac{3}{6}}{\frac{9}{12} - \frac{1}{12}} = \frac{\frac{7}{6}}{\frac{8}{12}} = \frac{7}{6} \div \frac{2}{3} = \frac{7}{6} \cdot \frac{3}{2} = \frac{7}{4}$$

2: 2, 4, 6, 8, 10, 12, ...  
 4: 4, 8, 12, 16, 20, ...  
 6: 6, 12, 18, 24, 30, ...  
 12: 12, 24, 36, ...

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# Complex Rational Expressions

A **complex rational expression** is a quotient with rational expressions in the dividend, divisor, or in both. Simplify these in the exact same way as you would a complex fraction.

## Example

Simplify.

$$\frac{\frac{x+5}{x^2-16}}{\frac{x^2-25}{x-4}}$$

**Answer**

Rewrite the complex rational expression as a division problem.

$$= \frac{x+5}{x^2-16} \div \frac{x^2-25}{x-4}$$

Rewrite the division as multiplication and take the reciprocal of the divisor. Note that the excluded values for this are  $-4$ ,  $4$ ,  $-5$ , and  $5$ , because those values make the denominators of one of the fractions zero.

$$= \frac{x+5}{x^2-16} \cdot \frac{x-4}{x^2-25}$$

Factor the numerator and denominator, looking for common factors. In this case,  $x+5$  and  $x-4$  are common factors of the numerator and denominator.

$$= \frac{(x+5)(x-4)}{(x+4)(x-4)(x+5)(x-5)}$$

$$= \frac{1}{(x+4)(x-5)}$$

$$\frac{\frac{x+5}{x^2-16}}{\frac{x^2-25}{x-4}} = \frac{1}{(x+4)(x-5)}, x \neq -4, 4, -5, 5$$

In the next video example, we will show that simplifying a complex fraction may require factoring first.

The video shows the following steps:

$$\frac{\frac{x-3}{x^2-25}}{\frac{x^2-9}{x-5}} = \frac{x-3}{x^2-25} \div \frac{x^2-9}{x-5}$$

$$= \frac{x-3}{x^2-25} \cdot \frac{x-5}{x^2-9}$$

$$= \frac{(x-3)(x-5)}{(x+5)(x-5)(x+3)(x-5)}$$

$$= \frac{x-3}{(x+5)(x+3)}$$

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The same ideas can be used when simplifying complex rational expressions that include more than one rational expression in the numerator or denominator. However, there is a shortcut that can be used. Compare these two examples of simplifying a complex fraction.

### Example

Simplify.

$$\frac{1 - \frac{9}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}}$$

**Answer**

Combine the expressions in the numerator and denominator. To do this, rewrite the expressions using a common denominator. There is an excluded value of 0 because this makes the denominators of the fractions zero.

$$\begin{aligned} &= \frac{\frac{x^2}{x^2} - \frac{9}{x^2}}{\frac{x^2}{x^2} + \frac{5x}{x^2} + \frac{6}{x^2}} \\ &= \frac{\frac{x^2 - 9}{x^2}}{\frac{x^2 + 5x + 6}{x^2}} \end{aligned}$$

Rewrite the complex rational expression as a division problem. (When you are comfortable with the step of rewriting the complex rational fraction as a division problem, you might skip this step and go straight to rewriting it as multiplication.)

$$= \frac{x^2 - 9}{x^2} \div \frac{x^2 + 5x + 6}{x^2}$$

Rewrite the division as multiplication and take the reciprocal of the divisor.

$$= \frac{x^2 - 9}{x^2} \cdot \frac{x^2}{x^2 + 5x + 6}$$

Factor the numerator and denominator looking for common factors. In this case,  $x + 3$  and  $x^2$  are common factors. We can now see there are two additional excluded values,  $-2$  and  $-3$ .

$$\begin{aligned} &= \frac{(x+3)(x-3)x^2}{x^2(x+3)(x+2)} \\ &= \frac{(x-3)}{(x+2)} \cdot \frac{x^2(x+3)}{x^2(x+3)} \end{aligned}$$

$$\frac{1 - \frac{9}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}} = \frac{x-3}{x+2}, x \neq -3, -2, 0$$

### Example

Simplify.

$$\frac{1 - \frac{9}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}}$$

**Answer**

Before combining the expressions, find a common denominator for all of the rational expressions. In this case,  $x^2$  is a common denominator. Multiply by 1 in the form of a fraction with the common denominator in both the numerator and denominator. In this case, multiply by  $\frac{x^2}{x^2}$ . There is an excluded value of 0 because this makes the denominators of the fractions zero.

$$= \frac{1 - \frac{9}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}} \cdot \frac{x^2}{x^2}$$

$$= \frac{\left(1 - \frac{9}{x^2}\right)x^2}{\left(1 + \frac{5}{x} + \frac{6}{x^2}\right)x^2}$$

$$= \frac{x^2 - 9}{x^2 + 5x + 6}$$

Notice that the expression is no longer complex! You can simplify by factoring and identifying common factors. We can now see there are two additional excluded values,  $-2$  and  $-3$ .

$$= \frac{(x+3)(x-3)}{(x+3)(x+2)}$$

$$= \frac{x+3}{x+3} \cdot \frac{x-3}{x+2}$$

$$= 1 \cdot \frac{x-3}{x+2}$$

$$\frac{1 - \frac{9}{x^2}}{1 + \frac{5}{x} + \frac{6}{x^2}} = \frac{x-3}{x+2}, x \neq -3, -2, 0$$

You may find the second method easier to use, but do try both ways to see what you prefer.

In our last example, we show a similar example as the one above.

**Complex Fractions**

Simplify:

$$\frac{1 - \frac{9}{x^2}}{1 - \frac{1}{x} - \frac{12}{x^2}} = \frac{\frac{1x^2 - 9}{x^2}}{\frac{1x^2 - 1x - 12}{x^2}} = \frac{x^2 - 9}{x^2 - x - 12}$$

$$= \frac{x^2 - 9}{x^2} \div \frac{x^2 - x - 12}{x^2} = \frac{x^2 - 9}{x^2} \cdot \frac{x^2}{x^2 - x - 12}$$

$$= \frac{(x-3)(x+3)}{x^2}$$

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## Summary

Complex rational expressions are quotients with rational expressions in the divisor, dividend, or both. When written in fraction form, they appear to be fractions within a fraction. These can be simplified by first treating

the quotient as a division problem. Then you can rewrite the division as multiplication and take the reciprocal of the divisor. Or you can simplify the complex rational expression by multiplying both the numerator and denominator by a denominator common to all rational expressions within the complex expression. This can help simplify the complex expression even faster.

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# INTRODUCTION TO RATIONAL EQUATIONS AND THEIR APPLICATIONS

In this section, we will learn techniques for solving equations that contain rational expressions. There are many applications for rational equations, from figuring out how many drinks to bring to a party to how best to plan driving time for a road trip. We will explore some of these applications, including rates, proportions, and variation.

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# SOLVE RATIONAL EQUATIONS

## LEARNING OUTCOMES

- Solve rational equations by clearing denominators
- Identify extraneous solutions in a rational expression

Equations that contain rational expressions are called **rational equations**. For example,  $\frac{2x+1}{4} = \frac{x}{3}$  is a rational equation. Rational equations can be useful for representing real-life situations and for finding answers to real problems. In particular, they are quite good for describing a variety of proportional relationships.

One of the most straightforward ways to solve a rational equation is to eliminate denominators with the common denominator and then use properties of equality to isolate the variable. This method is often used to solve linear equations that involve fractions as in the following example:

Solve  $\frac{1}{2}x - 3 = 2 - \frac{3}{4}x$  by clearing the fractions in the equation first.

Multiply both sides of the equation by 4, the common denominator of the fractional coefficients.

$$\frac{1}{2}x - 3 = 2 - \frac{3}{4}x$$

$$4\left(\frac{1}{2}x - 3\right) = 4\left(2 - \frac{3}{4}x\right)$$

$$4\left(\frac{1}{2}x\right) - 4(3) = 4(2) + 4\left(-\frac{3}{4}x\right)$$

$$2x - 12 = 8 - 3x$$

$$\begin{array}{r} +3x \\ 2x - 12 = 8 - 3x \end{array}$$

$$5x - 12 = 8$$

$$\begin{array}{r} +12 \\ 5x - 12 = 8 \end{array}$$

$$5x = 20$$

$$x = 4$$

We could have found a common denominator and worked with fractions, but that often leads to more mistakes. We can apply the same idea to solving rational equations. The difference between a linear equation and a rational equation is that rational equations can have polynomials in the numerator and denominator of the fractions. This means that clearing the denominator may sometimes mean multiplying the whole rational equation by a polynomial. In the next example, we will clear the denominators of a rational equation with a term that has a polynomial in the numerator.

### Example

Solve the equation  $\frac{x+5}{8} = \frac{7}{4}$ .

#### Answer

Find the least common denominator of 4 and 8. Remember, to find the LCD, identify the greatest number of times each factor appears in each factorization. Here, 2 appears 3 times, so  $2 \cdot 2 \cdot 2$ , or 8, will be the LCD.

Multiply both sides of the equation by the common denominator, 8, to keep the equation balanced and to eliminate the denominators.

$$8 \cdot \frac{x+5}{8} = \frac{7}{4} \cdot 8$$

$$\frac{8(x+5)}{8} = \frac{7(8)}{4}$$

$$\frac{8}{8} \cdot (x+5) = \frac{7(4 \cdot 2)}{4}$$

$$\frac{8}{8} \cdot (x+5) = 7 \cdot 2 \cdot \frac{4}{4}$$

$$1 \cdot (x+5) = 14 \cdot 1$$

Simplify and solve for  $x$ .

$$x + 5 = 14$$

$$x = 9$$

Check the solution by substituting 9 for  $x$  in the original equation.

$$\frac{x+5}{8} = \frac{7}{4}$$

$$\frac{9+5}{8} = \frac{7}{4}$$

$$\frac{14}{8} = \frac{7}{4}$$

$$\frac{7}{4} = \frac{7}{4}$$

Therefore,  $x = 9$ .

In the next example, we show how to solve a rational equation with a binomial in the denominator of one term. We will use the common denominator to eliminate the denominators from both fractions. Note that the LCD is the product of both denominators because they do not share any common factors.

### Example

Solve the equation  $\frac{8}{x+1} = \frac{4}{3}$ .

**Answer**

Clear the denominators by multiplying each side by the common denominator. The common denominator is  $3(x+1)$  since 3 and  $x+1$  do not have any common factors.

$$3(x+1)\left(\frac{8}{x+1}\right) = 3(x+1)\left(\frac{4}{3}\right)$$

Simplify common factors.

$$\cancel{3}(x+1)\left(\frac{8}{\cancel{x+1}}\right) = \cancel{3}(x+1)\left(\frac{4}{\cancel{3}}\right)$$

$$24 = 4(x+1)$$

$$24 = 4x + 4$$

Now this looks like a linear equation, and we can use the addition and multiplication properties of equality to solve it.

$$24 = 4x + 4$$

$$\underline{-4} \quad \underline{-4}$$

$$20 = 4x$$

$$x = 5$$

Check the solution in the original equation.

$$\frac{8}{(x+1)} = \frac{4}{3}$$

$$\frac{8}{(5+1)} = \frac{4}{3}$$

$$\frac{8}{6} = \frac{4}{3}$$

$$\frac{4}{3} = \frac{4}{3}$$

Therefore,  $x = 5$ .

### Example

Solve the equation  $\frac{x}{3} + 1 = \frac{4}{3}$ .

**Answer**

Both fractions in the equation have a denominator of 3. Multiply every term on both sides of the equation (not just the fractions!) by 3.

$$3\left(\frac{x}{3} + 1\right) = 3\left(\frac{4}{3}\right)$$

Apply the distributive property and multiply 3 by each term within the parentheses. Then simplify and solve for  $x$ .

$$3\left(\frac{x}{3}\right) + 3(1) = 3\left(\frac{4}{3}\right)$$

$$\cancel{3}\left(\frac{x}{\cancel{3}}\right) + 3(1) = \cancel{3}\left(\frac{4}{\cancel{3}}\right)$$

$$x + 3 = 4$$

$$\underline{-3} \quad \underline{-3}$$

$$x = 1$$

Therefore,  $x = 1$ .

In the video that follows, we present two ways to solve rational equations with both integer and variable denominators.

The video shows a math lesson on rational equations. The title is "Rational Equations" and it shows an example of solving a rational equation. The equation is  $\frac{x+8}{6} + 3 = \frac{3x+8}{8}$ . The video shows two methods: one using a common denominator of 24 and another using a common denominator of 8. The solution is  $x=16$ . The video also shows excluded values: none.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2934>

## Excluded Values and Extraneous Solutions

As you have seen, some rational expressions have a variable in the denominator. When this is the case, there is an extra step in solving them. Since division by 0 is undefined, you must exclude values of the variable that would result in a denominator of 0. These values are called **excluded values**. Let us look at an example.

### Example

Solve the equation  $\frac{2x-5}{x-5} = \frac{15}{x-5}$ .

**Answer**

Determine any values for  $x$  that would make the denominator 0.

$$\frac{2x-5}{x-5} = \frac{15}{x-5}$$

5 is an excluded value because it makes the denominator  $x - 5$  equal to 0.

Since the denominator of each expression in the equation is the same, the numerators must be equal. Set the numerators equal to one another and solve for  $x$ .

$$2x - 5 = 15$$

$$2x = 20$$

$$x = 10$$

Check the solution in the original equation.

$$\frac{2x-5}{x-5} = \frac{15}{x-5}$$

$$\frac{2(10)-5}{10-5} = \frac{15}{10-5}$$

$$\frac{20-5}{10-5} = \frac{15}{10-5}$$

$$\frac{15}{5} = \frac{15}{5}$$

Therefore,  $x = 10$ .

In the following video, we present an example of solving a rational equation with variables in the denominator.

The screenshot shows a video player interface with a grid background. At the top, it says "Rational Equations - Same Denominators" and "EXAMPLE". Below that, it says "Solve." and shows the equation  $\frac{4x-5}{x-3} = \frac{23}{x-3}$  and  $\frac{36}{x-6} = \frac{x^2}{x-6}$ . The video shows the following steps:

$$36 = x^2$$
$$\begin{array}{r} 36 = x^2 \\ -36 \quad -36 \\ \hline 0 = x^2 - 36 \end{array}$$
$$0 = (x - 6)(x + 6)$$

At the bottom, it says "excluded values: 6 (x ≠ 6)".

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=2934>

You have seen that there is more than one way to solve rational equations. Because both of these techniques manipulate and rewrite terms, sometimes they can produce solutions that do not work in the

original form of the equation. These types of answers are called **extraneous solutions**. That is why it is always important to check all solutions in the original equations—you may find that they yield untrue statements or produce undefined expressions.

### Example

Solve the equation  $\frac{16}{m+4} = \frac{m^2}{m+4}$ .

#### Answer

Determine any values for  $m$  that would make the denominator 0.  $-4$  is an excluded value because it makes  $m + 4$  equal to 0.

Since the denominator of each expression in the equation is the same, the numerators must be equal. Set the numerators equal to one another and solve for  $m$ .

$$16 = m^2$$

$$0 = m^2 - 16$$

$$0 = (m + 4)(m - 4)$$

$$0 = m + 4 \quad \text{or} \quad 0 = m - 4$$

$$m = -4 \quad \text{or} \quad m = 4$$

$$m = 4, -4$$

Check the solutions in the original equation.

Since  $m = -4$  leads to division by 0, it is an extraneous solution.

Notice, however, that  $m=4$  is a solution that results in a true statement.

$$\frac{16}{m+4} = \frac{m^2}{m+4}$$

$$\frac{16}{-4+4} = \frac{(-4)^2}{-4+4}$$

$$\frac{16}{0} = \frac{16}{0}$$

$-4$  is excluded because it leads to division by 0.

$$\frac{16}{4+4} = \frac{(4)^2}{4+4}$$

$$\frac{16}{8} = \frac{16}{8}$$

Therefore,  $m = 4$ .

## Summary

You can solve rational equations by finding a common denominator. By rewriting the equation so that all terms have the common denominator, you can solve for the variable using just the numerators. Or, you can multiply both sides of the equation by the least common denominator of all fractions so that all terms become polynomials instead of rational expressions.

An important step in solving rational equations is to reject any extraneous solutions from the final answer. Extraneous solutions are solutions that do not satisfy the original form of the equation because they produce untrue statements or are excluded values that cause a denominator to equal 0.

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## PROPORTIONS

### LEARNING OUTCOME

- Write proportions to solve proportion problems



*Matryoshka or nesting dolls.*

A proportion is a statement that two ratios are equal to each other. There are many things that can be represented with ratios, including the actual distance on the earth that is represented on a map. In fact, you probably use proportional reasoning on a regular basis and do not realize it. For example, say you have volunteered to provide drinks for a community event. You are asked to bring enough drinks for 35 – 40

people. At the store you see that drinks come in packages of 12. You multiply 12 by 3 and get 36 – this may not be enough if 40 people show up, so you decide to buy 4 packages of drinks just to be sure.

This process can also be expressed as a proportional equation and solved using mathematical principles. First, we can express the number of drinks in a package as a ratio:

$$\frac{12 \text{ drinks}}{1 \text{ package}}$$

Then we express the number of people who we are buying drinks for as a ratio with the unknown number of packages we need. We will use the maximum so we have enough.

$$\frac{40 \text{ people}}{x \text{ packages}}$$

We can find out how many packages to purchase by setting the expressions equal to each other:

$$\frac{12 \text{ drinks}}{1 \text{ package}} = \frac{40 \text{ people}}{x \text{ packages}}$$

To solve for  $x$ , we can use techniques for solving linear equations, or we can cross multiply as a shortcut.

$$\frac{12 \text{ drinks}}{1 \text{ package}} = \frac{40 \text{ people}}{x \text{ packages}}$$

$$x \cdot \frac{12 \text{ drinks}}{1 \text{ package}} = \frac{40 \text{ people}}{x \text{ packages}} \cdot x$$

$$12x = 40$$

$$x = \frac{40}{12} = \frac{10}{3} = 3.33$$

We can round up to 4 since it does not make sense to buy 0.33 of a package of drinks. Of course, you do not write out your thinking this way when you are in the grocery store, but doing so helps you to be able to apply the concepts to less obvious problems. In the following example, we will show how to use a proportion to find the number of people on the planet who do not have access to a toilet. Because, why not?

In the next example, we will use the length of a person's femur to estimate their height. This process is used in forensic science and anthropology. It has been found in many scientific studies to be a very good estimate.

Another way to describe the ratio of femur length to height that we found in the last example is to say there is a 1 : 4 ratio between femur length and height, or 1 to 4.

Ratios are also used in scale drawings. Scale drawings are enlarged or reduced drawings of objects, buildings, roads, and maps. Maps are smaller than what they represent and a drawing of dendritic cells in your brain is most likely larger than what it represents. The scale of the drawing is a ratio that represents a comparison of the length of the actual object and its representation in the drawing. The image below shows a map of the United States with a scale of 1 inch representing 557 miles. We could write the scale factor as a fraction  $\frac{1}{557}$  or as we did with the femur-height relationship, 1 : 557.

## EXAMPLE

As of March, 2016 the world's population was estimated at 7.4 billion. (Note: "Current World Population." World Population Clock: 7.4 Billion People (2016). Accessed June 21, 2016.

<http://www.worldometers.info/world-population/>. "Current World Population." World Population Clock: 7.4 Billion People (2016). Accessed June 21, 2016. <http://www.worldometers.info/world-population/>. "Current World Population." World Population Clock: 7.4 Billion People (2016). Accessed June 21, 2016.

<http://www.worldometers.info/world-population/>). According to [water.org](http://water.org), 1 out of every 3 people on the planet lives without access to a toilet. Find the number of people on the planet that do not have access to a toilet.

### Answer

We can use a proportion to find the unknown number of people who live without a toilet since we are given that 1 in 3 do not have access, and we are given the population of the planet.

We know that 1 out of every 3 people do not have access, so we can write that as a ratio (fraction).

$$\frac{1}{3}$$

Let the number of people without access to a toilet be  $x$ . The ratio of people with and without toilets is then

$$\frac{x}{7.4 \text{ billion}}$$

Equate the two ratios since they are representing the same fractional amount of the population.

$$\frac{1}{3} = \frac{x}{7.4 \text{ billion}}$$

Solve:

$$\frac{1}{3} = \frac{x}{7.4}$$

$$7.4 \cdot \frac{1}{3} = \frac{x}{7.4} \cdot 7.4$$

$$2.46 = x$$

The original units were billions of people, so our answer is 2.46 billion people do not have access to a toilet. Wow, that is a lot of people.

## EXAMPLE

It has been shown that a person's height is proportional to the length of their femur (Note: Obialor, Ambrose, Churchill Ihentuge, and Frank Akapuaka. "Determination of Height Using Femur Length in Adult Population of Oguta Local Government Area of Imo State Nigeria." Federation of American Societies for Experimental Biology, April 2015. Accessed June 22, 2016.

[http://www.fasebj.org/content/29/1\\_Supplement/LB19.short](http://www.fasebj.org/content/29/1_Supplement/LB19.short)). Given that a person who is 71 inches tall has a femur length of 17.75 inches, how tall is someone with a femur length of 16 inches?

### Answer

Height and femur length are proportional for everyone, so we can define a ratio with the given height and femur length. We can then use this to write a proportion to find the unknown height.

Let  $x$  be the unknown height. Define the ratio of femur length and height for both people using the given measurements.

$$\text{Person 1: } \frac{\text{femur length}}{\text{height}} = \frac{17.75 \text{ inches}}{71 \text{ inches}}$$

$$\text{Person 2: } \frac{\text{femur length}}{\text{height}} = \frac{16 \text{ inches}}{x \text{ inches}}$$

Equate the ratios since we are assuming height and femur length are proportional for everyone.

$$\frac{17.75 \text{ inches}}{71 \text{ inches}} = \frac{16 \text{ inches}}{x \text{ inches}}$$

Solve by using the common denominator to clear fractions. The common denominator is  $71x$ .

$$\frac{17.75}{71} = \frac{16}{x}$$

$$71x \cdot \frac{17.75}{71} = \frac{16}{x} \cdot 71x$$

$$17.75 \cdot x = 16 \cdot 71$$

$$17.75 \cdot x = 1136$$

$$x = \frac{1136}{17.75} = 64$$

The unknown height of person 2 is 64 inches. In general, we can simplify the fraction  $\frac{17.75}{71} = 0.25 = \frac{1}{4}$  to find a general rule for everyone. This would translate to saying a person's height is 4 times the length of their femur.



A map of the United States with a scale factor.

In the next example, we will use the scale factor given in the image above to find the distance between Seattle, Washington and San Jose, California.

### EXAMPLE

Given a scale factor of 1 : 557 on a map of the US, if the distance from Seattle, WA to San Jose, CA is 1.5 inches on the map, define a proportion to find the actual distance between them.

**Answer**

We need to define a proportion to solve for the unknown distance between Seattle and San Jose.

The scale factor is 1 : 557, and we will call the unknown distance  $x$ . The ratio of inches to miles is  $\frac{1}{557}$ .

We know the inches between the two cities, but we do not know miles, so the ratio that describes the distance between them is  $\frac{1.5}{x}$ .

The proportion that will help us solve this problem is  $\frac{1}{557} = \frac{1.5}{x}$ .

Solve using the common denominator  $557x$  to clear fractions.

$$\frac{1}{557} = \frac{1.5}{x}$$

$$557x \cdot \frac{1}{557} = \frac{1.5}{x} \cdot 557x$$

$$x = 1.5 \cdot 557 = 835.5$$

We used the scale factor 1 : 557 to find an unknown distance between Seattle and San Jose. We also checked our answer of 835.5 miles with Google maps and found that the distance is 839.9 miles, so we did pretty well!

In the next example, we will find a scale factor given the length between two cities on a map and their actual distance from each other.

In the video that follows, we present an example of using proportions to obtain the correct amount of

## EXAMPLE

Two cities are 2.5 inches apart on a map. Their actual distance from each other is 325 miles. Write a proportion to represent and solve for the scale factor for one inch of the map.

### Answer

We know that for each 2.5 inches on the map, it represents 325 actual miles. We are looking for the scale factor for one inch of the map.

The ratio we want is  $\frac{1}{x}$  where  $x$  is the actual distance represented by one inch on the map. We know that for every 2.5 inches, there are 325 actual miles, so we can define that relationship as  $\frac{2.5}{325}$

We can use a proportion to equate the two ratios and solve for the unknown distance.

$$\frac{1}{x} = \frac{2.5}{325}$$

$$325x \cdot \frac{1}{x} = \frac{2.5}{325} \cdot 325x$$

$$325 = 2.5x$$

$$x = 130$$

The scale factor for one inch on the map is 1 : 130 or for every inch of map there are 130 actual miles.

medication for a patient as well as finding a desired mixture of coffees.

**Proportion Application: Medication**

As a nurse, part of your daily duties is to mix medications in the proper proportions for your patients. For one patient, you always mix Medication A with Medication B in the same proportion. Last week, your patient's doctor indicated that you should mix 60 milligrams of Medication A with 48 milligrams of Medication B. However this week, the doctor said to only use 40 milligrams of Medication B. How many milligrams of Medication A should be mixed this week?

$\frac{60}{48} = \frac{A}{40} \rightarrow 48$

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## VARIATION

## LEARNING OUTCOMES

- Define direct variation and solve problems involving direct variation
- Define inverse variation and solve problems involving inverse variation
- Define joint variation and solve problems involving joint variation



*So many cars, so many tires.*

## Direct Variation

Variation equations are examples of rational formulas and are used to describe the relationship between variables. For example, imagine a parking lot filled with cars. The total number of tires in the parking lot is dependent on the total number of cars. Algebraically, you can represent this relationship with an equation.

$$\text{number of tires} = 4 \cdot \text{number of cars}$$

The number 4 tells you the rate at which cars and tires are related. You call the rate the **constant of variation**. It is a constant because this number does not change. Because the number of cars and the number of tires are linked by a constant, changes in the number of cars cause the number of tires to change in a proportional, steady way. This is an example of **direct variation**, where the number of tires varies directly with the number of cars.

You can use the car and tire equation as the basis for writing a general algebraic equation that will work for all examples of direct variation. In the example, the number of tires is the output, 4 is the constant, and the number of cars is the input. Let us enter those generic terms into the equation. You get  $y = kx$ . That is the formula for all direct variation equations.

number of tires = 4 · number of cars  
output = constant · input

### Example

Solve for  $k$ , the constant of variation, in a direct variation problem where  $y = 300$  and  $x = 10$ .

**Answer**

Write the formula for a direct variation relationship.

$$y = kx$$

Substitute known values into the equation.

$$300 = k(10)$$

Solve for  $k$  by dividing both sides of the equation by 10.

$$\frac{300}{10} = \frac{10k}{10}$$

$$30 = k$$

In the video that follows, we present an example of solving a direct variation equation.

**Direct Variation**  
The number of aluminum cans used each year varies directly as the number of people using the cans. If 500 people use 12,000 cans in one year, how many cans are used each year in a city which has a population of 130,000?

$x =$  the number of people  
 $y =$  the number of cans in one year

$$y = kx$$
$$x = 500 \rightarrow y = 12,000$$
$$12,000 = k(500)$$
$$\frac{12,000}{500} = \frac{k(500)}{500}$$
$$240 = k$$

**Direct Variation**  
"y varies directly as x"  
 $y = kx$   
 $k = \frac{y}{x}$   
 $k =$  variation constant

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<https://courses.lumenlearning.com/intermediatealgebra/?p=2942>

## Inverse Variation

Another kind of variation is called **inverse variation**. In these equations, the output equals a constant divided by the input variable that is changing. In symbolic form, this is the equation  $y = \frac{k}{x}$ .

One example of an inverse variation is the speed required to travel between two cities in a given amount of time.

Let us say you need to drive from Boston to Chicago which is about 1,000 miles. The more time you have, the slower you can go. If you want to get there in 20 hours, you need to go 50 miles per hour (assuming you do not stop driving!) because  $\frac{1,000}{20} = 50$ . But if you can take 40 hours to get there, you only have to average 25 miles per hour since  $\frac{1,000}{40} = 25$ .

The equation for figuring out how fast to travel from the amount of time you have is  $speed = \frac{miles}{time}$ . This equation should remind you of the distance formula  $d = rt$ . If you solve  $d = rt$  for  $r$ , you get  $r = \frac{d}{t}$ , or  $speed = \frac{miles}{time}$ .

In the case of the Boston to Chicago trip, you can write  $s = \frac{1,000}{t}$ . Notice that this is the same form as the inverse variation function formula  $y = \frac{k}{x}$ .

#### Example

Solve for  $k$ , the constant of variation, in an inverse variation problem where  $x = 5$  and  $y = 25$ .

**Answer**

Write the formula for an inverse variation relationship.

$$y = \frac{k}{x}$$

Substitute known values into the equation.

$$25 = \frac{k}{5}$$

Solve for  $k$  by multiplying both sides of the equation by 5.

$$5 \cdot 25 = \frac{k}{5} \cdot 5$$

$$125 = \frac{5k}{5}$$

$$125 = k$$

In the next example, we will find the water temperature in the ocean at a depth of 500 meters. Water temperature is inversely proportional to depth in the ocean.



*Water temperature in the ocean varies inversely with depth.*

In the video that follows, we present an example of inverse variation.

## EXAMPLE

The water temperature in the ocean varies inversely with the depth of the water. The deeper a person dives, the colder the water becomes. At a depth of 1,000 meters, the water temperature is 5° Celsius. What is the water temperature at a depth of 500 meters?

### Answer

You are told that this is an inverse relationship and that the water temperature ( $y$ ) varies inversely with the depth of the water ( $x$ ).

$$y = \frac{k}{x}$$

$$\text{temp} = \frac{k}{\text{depth}}$$

Substitute known values into the equation.

$$5 = \frac{k}{1,000}$$

Solve for  $k$ .

$$1,000 \cdot 5 = \frac{k}{1,000} \cdot 1,000$$

$$5,000 = \frac{1,000k}{1,000}$$

$$5,000 = k$$

Now that  $k$ , the constant of variation is known, use that information to solve the problem: find the water temperature at 500 meters.

$$\text{temp} = \frac{k}{\text{depth}}$$

$$\text{temp} = \frac{5,000}{500}$$

$$\text{temp} = 10$$

At 500 meters, the water temperature is 10° C.

**Examples: Inverse Variation**

The time,  $t$ , required to do a job varies inversely with the number of people,  $p$ , working on the job. If it takes 6 hour for 8 workers to complete a job. How long would it take if there are 9 workers?

$$t = \frac{k}{p}$$

$t = 6$     $p = 8$

Inverse Variation Equations

$$y = \frac{k}{x}$$
$$xy = k$$

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## Joint Variation

A third type of variation is called **joint variation**. Joint variation is the same as direct variation except there are two or more quantities. For example, the area of a rectangle can be found using the formula  $A = lw$ , where  $l$  is the length of the rectangle and  $w$  is the width of the rectangle. If you change the width of the rectangle, then the area changes and similarly if you change the length of the rectangle then the area will also change. You can say that the area of the rectangle “varies jointly with the length and the width of the rectangle.”

The formula for the volume of a cylinder,  $V = \pi r^2 h$ , is another example of joint variation. The volume of the cylinder varies jointly with the square of the radius and the height of the cylinder. The constant of variation is  $\pi$ .

### Example

The area of a triangle varies jointly with the lengths of its base and height. If the area of a triangle is 30 inches<sup>2</sup> when the base is 10 inches and the height is 6 inches, find the variation constant and the area of a triangle whose base is 15 inches and height is 20 inches.

#### Answer

You are told that this is a joint variation relationship and that the area of a triangle ( $A$ ) varies jointly with the lengths of the base ( $b$ ) and height ( $h$ ).

$$y = kxz$$

$$\text{Area} = k(\text{base})(\text{height})$$

Substitute known values into the equation and solve for  $k$ .

$$30 = k(10)(6)$$

$$30 = 60k$$

$$\frac{30}{60} = \frac{60k}{60}$$

$$\frac{1}{2} = k$$

Now that  $k$  is known, solve for the area of a triangle whose base is 15 inches and height is 20 inches.

$$\text{Area} = k(\text{base})(\text{height})$$

$$\text{Area} = (15)(20)\left(\frac{1}{2}\right)$$

$$\text{Area} = \frac{300}{2}$$

$$\text{Area} = 150 \text{ square inches}$$

The constant of variation,  $k$ , is  $\frac{1}{2}$ , and the area of the triangle is 150 square inches.

Finding  $k$  to be  $\frac{1}{2}$  should not be surprising. You know that the area of a triangle is one-half base times height,  $A = \frac{1}{2}bh$ . The  $\frac{1}{2}$  in this formula is exactly the same  $\frac{1}{2}$  that you calculated in this example!

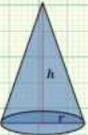
In the following video, we show an example of finding the constant of variation for a jointly varying relation.

## Summary

**Joint Variation**

**EXAMPLE**

The volume of a cone varies jointly with the square of the radius and the height. If the volume of a cone is  $V = 25\pi \text{ m}^3$  when the radius is 5 m and the height is 3 m, determine the variation constant.



$V = kr^2h$        $V = 25\pi \text{ m}^3$   
 $25\pi = k(5)^2(3)$        $r = 5 \text{ m}$   
 $25\pi = 75k$        $h = 3 \text{ m}$

**Joint Variation**  
 $y = kxz$   
 "y varies jointly with x and z."  
 k = constant of variation

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## DIRECT, JOINT, AND INVERSE VARIATION

$k$  is the constant of variation. In all cases,  $k \neq 0$ .

- Direct variation:  $y = kx$
- Inverse variation:  $y = \frac{k}{x}$
- Joint variation:  $y = kxz$

Rational formulas can be used to solve a variety of problems that involve rates, times, and work. Direct, inverse, and joint variation equations are examples of rational formulas. In direct variation, the variables have a direct relationship—as one quantity increases, the other quantity will also increase. As one quantity decreases, the other quantity decreases. In inverse variation, the variables have an inverse relationship—as one variable increases, the other variable decreases, and vice versa. Joint variation is the same as direct variation except there are two or more variables.

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# APPLICATIONS WITH RATIONAL EQUATIONS

## Learning Outcomes

- Solve a rational formula for a specified variable
- Solve work problems
- Define and solve an equation that represents the concentration of a mixture

## Rational Formulas

**Rational formulas** can be useful tools for representing real-life situations and for finding answers to real problems. Equations representing direct, inverse, and joint variation are examples of rational formulas that can model many real-life situations. As you will see, if you can find a formula, you can usually make sense of a situation.

When solving problems using rational formulas, it is often helpful to first solve the formula for the specified variable. For example, work problems ask you to calculate how long it will take different people working at different speeds to finish a task. The algebraic models of such situations often involve rational equations derived from the work formula,  $W = rt$ . The amount of work done ( $W$ ) is the product of the rate of work ( $r$ ) and the time spent working ( $t$ ). Using algebra, you can write the work formula 3 ways:

$$W = rt$$

Find the time ( $t$ ):  $t = \frac{W}{r}$  (divide both sides by  $r$ )

Find the rate ( $r$ ):  $r = \frac{W}{t}$  (divide both sides by  $t$ )

### Example

The formula for finding the density of an object is  $D = \frac{m}{v}$ , where  $D$  is the density,  $m$  is the mass of the object, and  $v$  is the volume of the object. Rearrange the formula to solve for the mass ( $m$ ) and then for the volume ( $v$ ).

#### Answer

Start with the formula for density.

$$D = \frac{m}{v}$$

Multiply both side of the equation by  $v$  to isolate  $m$ .

$$v \cdot D = \frac{m}{v} \cdot v$$

Simplify and rewrite the equation, solving for  $m$ .

$$v \cdot D = m \cdot \frac{v}{v}$$

$$v \cdot D = m \cdot 1$$

$$v \cdot D = m$$

To solve the equation  $D = \frac{m}{v}$  in terms of  $v$ , you will need do the same steps to this point and then divide both sides by  $D$ .

$$\frac{v \cdot D}{D} = \frac{m}{D}$$

$$\frac{D}{D} \cdot v = \frac{m}{D}$$

$$1 \cdot v = \frac{m}{D}$$

$$v = \frac{m}{D}$$

Therefore,  $m = D \cdot v$  and  $v = \frac{m}{D}$ .

Now let us look at an example using the formula for the volume of a cylinder.

### Example

The formula for finding the volume of a cylinder is  $V = \pi r^2 h$ , where  $V$  is the volume,  $r$  is the radius, and  $h$  is the height of the cylinder. Rearrange the formula to solve for the height ( $h$ ).

#### Answer

Start with the formula for the volume of a cylinder.

$$V = \pi r^2 h$$

Divide both sides by  $\pi r^2$  to isolate  $h$ .

$$\frac{V}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2}$$

Simplify. You find the height,  $h$ , is equal to  $\frac{V}{\pi r^2}$ .

$$\frac{V}{\pi r^2} = h$$

Therefore,  $h = \frac{V}{\pi r^2}$ .

In the following video, we give another example of solving for a variable in a formula, or as they are also called, a literal equation.

Solving Equations for Variables

Solve the following equations for the given variable.

$I = Prt$ for $P$ Answer: $P = \frac{I}{rt}$ Preview	$11x + 2y = 6$ for $y$ Answer: $y = -\frac{11}{2}x + 3$ Preview
$a = b(c+d)$ for $c$ Answer: <input type="text"/> Preview	$A = p + prt$ for $t$ Answer: <input type="text"/> Preview

$$\frac{a}{b} = \frac{b(c+d)}{b}$$
$$\frac{a}{b} = c + d$$
$$-d$$

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## Work

Rational equations can be used to solve a variety of problems that involve rates, times, and work. Using rational expressions and equations can help you answer questions about how to combine workers or machines to complete a job on schedule.



*Finished after a good day's work.*

A "work problem" is an example of a real life situation that can be modeled and solved using a rational equation. Work problems often ask you to calculate how long it will take different people working at different speeds to finish a task. The algebraic models of such situations often involve rational equations derived from

the work formula,  $W = rt$ . (Notice that the work formula is very similar to the relationship between distance, rate, and time, or  $d = rt$ .) The amount of work done ( $W$ ) is the product of the rate of work ( $r$ ) and the time spent working ( $t$ ). The work formula has 3 versions.

$$W = rt$$

$$t = \frac{W}{r}$$

$$r = \frac{W}{t}$$

Some work problems include multiple machines or people working on a project together for the same amount of time but at different rates. In that case, you can add their individual work rates together to get a total work rate. Let us look at an example.

### Example

Myra takes 2 hours to plant 50 flower bulbs. Francis takes 3 hours to plant 45 flower bulbs. Working together, how long should it take them to plant 150 bulbs?

#### Answer

Think about how many bulbs each person can plant in one hour. This is their planting rate.

Myra:  $\frac{50 \text{ bulbs}}{2 \text{ hours}}$  or  $\frac{25 \text{ bulbs}}{1 \text{ hour}}$

Francis:  $\frac{45 \text{ bulbs}}{3 \text{ hours}}$  or  $\frac{15 \text{ bulbs}}{1 \text{ hour}}$

Combine their hourly rates to determine the rate they work together.

Myra and Francis together:

$$\frac{25 \text{ bulbs}}{1 \text{ hour}} + \frac{15 \text{ bulbs}}{1 \text{ hour}} = \frac{40 \text{ bulbs}}{1 \text{ hour}}$$

Use one of the work formulas to write a rational equation, for example,  $r = \frac{W}{t}$ . You know  $r$ , the combined work rate, and you know  $W$ , the amount of work that must be done. What you do not know is how much time it will take to do the required work at the designated rate.

$$\frac{40}{1} = \frac{150}{t}$$

Solve the equation by multiplying both sides by the common denominator and then isolating  $t$ .

$$1t \cdot \frac{40}{1} = \frac{150}{t} \cdot 1t$$

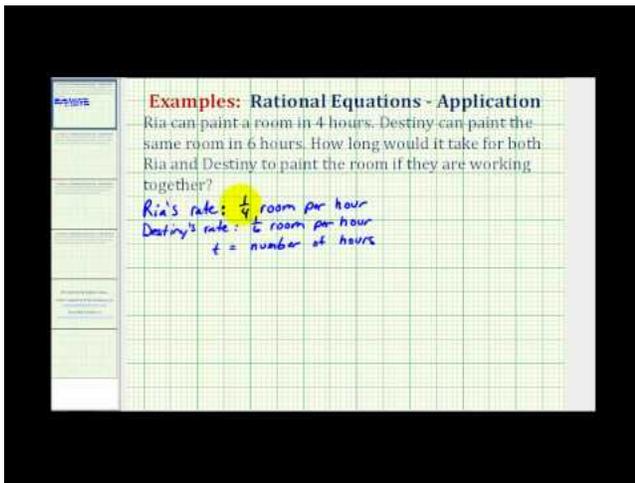
$$40t = 150$$

$$t = \frac{150}{40} = \frac{15}{4}$$

$$t = 3\frac{3}{4} \text{ hours}$$

It should take 3 hours 45 minutes for Myra and Francis to plant 150 bulbs together.

Other work problems go the other way. You can calculate how long it will take one person to do a job alone when you know how long it takes people working together to complete the job.



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### Example

Joe and John are planning to paint a house together. John thinks that if he worked alone, it would take him 3 times as long as it would take Joe to paint the entire house. Working together, they can complete the job in 24 hours. How long would it take each of them, working alone, to complete the job?

#### Answer

Choose variables to represent the unknowns. Since it takes John 3 times as long as Joe to paint the house, his time is represented as  $3x$ .

Let  $x$  = time it takes Joe to complete the job

$3x$  = time it takes John to complete the job

The work is painting 1 house or 1. Write an expression to represent each person's rate using the formula

$$r = \frac{W}{t}$$

Joe's rate:  $\frac{1}{x}$

John's rate:  $\frac{1}{3x}$

Their combined rate is the sum of their individual rates. Use this rate to write a new equation using the formula  $W = rt$ .

combined rate:  $\frac{1}{x} + \frac{1}{3x}$

The problem states that it takes them 24 hours together to paint a house, so if you multiply their combined hourly rate  $(\frac{1}{x} + \frac{1}{3x})$  by 24, you will get 1, which is the number of houses they can paint in 24 hours.

$$1 = (\frac{1}{x} + \frac{1}{3x}) 24$$

$$1 = \frac{24}{x} + \frac{24}{3x}$$

Now solve the equation for  $x$ . (Remember that  $x$  represents the number of hours it will take Joe to finish the job.)

$$1 = \frac{3}{3} \cdot \frac{24}{x} + \frac{24}{3x}$$

$$1 = \frac{3 \cdot 24}{3x} + \frac{24}{3x}$$

$$1 = \frac{72}{3x} + \frac{24}{3x}$$

$$1 = \frac{72+24}{3x}$$

$$1 = \frac{96}{3x}$$

$$3x = 96$$

$$x = 32$$

Check the solution in the original equation.

$$1 = \left(\frac{1}{x} + \frac{1}{3x}\right) 24$$

$$1 = \left[\frac{1}{32} + \frac{1}{3(32)}\right] 24$$

$$1 = \frac{24}{32} + \frac{24}{3(32)}$$

$$1 = \frac{24}{32} + \frac{24}{96}$$

$$1 = \frac{3}{3} \cdot \frac{24}{32} + \frac{24}{96}$$

$$1 = \frac{72}{96} + \frac{24}{96}$$

The solution checks. Since  $x = 32$ , it takes Joe 32 hours to paint the house by himself. John's time is  $3x$ , so it would take him 96 hours to do the same amount of work.

In the video that follows, we show another example of finding one person's work rate given a combined work rate.

**Applications of Rational Equations**

If Carlos does a job in 12 hours and with the help of Destiny they can do it together in 5 hours, how long would it take Destiny to do it alone?

	Work (job)	Rate per hour	Time
Carlos	1	$\frac{1}{12}$	12
Destiny	1	$\frac{1}{x}$	$x$
Together	1	$\frac{1}{5}$	5

$$\frac{1}{12} + \frac{1}{x} = \frac{1}{5}$$

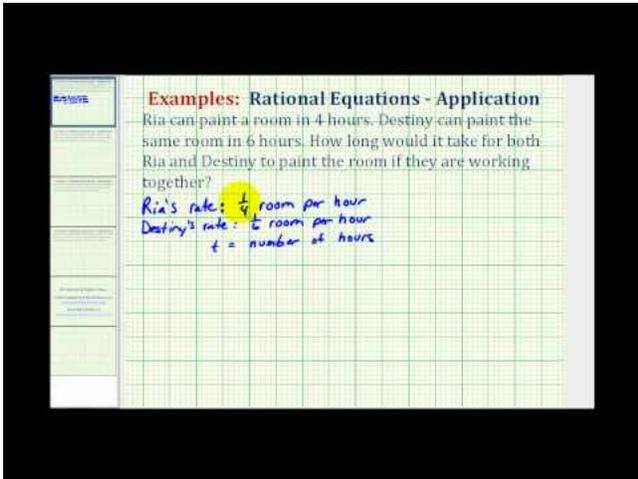
Work = Rate  $\times$  Time

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As shown above, many work problems can be represented by the equation  $\frac{t}{a} + \frac{t}{b} = 1$ , where  $t$  is the time to do the job together,  $a$  is the time it takes person A to do the job, and  $b$  is the time it takes person B to do the job. The 1 refers to the total work done—in this case, the work was to paint 1 house.

The key idea here is to figure out each worker's individual rate of work. Then, once those rates are identified, add them together, multiply by the time  $t$ , set it equal to the amount of work done, and solve the rational equation.

We present another example of two people painting at different rates in the following video.



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## Mixing

Mixtures are made of ratios of different substances that may include chemicals, foods, water, or gases. There are many different situations where mixtures may occur both in nature and as a means to produce a desired product or outcome. For example, chemical spills, manufacturing, and even biochemical reactions involve mixtures. The thing that can make mixtures interesting mathematically is when components of the mixture are added at different rates and concentrations. In our final example, we will define an equation that models the concentration – or ratio of sugar to water – in a large mixing tank over time. You are asked whether the final concentration of sugar is greater than the concentration at the beginning.

In the following video, we show another example of how to use rational functions to model mixing.

## EXAMPLE

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

### Answer

Let  $t$  be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is a linear equation, as is the amount of sugar in the tank. We can write an equation independently for each:

$$\begin{cases} \text{water: } W(t) = 100 + 10t \text{ in gallons} \\ \text{sugar: } S(t) = 5 + 1t \text{ in pounds} \end{cases}$$

The concentration,  $C$ , will be the ratio of pounds of sugar to gallons of water

$$C(t) = \frac{5+t}{100+10t}$$

The concentration after 12 minutes is given by evaluating  $C(t)$  at  $t = 12$ .

$$C(12) = \frac{5+12}{100+10(12)} = \frac{17}{220}$$

This means the concentration is 17 pounds of sugar to 220 gallons of water.

At the beginning, the concentration is

$$C(0) = \frac{5+0}{100+10(0)} = \frac{5}{100} = \frac{1}{20}$$

Since  $\frac{17}{220} \approx 0.08 > \frac{1}{20} = 0.05$ , the concentration is greater after 12 minutes than at the beginning.

The screenshot shows a video player with a title "Application of Rational Function - Concentration". The video content includes the following text:

**EXAMPLE**

A large mixing tank currently contains 200 gallons of water into which 40 pounds of sugar have been mixed. A tap will open pouring 15 gallons of water per minute into the tank at the same time sugar is being poured into the tank at the rate of 2 pounds per minute. Find the concentration (pounds per gallon) of sugar in the tank after 8 minutes. Is the concentration after 8 minutes greater than the beginning concentration?

$t$  = time in minutes  
 $W(t)$  = the amount of water in gallons after  $t$  minutes  
 $S(t)$  = the amount of sugar in pounds after  $t$  minutes

$W(t) = 200 + 15t$   
 $S(t) = 40 + 2t$

The concentration is the ratio of pounds of sugar to gallons of water.  
 $C(t)$  = the concentration after  $t$  minutes

$C(t) = \frac{S(t)}{W(t)} =$

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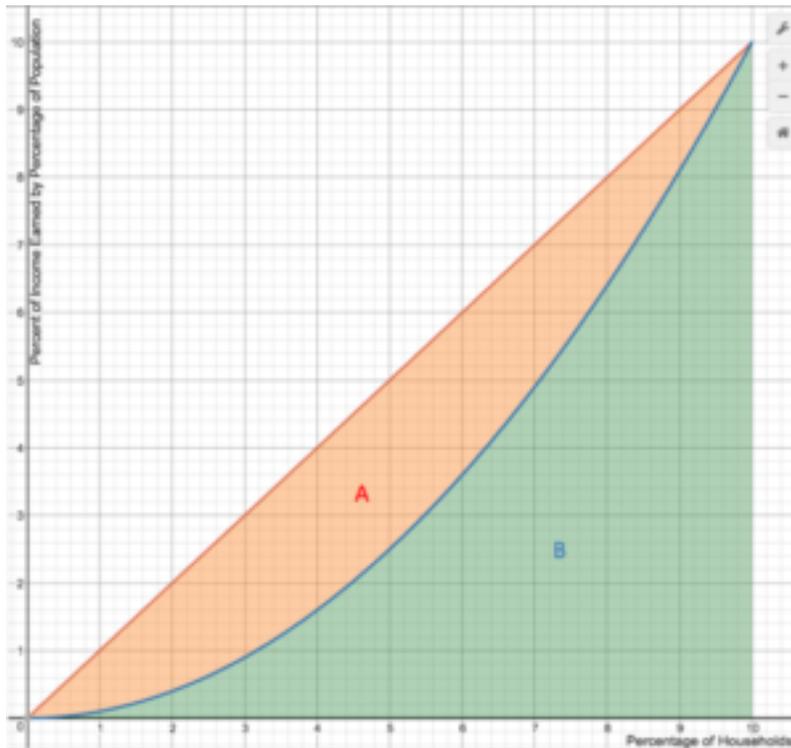
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# PUTTING IT TOGETHER: RATIONAL EXPRESSIONS AND EQUATIONS

In the beginning of this module, we presented the Gini coefficient. The Gini coefficient is the ratio of how far an economy departs from unequal distribution of wealth to a perfectly equal distribution of wealth as shown in the graph below.



It is great to have a tool to describe an issue like income inequality that ripples through social, economic, and political realms so pervasively. But, some may argue, the real power of the tool is how it is used to instruct those who make decisions that affect the ever widening rift between the top 1% and the rest of us. According to Andrew Fieldhouse, blogger for the Economic Policy Institute,

“Changes in tax and transfer policies are one of the more easily quantifiable contributors to income inequality, say compared with policies (or lack thereof) related to labor protections, collective bargaining, minimum wage erosion and trade.” (Note: "How Much Can Tax Policy Curb Income Inequality Growth? Maybe a Lot." Economic Policy Institute. Accessed June 29, 2016. [http://www.epi.org/blog/tax-policy-curb-income-inequality-growth/.](http://www.epi.org/blog/tax-policy-curb-income-inequality-growth/))

He states that market-based income inequality, as measured by the “Gini” index (the Gini coefficient multiplied by 100), rose 23.1 percent between 1979 and 2007. Fieldhouse argues that by raising top tax rates, policy makers could help to slow the growth of income inequality without hampering growth of the economy. He also claims that tax policy lacks what could be valuable guidance from economic indicators such as the Gini index. Using mathematics to inform policies also has the added benefit of being politically neutral. Math does not care what side of the aisle the policy may be coming from. When used wisely, mathematical facts offer an objective picture.

# MODULE 11: ROOTS AND RATIONAL EXPONENTS

## WHY IT MATTERS: ROOTS AND RATIONAL EXPONENTS

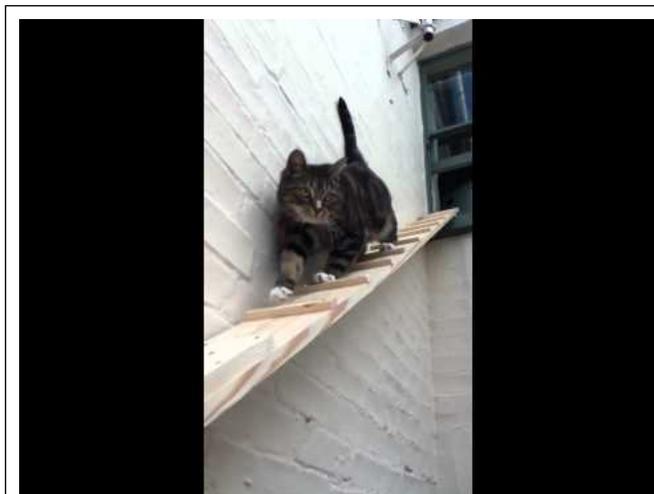
### Why Learn About Roots and Rational Exponents?

Joan has a cat who likes to come and go at night. He has been keeping her up, asking to go out and come back in, so she wants to devise a way for him to get in and out of her apartment without her needing to open the door for him. After watching the following video on YouTube one day, she decides she will get her brother, Jacob, to help her build a cat ladder so Hobbes can let her sleep at night.

Jacob is pretty clever, so she thinks he can help her with the complicated parts.

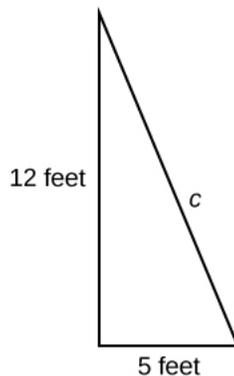


*Go to sleep!*



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Joan's window is located 12 feet above the ground, and the edge of the building is 5 feet from the edge of her window, so Joan needs to figure out how long of a ladder to make so it is confined within this space.



Even though Joan feels pretty confident about some of the things she has learned in her math class, she is not sure how to find the length for the ladder. She asks her math teacher, and she says that Joan can use the Pythagorean Theorem to find the length of the ladder she needs.

Joan looks up the Pythagorean Theorem and finds that it is an equation that describes the relationship between the lengths of the sides of a right triangle, such as the one a ladder will make with the side of her apartment building. It looks like this:

$$a^2 + b^2 = c^2$$

Joan learns that  $c^2$  represents the square of the length of the longest side of the right triangle, so in her case this is the length of ladder that she will need. She gets to work trying to solve the equation:

$$a^2 + b^2 = c^2$$

$$5^2 + 12^2 = c^2$$

$$169 = c^2$$

Joan gets this far and realizes she does not understand how to solve this kind of equation. She needs to find out the length that, when squared, is 169, to determine which ladder to choose.

What she does not realize is that she needs to find a square root.

In this module, we will investigate methods of finding solutions to problems such as this one. We will revisit Joan later and see if she and Jacob were successful in making a cat ladder for Hobbes.

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# INTRODUCTION TO SIMPLIFYING ROOTS AND RATIONAL EXPONENTS

Did you know that you can take the *6th* root of a number? You have probably heard of a square root, written  $\sqrt{\quad}$ , but you can also take a third, fourth and even a 5,000th root (if you really had to). In this lesson we will learn how a square root is defined and then we will build on that to form an understanding of *n*th roots. We will use factoring and rules for exponents to simplify mathematical expressions that contain roots.

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## DEFINE AND EVALUATE ROOTS

### LEARNING OUTCOMES

- Define and evaluate principal square roots
- Define and evaluate *n*th roots
- Estimate roots that are not perfect

The most common root is the **square root**. First, we will define what square roots are and how you find the square root of a number. Then we will apply similar ideas to define and evaluate *n*th roots.

Roots are the inverse of exponents, much like multiplication is the inverse of division. Recall that exponents are defined and written as repeated multiplication.

**Exponent:**  $3^2$ ,  $4^5$ ,  $x^3$ ,  $x^n$

**Name:** “Three squared” or “Three to the second power”, “Four to the fifth power”, “x cubed”, “x to the *n*th power”

**Repeated Multiplication:**  $3 \cdot 3$ ,  $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$ ,  $x \cdot x \cdot x$ ,  $\underbrace{x \cdot x \cdot x \dots x}_{n \text{ times}}$ .

Conversely, when you are trying to find the square root of a number (say, 25), you are trying to find a number that can be multiplied by itself to create that original number. In the case of 25, you find that  $5 \cdot 5 = 25$ , so 5 must be the square root.

### Square Roots

The symbol for the square root is called a **radical symbol** and looks like this:  $\sqrt{\quad}$ . The expression  $\sqrt{25}$  is read “the square root of twenty-five” or “radical twenty-five.” The number that is written under the radical symbol is called the **radicand**.



The following table shows different radicals and their equivalent written and simplified forms.

Radical	Name	Simplified Form
$\sqrt{36}$	“Square root of thirty-six” “Radical thirty-six”	$\sqrt{36} = \sqrt{6 \cdot 6} = 6$
$\sqrt{100}$	“Square root of one hundred” “Radical one hundred”	$\sqrt{100} = \sqrt{10 \cdot 10} = 10$
$\sqrt{225}$	“Square root of two hundred twenty-five” “Radical two hundred twenty-five”	$\sqrt{225} = \sqrt{15 \cdot 15} = 15$

Consider  $\sqrt{25}$  again. You may realize that there is another value that, when multiplied by itself, also results in 25. That number is  $-5$ .

$$5 \cdot 5 = 25$$

$$-5 \cdot -5 = 25$$

By definition, the square root symbol always means to find the positive root called the **principal root**. So while  $5 \cdot 5$  and  $-5 \cdot -5$  both equal 25, only 5 is the principal root. You should also know that zero is special because it has only one square root: itself (since  $0 \cdot 0 = 0$ ).

In our first example we will show you how to use radical notation to evaluate principal square roots.

## EXAMPLE

Find the principal root of each expression.

1.  $\sqrt{100}$
2.  $\sqrt{16}$
3.  $\sqrt{25 + 144}$
4.  $\sqrt{49} - \sqrt{81}$
5.  $-\sqrt{81}$
6.  $\sqrt{-9}$

### Answer

1.  $\sqrt{100} = 10$  because  $10^2 = 100$
2.  $\sqrt{16} = 4$  because  $4^2 = 16$
3. Recall that square roots act as grouping symbols in the order of operations, so addition and subtraction must be performed first when they occur under a radical.  $\sqrt{25 + 144} = \sqrt{169} = 13$  because  $13^2 = 169$
4. This problem is similar to the last one, but this time subtraction should occur after evaluating the root. Stop and think about why these two problems are different.  $\sqrt{49} - \sqrt{81} = 7 - 9 = -2$  because  $7^2 = 49$  and  $9^2 = 81$
5. The negative in front means to take the opposite of the value after you simplify the radical.  $-\sqrt{81} = -9$ .  

$$-\sqrt{9 \cdot 9}$$

The square root of 81 is 9. Then, take the opposite of 9 to get  $-(9)$
6. We are looking for a number that when it is squared returns  $-9$ . We can try  $(-3)^2$ , but that will give a positive result, and  $3^2$  will also give a positive result. This leads to an important fact – you cannot find the square root of a negative number.

In the following video, we present more examples of how to find a principal square root.

**Evaluate Principal Square Root Expressions**

**EXAMPLE**

Simplify.

1.  $\sqrt{16}$
2.  $\sqrt{81}$
3.  $\sqrt{9 + 16}$
4.  $\sqrt{36} - \sqrt{64}$
5.  $-\sqrt{49}$
6.  $\sqrt{-100}$

The square root of a square

For a nonnegative real number,  $a$ ,  $\sqrt{a^2} = a$

1.  $\sqrt{16} = \sqrt{4 \cdot 4} = \sqrt{4^2} = 4$

2.  $\sqrt{81} = \sqrt{9 \cdot 9} = \sqrt{9^2} = 9$

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1646>

The last example we showed leads to an important characteristic of square roots. You can only take the square root of values that are nonnegative.

## THINK ABOUT IT

Does  $\sqrt{25} = \pm 5$ ? Write your ideas and a sentence to defend them in the box below before you look at the answer.

**Answer**

*No. Although both  $5^2$  and  $(-5)^2$  are 25, the radical symbol implies only a nonnegative root, the principal square root. The principal square root of 25 is  $\sqrt{25} = 5$ .*

## Cube Roots

We know that  $5^2 = 25$ , and  $\sqrt{25} = 5$ , but what if we want to “undo”  $5^3 = 125$ , or  $5^4 = 625$ ? We can use higher order roots to answer these questions.

Suppose we know that  $a^3 = 8$ . We want to find what number raised to the 3rd power is equal to 8. Since  $2^3 = 8$ , we say that 2 is the cube root of 8. In the next example, we will evaluate the cube roots of some perfect cubes.

### Example

Evaluate the following:

1.  $\sqrt[3]{125}$
2.  $\sqrt[3]{-8}$
3.  $\sqrt[3]{27}$

**Answer**

1. You can read this as “the third root of 125” or “the cube root of 125.” To evaluate this expression, look for a number that, when multiplied by itself two times (for a total of three identical factors), equals 125.  $? \cdot ? \cdot ? = 125$ . Since 125 ends in 5, 5 is a good candidate.  $5 \cdot 5 \cdot 5 = 125$ .

2. We want to find a number whose cube is  $-8$ . We know 2 is the cube root of 8, so maybe we can try  $-2$  which gives  $-2 \cdot -2 \cdot -2 = -8$ , so the cube root of  $-8$  is  $-2$ . This is different from square roots because multiplying three negative numbers together results in a negative number.

3. We want to find a number whose cube is 27.  $3 \cdot 3 \cdot 3 = 27$ , so the cube root of 27 is 3.

As we saw in the last example, there is one interesting fact about cube roots that is not true of square roots. Negative numbers cannot have real number square roots, but negative numbers can have real number cube roots! What is the cube root of  $-8$ ?  $\sqrt[3]{-8} = -2$  because  $-2 \cdot -2 \cdot -2 = -8$ . Remember, when you are multiplying an odd number of negative numbers, the result is negative! Consider  $\sqrt[3]{(-1)^3} = -1$ .

In the following video, we show more examples of finding a cube root.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1646>

## Nth Roots

The cube root of a number is written with a small number 3, called the **index**, just outside and above the radical symbol. It looks like  $\sqrt[3]{}$ . This little 3 distinguishes cube roots from square roots which are written without a small number outside and above the radical symbol.

We can apply the same idea to any exponent and its corresponding root. The  $n$ th root of  $a$  is a number that, when raised to the  $n$ th power, gives  $a$ . For example, 3 is the 5th root of 243 because  $(3)^5 = 243$ . If  $a$  is a real number with at least one  $n$ th root, then the **principal  $n$ th root** of  $a$  is the number with the same sign as  $a$  that, when raised to the  $n$ th power, equals  $a$ .

The principal  $n$ th root of  $a$  is written as  $\sqrt[n]{a}$ , where  $n$  is a positive integer greater than or equal to 2. In the radical expression,  $n$  is called the **index** of the radical.

### DEFINITION: PRINCIPAL NTH ROOT

If  $a$  is a real number with at least one  $n$ th root, then the **principal  $n$ th root** of  $a$ , written as  $\sqrt[n]{a}$ , is the number with the same sign as  $a$  that, when raised to the  $n$ th power, equals  $a$ . The **index** of the radical is  $n$ .

In the following video, we show more examples of how to evaluate  $n$ th roots.

You can find the odd root of a negative number, but you cannot find the even root of a negative number. This means you can evaluate the radicals  $\sqrt[3]{-81}$ ,  $\sqrt[5]{-64}$ , and  $\sqrt[7]{-2187}$ , but you cannot evaluate the radicals  $\sqrt{-100}$ ,  $\sqrt[4]{-16}$ , or  $\sqrt[6]{-2,500}$ .

## Estimate Roots

## EXAMPLE

Evaluate each of the following:

1.  $\sqrt[5]{-32}$
2.  $\sqrt[4]{81}$
3.  $\sqrt[8]{-1}$

### Answer

1.  $\sqrt[5]{-32}$  Factor 32, which gives  $(-2)^5 = -32$
2.  $\sqrt[4]{81}$ . Factoring can help. We know that  $9 \cdot 9 = 81$  and we can further factor each 9:  
 $\sqrt[4]{81} = \sqrt[4]{3 \cdot 3 \cdot 3 \cdot 3} = \sqrt[4]{3^4} = 3$
3.  $\sqrt[8]{-1}$ . Since we have an 8th root – which is even- with a negative number as the radicand, this root has no real number solutions. In other words,  $-1 \cdot -1 = +1$

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<https://courses.lumenlearning.com/intermediatealgebra/?p=1646>

An approach to handling roots that are not perfect (squares, cubes, etc.) is to approximate them by comparing the values to perfect squares, cubes, or  $n$ th roots. Suppose you wanted to know the square root of 17. Let us look at how you might approximate it.

### Example

Estimate.  $\sqrt{17}$

### Answer

Think of two perfect squares that surround 17. 17 is in between the perfect squares 16 and 25. So,  $\sqrt{17}$  must be in between  $\sqrt{16}$  and  $\sqrt{25}$ .

Determine whether  $\sqrt{17}$  is closer to 4 or to 5 and make another estimate.

$$\sqrt{16} = 4 \text{ and } \sqrt{25} = 5$$

Since 17 is closer to 16 than 25,  $\sqrt{17}$  is probably about 4.1 or 4.2.

Use trial and error to get a better estimate of  $\sqrt{17}$ . Try squaring incrementally greater numbers, beginning with 4.1, to find a good approximation for  $\sqrt{17}$ .

$$(4.1)^2$$

$(4.1)^2$  gives a closer estimate than  $(4.2)^2$ .

$$4.1 \cdot 4.1 = 16.81$$

$$4.2 \cdot 4.2 = 17.64$$

Continue to use trial and error to get an even better estimate.

$$4.12 \cdot 4.12 = 16.9744$$

$$4.13 \cdot 4.13 = 17.0569$$

$$\sqrt{17} \approx 4.12$$

This approximation is pretty close. If you kept using this trial and error strategy, then you could continue to find the square root to the thousandths, ten-thousandths, and hundred-thousandths places, but eventually it would become too tedious to do by hand.

For this reason, when you need to find a more precise approximation of a square root, you should use a calculator. Most calculators have a square root key ( $\sqrt{\quad}$ ) that will give you the square root approximation quickly. On a simple 4-function calculator, you would likely key in the number that you want to take the square root of and then press the square root key.

Try to find  $\sqrt{17}$  using your calculator. Note that you will not be able to get an “exact” answer because  $\sqrt{17}$  is an irrational number, a number that cannot be expressed as a fraction, and the decimal never terminates or repeats. To nine decimal positions,  $\sqrt{17}$  is approximated as 4.123105626. A calculator can save a lot of time and yield a more precise square root when you are dealing with numbers that are not perfect squares.

### Example

Approximate  $\sqrt[3]{30}$  and also find its value using a calculator.

#### Answer

Find the cubes that surround 30.

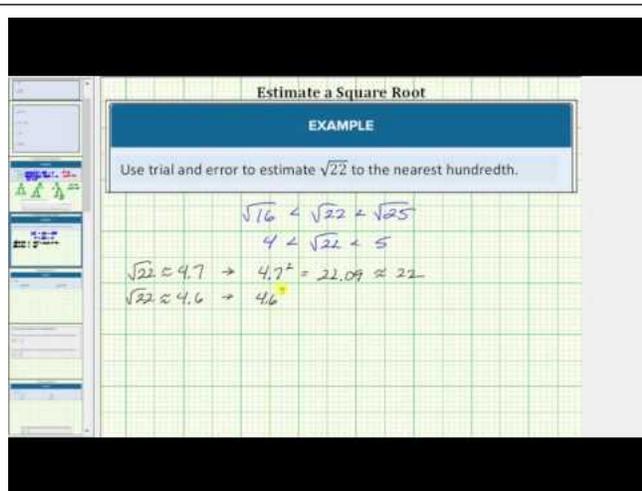
30 is in between the perfect cubes 27 and 81.

$\sqrt[3]{27} = 3$  and  $\sqrt[3]{81} = 4$ , so  $\sqrt[3]{30}$  is between 3 and 4.

Use a calculator.

$$\sqrt[3]{30} \approx 3.10723$$

The following video shows another example of how to estimate a square root.



A YouTube element has been excluded from this version of the text. You can view it online here:  
<https://courses.lumenlearning.com/intermediatealgebra/?p=1646>

## Summary

The square root of a number is the number which, when multiplied by itself, gives the original number. Principal square roots are always positive and the square root of 0 is 0. You can only take the square root of values that are greater than or equal to 0. The square root of a perfect square will be an integer. Other roots can be simplified by identifying factors that are perfect squares, cubes, etc. Nth roots can be approximated using trial and error or a calculator.

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# RADICAL EXPRESSIONS AND RATIONAL EXPONENTS

## Learning Outcomes

- Convert radicals to expressions with rational exponents
- Convert expressions with rational exponents to their radical equivalent

Square roots are most often written using a radical sign, like this,  $\sqrt{4}$ . But there is another way to represent them. You can use rational exponents instead of a radical. A **rational exponent** is an exponent that is a fraction. For example,  $\sqrt{4}$  can be written as  $4^{\frac{1}{2}}$ .

Having difficulty imagining a number being raised to a rational power? They may be hard to get used to, but rational exponents can actually help simplify some problems. Writing radicals with rational exponents will come in handy when we discuss techniques for simplifying more complex radical expressions.

**Radical expressions** are expressions that contain radicals. Radical expressions come in many forms, from simple and familiar, such as  $\sqrt{16}$ , to quite complicated, as in  $\sqrt[3]{250x^4y}$ .

## Write an Expression with a Rational Exponent as a Radical

Radicals and fractional exponents are alternate ways of expressing the same thing. In the table below we show equivalent ways to express radicals: with a root, with a rational exponent, and as a principal root.

Radical Form	Exponent Form	Principal Root
$\sqrt{16}$	$16^{\frac{1}{2}}$	4
$\sqrt{25}$	$25^{\frac{1}{2}}$	5
$\sqrt{100}$	$100^{\frac{1}{2}}$	10

Let us look at some more examples, but this time with cube roots. Remember, cubing a number raises it to the power of three. Notice that in the examples in the table below, the denominator of the rational exponent is the number 3.

Radical Form	Exponent Form	Principal Root
$\sqrt[3]{8}$	$8^{\frac{1}{3}}$	2
$\sqrt[3]{125}$	$125^{\frac{1}{3}}$	5
$\sqrt[3]{1000}$	$1000^{\frac{1}{3}}$	10

These examples help us model a relationship between radicals and rational exponents: namely, that the  $n$ th root of a number can be written as either  $\sqrt[n]{x}$  or  $x^{\frac{1}{n}}$ .

Radical Form	Exponent Form
$\sqrt{x}$	$x^{\frac{1}{2}}$
$\sqrt[3]{x}$	$x^{\frac{1}{3}}$
$\sqrt[4]{x}$	$x^{\frac{1}{4}}$
...	...
$\sqrt[n]{x}$	$x^{\frac{1}{n}}$

In the table above, notice how the denominator of the rational exponent determines the index of the root. So, an exponent of  $\frac{1}{2}$  translates to the square root, an exponent of  $\frac{1}{5}$  translates to the fifth root or  $\sqrt[5]{\quad}$ , and  $\frac{1}{8}$  translates to the eighth root or  $\sqrt[8]{\quad}$ .

#### Example

Express  $(2x)^{\frac{1}{3}}$  in radical form.

**Answer**

Rewrite the expression with the fractional exponent as a radical. The denominator of the fraction determines the root, in this case the cube root.

$$\sqrt[3]{2x}$$

The parentheses in  $(2x)^{\frac{1}{3}}$  indicate that the exponent refers to everything within the parentheses.

Remember that exponents only refer to the quantity immediately to their left unless a grouping symbol is used. The example below looks very similar to the previous example with one important difference—there are no parentheses! Look what happens.

#### Example

Express  $2x^{\frac{1}{3}}$  in radical form.

**Answer**

Rewrite the expression with the fractional exponent as a radical. The denominator of the fraction determines the root, in this case the cube root.

$$2\sqrt[3]{x}$$

The exponent refers only to the part of the expression immediately to the left of the exponent, in this case  $x$ , but not the 2.

## Write a Radical Expression as an Expression with a Rational Exponent

We can write radicals with rational exponents, and as we will see when we simplify more complex radical expressions, this can make things easier. Having different ways to express and write algebraic expressions allows us to have flexibility in solving and simplifying them. It is like having a thesaurus when you write. You want to have options for expressing yourself!



*Flexibility*

#### EXAMPLE

Write  $\sqrt[4]{81}$  as an expression with a rational exponent.

**Answer**

The radical form  $\sqrt[4]{\quad}$  can be rewritten as the exponent  $\frac{1}{4}$ . Remove the radical and place the exponent next to the base.

#### Example

Express  $4\sqrt[3]{xy}$  with rational exponents.

**Answer**

Rewrite the radical using a rational exponent. The root determines the fraction. In this case, the index of the radical is 3, so the rational exponent will be  $\frac{1}{3}$ .

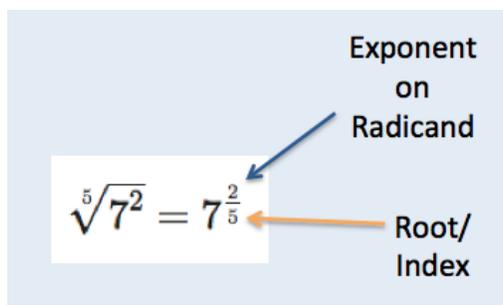
$$4(xy)^{\frac{1}{3}}$$

Since 4 is outside the radical, it is not included in the grouping symbol and the exponent does not refer to it.

# Rational Exponents Whose Numerator is Not Equal to One

All of the numerators for the fractional exponents in the examples above were 1. You can use fractional exponents that have numerators other than 1 to express roots, as shown below.

Radical	Exponent
$\sqrt{9}$	$9^{\frac{1}{2}}$
$\sqrt[3]{9^2}$	$9^{\frac{2}{3}}$
$\sqrt[4]{9^3}$	$9^{\frac{3}{4}}$
$\sqrt[5]{9^2}$	$9^{\frac{2}{5}}$
...	...
$\sqrt[n]{9^x}$	$9^{\frac{x}{n}}$



To rewrite a radical using a fractional exponent, the power to which the radicand is raised becomes the numerator and the root/ index becomes the denominator.

## WRITING RATIONAL EXPONENTS

Any radical in the form  $\sqrt[n]{a^x}$  can be written using a fractional exponent in the form  $a^{\frac{x}{n}}$ .

The relationship between  $\sqrt[n]{a^x}$  and  $a^{\frac{x}{n}}$  works for rational exponents that have a numerator of 1 as well. For example, the radical  $\sqrt[3]{8}$  can also be written as  $\sqrt[3]{8^1}$ , since any number remains the same value if it is raised to the first power. You can now see where the numerator of 1 comes from in the equivalent form of  $8^{\frac{1}{3}}$ .

In the next example, we practice writing radicals with rational exponents where the numerator is not equal to one.

In our last example we will rewrite expressions with rational exponents as radicals. This practice will help us when we simplify more complicated radical expressions and as we learn how to solve radical equations. Typically it is easier to simplify when we use rational exponents, but this exercise is intended to help you understand how the numerator and denominator of the exponent are the exponent of a radicand and index of a radical.

In the following video, we show more examples of writing radical expressions with rational exponents and expressions with rational exponents as radical expressions.

We will use this notation later, so come back for practice if you forget how to write a radical with a rational exponent.

## EXAMPLE

Rewrite the radicals using a rational exponent, then simplify your result.

1.  $\sqrt[3]{a^6}$
2.  $\sqrt[12]{16^3}$

### Answer

1.  $\sqrt[n]{a^x}$  can be rewritten as  $a^{\frac{x}{n}}$ , so in this case  $n = 3$ , and  $x = 6$ , therefore

$$\sqrt[3]{a^6} = a^{\frac{6}{3}}$$

Simplify the exponent.

$$a^{\frac{6}{3}} = a^2$$

2.  $\sqrt[n]{a^x}$  can be rewritten as  $a^{\frac{x}{n}}$ , so in this case  $n = 12$ , and  $x = 3$ , therefore

$$\sqrt[12]{16^3} = 16^{\frac{3}{12}} = 16^{\frac{1}{4}}$$

Simplify the expression using rules for exponents.

$$16^{\frac{1}{4}} = 2$$

## EXAMPLE

Rewrite the expressions using a radical.

1.  $x^{\frac{2}{3}}$
2.  $5^{\frac{4}{7}}$

### Answer

1.  $x^{\frac{2}{3}}$ , the numerator is 2 and the denominator is 3, therefore we will have the third root of x squared,  
 $\sqrt[3]{x^2}$

2.  $5^{\frac{4}{7}}$ , the numerator is 4 and the denominator is 7, so we will have the seventh root of 5 raised to the fourth power.  $\sqrt[7]{5^4}$

The screenshot shows a video player with a title "Radicals and Rational Exponents". It contains two examples:

**EXAMPLE**  
Write each radical expression with rational exponents.  
 $\sqrt{x} = x^{1/2}$ ,  $\sqrt[3]{4} = 4^{1/3}$ ,  $\sqrt[3]{2xy^3} = 2^{1/3}x^{1/3}y^{3/3} = 2^{1/3}x^{1/3}y$   
 $\sqrt[n]{x} = x^{1/n}$ ,  $\sqrt[n]{x^m} = x^{m/n}$

**EXAMPLE**  
Write each expression using radicals  
 $\sqrt{x} = x^{1/2}$ ,  $2x^{1/2}$ ,  $(3x^2)^{1/4}$   
 $\sqrt[n]{x} = x^{1/n}$ ,  $\sqrt[n]{x^m} = x^{m/n}$

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<https://courses.lumenlearning.com/intermediatealgebra/?p=1651>

# Summary

Any radical in the form  $\sqrt[n]{a^x}$  can be written using a fractional exponent in the form  $a^{\frac{x}{n}}$ . Rewriting radicals using fractional exponents can be useful when simplifying some radical expressions. When working with fractional exponents, remember that fractional exponents are subject to all of the same rules as other exponents when they appear in algebraic expressions.

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# SIMPLIFY RADICAL EXPRESSIONS

## LEARNING OUTCOMES

- Simplify radical expressions using factoring
- Simplify radical expressions using rational exponents and the laws of exponents
- Define  $\sqrt{x^2} = |x|$  and apply it when simplifying radical expressions

**Radical expressions** are expressions that contain radicals. Radical expressions come in many forms, from simple and familiar, such as  $\sqrt{16}$ , to quite complicated, as in  $\sqrt[3]{250x^4y}$ .

To simplify complicated radical expressions, we can use some definitions and rules from simplifying exponents. Recall the **Product Raised to a Power Rule** from when you studied exponents. This rule states that the product of two or more non-zero numbers raised to a power is equal to the product of each number raised to the same power. In math terms, it is written  $(ab)^x = a^x \cdot b^x$ . So, for example, you can use the rule to rewrite  $(3x)^2$  as  $3^2 \cdot x^2 = 9 \cdot x^2 = 9x^2$ .

Now instead of using the exponent 2, use the exponent  $\frac{1}{2}$ . The exponent is distributed in the same way.

$$(3x)^{\frac{1}{2}} = 3^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$$

And since you know that raising a number to the  $\frac{1}{2}$  power is the same as taking the square root of that number, you can also write it this way.

$$\sqrt{3x} = \sqrt{3} \cdot \sqrt{x}$$

Look at that—you can think of any number underneath a radical as the *product of separate factors*, each underneath its own radical.

## A PRODUCT RAISED TO A POWER RULE OR SOMETIMES CALLED THE SQUARE ROOT OF A PRODUCT RULE

For any real numbers  $a$  and  $b$ ,  $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ .

For example:  $\sqrt{100} = \sqrt{10} \cdot \sqrt{10}$ , and  $\sqrt{75} = \sqrt{25} \cdot \sqrt{3}$

This rule is important because it helps you think of one radical as the product of multiple radicals. If you can identify perfect squares within a radical, as with  $\sqrt{(2 \cdot 2)(2 \cdot 2)(3 \cdot 3)}$ , you can rewrite the expression as the product of multiple perfect squares:  $\sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{3^2}$ .

The square root of a product rule will help us simplify roots that are not perfect as is shown the following example.

## EXAMPLE

Simplify.  $\sqrt{63}$

**Answer**

63 is not a perfect square so we can use the square root of a product rule to simplify any factors that are perfect squares.

Factor 63 into 7 and 9.

$$\sqrt{7 \cdot 9}$$

9 is a perfect square,  $9 = 3^2$ , therefore we can rewrite the radicand.

$$\sqrt{7 \cdot 3^2}$$

Using the Product Raised to a Power rule, separate the radical into the product of two factors, each under a radical.

$$\sqrt{7} \cdot \sqrt{3^2}$$

Take the square root of  $3^2$ .

$$\sqrt{7} \cdot 3$$

Rearrange factors so the integer appears before the radical and then multiply. This is done so that it is clear that only the 7 is under the radical, not the 3.

$$3 \cdot \sqrt{7}$$

The answer is  $3\sqrt{7}$ .

The final answer  $3\sqrt{7}$  may look a bit odd, but it is in simplified form. You can read this as “three radical seven” or “three times the square root of seven.”

The following video shows more examples of how to simplify square roots that do not have perfect square radicands.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1649>

Before we move on to simplifying more complex radicals with variables, we need to learn about an important behavior of square roots with variables in the radicand.

Consider the expression  $\sqrt{x^2}$ . This looks like it should be equal to  $x$ , right? Test some values for  $x$  and see what happens.

In the chart below, look along each row and determine whether the value of  $x$  is the same as the value of  $\sqrt{x^2}$ . Where are they equal? Where are they not equal?

After doing that for each row, look again and determine whether the value of  $\sqrt{x^2}$  is the same as the value of  $|x|$ .

$x$	$x^2$	$\sqrt{x^2}$	$ x $
-5	25	5	5
-2	4	2	2
0	0	0	0
6	36	6	6
10	100	10	10

Notice—in cases where  $x$  is a negative number,  $\sqrt{x^2} \neq x$ ! However, in all cases  $\sqrt{x^2} = |x|$ . You need to consider this fact when simplifying radicals with an even index that contain variables, because by definition  $\sqrt{x^2}$  is always nonnegative.

## TAKING THE SQUARE ROOT OF A RADICAL EXPRESSION

When finding the square root of an expression that contains variables raised to an even power, remember that  $\sqrt{x^2} = |x|$ .

Examples:  $\sqrt{9x^2} = 3|x|$ , and  $\sqrt{16x^2y^2} = 4|xy|$

We will combine this with the square root of a product rule in our next example to simplify an expression with three variables in the radicand.

### Example

Simplify.  $\sqrt{a^3b^5c^2}$

**Answer**

Factor to find variables with even exponents.

$$\sqrt{a^2 \cdot a \cdot b^4 \cdot b \cdot c^2}$$

Rewrite  $b^4$  as  $(b^2)^2$ .

$$\sqrt{a^2 \cdot a \cdot (b^2)^2 \cdot b \cdot c^2}$$

Separate the squared factors into individual radicals.

$$\sqrt{a^2} \cdot \sqrt{(b^2)^2} \cdot \sqrt{c^2} \cdot \sqrt{a \cdot b}$$

Take the square root of each radical. Remember that  $\sqrt{a^2} = |a|$ .

$$|a| \cdot b^2 \cdot |c| \cdot \sqrt{a \cdot b}$$

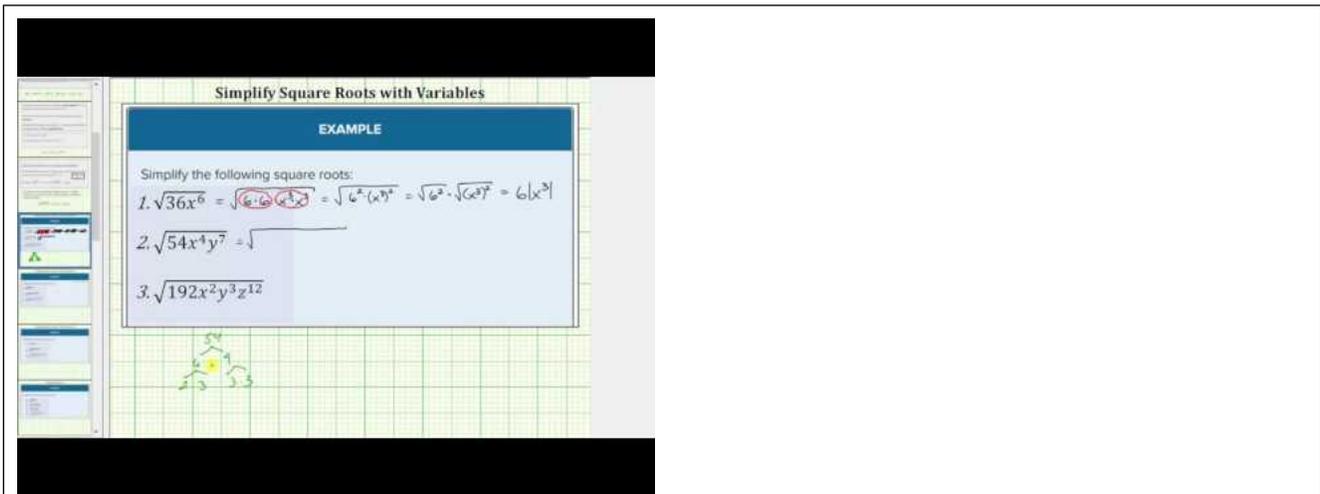
Simplify and multiply.

$$|ac| b^2 \sqrt{ab}$$

## Analysis of the Solution

Why did we not write  $b^2$  as  $|b^2|$ ? Because when you square a number, you will always get a positive result, so the principal square root of  $(b^2)^2$  will always be non-negative. One tip for knowing when to apply the absolute value after simplifying any even indexed root is to look at the final exponent on your variable terms. If the exponent is odd – including 1 – add an absolute value. This applies to simplifying any root with an even index, as we will see in later examples.

In the following video, you will see more examples of how to simplify radical expressions with variables.



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We will show another example where the simplified expression contains variables with both odd and even powers.

### Example

Simplify.  $\sqrt{9x^6y^4}$

**Answer**

Factor to find identical pairs.

$$\sqrt{3 \cdot 3 \cdot x^3 \cdot x^3 \cdot y^2 \cdot y^2}$$

Rewrite the pairs as perfect squares.

$$\sqrt{3^2 \cdot (x^3)^2 \cdot (y^2)^2}$$

Separate into individual radicals.

$$\sqrt{3^2} \cdot \sqrt{(x^3)^2} \cdot \sqrt{(y^2)^2}$$

Simplify.

$$3x^3y^2$$

Because  $x$  has an odd power, we will add the absolute value for our final solution.

$$3|x^3|y^2$$

In our next example, we will start with an expression written with a rational exponent. You will see that you can use a similar process – factoring and sorting terms into squares – to simplify this expression.

### Example

Simplify.  $(36x^4)^{\frac{1}{2}}$

**Answer**

Rewrite the expression with the fractional exponent as a radical.

$$\sqrt{36x^4}$$

Find the square root of both the coefficient and the variable.

$$\sqrt{6^2 \cdot x^4}$$

$$\sqrt{6^2} \cdot \sqrt{x^4}$$

$$\sqrt{6^2} \cdot \sqrt{(x^2)^2}$$

$$6 \cdot x^2$$

The answer is  $6x^2$ .

Here is one more example with perfect squares.

### Example

Simplify.  $\sqrt{49x^{10}y^8}$

**Answer**

Look for squared numbers and variables. Factor 49 into  $7 \cdot 7$ ,  $x^{10}$  into  $x^5 \cdot x^5$ , and  $y^8$  into  $y^4 \cdot y^4$ .

$$\sqrt{7 \cdot 7 \cdot x^5 \cdot x^5 \cdot y^4 \cdot y^4}$$

Rewrite the pairs as squares.

$$\sqrt{7^2 \cdot (x^5)^2 \cdot (y^4)^2}$$

Separate the squared factors into individual radicals.

$$\sqrt{7^2} \cdot \sqrt{(x^5)^2} \cdot \sqrt{(y^4)^2}$$

Take the square root of each radical using the rule that  $\sqrt{x^2} = x$ .

$$7 \cdot x^5 \cdot y^4$$

Multiply.

$$7x^5y^4$$

# Simplify Cube Roots

We can use the same techniques we have used for simplifying square roots to simplify higher order roots. For example, to simplify a cube root, the goal is to find factors under the radical that are perfect cubes so that you can take their cube root. We no longer need to be concerned about whether we have identified the principal root since we are now finding cube roots. Focus on finding identical trios of factors as you simplify.

## EXAMPLE

Simplify.  $\sqrt[3]{40m^5}$

**Answer**

Factor 40 into prime factors.

$$\sqrt[3]{5 \cdot 2 \cdot 2 \cdot 2 \cdot m^5}$$

Since you are looking for the cube root, you need to find factors that appear 3 times under the radical.

Rewrite  $2 \cdot 2 \cdot 2$  as  $2^3$ .

$$\sqrt[3]{2^3 \cdot 5 \cdot m^5}$$

Rewrite  $m^5$  as  $m^3 \cdot m^2$ .

$$\sqrt[3]{2^3 \cdot 5 \cdot m^3 \cdot m^2}$$

Rewrite the expression as a product of multiple radicals.

$$\sqrt[3]{2^3} \cdot \sqrt[3]{5} \cdot \sqrt[3]{m^3} \cdot \sqrt[3]{m^2}$$

Simplify and multiply.

$$2 \cdot \sqrt[3]{5} \cdot m \cdot \sqrt[3]{m^2}$$

The answer is  $2m\sqrt[3]{5m^2}$ .

Remember that you can take the cube root of a negative expression. In the next example, we will simplify a cube root with a negative radicand.

## Example

Simplify.  $\sqrt[3]{-27x^4y^3}$

**Answer**

Factor the expression into cubes.

Separate the cubed factors into individual radicals.

$$\sqrt[3]{-1 \cdot 27 \cdot x^4 \cdot y^3}$$

$$\sqrt[3]{(-1)^3 \cdot (3)^3 \cdot x^3 \cdot x \cdot y^3}$$

$$\sqrt[3]{(-1)^3} \cdot \sqrt[3]{(3)^3} \cdot \sqrt[3]{x^3} \cdot \sqrt[3]{x} \cdot \sqrt[3]{y^3}$$

Simplify the cube roots.

$$-1 \cdot 3 \cdot x \cdot y \cdot \sqrt[3]{x}$$

The answer is  $\sqrt[3]{-27x^4y^3} = -3xy\sqrt[3]{x}$ .

You could check your answer by performing the inverse operation. If you are right, when you cube  $-3xy\sqrt[3]{x}$  you should get  $-27x^4y^3$ .

$$\begin{aligned}
 &(-3xy\sqrt[3]{x})(-3xy\sqrt[3]{x})(-3xy\sqrt[3]{x}) \\
 &-3 \cdot -3 \cdot -3 \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot \sqrt[3]{x} \cdot \sqrt[3]{x} \cdot \sqrt[3]{x} \\
 &-27 \cdot x^3 \cdot y^3 \cdot \sqrt[3]{x^3} \\
 &-27x^3y^3 \cdot x \\
 &-27x^4y^3
 \end{aligned}$$

You can also skip the step of factoring out the negative one once you are comfortable with identifying cubes.

### Example

Simplify.  $\sqrt[3]{-24a^5}$

**Answer**

Factor  $-24$  to find perfect cubes. Here,  $-1$  and  $8$  are the perfect cubes.

$$\sqrt[3]{-1 \cdot 8 \cdot 3 \cdot a^5}$$

Factor variables. You are looking for cube exponents, so you factor  $a^5$  into  $a^3$  and  $a^2$ .

$$\sqrt[3]{(-1)^3 \cdot 2^3 \cdot 3 \cdot a^3 \cdot a^2}$$

Separate the factors into individual radicals.

$$\sqrt[3]{(-1)^3} \cdot \sqrt[3]{2^3} \cdot \sqrt[3]{a^3} \cdot \sqrt[3]{3 \cdot a^2}$$

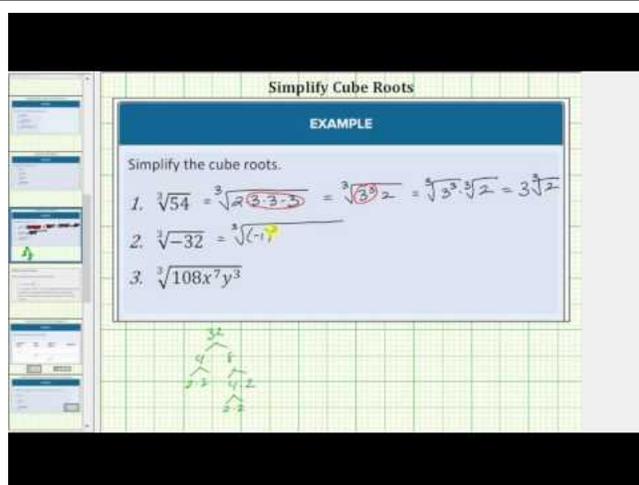
Simplify, using the property  $\sqrt[3]{x^3} = x$ .

$$-1 \cdot 2 \cdot a \cdot \sqrt[3]{3 \cdot a^2}$$

This is the simplest form of this expression; all cubes have been pulled out of the radical expression.

$$-2a\sqrt[3]{3a^2}$$

In the following video, we show more examples of simplifying cube roots.



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## Simplifying Fourth Roots

Now let us move to simplifying fourth degree roots. No matter what root you are simplifying, the same idea applies: find cubes for cube roots, powers of four for fourth roots, etc. Recall that when your simplified expression contains an even indexed radical and a variable factor with an odd exponent, you need to apply an absolute value.

### Example

Simplify.  $\sqrt[4]{81x^8y^3}$

**Answer**

Rewrite the expression.

$$\sqrt[4]{81} \cdot \sqrt[4]{x^8} \cdot \sqrt[4]{y^3}$$

Factor each radicand.

$$\sqrt[4]{3 \cdot 3 \cdot 3 \cdot 3} \cdot \sqrt[4]{x^2 \cdot x^2 \cdot x^2 \cdot x^2} \cdot \sqrt[4]{y^3}$$

Simplify.

$$\sqrt[4]{3^4} \cdot \sqrt[4]{(x^2)^4} \cdot \sqrt[4]{y^3}$$

$$3 \cdot x^2 \cdot \sqrt[4]{y^3}$$

The answer is  $\sqrt[4]{81x^8y^3} = 3x^2\sqrt[4]{y^3}$ .

An alternative method to factoring is to rewrite the expression with rational exponents, then use the rules of exponents to simplify. You may find that you prefer one method over the other. Either way, it is nice to have options. We will show the last example again, using this idea.

### Example

Simplify.  $\sqrt[4]{81x^8y^3}$

**Answer**

Rewrite the radical using rational exponents.

$$(81x^8y^3)^{\frac{1}{4}}$$

Use the rules of exponents to simplify the expression.

$$81^{\frac{1}{4}} \cdot x^{\frac{8}{4}} \cdot y^{\frac{3}{4}}$$

$$(3 \cdot 3 \cdot 3 \cdot 3)^{\frac{1}{4}} x^2 y^{\frac{3}{4}}$$

$$(3^4)^{\frac{1}{4}} x^2 y^{\frac{3}{4}}$$

$$3x^2 y^{\frac{3}{4}}$$

Change the expression with the rational exponent back to radical form.

$$3x^2 \sqrt[4]{y^3}$$

In the following video, we show another example of how to simplify a fourth and fifth root.

For our last example, we will simplify a more complicated expression,  $\frac{10b^2c^2}{c^3\sqrt[3]{8b^4}}$ . This expression has two

**Simplify Radicals**

**EXAMPLE**

Simplify.

$$\sqrt[4]{81x^8y^4} \qquad \sqrt[3]{64x^{10}y^6}$$

$$\sqrt[4]{81x^8y^4} = \sqrt[4]{\underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{81} \cdot \underbrace{x^2 \cdot x^2 \cdot x^2 \cdot x^2}_{x^8} \cdot \underbrace{y \cdot y \cdot y \cdot y}_{y^4}}$$

$$= \sqrt[4]{3^4 \cdot (x^2)^4 \cdot y^4} = 3x^2|y|$$

In general, when simplifying a radical expression, if the index is even and a variable outside the radical has an odd exponent, an absolute value is needed.

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variables, a fraction, and a radical. Let us take it step-by-step and see if using fractional exponents can help us simplify it.

We will start by simplifying the denominator since this is where the radical sign is located. Recall that an exponent in the denominator of a fraction can be rewritten as a negative exponent.

Example

Simplify.  $\frac{10b^2c^2}{c\sqrt[3]{8b^4}}$

**Answer**

Separate the factors in the denominator.

$$\frac{10b^2c^2}{c \cdot \sqrt[3]{8} \cdot \sqrt[3]{b^4}}$$

Take the cube root of 8, which is 2.

$$\frac{10b^2c^2}{c \cdot 2 \cdot \sqrt[3]{b^4}}$$

Rewrite the radical using a fractional exponent.

$$\frac{10b^2c^2}{c \cdot 2 \cdot b^{\frac{4}{3}}}$$

Rewrite the fraction as a series of factors in order to cancel factors (see next step).

$$\frac{10}{2} \cdot \frac{c^2}{c} \cdot \frac{b^2}{b^{\frac{4}{3}}}$$

Simplify the constant and  $c$  factors.

$$5 \cdot c \cdot \frac{b^2}{b^{\frac{4}{3}}}$$

Use the rule of negative exponents,  $n^{-x} = \frac{1}{n^x}$ , to rewrite  $\frac{1}{b^{\frac{4}{3}}}$  as  $b^{-\frac{4}{3}}$ .

$$5cb^2b^{-\frac{4}{3}}$$

Combine the  $b$  factors by adding the exponents.

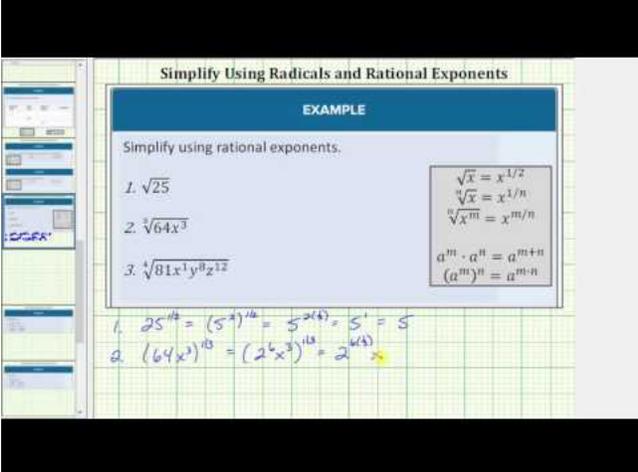
$$5cb^{\frac{2}{3}}$$

Change the expression with the fractional exponent back to radical form. By convention, an expression is not usually considered simplified if it has a fractional exponent or a radical in the denominator.

$$5c\sqrt[3]{b^2}$$

Well, that took a while, but you did it. You applied what you know about fractional exponents, negative exponents, and the rules of exponents to simplify the expression.

In our last video, we show how to use rational exponents to simplify radical expressions.



The screenshot shows a video player with a title "Simplify Using Radicals and Rational Exponents". The video content includes a box titled "EXAMPLE" with the text "Simplify using rational exponents." and three problems:

- $\sqrt{25}$
- $\sqrt[3]{64x^3}$
- $\sqrt[4]{81x^4y^8z^{12}}$

Handwritten solutions are shown below the problems:

- $25^{1/2} = (5^2)^{1/2} = 5^{2(1/2)} = 5^1 = 5$
- $(64x^3)^{1/3} = (2^6x^3)^{1/3} = 2^{6(1/3)}x^{3(1/3)}$

On the right side of the video frame, there is a box with the following properties:

$$\sqrt{x} = x^{1/2}$$
$$\sqrt[n]{x} = x^{1/n}$$
$$\sqrt[n]{x^m} = x^{m/n}$$
$$a^m \cdot a^n = a^{m+n}$$
$$(a^m)^n = a^{m \cdot n}$$

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## Summary

A radical expression is a mathematical way of representing the  $n$ th root of a number. Square roots and cube roots are the most common radicals, but a root can be any number. To simplify radical expressions, look for exponential factors within the radical, and then use the property  $\sqrt[n]{x^n} = x$  if  $n$  is odd and  $\sqrt[n]{x^n} = |x|$  if  $n$  is even to pull out quantities. All rules of integer operations and exponents apply when simplifying radical expressions.

The steps to consider when simplifying a radical are outlined below.

### SIMPLIFYING A RADICAL

When working with exponents and radicals:

- If  $n$  is odd,  $\sqrt[n]{x^n} = x$ .
- If  $n$  is even,  $\sqrt[n]{x^n} = |x|$ . (The absolute value accounts for the fact that if  $x$  is negative and raised to an even power, that number will be positive, as will the  $n$ th principal root of that number.)

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# INTRODUCTION TO ALGEBRAIC OPERATIONS WITH RADICAL EXPRESSIONS

When you learned how to solve linear equations, you probably learned about like terms first. We can only combine terms that are alike; otherwise, the terms will lose their meaning. When you learn how to perform algebraic operations on radical expressions in this section, you will use the concept of like terms in a new way.

You will also use the distributive property, rules for exponents, and methods for multiplying binomials to perform algebraic operations on radical expressions.

## MULTIPLY AND DIVIDE RADICAL EXPRESSIONS

### LEARNING OUTCOMES

- Multiply and divide radical expressions
- Use the product raised to a power rule to multiply radical expressions
- Use the quotient raised to a power rule to divide radical expressions

You can do more than just simplify **radical expressions**. You can multiply and divide them, too. The product raised to a power rule that we discussed previously will help us find products of radical expressions. Recall the rule:

### A PRODUCT RAISED TO A POWER RULE

For any numbers  $a$  and  $b$  and any integer  $x$ :  $(ab)^x = a^x \cdot b^x$

For any numbers  $a$  and  $b$  and any positive integer  $x$ :  $(ab)^{\frac{1}{x}} = a^{\frac{1}{x}} \cdot b^{\frac{1}{x}}$

For any numbers  $a$  and  $b$  and any positive integer  $x$ :  $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$

The Product Raised to a Power Rule is important because you can use it to multiply radical expressions. Note that you cannot multiply a square root and a cube root using this rule. The indices of the radicals must match in order to multiply them. In our first example, we will work with integers, and then we will move on to expressions with variable radicands.

### Example

Simplify.  $\sqrt{18} \cdot \sqrt{16}$

**Answer**

Use the rule  $\sqrt[x]{a} \cdot \sqrt[x]{b} = \sqrt[x]{ab}$  to multiply the radicands.

$$\sqrt{18 \cdot 16}$$

$$\sqrt{288}$$

Look for perfect squares in the radicand, and rewrite the radicand as the product of two factors.

$$\sqrt{144 \cdot 2}$$

Identify perfect squares.

$$\sqrt{(12)^2 \cdot 2}$$

Rewrite as the product of two radicals.

$$\sqrt{(12)^2} \cdot \sqrt{2}$$

Simplify, using  $\sqrt{x^2} = |x|$ .

$$|12| \cdot \sqrt{2}$$

$$12 \cdot \sqrt{2}$$

The answer is  $12\sqrt{2}$ .

You may have also noticed that both  $\sqrt{18}$  and  $\sqrt{16}$  can be written as products involving perfect square factors. How would the expression change if you simplified each radical first, *before* multiplying? In the next example, we will use the same product from above to show that you can simplify before multiplying and get the same result.

### Example

Simplify.  $\sqrt{18} \cdot \sqrt{16}$

**Answer**

Look for perfect squares in each radicand, and rewrite as the product of two factors.

$$\sqrt{9 \cdot 2} \cdot \sqrt{4 \cdot 4}$$

$$\sqrt{3 \cdot 3 \cdot 2} \cdot \sqrt{4 \cdot 4}$$

Identify perfect squares.

$$\sqrt{(3)^2 \cdot 2} \cdot \sqrt{(4)^2}$$

Rewrite as the product of radicals.

$$\sqrt{(3)^2} \cdot \sqrt{2} \cdot \sqrt{(4)^2}$$

Simplify, using  $\sqrt{x^2} = |x|$ .

$$|3| \cdot \sqrt{2} \cdot |4|$$

$$3 \cdot \sqrt{2} \cdot 4$$

Multiply.

$$12\sqrt{2}$$

In both cases, you arrive at the same product,  $12\sqrt{2}$ . It does not matter whether you multiply the radicands or simplify each radical first.

You multiply radical expressions that contain variables in the same manner. As long as the roots of the radical expressions are the same, you can use the Product Raised to a Power Rule to multiply and simplify. Look at the two examples that follow. In both problems, the Product Raised to a Power Rule is used right away and then the expression is simplified. Note that we specify that the variable is non-negative,  $x \geq 0$ , thus allowing us to avoid the need for absolute value.

### Example

Simplify.  $\sqrt{12x^4} \cdot \sqrt{3x^2}$ ,  $x \geq 0$

**Answer**

Use the rule  $\sqrt[x]{a} \cdot \sqrt[x]{b} = \sqrt[x]{ab}$  to multiply the radicands.

$$\frac{\sqrt{12x^4 \cdot 3x^2}}{\sqrt{12 \cdot 3 \cdot x^4 \cdot x^2}}$$

Recall that  $x^4 \cdot x^2 = x^{4+2}$ .

$$\sqrt{36 \cdot x^{4+2}}$$

$$\sqrt{36 \cdot x^6}$$

Look for perfect squares in the radicand.

$$\sqrt{(6)^2 \cdot (x^3)^2}$$

Rewrite as the product of radicals.

$$\frac{\sqrt{(6)^2} \cdot \sqrt{(x^3)^2}}{6 \cdot x^3}$$

The answer is  $6x^3$ .

## Analysis of the Solution

Even though our answer contained a variable with an odd exponent that was simplified from an even indexed root, we don't need to write our answer with absolute value because we specified before we simplified that  $x \geq 0$ . It is important to read the problem very well when you are doing math. Even the smallest statement like  $x \geq 0$  can influence the way you write your answer.

In our next example, we will multiply two cube roots.

### Example

Simplify.  $\sqrt[3]{x^5y^2} \cdot 5\sqrt[3]{8x^2y^4}$

**Answer**

Notice that *both* radicals are cube roots, so you can use the rule to multiply the radicands.

$$5\sqrt[3]{x^5y^2 \cdot 8x^2y^4}$$

$$5\sqrt[3]{8 \cdot x^5 \cdot x^2 \cdot y^2 \cdot y^4}$$

$$5\sqrt[3]{8 \cdot x^{5+2} \cdot y^{2+4}}$$

$$5\sqrt[3]{8 \cdot x^7 \cdot y^6}$$

Look for perfect cubes in the radicand. Since  $x^7$  is not a perfect cube, it has to be rewritten as  $x^{6+1} = (x^2)^3 \cdot x$ .

$$5\sqrt[3]{(2)^3 \cdot (x^2)^3 \cdot x \cdot (y^2)^3}$$

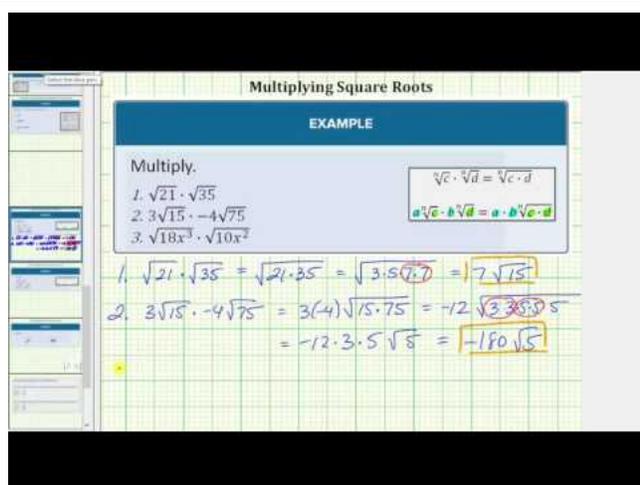
Rewrite as the product of radicals.

$$5\sqrt[3]{(2)^3} \cdot \sqrt[3]{(x^2)^3} \cdot \sqrt[3]{(y^2)^3} \cdot \sqrt[3]{x}$$

$$5 \cdot 2 \cdot x^2 \cdot y^2 \cdot \sqrt[3]{x}$$

The answer is  $10x^2y^2\sqrt[3]{x}$ .

In the following video, we present more examples of how to multiply radical expressions.



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<https://courses.lumenlearning.com/intermediatealgebra/?p=1653>

This next example is slightly more complicated because there are more than two radicals being multiplied. In this case, notice how the radicals are simplified before multiplication takes place. Remember that the order you choose to use is up to you—you will find that sometimes it is easier to multiply before simplifying, and other times it is easier to simplify before multiplying. With some practice, you may be able to tell which is easier before you approach the problem, but either order will work for all problems.

### Example

Simplify.  $2\sqrt[4]{16x^9} \cdot \sqrt[4]{y^3} \cdot \sqrt[4]{81x^3y}$ ,  $x \geq 0$ ,  $y \geq 0$

**Answer**

Notice this expression is multiplying three radicals with the same (fourth) root. Simplify each radical, if possible, before multiplying. Be looking for powers of 4 in each radicand.

$$2\sqrt[4]{(2)^4 \cdot (x^2)^4} \cdot x \cdot \sqrt[4]{y^3} \cdot \sqrt[4]{(3)^4 \cdot x^3y}$$

Rewrite as the product of radicals.

$$2\sqrt[4]{(2)^4} \cdot \sqrt[4]{(x^2)^4} \cdot \sqrt[4]{x} \cdot \sqrt[4]{y^3} \cdot \sqrt[4]{(3)^4} \cdot \sqrt[4]{x^3y}$$

Identify and pull out powers of 4, using the fact that  $\sqrt[4]{x^4} = |x|$ .

$$2 \cdot |2| \cdot |x^2| \cdot \sqrt[4]{x} \cdot \sqrt[4]{y^3} \cdot |3| \cdot \sqrt[4]{x^3y}$$

$$2 \cdot 2 \cdot x^2 \cdot \sqrt[4]{x} \cdot \sqrt[4]{y^3} \cdot 3 \cdot \sqrt[4]{x^3y}$$

Since all the radicals are fourth roots, you can use the rule  $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$  to multiply the radicands.

$$2 \cdot 2 \cdot 3 \cdot x^2 \cdot \sqrt[4]{x \cdot y^3 \cdot x^3y}$$

$$12x^2 \sqrt[4]{x^{1+3} \cdot y^{3+1}}$$

Now that the radicands have been multiplied, look again for powers of 4, and pull them out. We can drop the absolute value signs in our final answer because at the start of the problem we were told  $x \geq 0$ ,  $y \geq 0$ .

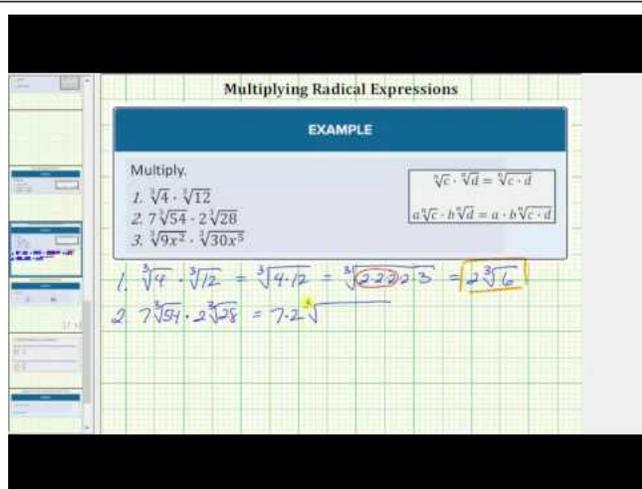
$$12x^2 \sqrt[4]{x^4 \cdot y^4}$$

$$12x^2 \sqrt[4]{x^4} \cdot \sqrt[4]{y^4}$$

$$12x^2 \cdot |x| \cdot |y|$$

The answer is  $12x^3y$ ,  $x \geq 0$ ,  $y \geq 0$ .

In the following video, we show more examples of multiplying cube roots.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1653>

## Dividing Radical Expressions

You can use the same ideas to help you figure out how to simplify and divide radical expressions. Recall that the Product Raised to a Power Rule states that  $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$ . Well, what if you are dealing with a quotient instead of a product?

There is a rule for that, too. The **Quotient Raised to a Power Rule** states that  $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$ . Again, if you imagine that the exponent is a rational number, then you can make this rule applicable for roots as well:

$$\left(\frac{a}{b}\right)^{\frac{1}{x}} = \frac{a^{\frac{1}{x}}}{b^{\frac{1}{x}}}, \text{ so } \sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}.$$

## A QUOTIENT RAISED TO A POWER RULE

For any real numbers  $a$  and  $b$  ( $b \neq 0$ ) and any positive integer  $x$ :  $\left(\frac{a}{b}\right)^{\frac{1}{x}} = \frac{a^{\frac{1}{x}}}{b^{\frac{1}{x}}}$

For any real numbers  $a$  and  $b$  ( $b \neq 0$ ) and any positive integer  $x$ :  $\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$

As you did with multiplication, you will start with some examples featuring integers before moving on to more complex expressions like  $\frac{\sqrt[3]{24xy^4}}{\sqrt[3]{8y}}$ .

### Example

Simplify.  $\sqrt{\frac{48}{25}}$

**Answer**

Use the rule  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$  to create two radicals; one in the numerator and one in the denominator.

$$\frac{\sqrt{48}}{\sqrt{25}}$$

Simplify each radical. Look for perfect square factors in the radicand, and rewrite the radicand as a product of factors.

$$\frac{\sqrt{16 \cdot 3}}{\sqrt{25}}$$

or

$$\frac{\sqrt{4 \cdot 4 \cdot 3}}{\sqrt{5 \cdot 5}}$$

Identify and pull out perfect squares.

$$\frac{\sqrt{(4)^2 \cdot 3}}{\sqrt{(5)^2}}$$

$$\frac{\sqrt{(4)^2 \cdot \sqrt{3}}}{\sqrt{(5)^2}}$$

Simplify.

$$\frac{4 \cdot \sqrt{3}}{5}$$

The answer is  $\frac{4\sqrt{3}}{5}$ .

### Example

Simplify.  $\sqrt[3]{\frac{640}{40}}$

**Answer**

Rewrite using the Quotient Raised to a Power Rule.

$$\frac{\sqrt[3]{640}}{\sqrt[3]{40}}$$

Simplify each radical. Look for perfect cubes in the radicand, and rewrite the radicand as a product of factors.

$$\frac{\sqrt[3]{64 \cdot 10}}{\sqrt[3]{8 \cdot 5}}$$

Identify and pull out perfect cubes.

$$\frac{\sqrt[3]{(4)^3 \cdot 10}}{\sqrt[3]{(2)^3 \cdot 5}}$$

$$\frac{\sqrt[3]{(4)^3} \cdot \sqrt[3]{10}}{\sqrt[3]{(2)^3} \cdot \sqrt[3]{5}}$$

$$\frac{4 \cdot \sqrt[3]{10}}{2 \cdot \sqrt[3]{5}}$$

You can simplify this expression even further by looking for common factors in the numerator and denominator.

$$\frac{4 \sqrt[3]{10}}{2 \sqrt[3]{5}}$$

Rewrite the numerator as a product of factors.

$$\frac{2 \cdot 2 \sqrt[3]{5} \cdot \sqrt[3]{2}}{2 \sqrt[3]{5}}$$

Identify factors of 1, and simplify.

$$2 \cdot \frac{2 \sqrt[3]{5}}{2 \sqrt[3]{5}} \cdot \sqrt[3]{2}$$

$$2 \cdot 1 \cdot \sqrt[3]{2}$$

The answer is  $2\sqrt[3]{2}$ .

That was a lot of effort, but you were able to simplify using the Quotient Raised to a Power Rule. What if you found the quotient of this expression by dividing within the radical first and then took the cube root of the quotient?

Now take another look at that problem using this approach.

### Example

Simplify.  $\frac{\sqrt[3]{640}}{\sqrt[3]{40}}$

## Answer

Since both radicals are cube roots, you can use the rule  $\frac{\sqrt[x]{a}}{\sqrt[x]{b}} = \sqrt[x]{\frac{a}{b}}$  to create a single rational expression underneath the radical.

$$\sqrt[3]{\frac{640}{40}}$$

Within the radical, divide 640 by 40.

$$640 \div 40 = 16$$
$$\sqrt[3]{16}$$

Look for perfect cubes in the radicand, and rewrite the radicand as a product of factors.

$$\sqrt[3]{8 \cdot 2}$$

Identify perfect cubes and pull them out.

$$\sqrt[3]{(2)^3 \cdot 2}$$
$$\sqrt[3]{(2)^3} \cdot \sqrt[3]{2}$$

Simplify.

$$2 \cdot \sqrt[3]{2}$$

The answer is  $2\sqrt[3]{2}$ .

Notice how much more straightforward the approach was.

In the next video, we show more examples of simplifying a radical that contains a quotient.

The screenshot shows a video player with a title "Dividing Radicals" and a subtitle "EXAMPLE". The video content displays the following steps on a grid background:

Simplify:

$$\frac{\sqrt{\frac{75}{16}}}{\sqrt{\frac{324}{4}}}$$
$$\frac{\sqrt{\frac{75}{16}}}{\sqrt{\frac{324}{4}}} = \frac{\sqrt{75}}{\sqrt{16}} = \frac{\sqrt{3 \cdot 5 \cdot 5}}{\sqrt{4 \cdot 4}} = \frac{5\sqrt{3}}{4}$$

Below the steps, a text box states: "For any real numbers a and b (b ≠ 0) and any positive integer n:  $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ "

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1653>

As with multiplication, the main idea here is that sometimes it makes sense to divide and then simplify, and other times it makes sense to simplify and then divide. Whichever order you choose, though, you should arrive at the same final expression.

Now let us turn to some radical expressions containing division. Notice that the process for dividing these is the same as it is for dividing integers.

### Example

Simplify.  $\frac{\sqrt{30x}}{\sqrt{10x}}$ ,  $x > 0$

**Answer**

Use the Quotient Raised to a Power Rule to rewrite this expression.

$$\sqrt{\frac{30x}{10x}}$$

Simplify  $\sqrt{\frac{30x}{10x}}$  by identifying similar factors in the numerator and denominator and then identifying factors of 1.

$$\sqrt{\frac{3 \cdot 10x}{10x}}$$

$$\sqrt{3 \cdot \frac{10x}{10x}}$$

$$\sqrt{3 \cdot 1}$$

The answer is  $\sqrt{3}$ .

### Example

Simplify.  $\frac{\sqrt[3]{24xy^4}}{\sqrt[3]{8y}}$ ,  $y \neq 0$

**Answer**

Use the Quotient Raised to a Power Rule to rewrite this expression.

$$\sqrt[3]{\frac{24xy^4}{8y}}$$

Simplify  $\sqrt[3]{\frac{24xy^4}{8y}}$  by identifying similar factors in the numerator and denominator and then identifying factors of 1.

$$\sqrt[3]{\frac{8 \cdot 3 \cdot x \cdot y^3 \cdot y}{8 \cdot y}}$$

$$\sqrt[3]{\frac{3 \cdot x \cdot y^3}{1} \cdot \frac{8y}{8y}}$$

$$\sqrt[3]{\frac{3 \cdot x \cdot y^3}{1} \cdot 1}$$

Identify perfect cubes and pull them out of the radical.

$$\sqrt[3]{3xy^3}$$

$$\sqrt[3]{(y)^3 \cdot 3x}$$

Simplify.

$$\sqrt[3]{(y)^3} \cdot \sqrt[3]{3x}$$

The answer is  $y\sqrt[3]{3x}$ .

In our last video, we show more examples of simplifying radicals that contain quotients with variables.

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1653>

As you become more familiar with dividing and simplifying radical expressions, make sure you continue to pay attention to the roots of the radicals that you are dividing. For example, while you can think of  $\frac{\sqrt{8y^2}}{\sqrt{225y^4}}$  as being equivalent to  $\sqrt{\frac{8y^2}{225y^4}}$  since both the numerator and the denominator are square roots, notice that you cannot express  $\frac{\sqrt{8y^2}}{\sqrt[4]{225y^4}}$  as  $\sqrt[4]{\frac{8y^2}{225y^4}}$ . In this second case, the numerator is a square root and the denominator is a fourth root.

## Summary

The Product Raised to a Power Rule and the Quotient Raised to a Power Rule can be used to simplify radical expressions as long as the roots of the radicals are the same. The Product Rule states that the product of two or more numbers raised to a power is equal to the product of each number raised to the same power. The same is true of roots:  $\sqrt[x]{ab} = \sqrt[x]{a} \cdot \sqrt[x]{b}$ . When dividing radical expressions, the rules governing quotients are similar:  $\sqrt[x]{\frac{a}{b}} = \frac{\sqrt[x]{a}}{\sqrt[x]{b}}$ .

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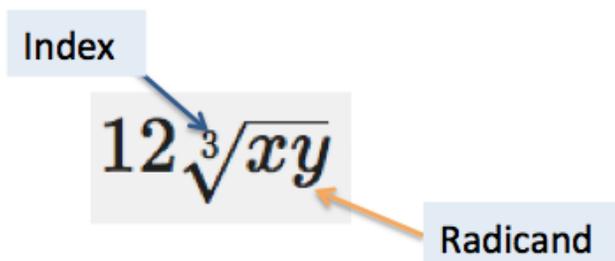
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# ADD AND SUBTRACT RADICAL EXPRESSIONS

## LEARNING OUTCOMES

- Determine when two radicals have the same index and radicand
- Recognize when a radical expression can be simplified either before or after addition or subtraction

There are two keys to combining radicals by addition or subtraction: look at the **index**, and look at the **radicand**. If these are the same, then addition and subtraction are possible. If not, then you cannot combine the two radicals. In the graphic below, the index of the expression  $12\sqrt[3]{xy}$  is 3 and the radicand is  $xy$ .



Making sense of a string of radicals may be difficult. One helpful tip is to think of radicals as variables, and treat them the same way. When you add and subtract variables, you look for like terms, which is the same thing you will do when you add and subtract radicals.

In this first example, both radicals have the same radicand and index.

### Example

Add.  $3\sqrt{11} + 7\sqrt{11}$

**Answer**

The two radicals are the same, . This means you can combine them as you would combine the terms  $3a + 7a$ .

$$3\sqrt{11} + 7\sqrt{11}$$

The answer is  $10\sqrt{11}$ .

This next example contains more addends, or terms that are being added together. Notice how you can combine *like* terms (radicals that have the same root and index), but you cannot combine *unlike* terms.

### Example

Add.  $5\sqrt{2} + \sqrt{3} + 4\sqrt{3} + 2\sqrt{2}$

**Answer**

Rearrange terms so that like radicals are next to each other. Then add.

$$5\sqrt{2} + 2\sqrt{2} + \sqrt{3} + 4\sqrt{3}$$

The answer is  $7\sqrt{2} + 5\sqrt{3}$

Notice that the expression in the previous example is simplified even though it has two terms:  $7\sqrt{2}$  and  $5\sqrt{3}$ . It would be a mistake to try to combine them further! Some people make the mistake that  $7\sqrt{2} + 5\sqrt{3} = 12\sqrt{5}$ . This is incorrect because  $\sqrt{2}$  and  $\sqrt{3}$  are not like radicals so they cannot be added.

### Example

Add.  $3\sqrt{x} + 12\sqrt[3]{xy} + \sqrt{x}$

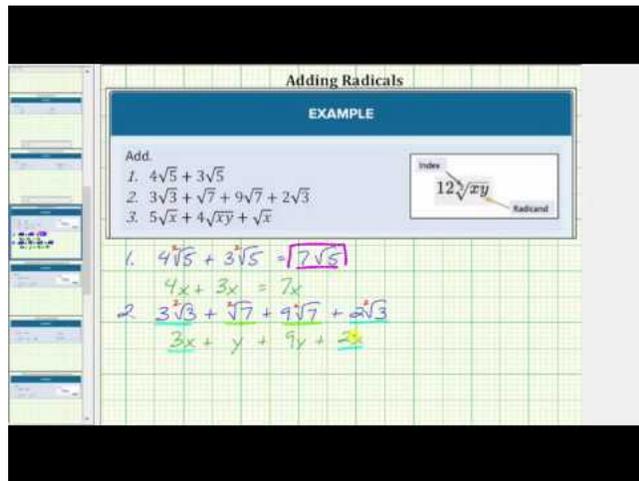
**Answer**

Rearrange terms so that like radicals are next to each other. Then add.

$$3\sqrt{x} + \sqrt{x} + 12\sqrt[3]{xy}$$

The answer is  $4\sqrt{x} + 12\sqrt[3]{xy}$ .

In the following video, we show more examples of how to identify and add like radicals.



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<https://courses.lumenlearning.com/intermediatealgebra/?p=1655>

Sometimes you may need to add *and* simplify the radical. If the radicals are different, try simplifying first—you may end up being able to combine the radicals at the end as shown in these next two examples.

### Example

Add and simplify.  $2\sqrt[3]{40} + \sqrt[3]{135}$

**Answer**

Simplify each radical by identifying perfect cubes.

$$\begin{aligned} & 2\sqrt[3]{8 \cdot 5} + \sqrt[3]{27 \cdot 5} \\ & 2\sqrt[3]{(2)^3 \cdot 5} + \sqrt[3]{(3)^3 \cdot 5} \\ & 2\sqrt[3]{(2)^3} \cdot \sqrt[3]{5} + \sqrt[3]{(3)^3} \cdot \sqrt[3]{5} \end{aligned}$$

Simplify.

$$2 \cdot 2 \cdot \sqrt[3]{5} + 3 \cdot \sqrt[3]{5}$$

$$4\sqrt[3]{5} + 3\sqrt[3]{5}$$

The answer is  $7\sqrt[3]{5}$ .

### Example

Add and simplify.  $x\sqrt[3]{xy^4} + y\sqrt[3]{x^4y}$

**Answer**

Simplify each radical by identifying perfect cubes.

$$x\sqrt[3]{x \cdot y^3 \cdot y} + y\sqrt[3]{x^3 \cdot x \cdot y}$$

$$x\sqrt[3]{y^3} \cdot \sqrt[3]{xy} + y\sqrt[3]{x^3} \cdot \sqrt[3]{xy}$$

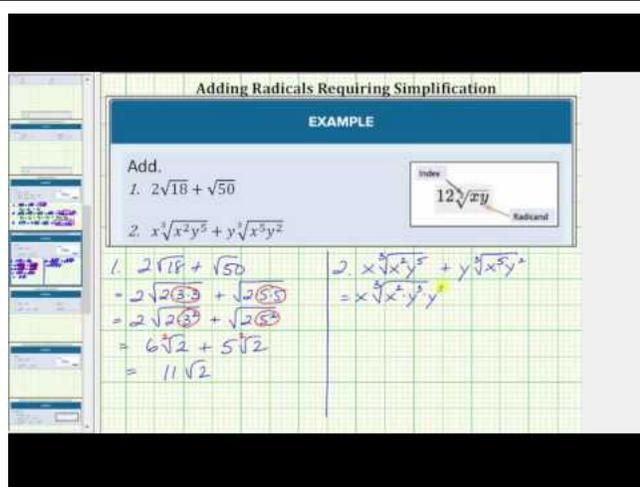
$$xy \cdot \sqrt[3]{xy} + xy \cdot \sqrt[3]{xy}$$

Add like radicals.

$$xy\sqrt[3]{xy} + xy\sqrt[3]{xy}$$

The answer is  $2xy\sqrt[3]{xy}$ .

The following video shows more examples of adding radicals that require simplification.



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## Subtract Radicals

Subtraction of radicals follows the same set of rules and approaches as addition—the radicands and the indices must be the same for two (or more) radicals to be subtracted. In the three examples that follow, subtraction has been rewritten as addition of the opposite.

### Example

Subtract.  $5\sqrt{13} - 3\sqrt{13}$

**Answer**

The radicands and indices are the same, so these two radicals can be combined.

$$5\sqrt{13} - 3\sqrt{13}$$

The answer is  $2\sqrt{13}$ .

### Example

Subtract.  $4\sqrt[3]{5a} - \sqrt[3]{3a} - 2\sqrt[3]{5a}$

**Answer**

Two of the radicals have the same index and radicand, so they can be combined. Rewrite the expression so that like radicals are next to each other.

$$4\sqrt[3]{5a} + (-\sqrt[3]{3a}) + (-2\sqrt[3]{5a})$$

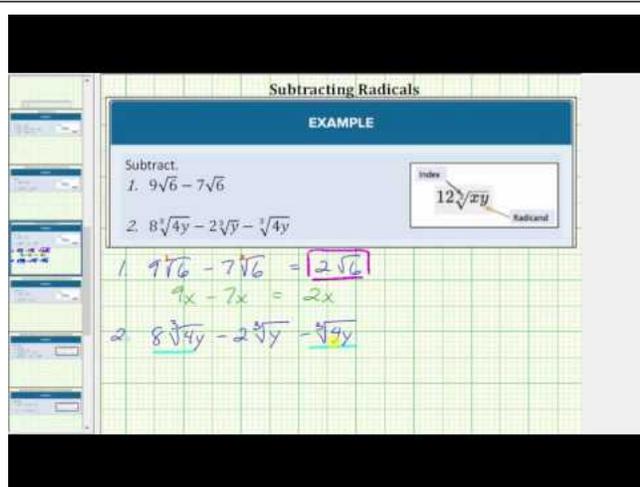
$$4\sqrt[3]{5a} + (-2\sqrt[3]{5a}) + (-\sqrt[3]{3a})$$

Combine. Although the indices of  $2\sqrt[3]{5a}$  and  $-\sqrt[3]{3a}$  are the same, the radicands are not—so they cannot be combined.

$$2\sqrt[3]{5a} + (-\sqrt[3]{3a})$$

The answer is  $2\sqrt[3]{5a} - \sqrt[3]{3a}$ .

In the following video, we show more examples of subtracting radical expressions when no simplifying is required.



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1655>

### Example

Subtract and simplify.  $5\sqrt[4]{a^5b} - a\sqrt[4]{16ab}$ , where  $a \geq 0$  and  $b \geq 0$

**Answer**

Simplify each radical by identifying and pulling out powers of 4.

$$5\sqrt[4]{a^4 \cdot a \cdot b} - a\sqrt[4]{(2)^4 \cdot a \cdot b}$$

$$5 \cdot a\sqrt[4]{a \cdot b} - a \cdot 2\sqrt[4]{a \cdot b}$$

$$5a\sqrt[4]{ab} - 2a\sqrt[4]{ab}$$

The answer is  $3a\sqrt[4]{ab}$ .

In our last video, we show more examples of subtracting radicals that require simplifying.

## Summary

Combining radicals is possible when the index and the radicand of two or more radicals are the same.

Radicals with the same index and radicand are known as like radicals. It is often helpful to treat radicals just as you would treat variables: like radicals can be added and subtracted in the same way that like variables

**Subtracting Radicals Requiring Simplification**

**EXAMPLE**

Subtract.

1.  $3\sqrt{12} - \sqrt{48}$

2.  $2\sqrt{x^4y^2} - x\sqrt{xy^2}$

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can be added and subtracted. Sometimes, you will need to simplify a radical expression before it is possible to add or subtract like terms.

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## MULTIPLE TERM RADICALS

### LEARNING OUTCOME

- Use the distributive property to multiply multiple term radicals and then simplify

When multiplying multiple term radical expressions, it is important to follow the **Distributive Property of Multiplication**, as when you are multiplying regular, non-radical expressions.

Radicals follow the same mathematical rules that other real numbers do. So, although the expression  $\sqrt{x}(3\sqrt{x} - 5)$  may look different than  $a(3a - 5)$ , you can treat them the same way.

Let us have a look at how to apply the Distributive Property. First let us do a problem with the variable  $a$ , and then solve the same problem replacing  $a$  with  $\sqrt{x}$ .

#### Example

Simplify.  $a(3a - 5)$

**Answer**

Use the Distributive Property of Multiplication over Subtraction.

$$a(3a) - a(5)$$

$$= 3a^2 - 5a$$

### Example

Simplify.  $\sqrt{x}(3\sqrt{x} - 5)$

**Answer**

Use the Distributive Property of Multiplication over Subtraction.

$$\sqrt{x}(3\sqrt{x}) - \sqrt{x}(5)$$

Apply the rules of multiplying radicals:  $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$  to multiply  $\sqrt{x}(3\sqrt{x})$ .

$$3\sqrt{x^2} - 5\sqrt{x}$$

Be sure to simplify radicals when you can:  $\sqrt{x^2} = |x|$ , so  $3\sqrt{x^2} = 3|x|$ .

The answer is  $3|x| - 5\sqrt{x}$

The answers to the previous two problems should look similar to you. The only difference is that in the second problem,  $\sqrt{x}$  has replaced the variable  $a$  (and so  $|x|$  has replaced  $a^2$ ). The process of multiplying is very much the same in both problems.

In these next two problems, each term contains a radical.

### Example

Simplify.  $7\sqrt{x}(2\sqrt{xy} + \sqrt{y})$

**Answer**

Use the Distributive Property of Multiplication over Addition to multiply each term within parentheses by  $7\sqrt{x}$ .

$$7\sqrt{x}(2\sqrt{xy}) + 7\sqrt{x}(\sqrt{y})$$

Apply the rules of multiplying radicals.

$$7 \cdot 2\sqrt{x^2y} + 7\sqrt{xy}$$

$\sqrt{x^2} = |x|$ , so  $|x|$  can be pulled out of the radical.

$$14|x|\sqrt{y} + 7\sqrt{xy}$$

### Example

Simplify.  $\sqrt[3]{a}(2\sqrt[3]{a^2} - 4\sqrt[3]{a^5} + 8\sqrt[3]{a^8})$

**Answer**

Use the Distributive Property.

$$\sqrt[3]{a}(2\sqrt[3]{a^2}) - \sqrt[3]{a}(4\sqrt[3]{a^5}) + \sqrt[3]{a}(8\sqrt[3]{a^8})$$

Apply the rules of multiplying radicals.

$$2\sqrt[3]{a \cdot a^2} - 4\sqrt[3]{a \cdot a^5} + 8\sqrt[3]{a \cdot a^8}$$
$$2\sqrt[3]{a^3} - 4\sqrt[3]{a^6} + 8\sqrt[3]{a^9}$$

Identify cubes in each of the radicals.

$$2\sqrt[3]{a^3} - 4\sqrt[3]{(a^2)^3} + 8\sqrt[3]{(a^3)^3}$$

The solution is  $2a - 4a^2 + 8a^3$ .

In the following video, we show more examples of how to multiply radical expressions using the distributive property.

The screenshot shows a video player with a title "Multiply Radical Expressions Using Distribution". Below the title is a blue box labeled "EXAMPLE" with the text "Multiply." and three problems:

- $\sqrt{x}(2\sqrt{x}-3)$
- $4\sqrt{y}(5\sqrt{xy^3}-\sqrt{y^3})$
- $\sqrt{z}(\sqrt{z^2}-7\sqrt{z^3}+2\sqrt{z^6})$

Below the problems are two columns of handwritten work. The first column shows the first problem being solved:

$$\begin{aligned} &\sqrt{x}(2\sqrt{x}-3) \\ &= \sqrt{x}(2\sqrt{x}) - \sqrt{x}(3) \\ &= 2\sqrt{x \cdot x} - 3\sqrt{x} \\ &= 2\sqrt{x^2} - 3\sqrt{x} \\ &= 2|x| - 3\sqrt{x} \end{aligned}$$

The second column shows the second problem being solved:

$$\begin{aligned} &4\sqrt{y}(5\sqrt{xy^3}-\sqrt{y^3}) \\ &= 4\sqrt{y}(5\sqrt{xy^3}) - 4\sqrt{y}(\sqrt{y^3}) \\ &= 20\sqrt{xy^4} - 4\sqrt{y^4} \\ &= 20\sqrt{x(y^2)^2} - 4\sqrt{(y^2)^2} \\ &= 20y^2\sqrt{x} - 4y^2 \end{aligned}$$

There are also two general formulas shown in boxes:  $\sqrt{c} \cdot \sqrt{d} = \sqrt{c \cdot d}$  and  $a\sqrt{c} \cdot b\sqrt{d} = a \cdot b\sqrt{c \cdot d}$ .

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=1657>

In all of these examples, multiplication of radicals has been shown following the pattern  $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ . Then, only after multiplying, some radicals have been simplified—like in the last problem. After you have worked with radical expressions a bit more, you may feel more comfortable identifying quantities such as  $\sqrt{x} \cdot \sqrt{x} = x$  without going through the intermediate step of finding that  $\sqrt{x} \cdot \sqrt{x} = \sqrt{x^2}$ . In the rest of the examples that follow, though, each step is shown.

## Multiply Binomial Expressions That Contain Radicals

You can use the same technique for multiplying binomials to multiply binomial expressions with radicals.

As a refresher, here is the process for multiplying two binomials. If you like using the expression “FOIL” (First, Outside, Inside, Last) to help you figure out the order in which the terms should be multiplied, you can use it here, too.

### Example

Multiply.  $(2x + 5)(3x - 2)$

**Answer**

Use the Distributive Property.

First :  $2x \cdot 3x = 6x^2$

Outside :  $2x \cdot (-2) = -4x$

Inside :  $5 \cdot 3x = 15x$

Last :  $5 \cdot (-2) = -10$

Record the terms, and then combine like terms.

$$6x^2 - 4x + 15x - 10$$

$$6x^2 + 11x - 10$$

Here is the same problem, with  $\sqrt{b}$  replacing the variable  $x$ .

### Example

Multiply.  $(2\sqrt{b} + 5)(3\sqrt{b} - 2)$ ,  $b \geq 0$

**Answer**

Use the Distributive Property to multiply. Simplify using  $\sqrt{x} \cdot \sqrt{x} = x$ .

$$\text{First : } 2\sqrt{b} \cdot 3\sqrt{b} = 2 \cdot 3 \cdot \sqrt{b} \cdot \sqrt{b} = 6b$$

$$\text{Outside : } 2\sqrt{b} \cdot (-2) = -4\sqrt{b}$$

$$\text{Inside : } 5 \cdot 3\sqrt{b} = 15\sqrt{b}$$

$$\text{Last : } 5 \cdot (-2) = -10$$

Record the terms, and then combine like terms.

$$6b - 4\sqrt{b} + 15\sqrt{b} - 10$$

$$6b + 11\sqrt{b} - 10$$

The multiplication works the same way in both problems; you just have to pay attention to the index of the radical (that is, whether the roots are square roots, cube roots, etc.) when multiplying radical expressions.

## MULTIPLYING TWO-TERM RADICAL EXPRESSIONS

To multiply radical expressions, use the same method as used to multiply polynomials.

- Use the Distributive Property (or, if you prefer, the shortcut FOIL method)
- Remember that  $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$
- Combine like terms

### Example

Multiply.  $(4x^2 + \sqrt[3]{x})(\sqrt[3]{x^2} + 2)$

**Answer**

Use FOIL to multiply.

$$\text{First : } 4x^2 \cdot \sqrt[3]{x^2} = 4x^2 \sqrt[3]{x^2}$$

$$\text{Outside : } 4x^2 \cdot 2 = 8x^2$$

$$\text{Inside : } \sqrt[3]{x} \cdot \sqrt[3]{x^2} = \sqrt[3]{x^2 \cdot x} = \sqrt[3]{x^3} = x$$

$$\text{Last : } \sqrt[3]{x} \cdot 2 = 2\sqrt[3]{x}$$

Record the terms, and then combine like terms (if possible). Here, there are no like terms to combine.

$$4x^2 \sqrt[3]{x^2} + 8x^2 + x + 2\sqrt[3]{x}$$

In the following video, we show more examples of how to multiply two binomials that contain radicals.

The screenshot shows a video player with a title "Multiply Radical Expressions Using Distribution". Below the title is a blue box labeled "EXAMPLE" containing two problems:

- $(2\sqrt{x} + 3)(5\sqrt{x} - 4)$
- $(3x^2 + \sqrt{x^2})(2\sqrt{x} - 1)$

Next to the problems are two properties of radicals:

$$\sqrt{c} \cdot \sqrt{d} = \sqrt{c \cdot d}$$
$$a\sqrt{c} \cdot b\sqrt{d} = a \cdot b\sqrt{c \cdot d}$$

Below these is a whiteboard showing the solution for the first problem:

$$\begin{aligned} &(2\sqrt{x} + 3)(5\sqrt{x} - 4) \\ &= 2\sqrt{x}(5\sqrt{x}) + 2\sqrt{x}(-4) + 3(5\sqrt{x}) + 3(-4) \\ &= 10\sqrt{x} - 8\sqrt{x} + 15\sqrt{x} - 12 \\ &= 10|x| + 7\sqrt{x} - 12 \end{aligned}$$

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## Summary

To multiply radical expressions that contain more than one term, use the same method that you use to multiply polynomials. First, use the Distributive Property (or, if you prefer, the shortcut FOIL method) to multiply the terms. Then, apply the rules  $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$ , and  $\sqrt{x} \cdot \sqrt{x} = x$  to multiply and simplify. Finally, combine like terms.

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# RATIONALIZE DENOMINATORS

## LEARNING OUTCOME

- Rationalize denominators with one or multiple terms

Although radicals follow the same rules that integers do, it is often difficult to figure out the value of an expression containing radicals. For example, you probably have a good sense of how much  $\frac{4}{8}$ , 0.75 and  $\frac{6}{9}$  are, but what about the quantities  $\frac{1}{\sqrt{2}}$  and  $\frac{1}{\sqrt{5}}$ ? These are much harder to visualize.

That said, sometimes you have to work with expressions that contain many radicals. Often the value of these expressions is not immediately clear. In cases where you have a fraction with a radical in the denominator, you can use a technique called **rationalizing a denominator** to eliminate the radical. The

point of rationalizing a denominator is to make it easier to understand what the quantity really is by removing radicals from the denominators.

The idea of rationalizing a denominator makes a bit more sense if you consider the definition of “rationalize.” Recall that the numbers 5,  $\frac{1}{2}$ , and 0.75 are all known as rational numbers—they can each be expressed as a ratio of two integers ( $\frac{5}{1}$ ,  $\frac{1}{2}$ , and  $\frac{3}{4}$  respectively). Some radicals are irrational numbers because they cannot be represented as a ratio of two integers. As a result, the point of rationalizing a denominator is to change the expression so that the denominator becomes a rational number.

Here are some examples of irrational and rational denominators.

Irrational		Rational
$\frac{1}{\sqrt{2}}$	=	$\frac{\sqrt{2}}{2}$
$\frac{2+\sqrt{3}}{\sqrt{3}}$	=	$\frac{2\sqrt{3}+3}{3}$

Now examine how to get from irrational to rational denominators.

## Rationalizing Denominators with One Term

Let us start with the fraction  $\frac{1}{\sqrt{2}}$ . Its denominator is  $\sqrt{2}$ , an irrational number. This makes it difficult to figure out what the value of  $\frac{1}{\sqrt{2}}$  is.

You can rename this fraction without changing its value if you multiply it by a quantity equal to 1. In this case, let that quantity be  $\frac{\sqrt{2}}{\sqrt{2}}$ . Watch what happens.

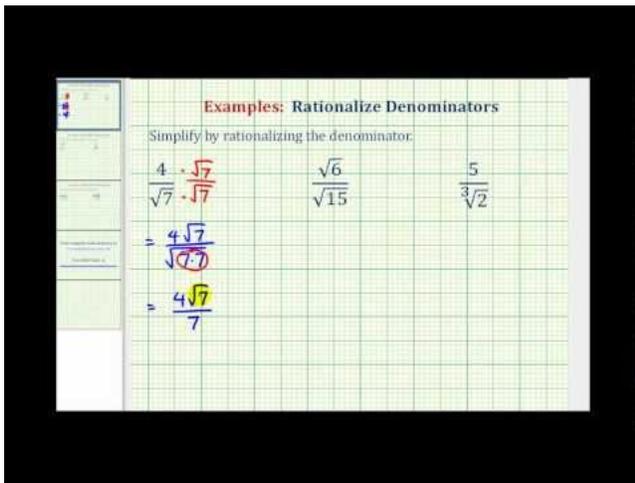
$$\frac{1}{\sqrt{2}} \cdot 1 = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2 \cdot 2}} = \frac{\sqrt{2}}{\sqrt{4}} = \frac{\sqrt{2}}{2}$$

The denominator of the new fraction is no longer a radical (notice, however, that the numerator is).

So why choose to multiply  $\frac{1}{\sqrt{2}}$  by  $\frac{\sqrt{2}}{\sqrt{2}}$ ? You knew that the square root of a number times itself will be a whole number. In algebraic terms, this idea is represented by  $\sqrt{x} \cdot \sqrt{x} = x$ . Look back to the denominators in the multiplication of  $\frac{1}{\sqrt{2}} \cdot 1$ . Do you see where  $\sqrt{2} \cdot \sqrt{2} = \sqrt{4} = 2$ ?

In the following video, we show examples of rationalizing the denominator of a radical expression that contains integer radicands.

Here are some more examples. Notice how the value of the fraction is not changed at all; it is simply being multiplied by another quantity equal to 1.



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### Example

Rationalize the denominator.

$$\frac{2+\sqrt{3}}{\sqrt{3}}$$

**Answer**

The denominator of this fraction is  $\sqrt{3}$ . To make it into a rational number, multiply it by  $\sqrt{3}$ , since  $\sqrt{3} \cdot \sqrt{3} = 3$ .

$$\frac{2+\sqrt{3}}{\sqrt{3}}$$

Multiply the entire fraction by a quantity which simplifies to 1:  $\frac{\sqrt{3}}{\sqrt{3}}$ .

$$\frac{2+\sqrt{3}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$$

$$\frac{\sqrt{3}(2+\sqrt{3})}{\sqrt{3} \cdot \sqrt{3}}$$

Use the Distributive Property to multiply  $\sqrt{3}(2 + \sqrt{3})$ .

$$\frac{2\sqrt{3} + \sqrt{3} \cdot \sqrt{3}}{\sqrt{9}}$$

$$\frac{2\sqrt{3} + \sqrt{9}}{\sqrt{9}}$$

Simplify the radicals, where possible.  $\sqrt{9} = 3$ .

The answer is  $\frac{2\sqrt{3}+3}{3}$ .

You can use the same method to rationalize denominators to simplify fractions with radicals that contain a variable. As long as you multiply the original expression by a quantity that simplifies to 1, you can eliminate a radical in the denominator without changing the value of the expression itself.

### Example

Rationalize the denominator.

$$\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x}}, \text{ where } x \neq 0$$

**Answer**

The denominator is  $\sqrt{x}$ , so the entire expression can be multiplied by  $\frac{\sqrt{x}}{\sqrt{x}}$  to get rid of the radical in the denominator.

$$\frac{\sqrt{x} + \sqrt{y}}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}$$

$$\frac{\sqrt{x}(\sqrt{x} + \sqrt{y})}{\sqrt{x} \cdot \sqrt{x}}$$

Use the Distributive Property. Simplify the radicals where possible. Remember that  $\sqrt{x} \cdot \sqrt{x} = x$ .

$$\frac{\sqrt{x} \cdot \sqrt{x} + \sqrt{x} \cdot \sqrt{y}}{\sqrt{x} \cdot \sqrt{x}}$$

The answer is  $\frac{x + \sqrt{xy}}{x}$ .

### Example

Rationalize the denominator and simplify.

$$\sqrt{\frac{100x}{11y}}, \text{ where } y \neq 0$$

**Answer**

Use the property  $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$  to rewrite the radical.

$$\frac{\sqrt{100x}}{\sqrt{11y}}$$

The denominator is  $\sqrt{11y}$ , so multiplying the entire expression by  $\frac{\sqrt{11y}}{\sqrt{11y}}$  will rationalize the denominator.

$$\frac{\sqrt{100x} \cdot \sqrt{11y}}{\sqrt{11y} \cdot \sqrt{11y}}$$

Multiply and simplify the radicals where possible.

$$\frac{\sqrt{100 \cdot 11xy}}{\sqrt{11y} \cdot \sqrt{11y}}$$

100 is a perfect square. Remember that  $\sqrt{100} = 10$  and  $\sqrt{x} \cdot \sqrt{x} = x$ .

$$\frac{\sqrt{100} \cdot \sqrt{11xy}}{\sqrt{11y} \cdot \sqrt{11y}}$$

The answer is  $\frac{10\sqrt{11xy}}{11y}$ .

## Rationalizing Denominators with Two Terms

Denominators do not always contain just one term as shown in the previous examples. Sometimes, you will see expressions like  $\frac{3}{\sqrt{2}+3}$  where the denominator is composed of two terms,  $\sqrt{2}$  and  $+3$ .

Unfortunately, you cannot rationalize these denominators the same way you rationalize single-term denominators. If you multiply  $\sqrt{2} + 3$  by  $\sqrt{2}$ , you get  $2 + 3\sqrt{2}$ . The original  $\sqrt{2}$  is gone, but now the quantity  $3\sqrt{2}$  has appeared...this is no better!

In order to rationalize this denominator, you want to square the radical term and somehow prevent the integer term from being multiplied by a radical. Is this possible?

It is possible—and you have already seen how to do it!

Recall what the product is when binomials of the form  $(a + b)(a - b)$  are multiplied. So, for example,  $(x + 3)(x - 3) = x^2 - 3x + 3x - 9 = x^2 - 9$ ; notice that the terms  $-3x$  and  $+3x$  combine to 0. Now for the connection to rationalizing denominators: what if you replaced  $x$  with  $\sqrt{2}$ ?

Look at the side by side examples below. Just as  $-3x + 3x$  combines to 0 on the left,  $-3\sqrt{2} + 3\sqrt{2}$  combines to 0 on the right.

	$(\sqrt{2} + 3)(\sqrt{2}$
	$- 3)$
$(x + 3)(x$	$= (\sqrt{2})^2$
$- 3)$	$- 3\sqrt{2}$
$= x^2 - 3x$	$+ 3\sqrt{2} - 9$
$+ 3x - 9$	$= (\sqrt{2})^2 - 9$
$= x^2 - 9$	$= 2 - 9$
	$= -7$

There you have it! Multiplying  $\sqrt{2} + 3$  by  $\sqrt{2} - 3$  removed one radical without adding another.

In this example,  $\sqrt{2} - 3$  is known as a **conjugate**, and  $\sqrt{2} + 3$  and  $\sqrt{2} - 3$  are known as a **conjugate pair**. To find the conjugate of a binomial that includes radicals, change the sign of the second term to its opposite as shown in the table below.

Term	Conjugate	Product
$\sqrt{2} + 3$	$\sqrt{2} - 3$	$(\sqrt{2} + 3)(\sqrt{2} - 3)$ $= (\sqrt{2})^2$ $- (3)^2 = 2 - 9$ $= -7$
$\sqrt{x} - 5$	$\sqrt{x} + 5$	$(\sqrt{x} - 5)(\sqrt{x}$ $+ 5) = (\sqrt{x})^2$ $- (5)^2 = x$ $- 25$
$8 - 2\sqrt{x}$	$8 + 2\sqrt{x}$	$(8 - 2\sqrt{x})(8$ $+ 2\sqrt{x}) = (8)^2$ $- (2\sqrt{x})^2 = 64$ $- 4x$
$1 + \sqrt{xy}$	$1 - \sqrt{xy}$	$(1 + \sqrt{xy})(1$ $- \sqrt{xy}) = (1)^2$ $- (\sqrt{xy})^2 = 1$ $- xy$

### Example

Rationalize the denominator and simplify.

$$\frac{5-\sqrt{7}}{3+\sqrt{5}}$$

**Answer**

Find the conjugate of  $3 + \sqrt{5}$ . Then multiply the entire expression by  $\frac{3-\sqrt{5}}{3-\sqrt{5}}$ .

$$\frac{5-\sqrt{7}}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}}$$

$$\frac{(5-\sqrt{7})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})}$$

Use the Distributive Property to multiply the binomials in the numerator and denominator.

$$\frac{5 \cdot 3 - 5\sqrt{5} - 3\sqrt{7} + \sqrt{7} \cdot \sqrt{5}}{3 \cdot 3 - 3\sqrt{5} + 3\sqrt{5} - \sqrt{5} \cdot \sqrt{5}}$$

Since you multiplied by the conjugate of the denominator, the radical terms in the denominator will combine to 0.

$$\frac{15 - 5\sqrt{5} - 3\sqrt{7} + \sqrt{35}}{9 - 3\sqrt{5} + 3\sqrt{5} - \sqrt{25}}$$

Simplify radicals where possible.

$$\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{9-\sqrt{25}}$$

$$\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{9-5}$$

The answer is  $\frac{15-5\sqrt{5}-3\sqrt{7}+\sqrt{35}}{4}$ .

### Example

Rationalize the denominator and simplify.

$$\frac{\sqrt{x}}{\sqrt{x+2}}$$

**Answer**

Find the conjugate of  $\sqrt{x} + 2$ . Then multiply the numerator and denominator by  $\frac{\sqrt{x}-2}{\sqrt{x}-2}$ .

$$\frac{\sqrt{x}}{\sqrt{x+2}} \cdot \frac{\sqrt{x}-2}{\sqrt{x}-2}$$

$$\frac{\sqrt{x}(\sqrt{x}-2)}{(\sqrt{x+2})(\sqrt{x}-2)}$$

Use the Distributive Property to multiply the binomials in the numerator and denominator.

$$\frac{\sqrt{x} \cdot \sqrt{x} - 2\sqrt{x}}{\sqrt{x} \cdot \sqrt{x} - 2\sqrt{x} + 2\sqrt{x} - 2 \cdot 2}$$

Simplify. Remember that  $\sqrt{x} \cdot \sqrt{x} = x$ . Since you multiplied by the conjugate of the denominator, the radical terms in the denominator will combine to 0.

$$\frac{\sqrt{x} \cdot \sqrt{x} - 2\sqrt{x}}{\sqrt{x} \cdot \sqrt{x} - 2\sqrt{x} + 2\sqrt{x} - 4}$$

The answer is  $\frac{x-2\sqrt{x}}{x-4}$ .

One word of caution: this method will work for binomials that include a square root, but not for binomials with roots greater than 2. This is because squaring a root that has an index greater than 2 does not remove the root, as shown below.

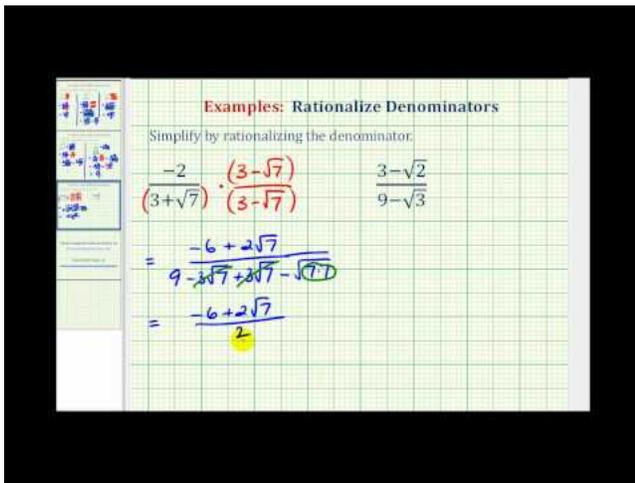
$$\begin{aligned} & (\sqrt[3]{10} + 5)(\sqrt[3]{10} - 5) \\ &= (\sqrt[3]{10})^2 - 5\sqrt[3]{10} + 5\sqrt[3]{10} - 25 \\ &= (\sqrt[3]{10})^2 - 25 \\ &= \sqrt[3]{100} - 25 \end{aligned}$$

$\sqrt[3]{100}$  cannot be simplified any further since its prime factors are  $2 \cdot 2 \cdot 5 \cdot 5$ . There are no cubed numbers to pull out! Multiplying  $\sqrt[3]{10} + 5$  by its conjugate does not result in a radical-free expression.

In the following video, we show more examples of how to rationalize a denominator using the conjugate.

## Summary

When you encounter a fraction that contains a radical in the denominator, you can eliminate the radical by using a process called rationalizing the denominator. To rationalize a denominator, you need to find a quantity that, when multiplied by the denominator, will create a rational number (no radical terms) in the



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denominator. When the denominator contains a single term, as in  $\frac{1}{\sqrt{5}}$ , multiplying the fraction by  $\frac{\sqrt{5}}{\sqrt{5}}$  will remove the radical from the denominator. When the denominator contains two terms, as in  $\frac{2}{\sqrt{5}+3}$ , identify the conjugate of the denominator, here  $\sqrt{5} - 3$ , and multiply both numerator and denominator by the conjugate.

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## INTRODUCTION TO EQUATIONS WITH RADICALS

Now that you understand how to combine radical terms and expressions using algebraic operations, you can solve equations that contain radicals. Soon, we will be able to help Joan figure out how tall the ladder needs to be! In this lesson you will use the addition and multiplication properties of equality as well as the rules for exponents.

## SOLVE RADICAL EQUATIONS

### LEARNING OUTCOMES

- Isolate a radical term in an equation using algebra and rules for radicals and exponents
- Identify a radical equation with no solution or extraneous solutions

An equation that contains a **radical expression** is called a **radical equation**. Solving radical equations requires applying the rules of exponents and following some basic algebraic principles. In some cases, it also requires looking out for errors generated by raising unknown quantities to an even power.

## Isolate a Radical Term

A basic strategy for solving radical equations is to isolate the radical term first, and then raise both sides of the equation to a power to remove the radical. The reason for using powers will become clear in a moment. This is the same type of strategy you used to solve other, non-radical equations—rearrange the expression to isolate the variable you want to know, and then solve the resulting equation.

There are two key ideas that you will be using to solve radical equations. The first is that if  $a = b$ , then  $a^2 = b^2$ . This property allows you to square both sides of an equation and remain certain that the two sides are still equal. The second is that if the square root of any nonnegative number  $x$  is squared, then you get  $x$ :  $(\sqrt{x})^2 = x$ . This property allows you to “remove” the radicals from your equations.

Let us start with a radical equation that you can solve in a few steps:  $\sqrt{x} - 3 = 5$ .

### Example

Solve.  $\sqrt{x} - 3 = 5$

**Answer**

Add 3 to both sides to isolate the variable term on the left side of the equation.

$$\begin{array}{r} \sqrt{x} - 3 = 5 \\ +3 \quad +3 \\ \hline \end{array}$$

Combine like terms.

$$\sqrt{x} = 8$$

Square both sides to remove the radical since  $(\sqrt{x})^2 = x$ . Make sure to square the 8 also! Then simplify.

$$\begin{array}{l} (\sqrt{x})^2 = 8^2 \\ x = 64 \end{array}$$

To check your solution, you can substitute 64 in for  $x$  in the original equation. Does  $\sqrt{64} - 3 = 5$ ? Yes—the square root of 64 is 8, and  $8 - 3 = 5$ .

Notice how you combined like terms and then squared both *sides* of the equation in this problem. This is a standard method for removing a radical from an equation. It is important to isolate a radical on one side of the equation and simplify as much as possible *before* squaring. The fewer terms there are before squaring, the fewer additional terms will be generated by the process of squaring.

In the example above, only the variable  $x$  was underneath the radical. Sometimes you will need to solve an equation that contains multiple terms underneath a radical. Follow the same steps to solve these, but pay attention to a critical point—square both *sides* of an equation, not individual *terms*. Watch how the next two problems are solved.

### Example

Solve.  $\sqrt{x + 8} = 3$

**Answer**

Notice how the radical contains a binomial:  $x + 8$ . Square both sides to remove the radical.

$$(\sqrt{x+8})^2 = (3)^2$$

$(\sqrt{x+8})^2 = x + 8$ . Now simplify the equation and solve for  $x$ .

$$x + 8 = 9$$

$$x = 1$$

Check your answer. Substituting 1 for  $x$  in the original equation yields a true statement, so the solution is correct.

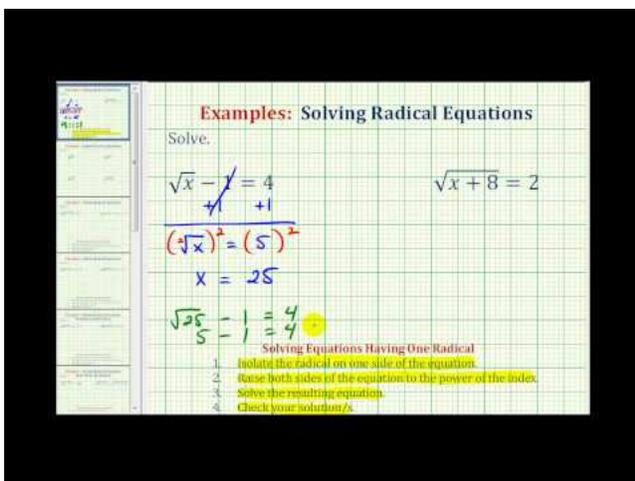
$$\sqrt{1+8} = 3$$

$$\sqrt{9} = 3$$

$$3 = 3$$

$x = 1$  is the solution to  $\sqrt{x+8} = 3$ .

In the following video example, you will see two more examples that are similar to the ones above.



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### Example

Solve.  $1 + \sqrt{2x+3} = 6$

**Answer**

Begin by subtracting 1 from both sides in order to isolate the radical term. Then square both sides to remove the binomial from the radical.

$$1 + \sqrt{2x+3} - 1 = 6 - 1$$

$$\sqrt{2x+3} = 5$$

$$(\sqrt{2x+3})^2 = (5)^2$$

Simplify the equation and solve for  $x$ .

$$2x + 3 = 25$$

$$2x = 22$$

$$x = 11$$

Check your answer. Substituting 11 for  $x$  in the original equation yields a true statement, so the solution is correct.

$$1 + \sqrt{2(11) + 3} = 6$$

$$1 + \sqrt{22 + 3} = 6$$

$$1 + \sqrt{25} = 6$$

$$1 + 5 = 6$$

$$6 = 6$$

$x = 11$  is the solution for  $1 + \sqrt{2x + 3} = 6$ .

## SOLVING RADICAL EQUATIONS

Follow the following four steps to solve radical equations.

1. Isolate the radical expression.
2. Square both sides of the equation: If  $x = y$ , then  $x^2 = y^2$ .
3. Once the radical is removed, solve for the unknown.
4. Check all answers.

## Identify a Radical Equation with No Solutions or Extraneous Solutions

Following rules is important, but so is paying attention to the math in front of you—especially when solving radical equations. Take a look at this next problem that demonstrates a potential pitfall of squaring both sides to remove the radical.

### Example

Solve.  $\sqrt{a - 5} = -2$

**Answer**

Square both sides to remove the term  $a - 5$  from the radical.

$$(\sqrt{a - 5})^2 = (-2)^2$$

Write the simplified equation, and solve for  $a$ .

$$a - 5 = 4$$

$$a = 9$$

Now check the solution by substituting  $a = 9$  into the original equation.

It does not check!

$$\sqrt{9 - 5} = -2$$

$$\sqrt{4} = -2$$

$$2 \neq -2$$

There is no solution.

Look at that—the answer  $a = 9$  does not produce a true statement when substituted back into the original equation. What happened?

Check the original problem:  $\sqrt{a - 5} = -2$ . Notice that the radical is set equal to  $-2$ , and recall that the principal square root of a number can only be *positive*. This means that no value for  $a$  will result in a radical expression whose positive square root is  $-2$ ! You might have noticed that right away and concluded that there were no solutions for  $a$ . But why did the process of squaring create an answer,  $a = 9$ , that proved to be incorrect?

The answer lies in the process of *squaring* itself. When you raise a number to an even power—whether it is the second, fourth, or 50th power—you can introduce a false solution because the result of an even power is always a positive number. Think about it:  $3^2$  and  $(-3)^2$  are both 9, and  $2^4$  and  $(-2)^4$  are both 16. So when you squared  $-2$  and got 4 in this problem, you artificially turned the quantity positive. This is why you were still able to find a value for  $a$ —you solved the problem as if you were solving  $\sqrt{a - 5} = 2$ ! (The correct solution to  $\sqrt{a - 5} = -2$  is actually “no solution.”)

Incorrect values of the variable, such as those that are introduced as a result of the squaring process are called **extraneous solutions**. Extraneous solutions may look like the real solution, but you can identify them because they will not create a true statement when substituted back into the original equation. This is one of the reasons why checking your work is so important—if you do not check your answers by substituting them back into the original equation, you may be introducing extraneous solutions into the problem.

Have a look at the following problem. Notice how the original problem is  $x + 4 = \sqrt{x + 10}$ , but after both sides are squared, it becomes  $x^2 + 8x + 16 = x + 10$ . Squaring both sides may have introduced an extraneous solution.

In the following video, we present more examples of solving radical equations by isolating a radical term on one side.

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### Example

Solve.  $x + 4 = \sqrt{x + 10}$

**Answer**

Square both sides to remove the term  $x + 10$  from the radical.

$$(x + 4)^2 = (\sqrt{x + 10})^2$$

Now simplify and solve the equation. Combine like terms, and then factor.

$$(x + 4)(x + 4) = x + 10$$

$$x^2 + 8x + 16 = x + 10$$

$$x^2 + 8x - x + 16 - 10 = 0$$

$$x^2 + 7x + 6 = 0$$

$$(x + 6)(x + 1) = 0$$

Set each factor equal to zero and solve for  $x$ .

$$(x + 6) = 0 \text{ or } (x + 1) = 0$$

$$x = -6 \text{ or } x = -1$$

Now check both solutions by substituting them into the original equation.

Since  $x = -6$  produces a false statement, it is an extraneous solution.

$$-6 + 4 = \sqrt{-6 + 10}$$

$$-2 = \sqrt{4}$$

$$-2 = 2$$

FALSE!

$$-1 + 4 = \sqrt{-1 + 10}$$

$$3 = \sqrt{9}$$

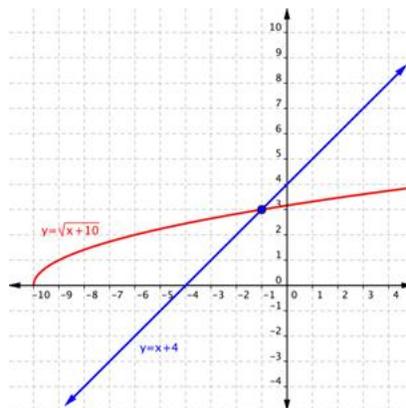
$$3 = 3$$

TRUE!

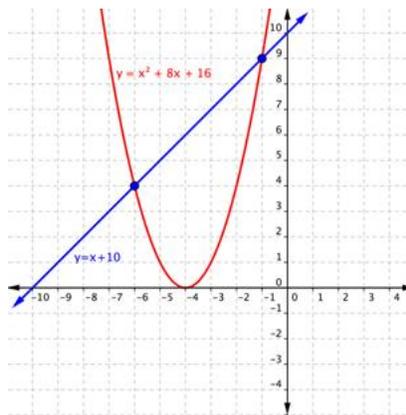
$x = -1$  is the only solution

It may be difficult to understand why extraneous solutions exist at all. Thinking about extraneous solutions by graphing the equation may help you make sense of what is going on.

You can graph  $x + 4 = \sqrt{x + 10}$  on a coordinate plane by breaking it into a system of two equations:  $y = x + 4$  and  $y = \sqrt{x + 10}$ . The graph is shown below. Notice how the two graphs intersect at one point, when the value of  $x$  is  $-1$ . This is the value of  $x$  that satisfies both equations, so it is the solution to the system.



Now, following the work we did in the example problem, let us square both of the expressions to remove the variable from the radical. Instead of solving the equation  $x + 4 = \sqrt{x + 10}$ , we are now solving the equation  $(x + 4)^2 = (\sqrt{x + 10})^2$ , or  $x^2 + 8x + 16 = x + 10$ . The graphs of  $y = x^2 + 8x + 16$  and  $y = x + 10$  are plotted below. Notice how the two graphs intersect at two points, when the values of  $x$  are  $-1$  and  $-6$ .



Although  $x = -1$  is shown as a solution in both graphs, squaring both sides of the equation had the effect of adding an extraneous solution,  $x = -6$ . Again, this is why it is so important to check your answers when solving radical equations!

### Example

Solve.  $4 + \sqrt{x + 2} = x$

**Answer**

Isolate the radical term.

$$\sqrt{x + 2} = x - 4$$

Square both sides to remove the term  $x + 2$  from the radical.

$$(\sqrt{x + 2})^2 = (x - 4)^2$$

Now simplify and solve the equation. Combine like terms, and then factor.

$$x + 2 = x^2 - 8x + 16$$

$$0 = x^2 - 8x - x + 16 - 2$$

$$0 = x^2 - 9x + 14$$

$$0 = (x - 7)(x - 2)$$

Set each factor equal to zero and solve for  $x$ .

$$(x - 7) = 0 \text{ or } (x - 2) = 0$$

$$x = 7 \text{ or } x = 2$$

Now check both solutions by substituting them into the original equation.

Since  $x = 2$  produces a false statement, it is an extraneous solution.

$$4 + \sqrt{7 + 2} = 7$$

$$4 + \sqrt{9} = 7$$

$$4 + 3 = 7$$

$$7 = 7$$

TRUE!

$$4 + \sqrt{2 + 2} = 2$$

$$4 + \sqrt{4} = 2$$

$$4 + 2 = 2$$

$$6 = 2$$

FALSE!

$x = 7$  is the only solution.

In the next video, we show another example of radical equations that have extraneous solutions. Always remember to check your answers for radical equations.

**Examples: Solving Radical Equations:  
Variables on Both Sides**

Solve:

$(\sqrt{x+6})^2 = (x+4)^2$   
 $x+6 = (x+4)(x+4)$   
 $x+6 = x^2+8x+16$   
 $-x-6 \quad -x \quad -16$   
 $0 = x^2+7x+10$   
 $0 = (x+5)(x+2)$

$\sqrt{4x+9}-1 = x$   
 $x+5=0$  or  $x+2=0$   
 $x=-5$  or  $x=-2$   
 $\sqrt{1} = -1$

**Solving Equations Having One Radical**

1. Isolate the radical on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Solve the resulting equation.
4. Check your solution/s.

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<https://courses.lumenlearning.com/intermediatealgebra/?p=1661>

In our last example, we will use a difference of squares to solve a radical equation with two radicals. You will see that we have to isolate and eliminate one radical first, and then tackle isolating and eliminating the other. We will also use factoring techniques for trinomials and the zero product principle to solve this equation.

In our last video, we show the solution for another radical equation that has square roots on both sides of the equation. Pay special attention to how you square the side that has the sum of a radical term and a constant.

Radical equations play a significant role in science, engineering, and even music. Sometimes you may need to use what you know about radical equations to solve for different variables in these types of problems.

## Summary

A common method for solving radical equations is to raise both sides of an equation to whatever power will eliminate the radical sign from the equation. But be careful—when both sides of an equation are raised to an *even* power, the possibility exists that extraneous solutions will be introduced. When solving a radical equation, it is important to always check your answer by substituting the value back into the original equation. If you get a true statement, then that value is a solution; if you get a false statement, then that value is not a solution.

## EXAMPLE

$$\text{Solve } \sqrt{2x+3} + \sqrt{x-2} = 4.$$

### Answer

As this equation contains two radicals, we isolate one radical, eliminate it, and then isolate the second radical.

$$\sqrt{2x+3} + \sqrt{x-2} = 4$$

$$\sqrt{2x+3} = 4 - \sqrt{x-2} \quad \text{Subtract } \sqrt{x-2} \text{ from both sides.}$$

$$(\sqrt{2x+3})^2 = (4 - \sqrt{x-2})^2 \quad \text{Square both sides.}$$

Use the perfect square formula to expand the right side:  $(a - b)^2 = a^2 - 2ab + b^2$ .

$$2x + 3 = (4)^2 - 2(4)\sqrt{x-2} + (\sqrt{x-2})^2$$

$$2x + 3 = 16 - 8\sqrt{x-2} + (x - 2)$$

$$2x + 3 = 14 + x - 8\sqrt{x-2} \quad \text{Combine like terms.}$$

$$x - 11 = -8\sqrt{x-2} \quad \text{Isolate the second radical.}$$

$$(x - 11)^2 = (-8\sqrt{x-2})^2 \quad \text{Square both sides.}$$

$$x^2 - 22x + 121 = 64(x - 2)$$

Now that both radicals have been eliminated, set the quadratic equal to zero and solve.

$$x^2 - 22x + 121 = 64x - 128$$

$$x^2 - 86x + 249 = 0$$

$$(x - 3)(x - 83) = 0 \quad \text{Factor and solve.}$$

$$x = 3$$

$$x = 83$$

The proposed solutions are  $x = 3$  and  $x = 83$ . Check each solution in the original equation.

$$\sqrt{2x+3} + \sqrt{x-2} = 4$$

$$\sqrt{2x+3} = 4 - \sqrt{x-2}$$

$$\sqrt{2(3)+3} = 4 - \sqrt{(3)-2}$$

$$\sqrt{9} = 4 - \sqrt{1}$$

$$3 = 3$$

One solution is  $x = 3$ .

Check  $x = 83$ .

$$\sqrt{2x+3} + \sqrt{x-2} = 4$$

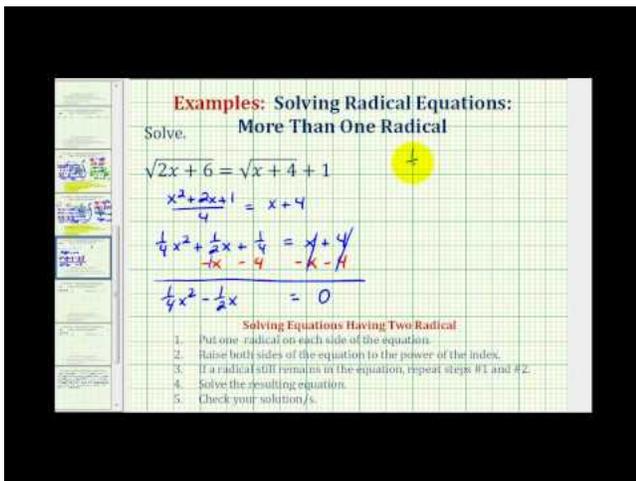
$$\sqrt{2x+3} = 4 - \sqrt{x-2}$$

$$\sqrt{2(83)+3} = 4 - \sqrt{(83)-2}$$

$$\sqrt{169} = 4 - \sqrt{81}$$

$$13 \neq -5$$

The only solution is  $x = 3$ . We see that  $x = 83$  is an extraneous solution.



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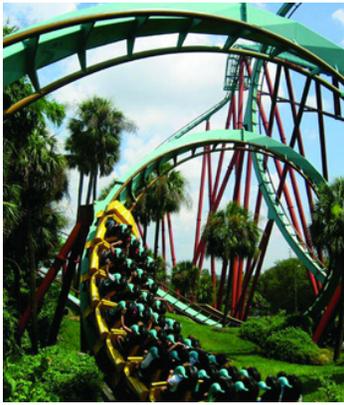
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## APPLICATIONS WITH RADICAL EQUATIONS

### LEARNING OUTCOME

- Solve application problems involving kinetic energy, volume, and free-fall

### Kinetic Energy



One way to measure the amount of energy that a moving object (such as a car or roller coaster) possesses is by finding its kinetic energy. The kinetic energy ( $E_k$ , measured in Joules) of an object depends on the object's mass ( $m$ , measured in kg) and velocity ( $v$ , measured in meters per second) and can be written as

$$v = \sqrt{\frac{2E_k}{m}}$$

### Example

What is the kinetic energy of an object with a mass of 1,000 kilograms that is traveling at 30 meters per second?

#### Answer

Identify variables and known values.

$$E_k = \text{unknown}$$

$$m = 1000$$

$$v = 30$$

Substitute values into the formula.

$$30 = \sqrt{\frac{2E_k}{1,000}}$$

Solve the radical equation for  $E_k$ .

$$(30)^2 = \left(\sqrt{\frac{2E_k}{1,000}}\right)^2$$

$$900 = \frac{2E_k}{1,000}$$

$$900 \cdot 1,000 = \frac{2E_k}{1,000} \cdot 1,000$$

$$900,000 = 2E_k$$

$$\frac{900,000}{2} = \frac{2E_k}{2}$$

$$450,000 = E_k$$

Now check the solution by substituting it into the original equation.

$$30 = \sqrt{\frac{2 \cdot 450,000}{1,000}}$$

$$30 = \sqrt{\frac{900,000}{1,000}}$$

$$30 = \sqrt{900}$$

$$30 = 30$$

The Kinetic Energy is 450,000 Joules.

Here is another example of finding the kinetic energy of an object in motion.

The screenshot shows a video player with a title "Applications of Radical Equations" and a sub-header "EXAMPLE". The text in the video reads: "What is the kinetic energy of an object with a mass of 500 kilograms that is traveling at 15 meters per second." Below this, the formula  $v = \sqrt{\frac{2E_k}{m}}$  is shown. Handwritten notes on a grid background show the solution:  $m = 500 \text{ kg}$ ,  $v = 15 \text{ m/s}$ ,  $E_k = ?$ , and the equation  $(15)^2 = \sqrt{\frac{2E_k}{500}}$  is manipulated to  $225 = \frac{2E_k}{500}$ .

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## Volume

Harvester ants found in the southwest of the U.S. create a vast interlocking network of tunnels for their nests. As a result of all this excavation, a very common above-ground hallmark of a harvester ant nest is a conical mound of small gravel or sand (Note: Taber, Stephen Welton. *The World of the Harvester Ants*. College Station: Texas A & M University Press, 1998.)



The radius of a cone whose height is equal to twice its radius is given as  $r = \sqrt[3]{\frac{3V}{2\pi}}$ .

Here is another example of finding volume given the radius of a cone.

## Free-Fall

## EXAMPLE

A mound of gravel is in the shape of a cone with the height equal to twice the radius. Calculate the volume of such a mound of gravel whose radius is 3.63 ft. Use  $\pi = 3.14$ .

**Answer**

The radius of a cone given its volume can be found with the following formula:  $r = \sqrt[3]{\frac{3V}{2\pi}}$ ,  $r \geq 0$

Cube both sides to eliminate the cube root.

$$3.63 = \sqrt[3]{\frac{3V}{2\pi}}$$

$$(3.63)^3 = \left(\sqrt[3]{\frac{3V}{2 \cdot 3.14}}\right)^3$$

$$47.83 = \frac{3V}{2\pi}$$

$$47.83 \cdot 2 \cdot 3.14 = 3V$$

$$100 \approx V$$

Therefore, the volume is about 100 cubic feet.

The screenshot shows a video player with a title "Applications of Radical Equations" and a sub-title "EXAMPLE". The video content includes a text box: "A mound of sand is in the shape of a cone with a height equal to the radius. Calculate the volume of a mound of sand with radius 2 feet." Below this is a small image of a sand mound. Handwritten notes on a grid background show the formula  $r = \sqrt[3]{\frac{3V}{2\pi}}$ , the substitution  $h = r$ , and the final values  $r = 2 \text{ ft}$  and  $h = 2 \text{ ft}$ . There are also some intermediate steps like  $(2)^3 = \left(\sqrt[3]{\frac{3V}{2\pi}}\right)^3$  and  $4 = \frac{3V}{2\pi}$ .

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When you drop an object from a height, the only force acting on it is gravity (and some air friction) and it is said to be in free-fall. We can use math to describe the height of an object in free fall after a given time because we know how to quantify the force of Earth pulling on us – the force of gravity.

An object dropped from a height of 600 feet has a height,  $h$ , in feet after  $t$  seconds have elapsed, such that  $h = 600 - 16t^2$ . In our next example we will find the time at which the object is at a given height by first

solving for t.

## EXAMPLE

Find the time it takes to reach a height of 400 feet by first finding an expression for t.

### Answer

We are asked two things. First, we will solve the height equation for t, then we will find how long it takes for the object's height above the ground to be 400 feet.

Solve for t:

$$h = 600 - 16t^2$$

$$h - 600 = -16t^2$$

$$\frac{h-600}{-16} = t^2$$

$$\pm \sqrt{\frac{h-600}{-16}} = t$$

At this point, let us stop and talk about whether it makes sense to include both  $\pm \sqrt{\frac{h-600}{-16}}$ . We want time to be only positive since we are talking about a measurable quantity, so we will restrict our answers to just  $+\sqrt{\frac{h-600}{-16}}$ .

We want to know at what time the height will be 400 ft., so let us substitute 400 for h.

$$\sqrt{\frac{400-600}{-16}} = t$$

$$\sqrt{\frac{-200}{-16}} = t$$

$$\sqrt{12.5} = t$$

$$3.54 = t$$

It takes 3.54 seconds for the object to be at a height of 400 ft.

## Analysis of the Solution

We have made a point of restricting the radicand of radical expressions to non-negative numbers. In the previous example, we divided by a negative number then took the square root to solve for t. In this example, is it possible to get a negative number in the radicand? In other words, for what values for height would we have an issue where we may be taking the square root of a negative number? We can use algebra to answer this question.

Let us translate our question into an inequality. Again, for what values of h would we get a negative quantity under the radical?

The radicand is  $\frac{h-600}{-16}$ , so if we set up an inequality we can solve it for h:

$$\frac{h-600}{-16} < 0$$

$$-16 \cdot \frac{h-600}{-16} < 0 \cdot -16$$

$$h - 600 > 0$$

$$h > 600$$

We can interpret this as “when the height is greater than 600 ft. the radicand will be negative and therefore not a real number.” If you re-read the question, you will see that heights greater than 600 do not even make sense, because the object starts at a height of 600 feet and is falling toward the ground, so height is decreasing. Understanding what domain our variables have is important in application problems so we can get answers that make sense.

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## PUTTING IT TOGETHER: ROOTS AND RATIONAL EXPONENTS

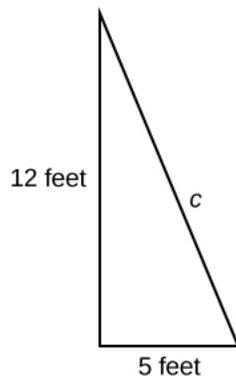
At the beginning of this lesson, we left Joan and her brother who were planning to make a ramp for Joan’s cat Hobbes from her window. She used the Pythagorean Theorem to get this far, but was left with an equation she did not know how to solve:

$$a^2 + b^2 = c^2$$

$$5^2 + 12^2 = c^2$$

$$169 = c^2$$

The goal was to find the unknown length in the following triangle, where  $c$  represents the length of the cat ladder.



Cat heaven

Joan’s brother, Jacob, suggested that she use a square root to find the solution to her equation. Joan remembered that roots and exponents are related – one “undoes” the other. She tried it out in the following way:

$$169 = c^2$$

$$\sqrt{169} = \sqrt{c^2}$$

$$13 = c$$

Hooray! Joan has the length she needs to create her cat's ladder. Hopefully she will be able to get some sleep now.

# MODULE 12: QUADRATIC EQUATIONS AND COMPLEX NUMBERS

## WHY IT MATTERS: QUADRATIC EQUATIONS AND COMPLEX NUMBERS

### Why Learn About Quadratic Equations and Complex Numbers?

In algebra, a quadratic equation (from the Latin *quadratus* for “square”) is any equation having the form

$$ax^2 + bx + c = 0$$

where  $x$  represents an unknown, and  $a$ ,  $b$ , and  $c$  represent known numbers such that  $a$  is not equal to  $0$ . If  $a = 0$ , then the equation is linear, not quadratic. The numbers  $a$ ,  $b$ , and  $c$  are the *coefficients* of the equation and may be distinguished by calling them, respectively, the *quadratic coefficient*, the *linear coefficient*, and the *constant or free term*.

Joan and her friend Hazel have decided to go cliff jumping at their favorite spot along the river. Hazel is an amateur photographer who just got a new camera that has a continuous shooting mode. Hazel takes a picture of Joan jumping off a cliff into the river that looks like this:



Cliff jumping off a rock into the ocean. The trajectory of the cliff jumper is parabolic.

When she sees the picture, Joan is reminded of what she learned in her math class about the shape of graphs of quadratic functions. She tells Hazel that the path of her jump reminds her of an upside down parabola. "Oh yeah!" responds Hazel, "I remember that from high school physics. We learned how to calculate how long it takes something to fall due to gravity."

Joan wonders if she could calculate how long she was in the air when she made the jump off the cliff, so when they are done playing in the river, they go back to Hazel's house to see if she still has her old physics notes.

In her notes, Hazel finds the following equation:

$$h(t) = -10 \frac{m}{s^2} t^2 + v_0 t + h_0$$

$$h(t) = -10 \frac{m}{s^2} t^2 + v_0 t + h_0$$

She had labeled the equation with arrows in the following way:  $-10 \frac{m}{s^2}$  is gravity,  $v_0$  is the initial speed of the object, and  $h_0$  is the initial height of the object.

Hazel and Joan think they can figure out how long they were in the air as they jumped from the cliff into the river. Stay tuned for the end of the module when we see how they do it.

# INTRODUCTION TO QUADRATIC EQUATIONS

An equation containing a second-degree polynomial is called a **quadratic equation**. For example, equations such as  $2x^2 + 3x - 1 = 0$  and  $x^2 - 4 = 0$  are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

## QUADRATIC EQUATIONS

### LEARNING OUTCOMES

- Recognize a quadratic equation
- Use the zero product principle to solve quadratic equations that can be factored

Often the easiest method of solving a quadratic equation is by **factoring**. Factoring means finding expressions that can be multiplied together to give the expression on one side of the equation. Note that we will not spend a lot of time explaining how to factor in this section. You may want to seek help if you do not feel confident about factoring.

Solving by factoring depends on the zero-product property which states that if  $a \cdot b = 0$ , then  $a = 0$  or  $b = 0$ , where  $a$  and  $b$  are real numbers or algebraic expressions.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is 1 or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form,  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ . The equation  $x^2 + x - 6 = 0$  is in standard form.

We can use the zero-product property to solve quadratic equations in which we first have to factor out the **greatest common factor** (GCF) and for equations that have special factoring formulas as well, such as the difference of squares, which we will see later in this section.

### THE ZERO-PRODUCT PROPERTY AND QUADRATIC EQUATIONS

The **zero-product property** states

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0$$

where  $a$  and  $b$  are real numbers or algebraic expressions.

A **quadratic equation** is an equation containing a second-degree polynomial; for example,

$$ax^2 + bx + c = 0$$

where  $a$ ,  $b$ , and  $c$  are real numbers, and if  $a \neq 0$ , it is in standard form.

## Solving Quadratics with a Leading Coefficient of 1

In the quadratic equation  $x^2 + x - 6 = 0$ , the leading coefficient, or the coefficient of  $x^2$ , is 1. We have one method of factoring quadratic equations in this form.

### REMINDER: GIVEN A QUADRATIC EQUATION WITH THE LEADING COEFFICIENT OF 1, FACTOR IT

1. Find two numbers whose product equals  $c$  and whose sum equals  $b$ .
2. Use those numbers to write two factors of the form  $(x + k)$  or  $(x - k)$ , where  $k$  is one of the numbers found in step 1. Use the numbers exactly as they are. In other words, if the two numbers are 1 and  $-2$ , the factors are  $(x + 1)(x - 2)$ .
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

In the following video, we provide more examples of factoring to solve quadratic equations.

In our next example, we will solve a quadratic equation that is written as a difference of squares.

## Solving Quadratics with a Leading Coefficient $\neq 1$

Recall that when the leading coefficient is not 1, we factor a quadratic equation using a method called grouping, which requires four terms. With the equation in standard form, let us review the grouping procedures:

1. With the quadratic in standard form,  $ax^2 + bx + c = 0$ , multiply  $a \cdot c$ .
2. Find two numbers whose product equals  $ac$  and whose sum equals  $b$ .
3. Rewrite the equation replacing the  $bx$  term with two terms using the numbers found in step 2 as coefficients of  $x$ .
4. Factor the first two terms and then factor the last two terms. The expressions in parentheses must be exactly the same to use grouping.
5. Factor out the expression in parentheses.
6. Set the expressions equal to zero and solve for the variable.

The following video contains another example of solving a quadratic equation using factoring by grouping.

Sometimes, we may be given an equation that does not look like a quadratic at first glance. In our next examples we will solve a cubic polynomial equation where the GCF of each term is  $x$  and can be factored. The result is a quadratic equation that we can solve.

In this last video example, we solve a quadratic equation with a leading coefficient of  $-1$  using a shortcut method of factoring and the zero product principle.

## Summary

You can find the solutions, or roots, of quadratic equations by setting one side equal to zero, factoring the polynomial, and then applying the Zero Product Property. The Principle of Zero Products states that if  $ab = 0$ , then either  $a = 0$  or  $b = 0$ , or both  $a$  and  $b$  are 0. Once the polynomial is factored, set each factor equal to zero and solve them separately. The answers will be the set of solutions for the original equation.

Not all solutions are appropriate for some applications. In many real-world situations, negative solutions are not appropriate and must be discarded.

## EXAMPLE

Factor and solve the equation:  $x^2 + x - 6 = 0$ .

### Answer

To factor  $x^2 + x - 6 = 0$ , we look for two numbers whose product equals  $-6$  and whose sum equals  $1$ . Begin by looking at the possible factors of  $-6$ .

$$1 \cdot (-6)$$

$$(-6) \cdot 1$$

$$2 \cdot (-3)$$

$$3 \cdot (-2)$$

The last pair,  $3 \cdot (-2)$  sums to  $1$ , so these are the numbers. Note that only one pair of numbers will work. Then, write the factors.

$$(x - 2)(x + 3) = 0$$

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.

$$(x - 2)(x + 3) = 0$$

$$(x - 2) = 0$$

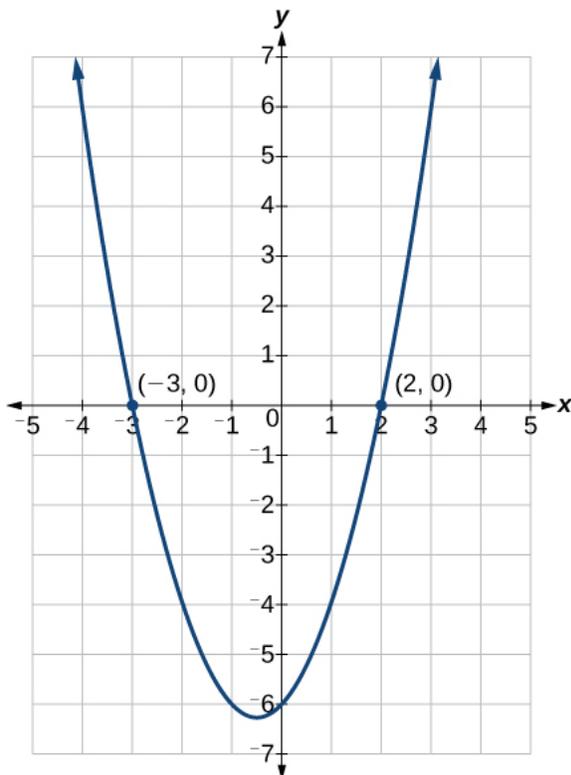
$$(x + 3) = 0$$

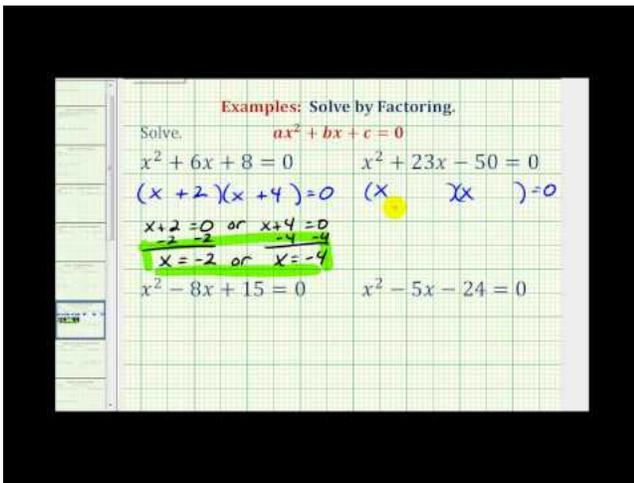
$$x = 2$$

$$x = -3$$

The two solutions are  $x = 2$  and  $x = -3$ .

If we graph the function  $f(x) = x^2 + x - 6$ , we will get the parabola in the figure below. The solutions to the equation  $x^2 + x - 6 = 0$  are the  $x$ -intercepts of the function  $f(x) = x^2 + x - 6$ . Recall that  $x$ -intercepts are where the outputs, or  $y$  values are zero, therefore the points  $(-3, 0)$  and  $(2, 0)$  represent the places where the parabola crosses the  $x$  axis.





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## EXAMPLE

Solve the difference of squares equation using the zero-product property:  $x^2 - 9 = 0$ .

### Answer

Recognizing that the equation represents the difference of squares, we can write the two factors by taking the square root of each term, using a minus sign as the operator in one factor and a plus sign as the operator in the other. Solve using the zero-factor property.

$$x^2 - 9 = 0$$

$$(x - 3) = 0$$

$$(x + 3) = 0$$

$$x = 3$$

$$x = -3$$

The solutions are  $x = 3$  and  $x = -3$ .

## EXAMPLE

Use grouping to factor and solve the quadratic equation:  $4x^2 + 15x + 9 = 0$ .

### Answer

First, multiply  $ac$ :  $4(9) = 36$ . Then list the factors of 36.

$$1 \cdot 36$$

$$2 \cdot 18$$

$$3 \cdot 12$$

$$4 \cdot 9$$

$$6 \cdot 6$$

The only pair of factors that sums to 15 is  $3 + 12$ . Rewrite the equation replacing the  $b$  term,  $15x$ , with two terms using 3 and 12 as coefficients of  $x$ . Factor the first two terms, and then factor the last two terms.

$$4x^2 + 3x + 12x + 9 = 0$$

$$x(4x + 3) + 3(4x + 3) = 0$$

$$(4x + 3)(x + 3) = 0$$

Solve using the zero-product property.

$$(4x + 3)(x + 3) = 0$$

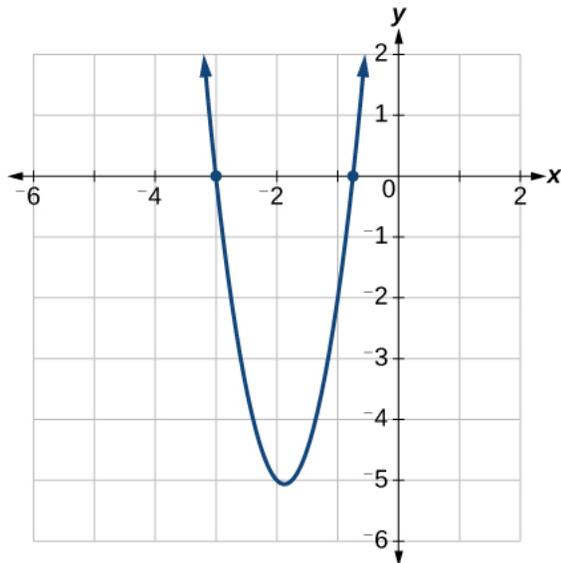
$$(4x + 3) = 0$$

$$(x + 3) = 0$$

$$x = -\frac{3}{4}$$

$$x = -3$$

The solutions are  $x = -\frac{3}{4}$ ,  $x = -3$ .



**Solve a Quadratic Equation Using Factor by Grouping**

$$12x^2 + 20x = -3$$

$$12x^2 + 20x + 3 = 0$$

$$12x^2 + 2x + 18x + 3 = 0$$

$a = 12$   $b = 20$   $c = 3$   
 $ac = 12(3) = 36$   
 $2 + 18 = 20$

1	36
2	18
3	12
4	9
6	6

**Steps to Factor a Quadratic Expression in the form  $ax^2 + bx + c$  by Grouping**

1. Find the factors of  $ac$  that add to  $b$ .
2. Write  $bx$  as a sum or difference using the factors from step #1.
3. Divide or group the polynomial into halves.
4. Factor out the GCF from the first half and second half.
5. Factor out the common binomial factor.

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## EXAMPLE

Solve the equation by factoring:  $-3x^3 - 5x^2 - 2x = 0$ .

**Answer**

This equation does not look like a quadratic, as the highest power is 3, not 2. Recall that the first thing we want to do when solving any equation is to factor out the GCF, if one exists. And it does here. We can factor out  $-x$  from all of the terms and then proceed with grouping.

$$-3x^3 - 5x^2 - 2x = 0$$

$$-x(3x^2 + 5x + 2) = 0$$

Use grouping on the expression in parentheses.

$$-x(3x^2 + 3x + 2x + 2) = 0$$

$$-x[3x(x + 1) + 2(x + 1)] = 0$$

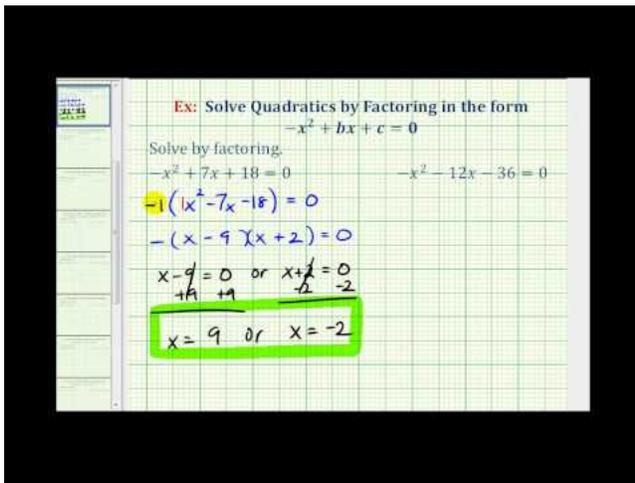
$$-x(3x + 2)(x + 1) = 0$$

Now, we use the zero-product property. Notice that we have three factors.

$$-x = 0 \quad (3x + 2) = 0 \quad (x + 1) = 0$$

$$x = 0 \quad x = -\frac{2}{3} \quad x = -1$$

The solutions are  $x = 0$ ,  $x = -\frac{2}{3}$ , and  $x = -1$ .



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# SQUARE ROOTS AND COMPLETING THE SQUARE

## LEARNING OUTCOMES

- Use the square root property to solve a quadratic equation
- Complete the square to solve a quadratic equation

Quadratic equations can be solved using many methods. You may already be familiar with factoring to solve some quadratic equations. However, not all quadratic equations can be factored. In this section, you will use square roots to learn another way to solve quadratic equations—and this method will work with *all* quadratic equations.

## Solve a Quadratic Equation by the Square Root Property

One way to solve the quadratic equation  $x^2 = 9$  is to subtract 9 from both sides to get one side equal to 0:  $x^2 - 9 = 0$ . The expression on the left can be factored; it is a difference of squares:  $(x + 3)(x - 3) = 0$ . Using the zero factor property, you know this means  $x + 3 = 0$  or  $x - 3 = 0$ , so  $x = -3$  or  $3$ .

Another property that would let you solve this equation more easily is called the square root property.

## THE SQUARE ROOT PROPERTY

If  $x^2 = a$ , then  $x = \sqrt{a}$  or  $-\sqrt{a}$ .

The property above says that you can take the square root of both sides of an equation, but you have to think about two cases: the positive square root of  $a$  and the negative square root of  $a$ .

A shortcut way to write " $\sqrt{a}$ " or " $-\sqrt{a}$ " is  $\pm\sqrt{a}$ . The symbol  $\pm$  is often read "positive or negative." If it is used as an operation (addition or subtraction), it is read "plus or minus."

### Example

Solve using the Square Root Property.  $x^2 = 9$

**Answer**

Since one side is simply  $x^2$ , you can take the square root of both sides to get  $x$  on one side. Do not forget to use both positive and negative square roots!

$$x^2 = 9$$

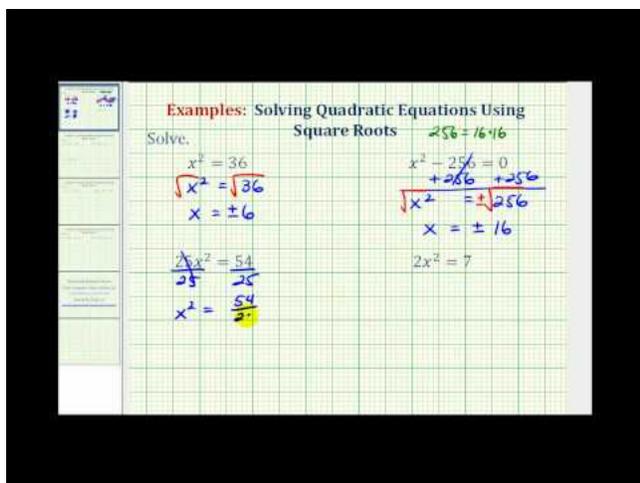
$$x = \pm\sqrt{9}$$

$$x = \pm 3 \text{ (that is, } x = 3 \text{ or } -3\text{)}$$

Notice that there is a difference here in solving  $x^2 = 9$  and finding  $\sqrt{9}$ . For  $x^2 = 9$ , you are looking for *all numbers* whose square is 9. For  $\sqrt{9}$ , you only want the *principal* (nonnegative) square root. The negative of the principal square root is  $-\sqrt{9}$ ; both would be  $\pm\sqrt{9}$ . *Unless there is a symbol in front of the radical sign, only the nonnegative value is wanted!*

In the example above, you can take the square root of both sides easily because there is only one term on each side. In some equations, you may need to do some work to get the equation in this form. You will find that this involves isolating  $x^2$ .

In our first video, we will show more examples of using the square root property to solve a quadratic equation.



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### Example

Solve.  $10x^2 + 5 = 85$

**Answer**

If you try taking the square root of both sides of the original equation, you will have  $\sqrt{10x^2 + 5}$  on the left, and you cannot simplify that. Subtract 5 from both sides to get the  $x^2$  term by itself.

$$10x^2 = 80$$

You could now take the square root of both sides, but you would have  $\sqrt{10}$  as a coefficient, and you would need to divide by that coefficient. Dividing by 10 before you take the square root will be a little easier.

$$x^2 = 8$$

Now you have only  $x^2$  on the left, so you can use the Square Root Property easily.

Be sure to simplify the radical if possible.

$$\begin{aligned}x^2 &= 8 \\x &= \pm\sqrt{8} \\&= \pm\sqrt{(4)(2)} \\&= \pm\sqrt{4}\sqrt{2} \\&= \pm 2\sqrt{2}\end{aligned}$$

The answer is  $x = \pm 2\sqrt{2}$ .

Sometimes more than just the  $x$  is being squared:

#### Example

Solve.  $(x-2)^2 - 50 = 0$

**Answer**

Again, taking the square root of both sides at this stage will leave something you cannot work with on the left. Start by adding 50 to both sides.

$$(x - 2)^2 = 50$$

Because  $(x-2)^2$  is a squared quantity, you can take the square root of both sides.

$$\begin{aligned}(x - 2)^2 &= 50 \\x - 2 &= \pm\sqrt{50}\end{aligned}$$

To isolate  $x$  on the left, you need to add 2 to both sides.

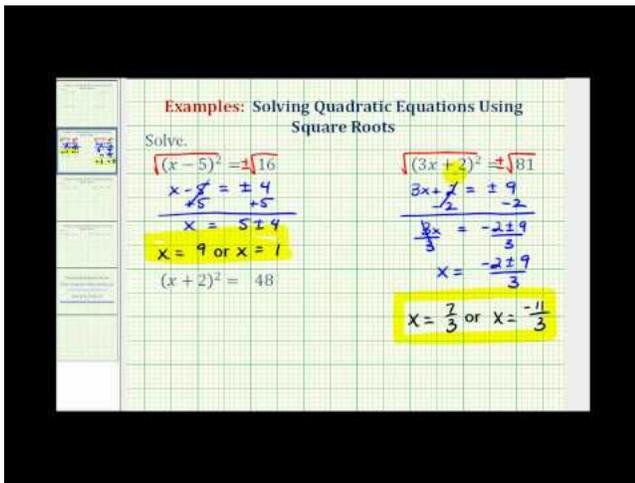
Be sure to simplify the radical if possible.

$$\begin{aligned}x &= 2 \pm \sqrt{50} \\&= 2 \pm \sqrt{(25)(2)} \\&= 2 \pm \sqrt{25}\sqrt{2} \\&= 2 \pm 5\sqrt{2}\end{aligned}$$

The answer is  $x = 2 \pm 5\sqrt{2}$ .

In the next video, you will see more examples of using square roots to solve quadratic equations.

## Solve a Quadratic Equation by Completing the Square



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Not all quadratic equations can be factored or solved in their original form using the square root property. In these cases, we may use a method for solving a **quadratic equation** known as **completing the square**. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property.

First, let us make sure we can recognize a perfect square trinomial and how to factor it.

## EXAMPLE

Factor  $9x^2 - 24x + 16$ .

**Answer**

First notice that the  $x^2$  term and the constant term are both perfect squares.

$$9x^2 = (3x)^2$$

$$16 = 4^2$$

Then notice that the middle term (ignoring the sign) is twice the product of the square roots of these squared terms.

$$24x = 2(3x)(4)$$

A trinomial in the form  $r^2 - 2rs + s^2$  can be factored as  $(r-s)^2$ .

In this case, the middle term is subtracted, so subtract  $r$  and  $s$  and square it to get  $(r-s)^2$ .

$$r = 3x$$

$$s = 4$$

$$9x^2 - 24x + 16 = (3x - 4)^2$$

If this were an equation, we could solve using either the square root property or the zero product property. If you do not start with a perfect square trinomial, you can complete the square to make what you have into one.

To complete the square, the leading coefficient,  $a$ , must equal 1. If it does not, then divide the entire equation by  $a$ . Then, we can use the following procedures to solve a quadratic equation by completing the square.

## STEPS FOR COMPLETING THE SQUARE

We will use the example  $x^2 + 4x + 1 = 0$  to illustrate each step.

1. Given a quadratic equation that cannot be factored and with  $a = 1$ , first add or subtract the constant term to the right side of the equal sign.

$$x^2 + 4x = -1$$

2. Multiply the  $b$  term by  $\frac{1}{2}$  and square it.

$$\frac{1}{2}(4) = 2$$

$$2^2 = 4$$

3. Add  $(\frac{1}{2}b)^2$  to both sides of the equal sign and simplify the right side. We have:

$$x^2 + 4x + 4 = -1 + 4$$

$$x^2 + 4x + 4 = 3$$

4. The left side of the equation can now be factored as a perfect square.

$$x^2 + 4x + 4 = 3$$

$$(x + 2)^2 = 3$$

5. Use the square root property and solve.

$$\sqrt{(x + 2)^2} = \pm\sqrt{3}$$

$$x + 2 = \pm\sqrt{3}$$

$$x = -2 \pm \sqrt{3}$$

6. The solutions are  $x = -2 + \sqrt{3}$ ,  $x = -2 - \sqrt{3}$ .

### Example

Solve by completing the square.  $x^2 - 12x - 4 = 0$

**Answer**

Since you cannot factor the trinomial on the left side, you will use completing the square to solve the equation.

First, move the constant term to the right side of the equal sign.

$$x^2 - 12x = 4$$

Identify  $b$ :  $b = -12$

Then take  $\frac{1}{2}$  of the  $b$  term and square it. Add  $(\frac{b}{2})^2$  to complete the square, so

$$\left(\frac{b}{2}\right)^2 = \left(\frac{-12}{2}\right)^2 = (-6)^2 = 36.$$

Add the value to *both* sides of the equation and simplify.

$$x^2 - 12x + 36 = 4 + 36$$

$$x^2 - 12x + 36 = 40$$

Rewrite the left side as a squared binomial.

$$(x - 6)^2 = 40$$

Use the Square Root Property. Remember to include both the positive and negative square root, or you will miss one of the solutions.

$$x - 6 = \pm\sqrt{40}$$

Solve for  $x$  by adding 6 to both sides. Simplify as needed.

$$\begin{aligned}x &= 6 \pm \sqrt{40} \\ &= 6 \pm \sqrt{4}\sqrt{10} \\ &= 6 \pm 2\sqrt{10}\end{aligned}$$

The answer is  $x = 6 \pm 2\sqrt{10}$ .

## EXAMPLE

Solve by completing the square:  $x^2 - 3x - 5 = 0$ .

**Answer**

First, move the constant term to the right side of the equal sign.

$$x^2 - 3x = 5$$

Identify  $b$ :  $b = -3$

Then, take  $\frac{1}{2}$  of the  $b$  term and square it.

$$\frac{1}{2}(-3) = -\frac{3}{2}$$

$$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$$

Add the result to both sides of the equal sign.

$$x^2 - 3x + \left(-\frac{3}{2}\right)^2 = 5 + \left(-\frac{3}{2}\right)^2$$

$$x^2 - 3x + \frac{9}{4} = 5 + \frac{9}{4}$$

Factor the left side as a perfect square and simplify the right side.

$$\left(x - \frac{3}{2}\right)^2 = \frac{29}{4}$$

Use the square root property and solve.

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \pm\sqrt{\frac{29}{4}}$$

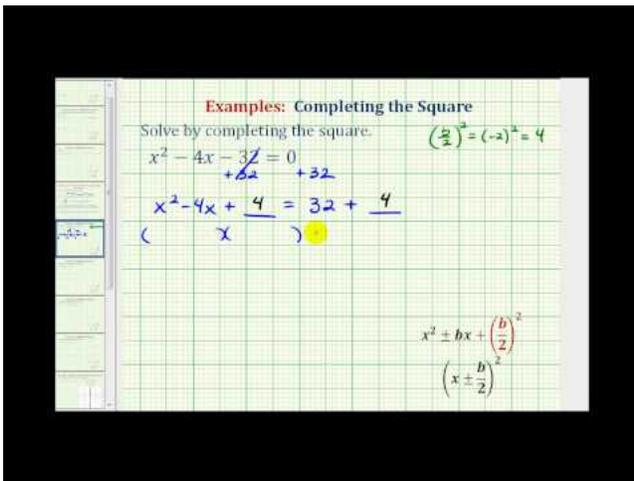
$$x - \frac{3}{2} = \pm\frac{\sqrt{29}}{2}$$

$$x = \frac{3}{2} \pm \frac{\sqrt{29}}{2}$$

The solutions are  $x = \frac{3+\sqrt{29}}{2}$ ,  $x = \frac{3-\sqrt{29}}{2}$ .

In the next video, you will see more examples of how to use completing the square to solve a quadratic equation.

You may have noticed that because you have to use both square roots, all the examples have two solutions. Here is another example that is slightly different.



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### Example

Solve by completing the square.  $x^2 + 16x + 17 = -47$ .

**Answer**

Rewrite the equation so the left side has the form  $x^2 + bx$ . Identify  $b$ .

$$x^2 + 16x = -64$$

$$b = 16$$

Add  $\left(\frac{b}{2}\right)^2$ , which is  $\left(\frac{16}{2}\right)^2 = 8^2 = 64$ , to both sides.

$$x^2 + 16x + 64 = -64 + 64$$

$$x^2 + 16x + 64 = 0$$

Write the left side as a squared binomial.

$$(x + 8)^2 = 0$$

Take the square roots of both sides. Normally both positive and negative square roots are needed, but 0 is neither positive nor negative. 0 has only one root.

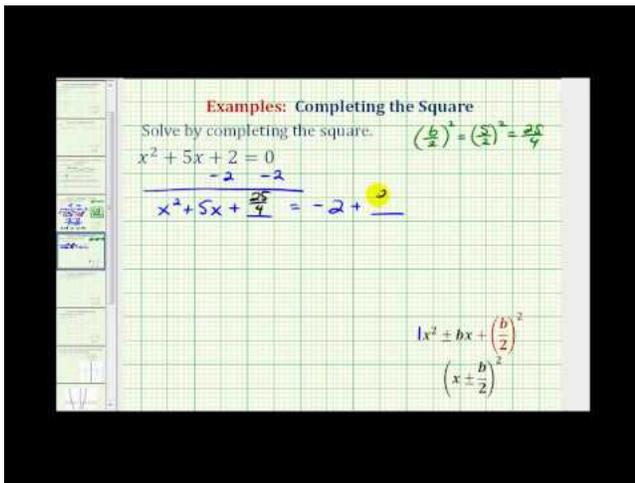
$$x + 8 = 0$$

$$x = -8$$

Take a closer look at this problem and you may see something familiar. Instead of completing the square, try adding 47 to both sides in the equation. The equation  $x^2 + 16x + 17 = -47$  becomes  $x^2 + 16x + 64 = 0$ . Can you factor this equation using grouping? (Think of two numbers whose product is 64 and whose sum is 16).

It can be factored as  $(x + 8)(x + 8) = 0$ , of course! Knowing how to complete the square is very helpful, but it is not always the only way to solve an equation.

In our last video, we show an example of how to use completing the square to solve a quadratic equation whose solutions are rational.



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## Summary

Completing the square is used to change a binomial of the form  $x^2 + bx$  into a perfect square trinomial  $x^2 + bx + \left(\frac{b}{2}\right)^2$  which can be factored to  $\left(x + \frac{b}{2}\right)^2$ . When solving quadratic equations by completing the square, be careful to add  $\left(\frac{b}{2}\right)^2$  to *both* sides of the equation to maintain equality. The Square Root Property can then be used to solve for  $x$ . With the Square Root Property, be careful to include both the principal square root and its opposite. Be sure to simplify as needed.

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# THE QUADRATIC FORMULA

## Learning Outcomes

- Write a quadratic equation in standard form and identify the values of  $a$ ,  $b$ , and  $c$  in a standard form quadratic equation.
- Use the Quadratic Formula to find all real solutions of a quadratic equation and recognize when there are no real solutions
- Solve application problems involving quadratic equations

You can solve any quadratic equation by **completing the square**—rewriting part of the equation as a perfect square trinomial. If you complete the square on the generic equation  $ax^2 + bx + c = 0$  and then solve for  $x$ , you find that  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . This equation is known as the Quadratic Formula.

We can derive the quadratic formula by completing the square. First, assume that the leading coefficient is positive; if it is negative, we can multiply the equation by  $-1$  and obtain a positive  $a$ . Given  $ax^2 + bx + c = 0$ ,  $a \neq 0$ , we will complete the square as follows:

1. First, move the constant term to the right side of the equal sign:

$$ax^2 + bx = -c$$

2. As we want the leading coefficient to equal 1, divide through by  $a$ :

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

3. Then, find  $\frac{1}{2}$  of the middle term, and add  $(\frac{1}{2} \frac{b}{a})^2 = \frac{b^2}{4a^2}$  to both sides of the equal sign:

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$$

4. Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

5. Now, use the square root property, which gives

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x + \frac{b}{2a} = \frac{\pm \sqrt{b^2 - 4ac}}{2a}$$

6. Finally, add  $-\frac{b}{2a}$  to both sides of the equation and combine the terms on the right side. Thus,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is very helpful for solving quadratic equations that are difficult or impossible to factor, and using it can be faster than completing the square. The Quadratic Formula can be used to solve any quadratic equation of the form  $ax^2 + bx + c = 0$ .

The form  $ax^2 + bx + c = 0$  is called standard form of a quadratic equation. Before solving a quadratic equation using the Quadratic Formula, it is *vital* that you be sure the equation is in this form. If you do not, you might use the wrong values for  $a$ ,  $b$ , or  $c$ , and then the formula will give incorrect solutions.

## Solving a Quadratic Equation using the Quadratic Formula

The Quadratic Formula will work with *any* quadratic equation, but *only* if the equation is in standard form,  $ax^2 + bx + c = 0$ . To use it, follow these steps.

1. Put the equation in standard form first.
2. Identify the coefficients,  $a$ ,  $b$ , and  $c$ . Be sure to include negative signs if the  $bx$  or  $c$  terms are subtracted.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Simplify as much as possible.
5. Use the  $\pm$  in front of the radical to separate the solution into two values: one in which the square root is added and one in which it is subtracted.
6. Simplify both values to get the possible solutions.

That is a lot of steps. Let us try using the Quadratic Formula to solve a relatively simple equation first; then you will go back and solve it again using another factoring method.

### Example

Use the Quadratic Formula to solve the equation  $x^2 + 4x = 5$ .

#### Answer

First write the equation in standard form.

$$x^2 + 4x = 5$$

$$x^2 + 4x - 5 = 0$$

$$a = 1, b = 4, c = -5$$

Note that the subtraction sign means the constant  $c$  is negative.

$$x^2 + 4x - 5 = 0$$

$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ ax^2 & + bx & + c = 0 \end{array}$$

Substitute the values into the Quadratic Formula.  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(-5)}}{2(1)}$$

Simplify, being careful to get the signs correct.

$$x = \frac{-4 \pm \sqrt{16 + 20}}{2}$$

Simplify some more.

$$x = \frac{-4 \pm \sqrt{36}}{2}$$

Simplify the radical:  $\sqrt{36} = 6$ .

$$x = \frac{-4 \pm 6}{2}$$

Separate and simplify to find the solutions to the quadratic equation. Note that in one, 6 is added and in the other, 6 is subtracted.

$$x = \frac{-4+6}{2} = \frac{2}{2} = 1$$

or

$$x = \frac{-4-6}{2} = \frac{-10}{2} = -5$$

The solutions are  $x = 1$  or  $-5$ .

You can check these solutions by substituting 1 and  $-5$  into the original equation.

$$x = 1$$

$$x^2 + 4x = 5$$

$$(1)^2 + 4(1) = 5$$

$$1 + 4 = 5$$

$$5 = 5$$

$$x = -5$$

$$x^2 + 4x = 5$$

$$(-5)^2 + 4(-5) = 5$$

$$25 - 20 = 5$$

$$5 = 5$$

You get two true statements, so you know that both solutions work:  $x = 1$  or  $-5$ . You have solved the equation successfully using the Quadratic Formula!

Watch this video to see an example of how to use the quadratic formula to solve a quadratic equation that has two real, rational solutions.

**Examples: The Quadratic Formula**  
Solve using the quadratic formula.  
 $6x^2 - x - 15 = 0$        $x = \frac{1 \pm \sqrt{361}}{12}$   
 $a = 6$     $b = -1$     $c = -15$   
 $ax^2 + bx + c = 0$        $x = \frac{1 \pm 19}{12}$   
 $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$   
 $x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(6)(-15)}}{2(6)}$   
 $x = \frac{1 \pm \sqrt{1 + 360}}{12}$

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Sometimes, it may be easier to solve an equation using conventional factoring methods like finding number pairs that sum to one number (in this example, 4) and that produce a specific product (in this example  $-5$ ) when multiplied. The power of the Quadratic Formula is that it can be used to solve *any* quadratic equation, even those where finding number combinations will not work.

In the next video example, we show that the quadratic formula is useful when a quadratic equation has two irrational solutions that could not have been obtained by factoring.

**Examples: The Quadratic Formula**  
Solve using the quadratic formula.  
 $x^2 + 8x - 16 = 0$        $x = \frac{-8 \pm \sqrt{128}}{2}$   
 $a = 1$     $b = 8$     $c = -16$   
 $ax^2 + bx + c = 0$   
 $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$   
 $x = \frac{-8 \pm \sqrt{(8)^2 - 4(1)(-16)}}{2(1)}$   
 $x = \frac{-8 \pm \sqrt{64 + 64}}{2}$

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Most of the quadratic equations you have looked at have two solutions, like the one above. The following example is a little different.

### Example

Use the Quadratic Formula to solve the equation  $x^2 - 2x = 6x - 16$ .

**Answer**

Subtract  $6x$  from each side and add 16 to both sides to put the equation in standard form.

$$x^2 - 2x = 6x - 16$$

$$x^2 - 2x - 6x + 16 = 0$$

$$x^2 - 8x + 16 = 0$$

Identify the coefficients  $a$ ,  $b$ , and  $c$ .  $x^2 = 1x^2$ , so  $a = 1$ . Since  $8x$  is subtracted,  $b$  is negative.  
 $a = 1$ ,  $b = -8$ ,  $c = 16$

$$x^2 - 8x + 16 = 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$ax^2 + bx + c = 0$$

Substitute the values into the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(16)}}{2(1)}$$

Simplify.

$$x = \frac{8 \pm \sqrt{64 - 64}}{2}$$

Since the square root of 0 is 0, and both adding and subtracting 0 give the same result, there is only one possible value.

$$x = \frac{8 \pm \sqrt{0}}{2} = \frac{8}{2} = 4$$

The answer is  $x = 4$ .

Again, check using the original equation.

$$x^2 - 2x = 6x - 16$$

$$(4)^2 - 2(4) = 6(4) - 16$$

$$16 - 8 = 24 - 16$$

$$8 = 8$$

In the next example, we will show that some quadratic equations do not have real solutions. As we simplify with the quadratic formula, we may end up with a negative number under a square root, which, as we know, is not defined for real numbers.

## Applying the Quadratic Formula

Quadratic equations are widely used in science, business, and engineering. Quadratic equations are commonly used in situations where two things are multiplied together and they both depend on the same variable. For example, when working with area, if both dimensions are written in terms of the same variable, you use a quadratic equation. Because the quantity of a product sold often depends on the price, you sometimes use a quadratic equation to represent revenue as a product of the price and the quantity sold. Quadratic equations are also used when gravity is involved, such as the path of a ball or the shape of cables in a suspension bridge.

## EXAMPLE

Use the Quadratic Formula to solve the equation  $x^2 + x = -x - 3$

### Answer

Add  $x$  to both sides and add 3 to both sides to get the quadratic equation in standard form.

$$x^2 + x = -x - 3$$

$$x^2 + 2x + 3 = 0$$

Identify  $a$ ,  $b$ ,  $c$ .

$$a = 1, b = 2, c = 3$$

Substitute values for  $a$ ,  $b$ ,  $c$  into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(3)}}{2(1)}$$

Simplify

$$x = \frac{-2 \pm \sqrt{-8}}{2}$$

Since the square root of a negative number is not defined for real numbers, there are no real number solutions to this equation.

A very common and easy-to-understand application is the height of a ball thrown at the ground off a building. Because gravity will make the ball speed up as it falls, a quadratic equation can be used to estimate its height any time before it hits the ground. *Note: The equation is not completely accurate, because friction from the air will slow the ball down a little. For our purposes, this is close enough.*

## Example

A ball is thrown off a building from 200 feet above the ground. Its starting velocity (also called *initial velocity*) is  $-10$  feet per second. The negative value means it is heading toward the ground.

The equation  $h = -16t^2 - 10t + 200$  can be used to model the height of the ball after  $t$  seconds. About how long does it take for the ball to hit the ground?

### Answer

When the ball hits the ground, the height is 0. Substitute 0 for  $h$ .

$$h = -16t^2 - 10t + 200$$

$$0 = -16t^2 - 10t + 200$$

$$-16t^2 - 10t + 200 = 0$$

This equation is difficult to solve by factoring or by completing the square, so solve it by applying the Quadratic Formula,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . In this case, the variable is  $t$  rather than  $x$ .  $a = -16$ ,  $b = -10$ , and  $c = 200$ .

$$t = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(-16)(200)}}{2(-16)}$$

Simplify. Be very careful with the signs.

$$t = \frac{10 \pm \sqrt{100 + 12800}}{-32}$$

$$= \frac{10 \pm \sqrt{12900}}{-32}$$

Use a calculator to find both roots.

$t$  is approximately  $-3.86$  or  $3.24$ .

Consider the roots logically. One solution,  $-3.86$ , cannot be the time because it is a negative number. The other solution,  $3.24$  seconds, must be when the ball hits the ground.

The ball hits the ground approximately  $3.24$  seconds after being thrown.

In the next video, we show another example of how the quadratic equation can be used to find the time it takes for an object in free fall to hit the ground.

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The area problem below does not look like it includes a Quadratic Formula of any type, and the problem seems to be something you have solved many times before by simply multiplying. But in order to solve it, you will need to use a quadratic equation.

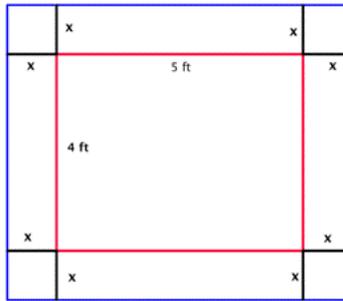
### Example

Bob made a quilt that is  $4 \text{ ft} \times 5 \text{ ft}$ . He has  $10 \text{ sq. ft.}$  of fabric he can use to add a border around the quilt. How wide should he make the border to use all the fabric? (The border must be the same width on all four sides.)

**Answer**

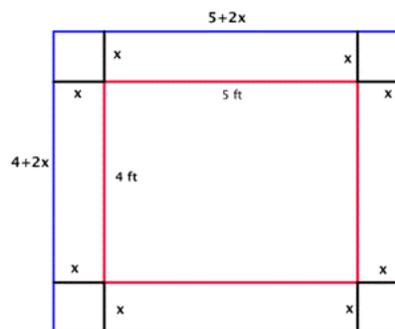
Sketch the problem. Since you do not know the width of the border, you will let the variable  $x$  represent the width.

In the diagram. The original quilt is indicated by the red rectangle. The border is the area between the red and blue lines.



Since each side of the original 4 by 5 quilt has the border of width  $x$  added, the length of the quilt with the border will be  $5 + 2x$ , and the width will be  $4 + 2x$ .

(Both dimensions are written in terms of the same variable, and you will multiply them to get an area! This is where you might start to think that a quadratic equation might be used to solve this problem.)



You are only interested in the area of the border strips. Write an expression for the area of the border.

Area of border = Area of the blue rectangle minus the area of the red rectangle

$$\text{Area of border} = (4 + 2x)(5 + 2x) - (4)(5)$$

There are 10 sq ft of fabric for the border, so set the area of border to be 10.

$$10 = (4 + 2x)(5 + 2x) - 20$$

Multiply  $(4 + 2x)(5 + 2x)$ .

$$10 = 20 + 8x + 10x + 4x^2 - 20$$

Simplify.

$$10 = 18x + 4x^2$$

Subtract 10 from both sides so that you have a quadratic equation in standard form and can apply the Quadratic Formula to find the roots of the equation.

$$0 = 18x + 4x^2 - 10$$

or

$$4x^2 + 18x - 10 = 0$$

$$2(2x^2 + 9x - 5) = 0$$

Factor out the greatest common factor, 2, so that you can work with the simpler equivalent equation,  $2x^2 + 9x - 5 = 0$ .

$$2(2x^2 + 9x - 5) = 0$$

$$\frac{2(2x^2 + 9x - 5)}{2} = \frac{0}{2}$$

$$2x^2 + 9x - 5 = 0$$

Use the Quadratic Formula. In this case,  $a = 2$ ,  $b = 9$ , and  $c = -5$ .

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-9 \pm \sqrt{9^2 - 4(2)(-5)}}{2(2)}$$

Simplify.

$$x = \frac{-9 \pm \sqrt{121}}{4} = \frac{-9 \pm 11}{4}$$

Find the solutions, making sure that the  $\pm$  is evaluated for both values.

$$x = \frac{-9 + 11}{4} = \frac{2}{4} = \frac{1}{2} = 0.5$$

or

$$x = \frac{-9 - 11}{4} = \frac{-20}{4} = -5$$

Ignore the solution  $x = -5$ , since the width could not be negative.

The width of the border should be 0.5 ft.

Our last video gives another example of using the quadratic formula for a geometry problem involving the border around a quilt.

**Quadratic Equation Application - The Quadratic Formula**

**EXAMPLE**

A quilt is 3 feet by 5 feet. If there is 22 square feet of fabric available to use as a border around the quilt, how wide should the border be. Assume all of the fabric must be used.

Diagram: A red quilt with dimensions  $(3+2x)$  and  $(5+2x)$ . The border width is  $x$ .

Equations shown:

$$(6+2x)(3+2x) - 3(6) = 22$$

$$18 + 12x + 6x + 4x^2 - 18 = 22$$

$$4x^2 + 18x = 22$$

$$\begin{array}{r} 4x^2 + 18x = 22 \\ -22 \quad -22 \\ \hline 4x^2 + 18x - 22 = 0 \end{array}$$

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<https://courses.lumenlearning.com/intermediatealgebra/?p=1665>

## Summary

Quadratic equations can appear in different applications. The Quadratic Formula is a useful way to solve these equations or any other quadratic equation! The Quadratic Formula,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ , is found by completing the square of the quadratic equation. When you simplify using the quadratic formula and your result is a negative number under a square root, there are no real number solutions to the equation.

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# INTRODUCTION TO COMPLEX NUMBERS

Up until now, you have known it was impossible to take a square root of a negative number. This is true when considering solutions that are only *real numbers*. But here you will learn about a new kind of number that lets you work with square roots of negative numbers! Complex numbers are made from both real and imaginary numbers. Imaginary numbers result from taking the square root of a negative number.

We will first define and perform algebraic operations on complex numbers; then we will provide examples of quadratic equations that have solutions that are complex numbers.

# IMAGINARY AND COMPLEX NUMBERS

## LEARNING OUTCOMES

- Express roots of negative numbers in terms of  $i$
- Express imaginary numbers as  $bi$  and complex numbers as  $a + bi$

You really need only one new number to start working with the square roots of negative numbers. That number is the square root of  $-1$ ,  $\sqrt{-1}$ . The *real numbers* are those that can be shown on a number line—they seem pretty *real* to us! When something is not real, we often say it is *imaginary*. So let us call this new number  $i$  and use it to represent the square root of  $-1$ .

$$i = \sqrt{-1}$$

Because  $\sqrt{x} \cdot \sqrt{x} = x$ , we can also see that  $\sqrt{-1} \cdot \sqrt{-1} = -1$  or  $i \cdot i = -1$ . We also know that  $i \cdot i = i^2$ , so we can conclude that  $i^2 = -1$ .

$$i^2 = -1$$

The number  $i$  allows us to work with roots of all negative numbers, not just  $\sqrt{-1}$ . There are two important rules to remember:  $\sqrt{-1} = i$ , and  $\sqrt{ab} = \sqrt{a}\sqrt{b}$ . You will use these rules to rewrite the square root of a negative number as the square root of a positive number times  $\sqrt{-1}$ . Next you will simplify the square root and rewrite  $\sqrt{-1}$  as  $i$ . Let us try an example.

### EXAMPLE

Simplify.  $\sqrt{-4}$

**Answer**

Use the rule  $\sqrt{ab} = \sqrt{a}\sqrt{b}$  to rewrite this as a product using  $\sqrt{-1}$ .

$$\sqrt{-4} = \sqrt{4 \cdot -1} = \sqrt{4}\sqrt{-1}$$

Since 4 is a perfect square ( $4 = 2^2$ ), you can simplify the square root of 4.

$$\sqrt{4}\sqrt{-1} = 2\sqrt{-1}$$

Use the definition of  $i$  to rewrite  $\sqrt{-1}$  as  $i$ .

$$2\sqrt{-1} = 2i$$

The answer is  $2i$ .

### EXAMPLE

Simplify.  $\sqrt{-18}$

**Answer**

Use the rule  $\sqrt{ab} = \sqrt{a}\sqrt{b}$  to rewrite this as a product using  $\sqrt{-1}$ .

$$\sqrt{-18} = \sqrt{18 \cdot -1} = \sqrt{18}\sqrt{-1}$$

Since 18 is *not* a perfect square, use the same rule to rewrite it using factors that are perfect squares. In this case, 9 is the only perfect square factor, and the square root of 9 is 3.

$$\sqrt{18}\sqrt{-1} = \sqrt{9}\sqrt{2}\sqrt{-1} = 3\sqrt{2}\sqrt{-1}$$

Use the definition of  $i$  to rewrite  $\sqrt{-1}$  as  $i$ .

$$3\sqrt{2}\sqrt{-1} = 3\sqrt{2}i = 3i\sqrt{2}$$

Remember to write  $i$  in front of the radical.

The answer is  $3i\sqrt{2}$ .

You may have wanted to simplify  $-\sqrt{-72}$  using different factors. Some may have thought of rewriting this radical as  $-\sqrt{-9}\sqrt{8}$ , or  $-\sqrt{-4}\sqrt{18}$ , or  $-\sqrt{-6}\sqrt{12}$  for instance. Each of these radicals would have eventually yielded the same answer of  $-6i\sqrt{2}$ .

In the following video, we show more examples of how to use imaginary numbers to simplify a square root with a negative radicand.

## Complex Numbers

$5 + 2i$   
Real part    Imaginary part

A **complex number** is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written  $a + bi$  where  $a$  is the real part and  $bi$  is the imaginary part. For example,  $5 + 2i$  is a complex number. So, too, is  $3 + 4i\sqrt{3}$ .

## EXAMPLE

Simplify.  $-\sqrt{-72}$

**Answer**

Use the rule  $\sqrt{ab} = \sqrt{a}\sqrt{b}$  to rewrite this as a product using  $\sqrt{-1}$ .

$$-\sqrt{-72} = -\sqrt{72 \cdot -1} = -\sqrt{72}\sqrt{-1}$$

Since 72 is *not* a perfect square, use the same rule to rewrite it using factors that are perfect squares.

Notice that 72 has three perfect squares as factors: 4, 9, and 36. It is easiest to use the largest factor that is a perfect square.

$$-\sqrt{72}\sqrt{-1} = -\sqrt{36}\sqrt{2}\sqrt{-1} = -6\sqrt{2}\sqrt{-1}$$

Use the definition of  $i$  to rewrite  $\sqrt{-1}$  as  $i$ .

$$-6\sqrt{2}\sqrt{-1} = -6\sqrt{2}i = -6i\sqrt{2}$$

Remember to write  $i$  in front of the radical.

The answer is  $-6i\sqrt{2}$

The screenshot shows a digital workspace titled "Imaginary Numbers" with an "EXAMPLE" section. The problem asks to simplify  $\sqrt{-25}$ ,  $\sqrt{-12}$ , and  $-\sqrt{-48}$ . Handwritten solutions are shown on a grid background:

- $\sqrt{-25} = \sqrt{-1 \cdot 25} = \sqrt{-1} \cdot \sqrt{25} = i \cdot 5 = 5i$
- $\sqrt{-12} = \sqrt{-1 \cdot 12} = \sqrt{-1} \cdot \sqrt{12} = i \cdot \sqrt{4 \cdot 3} = i \cdot 2\sqrt{3} = 2i\sqrt{3}$
- $-\sqrt{-48} = -\sqrt{-1 \cdot 48} = -\sqrt{-1} \cdot \sqrt{48} = -i \cdot \sqrt{16 \cdot 3} = -i \cdot 4\sqrt{3} = -4i\sqrt{3}$

At the bottom, a text box defines the imaginary unit: "An imaginary number is real number multiplied by the imaginary unit  $i$ .  $i = \sqrt{-1}$ ,  $i^2 = -1$ ,  $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ ".

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<https://courses.lumenlearning.com/intermediatealgebra/?p=237>

## REWRITING THE SQUARE ROOT OF A NEGATIVE NUMBER

- Find perfect squares within the radical.
- Rewrite the radical using the rule  $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ .
- Rewrite  $\sqrt{-1}$  as  $i$ .

Example:  $\sqrt{-18} = \sqrt{9}\sqrt{-2} = \sqrt{9}\sqrt{2}\sqrt{-1} = 3i\sqrt{2}$

Imaginary numbers are distinguished from real numbers because a squared imaginary number produces a negative real number. Recall, when a positive real number is squared, the result is a positive real number and when a negative real number is squared, again, the result is a positive real number. Complex numbers are a combination of real and imaginary numbers. You can use the usual operations (addition, subtraction, multiplication, and so on) with imaginary numbers. You will see more of that later.

Complex Number	Real Part	Imaginary Part
$3 + 7i$	3	$7i$
$18 - 32i$	18	$-32i$
$-\frac{3}{5} + i\sqrt{2}$	$-\frac{3}{5}$	$i\sqrt{2}$
$\frac{\sqrt{2}}{2} - \frac{1}{2}i$	$\frac{\sqrt{2}}{2}$	$-\frac{1}{2}i$

In a number with a radical as part of  $b$ , such as  $-\frac{3}{5} + i\sqrt{2}$  above, the imaginary  $i$  should be written in front of the radical. Though writing this number as  $-\frac{3}{5} + \sqrt{2}i$  is technically correct, it makes it much more difficult to tell whether  $i$  is inside or outside of the radical. Putting it before the radical, as in  $-\frac{3}{5} + i\sqrt{2}$ , clears up any confusion. Look at these last two examples.

Number	Complex Form: $a + bi$	Real Part	Imaginary Part
17	$17 + 0i$	17	$0i$
$-3i$	$0 - 3i$	0	$-3i$

By making  $b = 0$ , any real number can be expressed as a complex number. The real number  $a$  is written as  $a + 0i$  in complex form. Similarly, any imaginary number can be expressed as a complex number. By making  $a = 0$ , any imaginary number  $bi$  can be written as  $0 + bi$  in complex form.

### EXAMPLE

Write 83.6 as a complex number.

**Answer**

Remember that a complex number has the form  $a + bi$ . You need to figure out what  $a$  and  $b$  need to be.

$a + bi$

Since 83.6 is a real number, it is the real part ( $a$ ) of the complex number  $a + bi$ .

$83.6 + bi$

A real number does not contain any imaginary parts, so the value of  $b$  is 0.

The answer is  $83.6 + 0i$ .

### EXAMPLE

Write  $-3i$  as a complex number.

**Answer**

Remember that a complex number has the form  $a + bi$ . You need to figure out what  $a$  and  $b$  need to be.

$a + bi$

Since  $-3i$  is an imaginary number, it is the imaginary part  $bi$  of the complex number  $a + bi$ .

$a - 3i$

This imaginary number has no real parts, so the value of  $a$  is 0.

The answer is  $0 - 3i$ .

In the next video, we show more examples of how to write numbers as complex numbers.

## Summary

**Writing Complex Number**

**EXAMPLE**

Write the following numbers as complex numbers.

1.  $3.4$
2.  $-7i$
3.  $6 + \sqrt{-4}$
4.  $-3 - \sqrt{-18}$

$$5 + 2i$$

Real part      Imaginary part

1.  $3.4 = 3.4 + 0i$

2.  $-7i = 0 - 7i$

3.  $6 + \sqrt{-4} = 6 + 2i$

4.  $-3 - \sqrt{-18} = -3 - 3\sqrt{2}i$

A complex number is a number in the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i$  is the imaginary unit.  
 $i = \sqrt{-1}$

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Complex numbers have the form  $a + bi$ , where  $a$  and  $b$  are real numbers and  $i$  is the square root of  $-1$ . All real numbers can be written as complex numbers by setting  $b = 0$ . Imaginary numbers have the form  $bi$  and can also be written as complex numbers by setting  $a = 0$ . Square roots of negative numbers can be simplified using  $\sqrt{-1} = i$  and  $\sqrt{ab} = \sqrt{a}\sqrt{b}$ .

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# ADD AND SUBTRACT COMPLEX NUMBERS

## LEARNING OUTCOMES

- Add complex numbers
- Subtract complex numbers

Any time new kinds of numbers are introduced, one of the first questions that needs to be addressed is, “How do you add them?” In this section, you will learn how to add and subtract complex numbers.

First, consider the following expression.

$$(6x + 8) + (4x + 2)$$

To simplify this expression, you combine the like terms,  $6x$  and  $4x$ . These are like terms because they have the same variable with the same exponents. Similarly,  $8$  and  $2$  are like terms because they are both constants, with no variables.

$$(6x + 8) + (4x + 2) = 10x + 10$$

In the same way, you can simplify expressions with radicals.

$$(6\sqrt{3} + 8) + (4\sqrt{3} + 2) = 10\sqrt{3} + 10$$

You can add  $6\sqrt{3}$  to  $4\sqrt{3}$  because the two terms have the same radical,  $\sqrt{3}$ , just as  $6x$  and  $4x$  have the same variable and exponent.

The number  $i$  looks like a variable, but remember that it is equal to  $\sqrt{-1}$ . The great thing is you have no new rules to worry about—whether you treat it as a variable or a radical, the exact same rules apply to adding and subtracting **complex numbers**. You combine the imaginary parts (the terms with  $i$ ), and you combine the real parts.

### EXAMPLE

Add.  $(-3 + 3i) + (7 - 2i)$

**Answer**

Rearrange the sums to put like terms together.

$$-3 + 3i + 7 - 2i = -3 + 7 + 3i - 2i$$

Combine like terms.

$$-3 + 7 = 4$$

and

$$3i - 2i = (3 - 2)i = i$$

The answer is  $4 + i$ .

### EXAMPLE

Subtract.  $(-3 + 3i) - (7 - 2i)$

**Answer**

Be sure to distribute the subtraction sign to all terms in the set of parentheses that follows.

$$(-3 + 3i) - (7 - 2i) = -3 + 3i - 7 + 2i$$

Rearrange the terms to put like terms together.

$$-3 - 7 + 3i + 2i$$

Combine like terms.

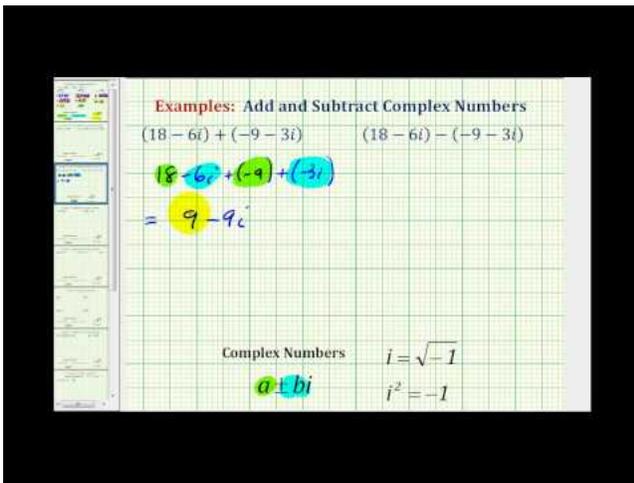
$$-3 - 7 = -10$$

and

$$3i + 2i = (3 + 2)i = 5i$$

The answer is  $-10 + 5i$ .

In the following video, we show more examples of how to add and subtract complex numbers.



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# MULTIPLY AND DIVIDE COMPLEX NUMBERS

## LEARNING OUTCOMES

- Multiply complex numbers
- Find conjugates of complex numbers
- Divide complex numbers
- Simplify powers of  $i$

Multiplying complex numbers is much like multiplying binomials. The major difference is that we work with the real and imaginary parts separately.

## Multiplying a Complex Number by a Real Number

$$\begin{aligned}
 3(6 + 2i) &= (3 \cdot 6) + (3 \cdot 2i) && \text{Distribute.} \\
 &= 18 + 6i && \text{Simplify.}
 \end{aligned}$$

Let us begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. So, for  $3(6+2i)$ , 3 is multiplied to both the real and imaginary parts. So we have  $(3)(6) + (3)(2i) = 18 + 6i$ .

## HOW TO: GIVEN A COMPLEX NUMBER AND A REAL NUMBER, MULTIPLY TO FIND THE PRODUCT

1. Use the distributive property.
2. Simplify.

### EXAMPLE

Find the product  $4(2 + 5i)$ .

**Answer**

Distribute the 4.

$$\begin{aligned}4(2 + 5i) &= (4 \cdot 2) + (4 \cdot 5i) \\ &= 8 + 20i\end{aligned}$$

## Multiplying Complex Numbers Together

Now, let us multiply two complex numbers. We can use either the distributive property or the FOIL method. Recall that FOIL is an acronym for multiplying First, Outer, Inner, and Last terms together. Using either the distributive property or the FOIL method, we get

$$(a + bi)(c + di) = ac + adi + bci + bdi^2$$

Because  $i^2 = -1$ , we have

$$(a + bi)(c + di) = ac + adi + bci - bd$$

To simplify, we combine the real parts, and we combine the imaginary parts.

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

## HOW TO: GIVEN TWO COMPLEX NUMBERS, MULTIPLY TO FIND THE PRODUCT

1. Use the distributive property or the FOIL method.
2. Simplify.

In the first video, we show more examples of multiplying complex numbers.

## Simplifying Powers of $i$

There is a pattern to evaluating powers of  $i$ . Let us look at what happens when we raise  $i$  to increasing powers.

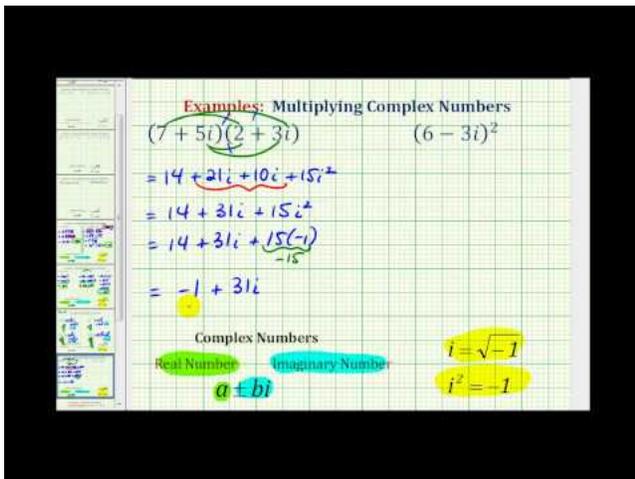
## EXAMPLE

Multiply  $(4 + 3i)(2 - 5i)$ .

**Answer**

Use FOIL.

$$\begin{aligned}(4 + 3i)(2 - 5i) &= (4 \cdot 2) + (4 \cdot (-5i)) + (3i \cdot 2) \\ &\quad + ((3i) \cdot (-5i)) \\ &= 8 - 20i + 6i - 15i^2 \\ &= 8 - 20i + 6i - 15(-1) && \text{Substituting -1 for } i^2. \\ &= 8 - 20i + 6i + 15 \\ &= (8 + 15) + (-20 + 6)i && \text{Grouping like terms together.} \\ &= 23 - 14i && \text{Simplifying.}\end{aligned}$$



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$$i^1 = i$$

$$i^2 = -1$$

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$

$$i^4 = i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1$$

$$i^5 = i^4 \cdot i = 1 \cdot i = i$$

We can see that when we get to the fifth power of  $i$ , it is equal to the first power. As we continue to multiply  $i$  by itself for increasing powers, we will see a cycle of 4. Let us examine the next 4 powers of  $i$ .

$$i^6 = i^5 \cdot i = i \cdot i = i^2 = -1$$

$$i^7 = i^6 \cdot i = i^2 \cdot i = i^3 = -i$$

$$i^8 = i^7 \cdot i = i^3 \cdot i = i^4 = 1$$

$$i^9 = i^8 \cdot i = i^4 \cdot i = i^5 = i$$

## EXAMPLE

Evaluate  $i^{35}$ .

### Answer

Since  $i^4 = 1$ , we can simplify the problem by factoring out as many factors of  $i^4$  as possible. To do so, first determine how many times 4 goes into 35:  $35 = 4 \cdot 8 + 3$ .

$$i^{35} = i^{4 \cdot 8 + 3} = i^{4 \cdot 8} \cdot i^3 = (i^4)^8 \cdot i^3 = 1^8 \cdot i^3 = i^3 = -i$$

## Q & A

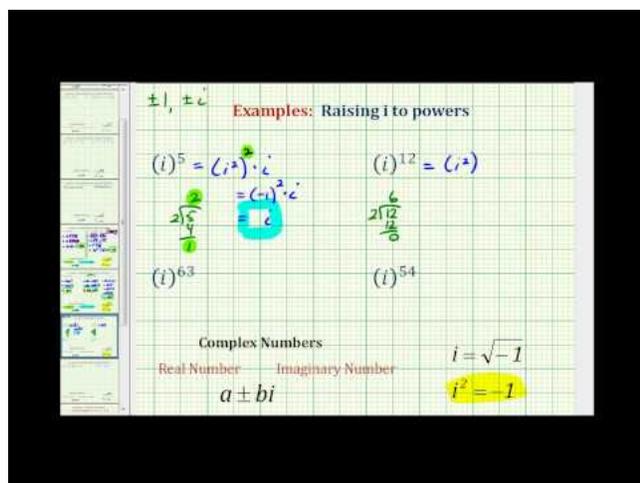
### Can we write $i^{35}$ in other helpful ways?

As we saw in the previous example, we reduced  $i^{35}$  to  $i^3$  by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of  $i^{35}$  may be more useful. The table below shows some other possible factorizations.

<b>Factorization of <math>i^{35}</math></b>	$i^{34} \cdot i$	$i^{33} \cdot i^2$	$i^{31} \cdot i^4$	$i^{19} \cdot i^{16}$
<b>Reduced form</b>	$(i^2)^{17} \cdot i$	$i^{33} \cdot (-1)$	$i^{31} \cdot 1$	$i^{19} \cdot (i^4)^4$
<b>Simplified form</b>	$(-1)^{17} \cdot i$	$-i^{33}$	$i^{31}$	$i^{19}$

Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

In the following video, you will see more examples of how to simplify powers of  $i$ .



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## Dividing Complex Numbers

Division of two complex numbers is more complicated than addition, subtraction, and multiplication because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator. This idea is similar to rationalizing the denominator of a fraction that contains a radical. To eliminate the complex or imaginary number in the denominator, you multiply by the **complex conjugate** of the

denominator which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of  $a + bi$  is  $a - bi$ .

Note that complex conjugates have a reciprocal relationship: The complex conjugate of  $a + bi$  is  $a - bi$ , and the complex conjugate of  $a - bi$  is  $a + bi$ . Furthermore, when a quadratic equation with real coefficients has complex solutions, the solutions are always complex conjugates of one another.

Suppose we want to divide  $c + di$  by  $a + bi$ , where neither  $a$  nor  $b$  equals zero. We first write the division as a fraction then find the complex conjugate of the denominator and multiply.

$$\frac{c+di}{a+bi} \text{ where } a \neq 0 \text{ and } b \neq 0$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

$$\frac{(c+di)}{(a+bi)} \cdot \frac{(a-bi)}{(a-bi)} = \frac{(c+di)(a-bi)}{(a+bi)(a-bi)}$$

Apply the distributive property.

$$= \frac{ca-cbi+adi-bdi^2}{a^2-abi+abi-b^2i^2}$$

Simplify, remembering that  $i^2 = -1$ .

$$\begin{aligned} &= \frac{ca-cbi+adi-bd(-1)}{a^2-abi+abi-b^2(-1)} \\ &= \frac{(ca+bd)+(ad-cb)i}{a^2+b^2} \end{aligned}$$

## A GENERAL NOTE: THE COMPLEX CONJUGATE

The **complex conjugate** of a complex number  $a + bi$  is  $a - bi$ . It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.

- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

## EXAMPLE

Find the complex conjugate of each number.

1.  $2 + i\sqrt{5}$
2.  $-\frac{1}{2}i$

**Answer**

1. The number is already in the form  $a + bi$ . The complex conjugate is  $a - bi$ , or  $2 - i\sqrt{5}$ .
2. We can rewrite this number in the form  $a - bi$  as  $0 - \frac{1}{2}i$ . The complex conjugate is  $a + bi$ , or  $0 + \frac{1}{2}i$ . This can be written simply as  $\frac{1}{2}i$ .

## Analysis of the Solution

Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. When dividing by a purely imaginary number, we can simply multiply the numerator and denominator by  $i$  instead of the complex conjugate.

**HOW TO: GIVEN TWO COMPLEX NUMBERS, DIVIDE ONE BY THE OTHER.**

1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

## EXAMPLE

Divide  $(2 + 5i)$  by  $(4 - i)$ .

### Answer

We begin by writing the problem as a fraction.

$$\frac{(2+5i)}{(4-i)}$$

Then we multiply the numerator and denominator by the complex conjugate of the denominator.

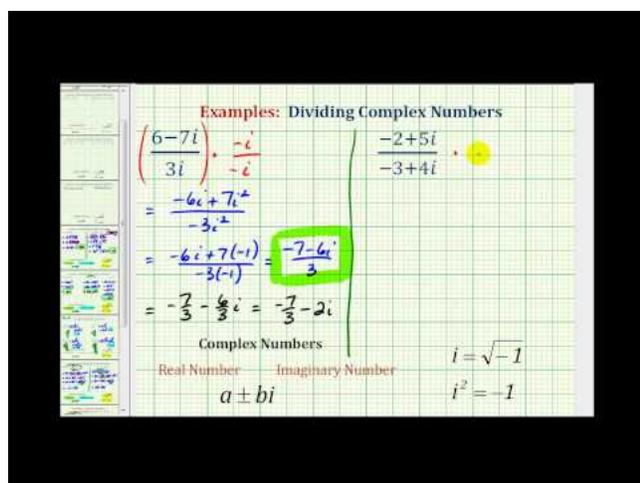
$$\frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}$$

To multiply two complex numbers, we expand the product as we would with polynomials (the process commonly called FOIL).

$$\begin{aligned} \frac{(2+5i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)} &= \frac{8+2i+20i+5i^2}{16+4i-4i-i^2} \\ &= \frac{8+2i+20i+5(-1)}{16+4i-4i-(-1)} && \text{Because } i^2 = -1 \\ &= \frac{3+22i}{17} \\ &= \frac{3}{17} + \frac{22}{17}i && \text{Separate real and imaginary parts} \end{aligned}$$

Note that this expresses the quotient in standard form.

In the last video, you will see more examples of dividing complex numbers.



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## Summary

Multiplying complex numbers is similar to multiplying polynomials. Remember that an imaginary number times another imaginary number gives a real result. When you divide complex numbers, you must first multiply by the complex conjugate to eliminate any imaginary parts, and then you can divide.

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# INTRODUCTION TO QUADRATIC EQUATIONS WITH COMPLEX SOLUTIONS

In this section we will bring the concepts of solving quadratic equations and complex numbers together to find solutions to quadratic equations that are imaginary. You will also learn about a really useful quality of the quadratic formula that can help you know whether a quadratic equation has real or complex solutions.

## QUADRATIC EQUATIONS WITH COMPLEX SOLUTIONS

### LEARNING OUTCOMES

- Use the quadratic formula to solve quadratic equations with complex solutions
- Connect complex solutions with the graph of a quadratic function that does not cross the x-axis

We have seen two outcomes for solutions to quadratic equations; either there was one or two real number solutions. We have also learned that it is possible to take the square root of a negative number by using imaginary numbers. Having this new knowledge allows us to explore one more possible outcome when we solve quadratic equations. Consider this equation:

$$2x^2 + 3x + 6 = 0$$

Using the Quadratic Formula to solve this equation, we first identify  $a$ ,  $b$ , and  $c$ .

$$a = 2, b = 3, c = 6$$

We can place  $a$ ,  $b$  and  $c$  into the quadratic formula and simplify to get the following result:

$$x = -\frac{3}{4} + \frac{\sqrt{-39}}{4}, x = -\frac{3}{4} - \frac{\sqrt{-39}}{4}$$

Up to this point, we would have said that  $\sqrt{-39}$  is not defined for real numbers and determine that this equation has no solutions. But, now that we have defined the square root of a negative number, we can

also define a solution to this equation as follows.

$$x = -\frac{3}{4} + i\frac{\sqrt{39}}{4}, x = -\frac{3}{4} - i\frac{\sqrt{39}}{4}$$

In this section we will practice simplifying and writing solutions to quadratic equations that are complex. We will then present a technique for classifying whether the solution(s) to a quadratic equation will be complex, and how many solutions there will be.

In our first example, we will work through the process of solving a quadratic equation with complex solutions. Take note that we will be simplifying complex numbers, so if you need a review of how to rewrite the square root of a negative number as an imaginary number, now is a good time.

### Example

Use the Quadratic Formula to solve the equation  $x^2 + 2x = -5$ .

**Answer**

First write the equation in standard form.

$$x^2 + 2x = -5$$

$$x^2 + 2x + 5 = 0$$

$$a = 1, b = 2, c = 5$$

$$x^2 + 2x + 5 = 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$ax^2 + bx + c = 0$$

Substitute the values into the Quadratic Formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2(1)}$$

Simplify, being careful to get the signs correct.

$$x = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

Simplify some more.

$$x = \frac{-2 \pm \sqrt{-16}}{2}$$

Simplify the radical, but notice that the number under the radical symbol is negative! The square root of  $-16$  is imaginary.  $\sqrt{-16} = 4i$ .

$$x = \frac{-2 \pm 4i}{2}$$

Separate and simplify to find the solutions to the quadratic equation.

$$x = \frac{-2 + 4i}{2} = \frac{-2}{2} + \frac{4i}{2} = -1 + 2i$$

or

$$x = \frac{-2 - 4i}{2} = \frac{-2}{2} - \frac{4i}{2} = -1 - 2i$$

Therefore,  $x = -1 + 2i$  or  $-1 - 2i$ .

We can check these solutions in the original equation. Be careful when you expand the squares, and replace  $i^2$  with  $-1$ .

$$x = -1 + 2i$$

$$x^2 + 2x = -5$$

$$(-1 + 2i)^2 + 2(-1 + 2i) = -5$$

$$1 - 4i + 4i^2 - 2 + 4i = -5$$

$$1 - 4i + 4(-1) - 2 + 4i = -5$$

$$1 - 4 - 2 = -5$$

$$-5 = -5$$

$$x = -1 - 2i$$

$$x^2 + 2x = -5$$

$$(-1 - 2i)^2 + 2(-1 - 2i) = -5$$

$$1 + 4i + 4i^2 - 2 - 4i = -5$$

$$1 + 4i + 4(-1) - 2 - 4i = -5$$

$$1 - 4 - 2 = -5$$

$$-5 = -5$$

## EXAMPLE

Use the quadratic formula to solve  $x^2 + x + 2 = 0$ .

**Answer**

First, we identify the coefficients:  $a = 1$ ,  $b = 1$ , and  $c = 2$ . Substitute these values into the quadratic formula.

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-(1) \pm \sqrt{(1)^2 - (4) \cdot (1) \cdot (2)}}{2 \cdot 1} \\&= \frac{-1 \pm \sqrt{1 - 8}}{2} \\&= \frac{-1 \pm \sqrt{-7}}{2} \\&= \frac{-1 \pm i\sqrt{7}}{2}\end{aligned}$$

Now we can separate the expression  $\frac{-1 \pm i\sqrt{7}}{2}$  into two solutions:

$$\begin{aligned}-\frac{1}{2} + \frac{i\sqrt{7}}{2} \\-\frac{1}{2} - \frac{i\sqrt{7}}{2}\end{aligned}$$

The solutions to the equation are  $x = \frac{-1}{2} + \frac{i\sqrt{7}}{2}$  and  $x = \frac{-1}{2} - \frac{i\sqrt{7}}{2}$ . It is important that you separate the real and imaginary part as this is proper complex number form.

Now that we have had a little practice solving quadratic equations whose solutions are complex, we can explore a related feature of quadratic functions. Consider the following function:  $f(x) = x^2 + 2x + 3$ . Recall that the x-intercepts of a function are found by setting the function equal to zero:

$$x^2 + 2x + 3 = 0$$

The function now looks like the type of quadratic equations we have been solving. In the next example, we will solve this equation, then graph the original function and see that it has no x-intercepts.

The following video gives another example of how to use the quadratic formula to find solutions to a

## EXAMPLE

Find the x-intercepts of the quadratic function.  $f(x) = x^2 + 2x + 3$

### Answer

The x-intercepts of the function  $f(x) = x^2 + 2x + 3$  are found by setting it equal to zero and solving for x since the y values of the x-intercepts are zero.

First, identify a, b, c.

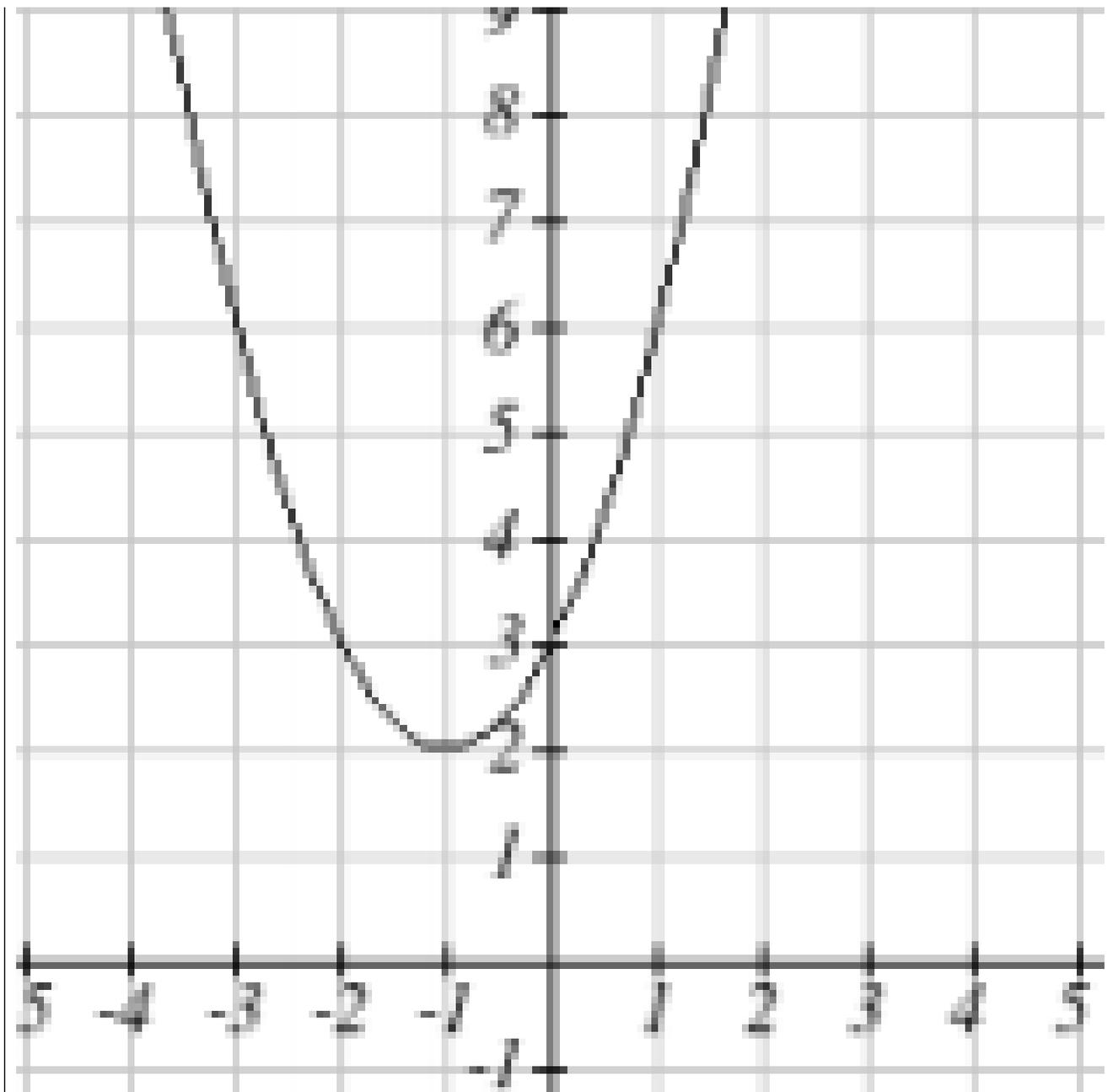
$$x^2 + 2x + 3 = 0$$

$$a = 1, b = 2, c = 3$$

Substitute these values into the quadratic formula.

$$\begin{aligned}x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\&= \frac{-2 \pm \sqrt{2^2 - 4(1)(3)}}{2(1)} \\&= \frac{-2 \pm \sqrt{4 - 12}}{2} \\&= \frac{-2 \pm \sqrt{-8}}{2} \\&= \frac{-2 \pm 2i\sqrt{2}}{2} \\&= -1 \pm i\sqrt{2} \\&= -1 + \sqrt{2}i, -1 - \sqrt{2}i\end{aligned}$$

The solutions to this equations are complex, therefore there are no x-intercepts for the function  $f(x) = x^2 + 2x + 3$  in the set of real numbers that can be plotted on the Cartesian Coordinate plane. The graph of the function is plotted on the Cartesian Coordinate plane below:



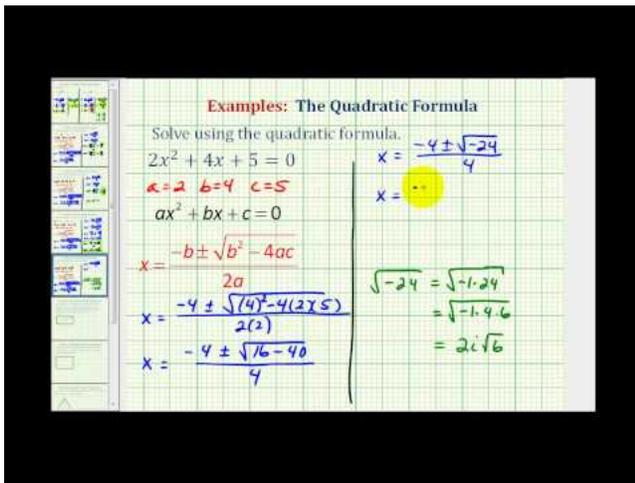
*Graph of quadratic function with no x-intercepts in the real numbers.*

Note how the graph does not cross the x-axis; therefore, there are no real x-intercepts for this function.

quadratic equation that has complex solutions.

## Summary

Quadratic equations can have complex solutions. Quadratic functions whose graphs do not cross the x-axis will have complex solutions for  $f(x) = 0$ .



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# THE DISCRIMINANT

## LEARNING OUTCOME

- Define the discriminant and use it to classify solutions to quadratic equations

## The Discriminant

The **quadratic formula** not only generates the solutions to a quadratic equation, but also tells us about the nature of the solutions. When we consider the **discriminant**, or the expression under the radical,  $b^2 - 4ac$ , it tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

Let us explore how the discriminant affects the evaluation of  $\sqrt{b^2 - 4ac}$  in the quadratic formula and how it helps to determine the solution set.

- If  $b^2 - 4ac > 0$ , then the number underneath the radical will be a positive value. You can always find the square root of a positive number, so evaluating the quadratic formula will result in two real solutions (one by adding the positive square root and one by subtracting it).
- If  $b^2 - 4ac = 0$ , then you will be taking the square root of 0, which is 0. Since adding and subtracting 0 both give the same result, the “±” portion of the formula does not matter. There will be one real repeated solution.

- If  $b^2 - 4ac < 0$ , then the number underneath the radical will be a negative value. Since you cannot find the square root of a negative number using real numbers, there are no real solutions. However, you can use imaginary numbers. You will then have two complex solutions, one by adding the imaginary square root and one by subtracting it.

The table below summarizes the relationship between the value of the discriminant and the solutions of a quadratic equation.

Value of Discriminant	Results
$b^2 - 4ac = 0$	One repeated rational solution
$b^2 - 4ac > 0$ , perfect square	Two rational solutions
$b^2 - 4ac > 0$ , not a perfect square	Two irrational solutions
$b^2 - 4ac < 0$	Two complex solutions

## A GENERAL NOTE: THE DISCRIMINANT

For  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are real numbers, the **discriminant** is the expression under the radical in the quadratic formula:  $b^2 - 4ac$ . It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

## EXAMPLE

Use the discriminant to find the nature of the solutions to the following quadratic equations:

1.  $x^2 + 4x + 4 = 0$
2.  $8x^2 + 14x + 3 = 0$
3.  $3x^2 - 5x - 2 = 0$
4.  $3x^2 - 10x + 15 = 0$

### Answer

Calculate the discriminant  $b^2 - 4ac$  for each equation and state the expected type of solutions.

1.  $x^2 + 4x + 4 = 0$   $b^2 - 4ac = (4)^2 - 4(1)(4) = 0$  There will be one repeated rational solution.
2.  $8x^2 + 14x + 3 = 0$   $b^2 - 4ac = (14)^2 - 4(8)(3) = 100$   
100 is a perfect square, so there will be two rational solutions.
3.  $3x^2 - 5x - 2 = 0$   $b^2 - 4ac = (-5)^2 - 4(3)(-2) = 49$   
49 is a perfect square, so there will be two rational solutions.
4.  $3x^2 - 10x + 15 = 0$   $b^2 - 4ac = (-10)^2 - 4(3)(15) = -80$  There will be two complex solutions.

### Example

Use the discriminant to determine how many and what kind of solutions the quadratic equation  $x^2 - 4x + 10 = 0$  has.

### Answer

Evaluate  $b^2 - 4ac$ . First note that  $a = 1$ ,  $b = -4$ , and  $c = 10$ .

$$b^2 - 4ac = (-4)^2 - 4(1)(10) = 16 - 40 = -24$$

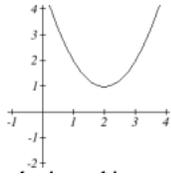
The result is a negative number. The discriminant is negative, so the quadratic equation has two complex solutions.

In the last example, we will draw a correlation between the number and type of solutions to a quadratic equation and the graph of its corresponding function.

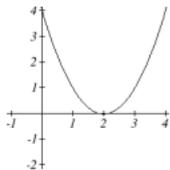
## EXAMPLE

Use the following graphs of quadratic functions to determine how many and what type of solutions the corresponding quadratic equation  $f(x) = 0$  will have. Determine whether the discriminant will be greater than, less than, or equal to zero for each.

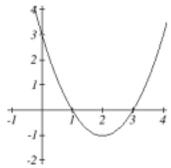
a.



b.



c.



### Answer

a. This quadratic function does not touch or cross the x-axis; therefore, the corresponding equation  $f(x) = 0$  will have complex solutions. This implies that  $b^2 - 4ac < 0$ .

b. This quadratic function touches the x-axis exactly once which implies there is one repeated solution to the equation  $f(x) = 0$ . We can then say that  $b^2 - 4ac = 0$ .

c. In our final graph, the quadratic function crosses the x-axis twice which tells us that there are two real number solutions to the equation  $f(x) = 0$ , and therefore  $b^2 - 4ac > 0$ .

We can summarize our results as follows:

Discriminant	Number and Type of Solutions	Graph of Quadratic Function
$b^2 - 4ac < 0$	two complex solutions	will not cross the x-axis
$b^2 - 4ac = 0$	one real repeated solution	will touch x-axis once
$b^2 - 4ac > 0$	two real solutions	will cross x-axis twice

In the following video, we show more examples of how to use the discriminant to describe the type of solutions of a quadratic equation.

## Summary

**Examples: The Discriminant**

Describe the type of solutions to the quadratic equation.

$$ax^2 + bx + c = 0$$

$$1x^2 - 3x - 28 = 0$$

$a = 1$     $b = -3$     $c = -28$

$$b^2 - 4ac = (-3)^2 - 4(1)(-28)$$

$$9 - (-112) = 9 + 112 = 121$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If  $b^2 - 4ac > 0$  and perfect square → two real rational solutions  
 If  $b^2 - 4ac > 0$  and not a perfect square → two real irrational solutions  
 If  $b^2 - 4ac = 0$  → one repeated real rational solution  
 If  $b^2 - 4ac < 0$  → two complex solutions

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The discriminant of the quadratic formula is the quantity under the radical,  $b^2 - 4ac$ . It determines the number and the type of solutions that a quadratic equation has. If the discriminant is positive, there are 2 real solutions. If it is 0, there is 1 real repeated solution. If the discriminant is negative, there are 2 complex solutions (but no real solutions).

The discriminant can also tell us about the behavior of the graph of a quadratic function.

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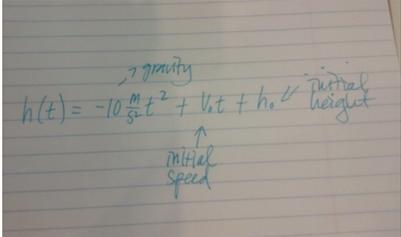
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# PUTTING IT TOGETHER: QUADRATIC EQUATIONS AND COMPLEX NUMBERS

In the beginning of this module, we found Joan and Hazel at the river jumping off a cliff into the water below. Joan recognized the path of their fall as parabolic in shape after seeing a picture Hazel took using continuous mode on her camera while Joan was jumping. They decided to see if they could use physics and math to find out how long they were in the air between the time they jumped and the time they hit the water.

Hazel had found this equation from some old high school physics notes:



$$h(t) = -10\frac{m}{s^2}t^2 + v_0t + h_0$$

She had labeled the equation with arrows in the following way:  $-10\frac{m}{s^2}$  is gravity,  $v_0$  is the initial speed of the object,  $h_0$  is the initial height of the object.

Joan and Hazel decide to define some of the unknowns in the equation. Since they started from a standing position, they decide that  $v_0 = 0$ . Then, they estimate that the height of the cliff was about 15 meters (or about 45 feet), so  $h_0 = 15$ . They use substitution to replace these values in the equation and simplify:

$$h(t) = -10\frac{m}{s^2}t^2 + 0 \cdot 0 \cdot t + 15m$$

$$h(t) = -10\frac{m}{s^2}t^2 + 15m$$

Removing the units:  $h(t) = -10t^2 + 15$

“Now what?” asks Hazel. “I don’t remember what we did from here.” Joan takes over because she has just finished a module on quadratic equations in her math class, and she knows this is a quadratic equation.

“Our original question was, how long did it take us to fall from the cliff to the water, right? So that means that our height above ground at that time would be equal to zero. I think we should set the equation equal to zero and solve for t.”

$$h(t) = -10t^2 + 15 = 0$$

$$-10t^2 = -15$$

$$t^2 = \frac{3}{2}$$

$$\sqrt{t^2} = \sqrt{\frac{3}{2}}$$

Using a calculator, they evaluate  $\sqrt{\frac{3}{2}}$  and get 1.22.

“Wow! When you are falling it seems longer than 1.22 seconds!” says Hazel.

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# MODULE 13: EXPONENTIAL, AND LOGARITHMIC FUNCTIONS

## WHY IT MATTERS: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

### Why Learn About Exponential and Logarithmic Functions?



Joan decided to try online dating since she has a strict rule about not dating people at work, and the guys in her math class are all taken. Not long after posting a profile on a site her friend recommended to her, she was asked out to dinner by a cute, young man who shared some of the same interests and music tastes as her.

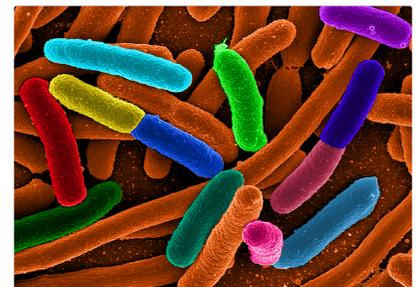
Joan nervously prepared for her first date from an online dating site; they had planned to meet at a sushi restaurant. After getting lectured by her dad about how to be safe on a “blind” date, Joan went to meet what she hoped was her next dreamy boyfriend.

Things did not get off to a good start, as it seemed that he had lied about his height. Joan is almost 6 feet tall and prefers to date men that are at least as tall as her. Maybe he cannot remember, she thought, wanting it to be OK.

The night got progressively worse for Joan from there. Her date ordered drink after drink and his volume increased with every one. The people around them began to get up and leave. As they were getting ready to pay the bill, he claimed to have forgotten his card. What a disaster!

Later that night, to add insult to injury, Joan found herself very ill with food poisoning from the sushi restaurant.

Bacteria commonly reproduce through a process called binary fission where one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly, as in the case of Joan’s food poisoning. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours, as opposed to days or years. (Note: Todar, PhD, Kenneth. *Todar's Online Textbook of Bacteriology*. [http://textbookofbacteriology.net/growth\\_3.html](http://textbookofbacteriology.net/growth_3.html).)



*Electron micrograph of E.Coli bacteria (credit: "Mattosaurus," Wikimedia Commons).*

For simplicity's sake, imagine that the bacterial cells that caused Joan's food poisoning can divide every hour. The table below shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! And if we were to extrapolate the table to twenty-four hours, we would have over 16 million! No wonder it did not take long for Joan to feel sick after eating sushi with her awful date.

Hour	0	1	2	3	4	5	6	7	8	9	10
Bacteria	1	2	4	8	16	32	64	128	256	512	1024

## Exponential Functions

In this chapter, we will explore exponential functions which can be used for, among other things, modeling growth patterns such as those found in bacteria and viruses. We will also investigate logarithmic functions which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

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# INTRODUCTION TO EXPONENTIAL FUNCTIONS

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about 1.2% each year. (Note: <http://www.worldometers.info/world-population/>. Accessed February 24, 2014.) If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential." To a mathematician, however, the term *exponential growth* has a very specific meaning. In this section, we will take a look at *exponential functions* which model this kind of rapid growth.

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# DEFINE AND EVALUATE EXPONENTIAL FUNCTIONS

## LEARNING OUTCOMES

- Identify and evaluate exponential functions
- Use compound interest formulas

Linear functions have a constant rate of change – a constant number that the output increases for each increase in input. For example, in the equation  $f(x) = 3x + 4$ , the slope tells us the output increases by three each time the input increases by one. Sometimes, on the other hand, quantities grow by a percent rate of change rather than by a fixed amount. In this lesson, we will define a function whose rate of change increases by a percent of the current value rather than a fixed quantity.

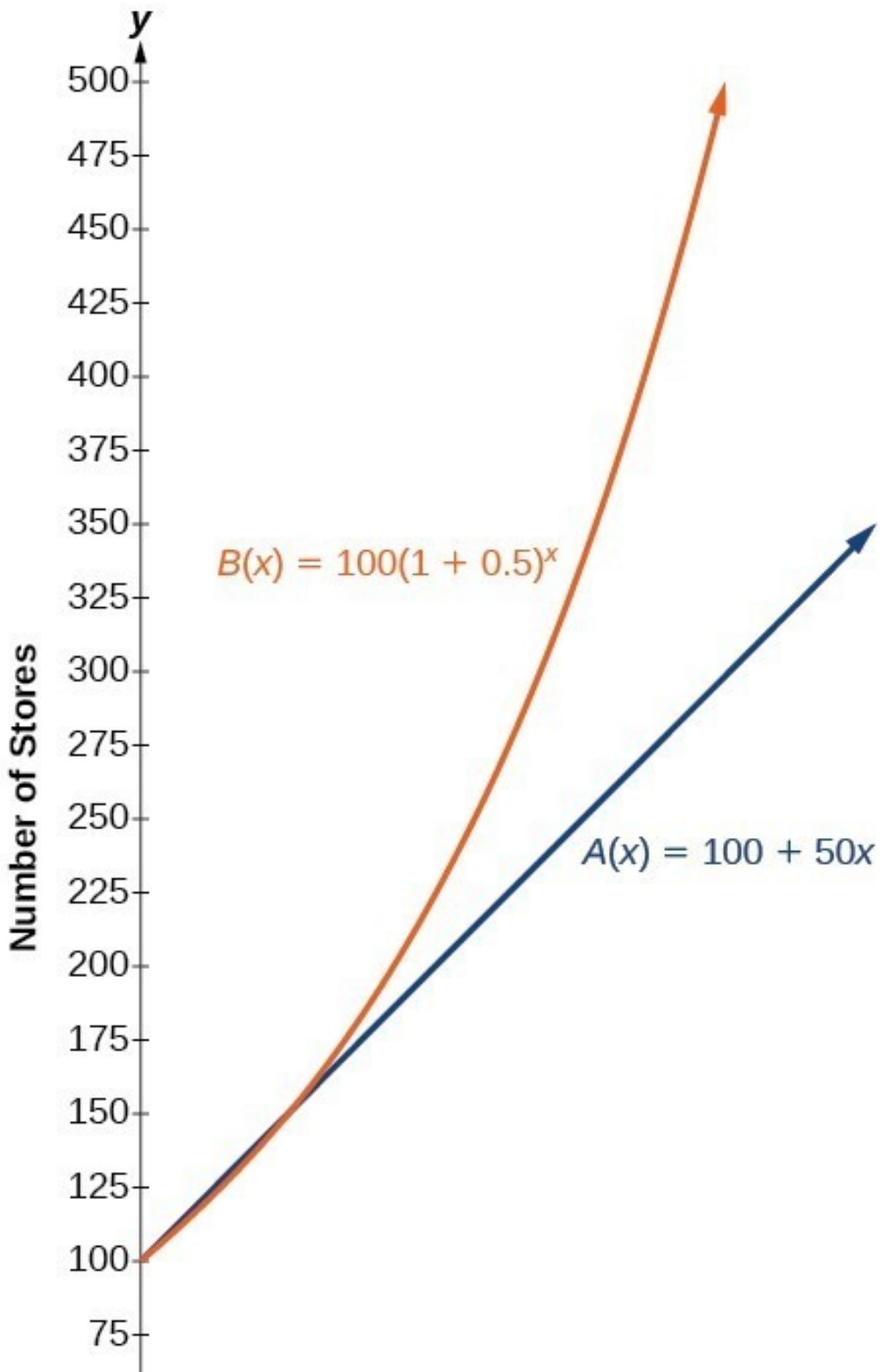
To illustrate this difference consider two companies whose business is expanding: Company A has 100 stores and expands by opening 50 new stores a year, while Company B has 100 stores and expands by increasing the number of stores by 50% of their total each year.

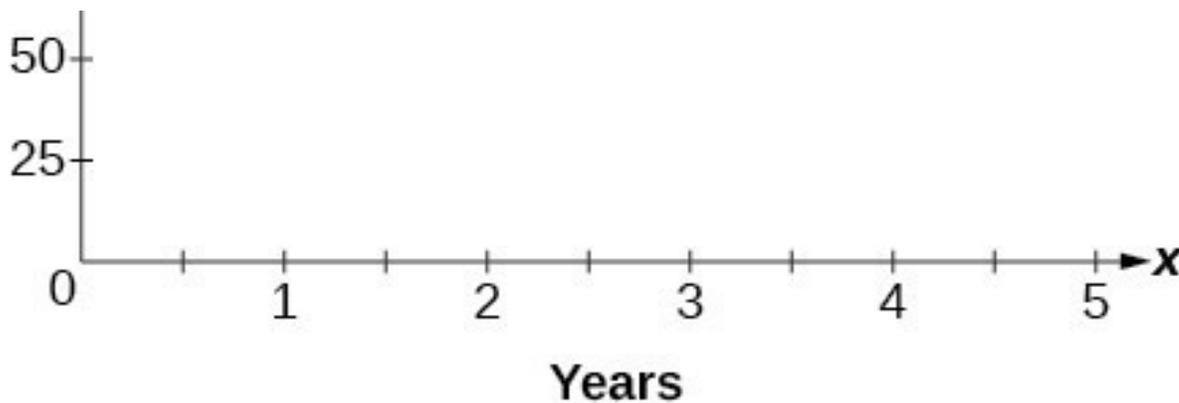
The table below compares the growth of each company where company A increases the number of stores linearly, and company B increases the number of stores by a rate of 50% each year.

Year	Stores, Company A	Description of Growth	Stores, Company B
0	100	Starting with 100 each	100
1	$100 + 50 = 150$	They both grow by 50 stores in the first year.	$100 + 50\%$ of 100 $100 + 0.50(100)$ $= 150$
2	$150 + 50 = 200$	Store A grows by 50, Store B grows by 75	$150 + 50\%$ of 150 $150 + 0.50(150)$ $= 225$
3	$200 + 50 = 250$	Store A grows by 50, Store B grows by 112.5	$225 + 50\%$ of 225 $225 + 0.50(225)$ $= 337.5$

Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function  $A(x) = 100 + 50x$ . Company B has 100 stores and expands by increasing the number of stores by 50% each year, so its growth can be represented by the function  $B(x) = 100(1 + 0.5)^x$ .

The graphs comparing the number of stores for each company over a five-year period are shown below. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.





The graph shows the number of stores Companies A and B opened over a five-year period.

Notice that the domain for both functions is  $[0, \infty)$ , and the range for both functions is  $[100, \infty)$ . After year 1, Company B always has more stores than Company A.

Consider the function representing the number of stores for Company B:

$$B(x) = 100(1 + 0.5)^x$$

In this exponential function, 100 represents the initial number of stores, 0.50 represents the growth rate, and  $1 + 0.5 = 1.5$  represents the growth factor. Generalizing further, we can write this function as  $B(x) = 100(1.5)^x$ , where 100 is the initial value, 1.5 is called the *base*, and  $x$  is called the *exponent*. This is an exponential function.

## EXPONENTIAL GROWTH

A function that models **exponential growth** grows by a rate proportional to the current amount. For any real number  $x$  and any positive real numbers  $a$  and  $b$  such that  $b \neq 1$ , an exponential growth function has the form

$$f(x) = ab^x$$

where

- $a$  is the initial or starting value of the function.
- $b$  is the growth factor or growth multiplier per unit  $x$ .

To evaluate an exponential function of the form  $f(x) = b^x$ , we simply substitute  $x$  with the given value, and calculate the resulting power. For example:

Let  $f(x) = 2^x$ . What is  $f(3)$ ?

$$f(x) = 2^x$$

$$f(3) = 2^3 \quad \text{Substitute } x = 3.$$

$$= 8 \quad \text{Evaluate the power.}$$

To evaluate an exponential function with a form other than the basic form, it is important to follow the order of operations. For example:

Let  $f(x) = 30(2)^x$ . What is  $f(3)$ ?

$$f(x) = 30(2)^x$$

$$f(3) = 30(2)^3 \quad \text{Substitute } x = 3.$$

$$= 30(8) \quad \text{Simplify the power first.}$$

= 240      Multiply.

Note that if the order of operations were not followed, the result would be incorrect:

$$f(3) = 30(2)^3 \neq 60^3 = 216,000$$

In our first example, we will evaluate an exponential function without the aid of a calculator.

## EXAMPLE

Let  $f(x) = 5(3)^{x+1}$ . Evaluate  $f(2)$  without using a calculator.

### Answer

Follow the order of operations. Be sure to pay attention to the parentheses.

$$f(x) = 5(3)^{x+1}$$

$$f(2) = 5(3)^{2+1} \quad \text{Substitute } x = 2.$$

$$= 5(3)^3 \quad \text{Add the exponents.}$$

$$= 5(27) \quad \text{Simplify the power.}$$

$$= 135 \quad \text{Multiply.}$$

In the following video, we present more examples of evaluating an exponential function at several different values.

The screenshot shows a video player with a title "Function Values of an Exponential Function". The video content displays the following text:

**EXAMPLE**

Given  $f(x) = 3(2)^{x+1}$ , determine the following function values.

$f(-2) = 3(2)^{-2+1} = 3(2)^{-1} = 3(\frac{1}{2}) = \frac{3}{2} = 1.5$

$f(-1) = 3(2)^{-1+1} = 3(2)^0 = 3(1) = 3$

$f(0) = 3(2)^{0+1} = 3(2)^1 = 3(2) = 6$

$f(1) = 3(2)^{1+1} = 3(2)^2 = 3(4) = 12$

$f(2) =$

Use these function values to graph the function.

At the bottom of the video frame, a formula is shown:  $a^{-n} = \frac{1}{a^n}, a \neq 0$

A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=3503>

In the next example, we will revisit the population of India.

In the following video, we show another example of using an exponential function to predict the population of a small town.

You may have seen formulas that are used to calculate compound interest rates. These formulas are another example of exponential growth. The term *compounding* refers to interest earned not only on the original value but on the accumulated value of the account.

The **annual percentage rate (APR)** of an account, also called the **nominal rate**, is the yearly interest rate earned by an investment account. The term *nominal* is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being *greater* than the nominal rate! This is a powerful tool for investing.

## EXAMPLE

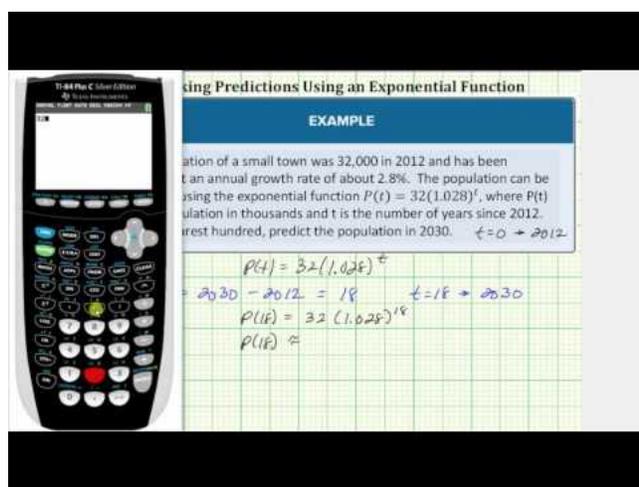
At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%. This situation is represented by the growth function  $P(t) = 1.25(1.012)^t$ , where  $t$  is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

### Answer

To estimate the population in 2031, we evaluate the model for  $t = 18$ , because 2031 is 18 years after 2013. Rounding to the nearest thousandth,

$$P(18) = 1.25(1.012)^{18} \approx 1.549$$

There will be about 1.549 billion people in India in the year 2031.



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We can calculate the compound interest using the compound interest formula, which is an exponential function of the variables time  $t$ , principal  $P$ , APR  $r$ , and number of compounding periods in a year  $n$ :

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

## THE COMPOUND INTEREST FORMULA

**Compound interest** can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

- $A(t)$  is the account value,
- $t$  is measured in years,
- $P$  is the starting amount of the account, often called the principal, or more generally present value,
- $r$  is the annual percentage rate (APR) expressed as a decimal, and
- $n$  is the number of compounding periods in one year.

In our next example, we will calculate the value of an account after 10 years of interest compounded quarterly.

The following video shows an example of using exponential growth to calculate interest compounded quarterly.

In our next example, we will use the compound interest formula to solve for the principal.

## EXAMPLE

If we invest \$3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

### Answer

Because we are starting with \$3,000,  $P = 3000$ . Our interest rate is 3%, so  $r = 0.03$ . Because we are compounding quarterly, we are compounding 4 times per year, so  $n = 4$ . We want to know the value of the account in 10 years, so we are looking for  $A(10)$  which is the value when  $t = 10$ .

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula.}$$

$$A(10) = 3000\left(1 + \frac{0.03}{4}\right)^{4 \cdot 10} \quad \text{Substitute using given values.}$$

$$\approx 4045.05 \quad \text{Round to two decimal places.}$$

The account will be worth about \$4,045.05 in 10 years.

**Ex: Compounded Interest**  
A bank features a savings account that has an annual percentage rate of 3.4% with interest compounded quarterly. Ashley deposits \$2,000 into the account.  
What equation can be used to determine the account balance after  $t$  years?  
 $A = 2000\left(1 + \frac{0.034}{4}\right)^{4t} \rightarrow A = 2000(1.0085)^{4t}$   
What is the account balance after 4 years?  
 $A = 2000(1.0085)^{16}$   
 $A \approx 2,290.05$

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In the following video, we show another example of finding the deposit amount necessary to obtain a future value from compounded interest.

## Summary

Exponential growth grows by a rate proportional to the current amount. For any real number  $x$  and any positive real numbers  $a$  and  $b$  such that  $b \neq 1$ , an exponential growth function has the form  $f(x) = ab^x$ .

Evaluating exponential functions requires careful attention to the order of operations. Compound interest is an example of exponential growth.

## EXAMPLE

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to \$40,000 over 18 years. She believes the account will earn 6% compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?

### Answer

The nominal interest rate is 6%, so  $r = 0.06$ . Interest is compounded twice a year, so  $k = 2$ .

We want to find the initial investment,  $P$ , needed so that the value of the account will be worth \$40,000 in 18 years. Substitute the given values into the compound interest formula, and solve for  $P$ .

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} \quad \text{Use the compound interest formula.}$$

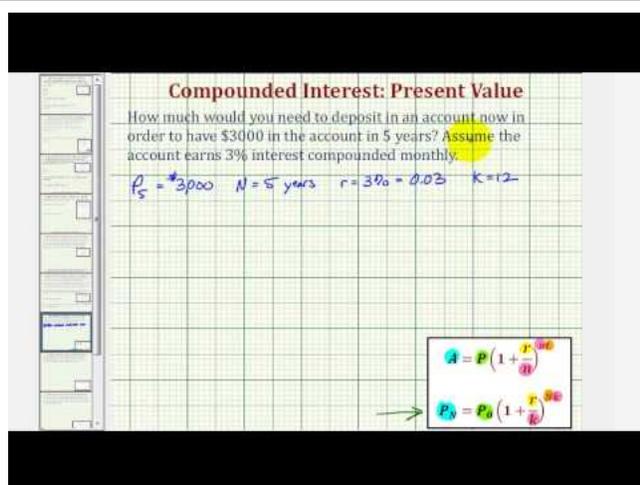
$$40,000 = P\left(1 + \frac{0.06}{2}\right)^{2(18)} \quad \text{Substitute using given values } A, r, n, \text{ and } t.$$

$$40,000 = P(1.03)^{36} \quad \text{Simplify.}$$

$$\frac{40,000}{(1.03)^{36}} = P \quad \text{Isolate } P.$$

$$P \approx 13,801 \quad \text{Divide and round to the nearest dollar.}$$

Lily will need to invest \$13,801 to have \$40,000 in 18 years.



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# EXPONENTIAL FUNCTIONS WITH BASE E

## LEARNING OUTCOME

- Evaluate exponential functions with base  $e$

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. The table below shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of \$1 invested at 100% interest for 1 year, compounded at various frequencies.

Frequency	$A(t) = \left(1 + \frac{1}{n}\right)^n$	Value
Annually	$\left(1 + \frac{1}{1}\right)^1$	\$2
Semiannually	$\left(1 + \frac{1}{2}\right)^2$	\$2.25
Quarterly	$\left(1 + \frac{1}{4}\right)^4$	\$2.441406
Monthly	$\left(1 + \frac{1}{12}\right)^{12}$	\$2.613035
Daily	$\left(1 + \frac{1}{365}\right)^{365}$	\$2.714567
Hourly	$\left(1 + \frac{1}{8766}\right)^{8766}$	\$2.718127
Once per minute	$\left(1 + \frac{1}{525960}\right)^{525960}$	\$2.718279
Once per second	$\left(1 + \frac{1}{31557600}\right)^{31557600}$	\$2.718282

These values appear to be approaching a limit as  $n$  increases. In fact, as  $n$  gets larger and larger, the expression  $\left(1 + \frac{1}{n}\right)^n$  approaches a number used so frequently in mathematics that it has its own name: the letter  $e$ . This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

## A GENERAL NOTE: THE NUMBER $e$

The letter  $e$  represents the irrational number

$$\left(1 + \frac{1}{n}\right)^n$$

as  $n$  increases without bound

The letter  $e$  is used as a base for many real-world exponential models. To work with base  $e$ , we use the approximation,  $e \approx 2.718282$ . The constant was named by the Swiss mathematician Leonhard Euler (1707–1783) who first investigated and discovered many of its properties.

In our first example, we will use a calculator to find powers of  $e$ .

## Investigating Continuous Growth

## EXAMPLE

Calculate  $e^{3.14}$ . Round to five decimal places.

### Answer

On a calculator, press the button labeled  $[e^x]$ . The window shows  $[e^{\wedge}()]$ . Type 3.14 and then close parenthesis,  $()$ . Press [ENTER]. Rounding to 5 decimal places,  $e^{3.14} \approx 23.10387$ . Caution: Many scientific calculators have an “Exp” button, which is used to enter numbers in scientific notation. It is not used to find powers of  $e$ .

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however,  $e$  is used as the base for exponential functions. Exponential models that use  $e$  as the base are called *continuous growth or decay models*. We see these models in finance, computer science, and most of the sciences, such as physics, toxicology, and fluid dynamics.

## THE CONTINUOUS GROWTH/DECAY FORMULA

For all real numbers  $r$ ,  $t$ , and all positive numbers  $a$ , continuous growth or decay is represented by the formula

$$A(t) = ae^{rt}$$

where

- $a$  is the initial value,
- $r$  is the continuous growth or decay rate per unit time,
- and  $t$  is the elapsed time.

If  $r > 0$ , then the formula represents continuous growth. If  $r < 0$ , then the formula represents continuous decay.

For business applications, the continuous growth formula is called the continuous compounding formula and takes the form

$$A(t) = Pe^{rt}$$

where

- $P$  is the principal or the initial amount invested,
- $r$  is the growth or interest rate per unit time,
- and  $t$  is the period or term of the investment.

In our next example, we will calculate continuous growth of an investment. It is important to note the language that is used in the instructions for interest rate problems. You will know to use the *continuous* growth or decay formula when you are asked to find an amount based on continuous compounding. In previous examples we asked that you find an amount based on quarterly or monthly compounding where, in that case, you used the *compound* interest formula.

In the following video, we show another example of interest compounded continuously.

In our next example, we will calculate continuous decay. Pay attention to the rate – it is negative which means we are considering a situation where an amount decreases or decays.

In the following video, we show an example of calculating the remaining amount of a radioactive substance after it decays for a length of time.

## Summary

Continuous growth or decay functions are of the form  $A(t) = ae^{rt}$ . If  $r > 0$ , then the formula represents continuous growth. If  $r < 0$ , then the formula represents continuous decay. For business applications, the

## EXAMPLE

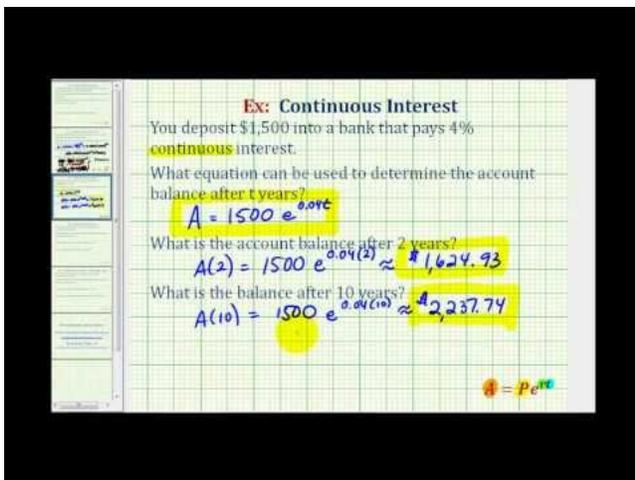
A person invested \$1,000 in an account earning a nominal 10% per year compounded continuously. How much was in the account at the end of one year?

### Answer

Since the account is growing in value, this is a continuous compounding problem with growth rate  $r = 0.10$ . The initial investment was \$1,000, so  $P = 1000$ . We use the continuous compounding formula to find the value after  $t = 1$  year:

$$\begin{aligned} A(t) &= Pe^{rt} && \text{Use the continuous compounding formula.} \\ &= 1000(e)^{0.1} && \text{Substitute known values for } P, r, \\ &&& \text{and } t. \\ &\approx 1105.17 && \text{Use a calculator to approximate.} \end{aligned}$$

The account is worth \$1,105.17 after one year.



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## HOW TO: GIVEN THE INITIAL VALUE, RATE OF GROWTH OR DECAY, AND TIME $t$ , SOLVE A CONTINUOUS GROWTH OR DECAY FUNCTION

1. Use the information in the problem to determine  $a$ , the initial value of the function.
2. Use the information in the problem to determine the growth rate  $r$ .
  1. If the problem refers to continuous growth, then  $r > 0$ .
  2. If the problem refers to continuous decay, then  $r < 0$ .
3. Use the information in the problem to determine the time  $t$ .
4. Substitute the given information into the continuous growth formula and solve for  $A(t)$ .

continuous growth formula is called the continuous compounding formula and takes the form  $A(t) = Pe^{rt}$ .

## EXAMPLE

Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?

### Answer

Since the substance is decaying, the rate, 17.3%, is negative. So,  $r = -0.173$ . The initial amount of radon-222 was 100 mg, so  $a = 100$ . We use the continuous decay formula to find the value after  $t = 3$  days:

$$A(t) = ae^{rt} \quad \text{Use the continuous growth formula.}$$

$$= 100e^{-0.173(3)} \quad \text{Substitute known values for } a, r, \text{ and } t.$$

$$\approx 59.5115 \quad \text{Use a calculator to approximate.}$$

So 59.5115 mg of radon-222 will remain.

The screenshot shows a video player interface. The main content is a slide titled "Continuous Growth/Decay Function" with a sub-heading "EXAMPLE". The text on the slide reads: "Certain medical tests involve injecting radioactive materials into the body. One medical test injects 0.5 ml of Technetium-99m that decays at a continuous rate of 11.55% per hour. How much Technetium-99m is left in the body after 24 hours?". Below the text, there is handwritten work:  $A(t) = ae^{rt}$ ,  $a = 0.5 \text{ ml}$ ,  $r = -11.55\% = -0.1155$ , and  $t = \text{time in hours}$ . At the bottom of the slide, there is a box titled "Continuous Exponential Growth/Decay Function" with the formula  $A(t) = ae^{rt}$  and three bullet points: 

- $a$  is the initial value or initial amount
- $r$  is the continuous growth or decay rate
- $t$  is time

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## GRAPH EXPONENTIAL FUNCTIONS

### LEARNING OUTCOME

- Graph exponential functions

We learn a lot about things by seeing their pictorial representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events. Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form  $f(x) = b^x$  whose base is greater than one. We will use the function  $f(x) = 2^x$ . Observe how the output values in the table below change as the input increases by 1.

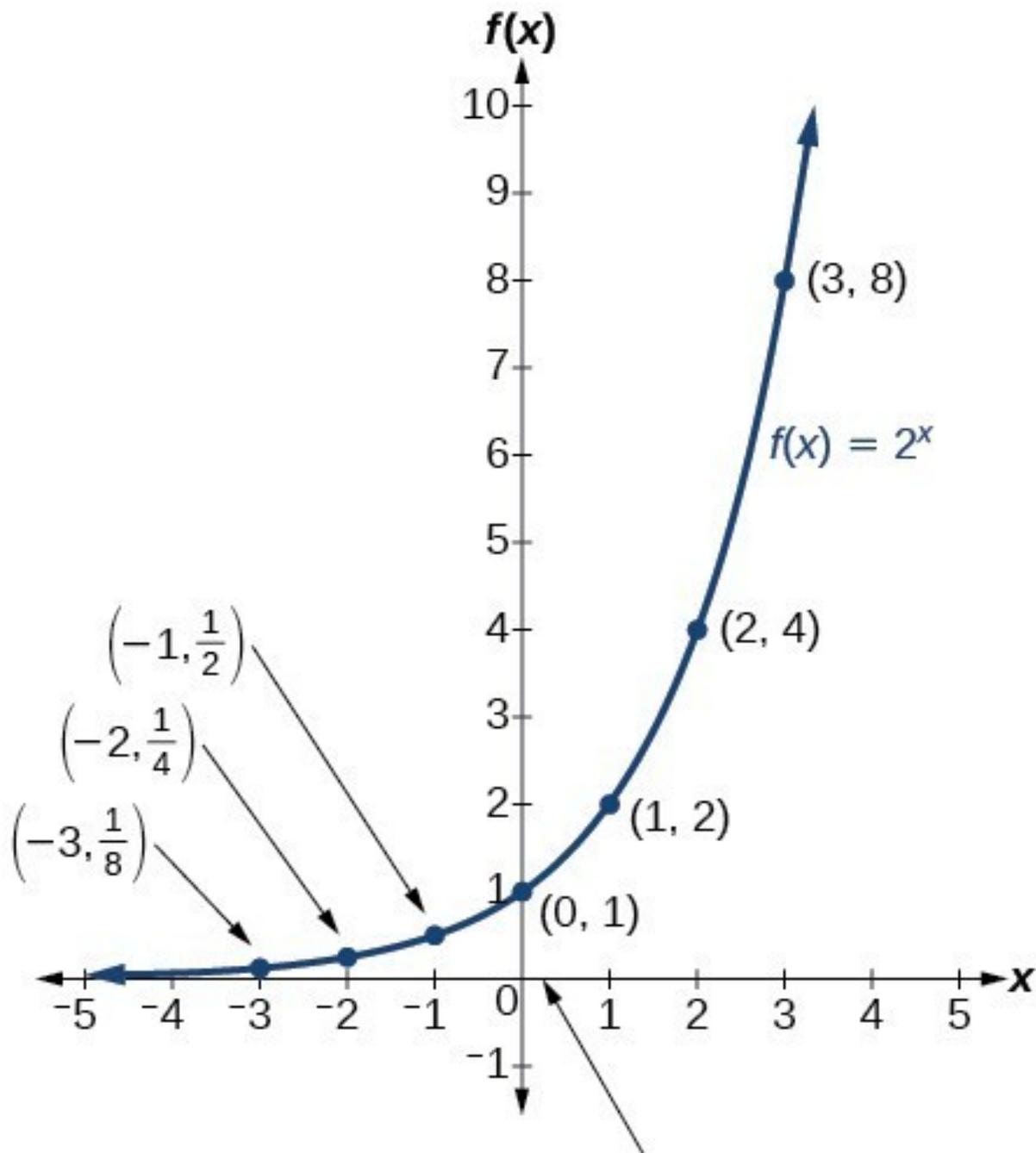
<b>x</b>	-3	-2	-1	0	1	2	3
$f(x)$ $= 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

Each output value is the product of the previous output and the base, 2. We call the base 2 the *constant ratio*. In fact, for any exponential function of the form  $f(x) = ab^x$ ,  $b$  is the constant ratio of the function. This means that as the input increases by 1, the output value will be the product of the base and the previous output, regardless of the value of  $a$ .

Notice from the table that

- the output values are positive for all values of  $x$ ;
- as  $x$  increases, the output values increase without bound; and
- as  $x$  decreases, the output values grow smaller, approaching zero.

The graph below shows the exponential growth function  $f(x) = 2^x$ .



The x-axis is an asymptote.

In the graph, notice that the graph gets close to the x-axis, but never touches it.

The domain of  $f(x) = 2^x$  is all real numbers; the range is  $(0, \infty)$ .

To get a sense of the behavior of **exponential decay**, we can create a table of values for a function of the form  $f(x) = b^x$  whose base is between zero and one. We will use the function  $g(x) = (\frac{1}{2})^x$ . Observe how the output values in the table below change as the input increases by 1.

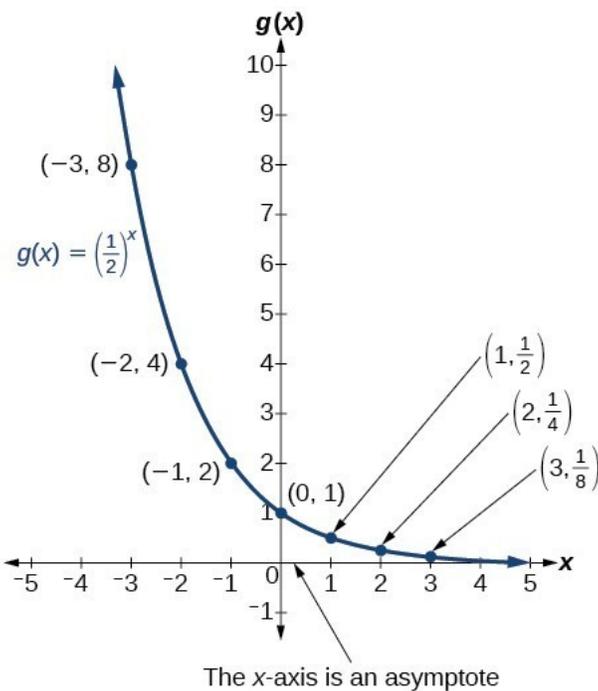
$x$	-3	-2	-1	0	1	2	3
$g(x) = (\frac{1}{2})^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

Again, because the input is increasing by 1, each output value is the product of the previous output and the base, or constant ratio  $\frac{1}{2}$ .

Notice from the table that:

- the output values are positive for all values of  $x$ ;
- as  $x$  increases, the output values grow smaller, approaching zero; and
- as  $x$  decreases, the output values grow without bound.

The graph shows the exponential decay function,  $g(x) = \left(\frac{1}{2}\right)^x$ .



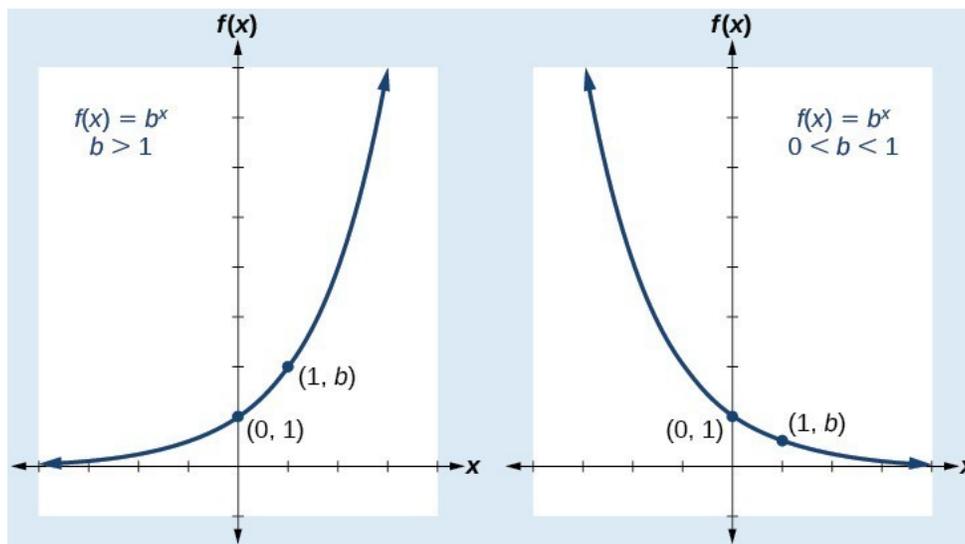
The domain of  $g(x) = \left(\frac{1}{2}\right)^x$  is all real numbers; the range is  $(0, \infty)$ .

## CHARACTERISTICS OF THE GRAPH OF $f(x) = b^x$

An exponential function of the form  $f(x) = b^x$ ,  $b > 0$ ,  $b \neq 1$ , has these characteristics:

- **one-to-one** function
- domain:  $(-\infty, \infty)$
- range:  $(0, \infty)$
- x-intercept: none
- y-intercept:  $(0, 1)$
- increasing if  $b > 1$
- decreasing if  $b < 1$

Compare the graphs of **exponential growth** and decay functions below.



1 and the second graph is of the same function when  $b$  is 0

In our first example, we will plot an exponential decay function where the base is between 0 and 1.

### EXAMPLE

Sketch a graph of  $f(x) = 0.25^x$ . State the domain, range.

#### Answer

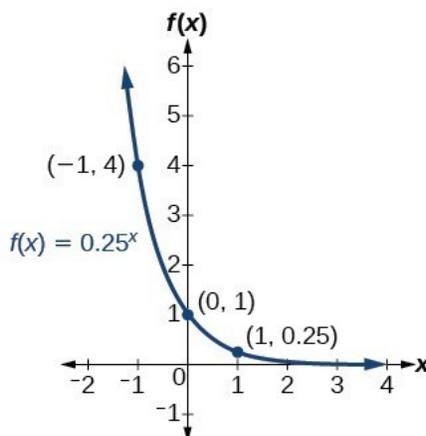
Before graphing, identify the behavior and create a table of points for the graph.

- Since  $b = 0.25$  is between zero and one, we know the function is decreasing, and we can verify this by creating a table of values. The left tail of the graph will increase without bound and the right tail will get really close to the  $x$ -axis.
- Create a table of points.

$x$	-3	-2	-1	0	1	2	3
$f(x) = 0.25^x$	64	16	4	1	0.25	0.0625	0.015625

- Plot the  $y$ -intercept,  $(0, 1)$ , along with two other points. We can use  $(-1, 4)$  and  $(1, 0.25)$ .

Draw a smooth curve connecting the points.



The domain is  $(-\infty, \infty)$ ; the range is  $(0, \infty)$ .

In the following video, we show another example of graphing an exponential function. The base of the exponential term is between 0 and 1, so this graph will represent decay.

**Graphing Exponential Functions by Plotting Points**  
**EXAMPLE**  
 $f(x) = \left(\frac{1}{2}\right)^x$   
 Sketch a graph of  $f(x) = 0.5^x$ . Then state the domain and range.

x	f(x)
-3	8
-2	4
-1	2
0	1
1	

$f(-3) = \left(\frac{1}{2}\right)^{-3} = \frac{1}{2^{-3}} = \frac{1}{\frac{1}{8}} = 8$   
 $f(-2) = \left(\frac{1}{2}\right)^{-2} = \frac{1}{2^{-2}} = \frac{1}{\frac{1}{4}} = 4$   
 $f(-1) = \left(\frac{1}{2}\right)^{-1} = \frac{1}{2^{-1}} = \frac{1}{\frac{1}{2}} = 2$   
 $f(0) = \left(\frac{1}{2}\right)^0 = 1$   
 $f(1) = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$

$a^{-m} = \frac{1}{a^m}$   
 $a^{-m} = a^m$   
 $a^{-m} = a^m$

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## HOW TO: GIVEN AN EXPONENTIAL FUNCTION OF THE FORM $f(x) = b^x$ , GRAPH THE FUNCTION

1. Create a table of points.
2. Plot at least 3 point from the table, including the y-intercept (0, 1).
3. Draw a smooth curve through the points.
4. State the domain,  $(-\infty, \infty)$ , and the range,  $(0, \infty)$ .

In our next example, we will plot an exponential growth function where the base is greater than 1.

Our next video example includes graphing an exponential growth function and defining the domain and range of the function.

## Summary

Graphs of exponential growth functions will have a right tail that increases without bound and a left tail that gets really close to the x-axis. On the other hand, graphs of exponential decay functions will have a left tail that increases without bound and a right tail that gets really close to the x-axis. Points can be generated with a table of values which can then be used to graph the function.

## EXAMPLE

Sketch a graph of  $f(x) = \sqrt{2}(\sqrt{2})^x$ . State the domain and range.

### Answer

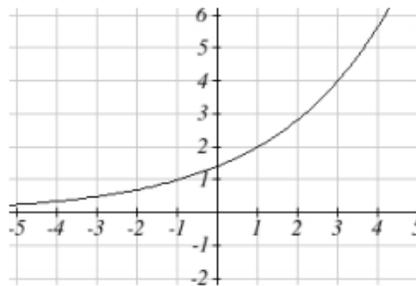
Before graphing, identify the behavior and create a table of points for the graph.

- Since  $b = \sqrt{2}$ , which is greater than one, we know the function is increasing, and we can verify this by creating a table of values. The left tail of the graph will get really close to the x-axis and the right tail will increase without bound.
- Create a table of points.

<b>x</b>	-3	-2	-1	0	1	2	3
<b>f(x)</b> $= \sqrt{2}(\sqrt{2})^x$	0.5	0.71	1	1.41	2	2.83	4

- Plot the y-intercept, (0, 1.41), along with two other points. We can use (-1, 1) and (1, 2).

Draw a smooth curve connecting the points.



The domain is  $(-\infty, \infty)$ ; the range is  $(0, \infty)$ .

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# INTRODUCTION TO LOGARITHMIC FUNCTIONS



*Devastation of the earthquake which happened on March 11, 2011 in Honshu, Japan. (credit: Daniel Pierce)*

In 2010, a major earthquake struck Haiti, destroying or damaging over 285, 000 homes. (Note: <http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/#summary>. Accessed 3/4/2013.) One year later, another, stronger earthquake devastated Honshu, Japan, destroying or damaging over 332, 000 buildings, (Note: <http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#summary>. Accessed 3/4/2013.) like those shown in the picture above. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale (Note: <http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/>. Accessed 3/4/2013.) whereas the Japanese earthquake registered a 9.0. (Note: <http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/#details>. Accessed 3/4/2013.)

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is  $10^{8-4} = 10^4 = 10,000$  times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

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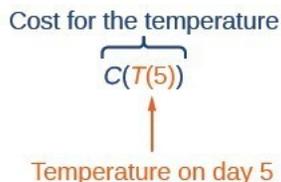
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# COMPOSITE AND INVERSE FUNCTIONS

## LEARNING OUTCOMES

- Find composite functions
- Use compositions of functions to verify inverses algebraically
- Identify the domain and range of inverse functions with tables

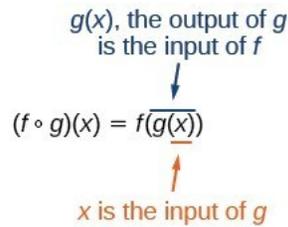
Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.



Using descriptive variables, we can notate these two functions. The function  $C(T)$  gives the cost  $C$  of heating a house for a given average daily temperature in  $T$  degrees Celsius. The function  $T(d)$  gives the average daily temperature on day  $d$  of the year. For any given day,  $\text{Cost} = C(T(d))$  means that the cost depends on the temperature which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature  $T(d)$ . For example, we could evaluate  $T(5)$  to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the **cost function** at that temperature. We would write  $C(T(5))$ . By combining these two relationships into one function, we have performed function composition.

We read the left-hand side as “ $f$  composed with  $g$  at  $x$ ,” and the right-hand side as “ $f$  of  $g$  of  $x$ .” The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol  $\circ$  is called the composition operator.

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first and then working to the outside.



## EXAMPLE

Using the functions provided, find  $f(g(x))$  and  $g(f(x))$ .

$$f(x) = 2x + 1$$

$$g(x) = 3 - x$$

**Answer**

$$f(x) = 2x + 1$$

$$g(x) = 3 - x$$

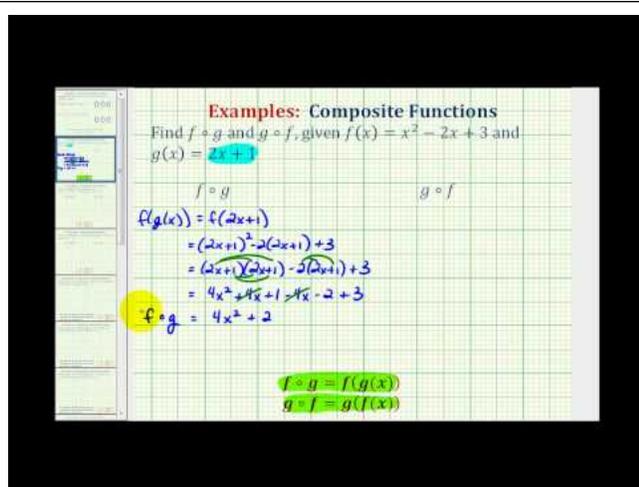
Let us begin by substituting  $g(x)$  into  $f(x)$ .

$$\begin{aligned} f(g(x)) &= 2(3 - x) + 1 \\ &= 6 - 2x + 1 \\ &= 7 - 2x \end{aligned}$$

Now we can substitute  $f(x)$  into  $g(x)$ .

$$\begin{aligned} g(f(x)) &= 3 - (2x + 1) \\ &= 3 - 2x - 1 \\ &= -2x + 2 \end{aligned}$$

In the following video, you will see another example of how to find the composition of two functions.



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## Inverse Functions

An **inverse function** is a function where the input of the original function becomes the output of the inverse function. This naturally leads to the output of the original function becoming the input of the inverse function. The reason we want to introduce inverse functions is because exponential and logarithmic functions are inverses of each other, and understanding this quality helps to make understanding logarithmic functions easier. And the reason we introduced composite functions is because you can verify, algebraically, whether two functions are inverses of each other by using a composition.

Given a function  $f(x)$ , we represent its inverse as  $f^{-1}(x)$ , read as “ $f$  inverse of  $x$ .” The raised  $-1$  is part of the notation. It is not an exponent; it does not imply a power of  $-1$ . In other words,  $f^{-1}(x)$  does *not* mean  $\frac{1}{f(x)}$  because  $\frac{1}{f(x)}$  is the reciprocal of  $f$  and not the inverse.

Just as zero does not have a **reciprocal**, some functions do not have inverses.

## INVERSE FUNCTION

For any **one-to-one function**  $f(x) = y$ , a function  $f^{-1}(x)$  is an **inverse function** of  $f$  if  $f^{-1}(y) = x$ . The notation  $f^{-1}$  is read “ $f$  inverse.” Like any other function, we can use any variable name as the input for  $f^{-1}$ , so we will often write  $f^{-1}(x)$ , which we read as “ $f$  inverse of  $x$ .” Keep in mind that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

and not all functions have inverses.

In our first example, we will identify an inverse function from ordered pairs.

## EXAMPLE

If for a particular one-to-one function  $f(2) = 4$  and  $f(5) = 12$ , what are the corresponding input and output values for the inverse function?

### Answer

The inverse function reverses the input and output quantities, so if

$$f(2) = 4, \text{ then } f^{-1}(4) = 2;$$

$$f(5) = 12, \text{ then } f^{-1}(12) = 5.$$

Alternatively, if we want to name the inverse function  $g$ , then  $g(4) = 2$  and  $g(12) = 5$ .

## Analysis of the Solution

Notice that if we show the coordinate pairs from the previous example in table form, the input and output are clearly reversed.

$(x, f(x))$	$(x, g(x))$
$(2, 4)$	$(4, 2)$
$(5, 12)$	$(12, 5)$

In the following video, we show an example of finding corresponding input and output values given two ordered pairs from functions that are inverses.

In our next example, we will test inverse relationships algebraically.

**Examples: Inverse Function Values**  
 Assume  $f$  is a one-to-one function. Determine the function values.

If  $f(2) = 5$ , then  $f^{-1}(5) = 2$        $2 \xrightarrow{f} 5 \xrightarrow{f^{-1}} 2$

If  $f^{-1}(-3) = -2$ , then  $f(-2) = -3$        $-2 \xrightarrow{f} -3 \xrightarrow{f^{-1}} -2$

A function and its inverse "undo" each other.  
 $f(x) = y \rightarrow f^{-1}(y) = x$

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## HOW TO: GIVEN TWO FUNCTIONS $f(x)$ AND $g(x)$ , TEST WHETHER THE FUNCTIONS ARE INVERSES OF EACH OTHER

1. Substitute  $g(x)$  into  $f(x)$ . The result must be  $x$ .  $f(g(x)) = x$
2. Substitute  $f(x)$  into  $g(x)$ . The result must be  $x$ .  $g(f(x)) = x$

If  $f(x)$  and  $g(x)$  are inverses, then  $f(x) = g^{-1}(x)$  and  $g(x) = f^{-1}(x)$

### EXAMPLE

If  $f(x) = x^2 - 3$ , for  $x \geq 0$  and  $g(x) = \sqrt{x+3}$ , is  $g$  the inverse of  $f$ ? In other words, does  $g = f^{-1}$ ?

**Answer**

Substitute  $g(x) = \sqrt{x+3}$  into  $f(x)$ ; this means the new variable in  $f(x)$  is  $\sqrt{x+3}$ , so you will substitute that expression where you see  $x$ . Using parentheses helps keep track of things.

$$\begin{aligned} f(\sqrt{x+3}) &= (\sqrt{x+3})^2 - 3 \\ &= x + 3 - 3 \\ &= x \end{aligned}$$

Our result implies that  $g(x)$  is indeed the inverse of  $f(x)$ .

$$g = f^{-1}, \text{ for } x \geq 0$$

In the following video, we use algebra to determine if two functions are inverses.

We will show one more example of how to verify whether you have an inverse algebraically.

We will show one more example of how to use algebra to determine whether two functions are inverses of each other.

## Domain and Range of a Function and Its Inverse

The outputs of the function  $f$  are the inputs to  $f^{-1}$ , so the range of  $f$  is also the domain of  $f^{-1}$ . Likewise,

**EXAMPLE**

Determine if the two functions are inverses of one another.  
 $f(x) = x^2 - 2$ , for  $x \geq 0$  and  $g(x) = \sqrt{x+2}$

How to determine if two functions are inverses of each other.

1. Substitute  $g(x)$  into  $f(x)$ . The result must be  $x$ .  $f(g(x)) = x$
2. Substitute  $f(x)$  into  $g(x)$ . The result must be  $x$ .  $g(f(x)) = x$

If  $f(x)$  and  $g(x)$  are inverses, then  
 $f(x) = g^{-1}(x)$  and  $g(x) = f^{-1}(x)$

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**EXAMPLE**

If  $f(x) = \frac{1}{x+2}$  and  $g(x) = \frac{1}{x} - 2$ , is  $g$  the inverse of  $f$ ? In other words, does  $g = f^{-1}$ ?

**Answer**

Substitute  $g(x) = \frac{1}{x} - 2$  into  $f(x)$ ; this means the new variable in  $f(x)$  is  $\frac{1}{x} - 2$ , so you will substitute that expression where you see  $x$ . Using parentheses helps keep track of things.

$$\begin{aligned}
 f\left(\frac{1}{x} - 2\right) &= \frac{1}{\left(\frac{1}{x} - 2\right) + 2} \\
 &= \frac{1}{\frac{1}{x}} \\
 &= x
 \end{aligned}$$

$$g = f^{-1}$$

**EXAMPLE**

Determine if the two functions are inverses of one another.  
 $f(x) = \frac{1}{x-4}$  and  $g(x) = \frac{1}{x} + 4$

$f(g(x))$

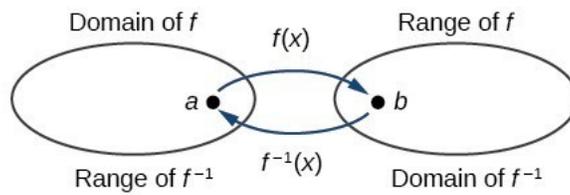
How to determine if two functions are inverses of each other.

1. Substitute  $g(x)$  into  $f(x)$ . The result must be  $x$ .  $f(g(x)) = x$
2. Substitute  $f(x)$  into  $g(x)$ . The result must be  $x$ .  $g(f(x)) = x$

If  $f(x)$  and  $g(x)$  are inverses, then  
 $f(x) = g^{-1}(x)$  and  $g(x) = f^{-1}(x)$

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because the inputs to  $f$  are the outputs of  $f^{-1}$ , the domain of  $f$  is the range of  $f^{-1}$ . We can visualize the situation.



In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on which it is one-to-one. For example, we can make a restricted version of the square function  $f(x) = x^2$  with its domain limited to  $[0, \infty)$ , which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

## DOMAIN AND RANGE OF INVERSE FUNCTIONS

The range of a function  $f(x)$  is the domain of the inverse function  $f^{-1}(x)$ .

The domain of  $f(x)$  is the range of  $f^{-1}(x)$ .

In our last example, we will define the domain and range of a function's inverse using a table of values, and evaluate the inverse at a specific value.

### EXAMPLE

A function  $f(t)$  is given below, showing distance in miles that a car has traveled in  $t$  minutes.

1. Define the domain and range of the function and its inverse.
2. Find and interpret  $f^{-1}(70)$ .

$t$ (minutes)	30	50	70	90
$f(t)$ ) (miles)	20	40	60	70

#### **Answer**

##### 1. Domain and Range of the Original Function

The domain of this tabular function,  $f(t)$ , is all the input values,  $t$ , in minutes: 30, 50, 70, 90

The range of this tabular function,  $f(t)$ , is all the output values  $f(t)$ , in miles: 20, 40, 60, 70

##### Domain and Range of the Inverse Function

The domain for the inverse will be the outputs from the original, so the domain of  $f^{-1}(x)$  is the output values from  $f(t)$ : 20, 40, 60, 70

The range for the inverse will be the inputs from the original: 30, 50, 70, 90

This translates to putting in a number of miles and getting out how long it took to drive that far in minutes.

2. So in the expression  $f^{-1}(70)$ , 70 is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function  $f$ , 90 minutes, so  $f^{-1}(70) = 90$ . The interpretation of this is that, to drive 70 miles, it took 90 minutes.

## Summary

The inverse of a function can be defined for one-to-one functions. If a function is not one-to-one, it can be possible to restrict its domain to make it so. The domain of a function will become the range of its inverse. The range of a function will become the domain of its inverse. Inverses can be verified using tabular data as well as algebraically.

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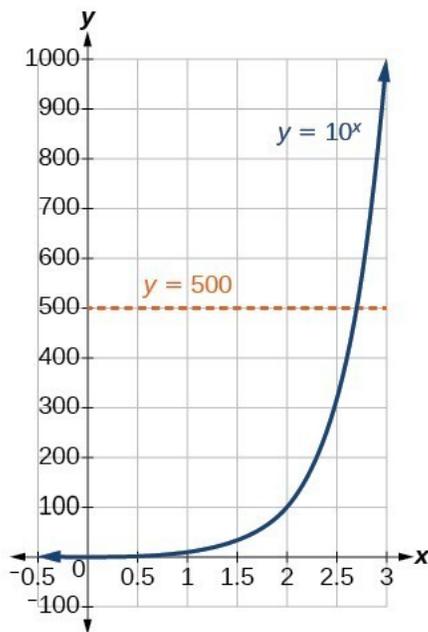
# DEFINE LOGARITHMIC FUNCTIONS

## LEARNING OUTCOMES

- Convert from logarithmic to exponential form
- Convert from exponential to logarithmic form

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake were 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is  $10^x = 500$ , where  $x$  represents the difference in magnitudes on the **Richter Scale**. How would we solve for  $x$ ?

We have not yet learned a method for solving exponential equations. None of the algebraic tools discussed so far is sufficient to solve  $10^x = 500$ . We know that  $10^2 = 100$  and  $10^3 = 1000$ , so it is clear that  $x$  must be some value between 2 and 3, since  $y = 10^x$  is increasing. We can examine a graph to better estimate the solution.



Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph above passes the horizontal line test. The exponential function  $y = b^x$  is

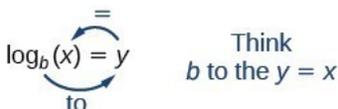
**one-to-one**, so its inverse,  $x = b^y$  is also a function. As is the case with all inverse functions, we simply interchange  $x$  and  $y$  and solve for  $y$  to find the inverse function. To represent  $y$  as a function of  $x$ , we use a logarithmic function of the form  $y = \log_b(x)$ . The base  $b$  **logarithm** of a number is the exponent by which we must raise  $b$  to get that number.

We read a logarithmic expression as, “The logarithm with base  $b$  of  $x$  is equal to  $y$ ,” or, simplified, “log base  $b$  of  $x$  is  $y$ .” We can also say, “ $b$  raised to the power of  $y$  is  $x$ ,” because logs are exponents. For example, the base 2 logarithm of 32 is 5, because 5 is the exponent we must apply to 2 to get 32. Since  $2^5 = 32$ , we can write  $\log_2 32 = 5$ . We read this as “log base 2 of 32 is 5.”

We can express the relationship between logarithmic form and its corresponding exponential form as follows:

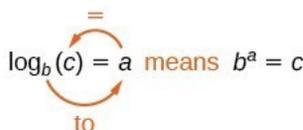
$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base  $b$  is always positive.



Because logarithms are functions, they are most correctly written as  $\log_b(x)$ , using parentheses to denote function evaluation, just as we would with  $f(x)$ . However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses, as  $\log_b x$ . Note that many calculators require parentheses around the  $x$ .

We can illustrate the notation of logarithms as follows:



Notice that, comparing the logarithm function and the exponential function, the input and the output are switched. This means  $y = \log_b(x)$  and  $y = b^x$  are inverse functions.

## DEFINITION OF THE LOGARITHMIC FUNCTION

A **logarithm** base  $b$  of a positive number  $x$  satisfies the following definition.

For  $x > 0, b > 0, b \neq 1$ ,

$$y = \log_b(x) \text{ is the same as } b^y = x$$

where

- we read  $\log_b(x)$  as, “the logarithm with base  $b$  of  $x$ ” or the “log base  $b$  of  $x$ .”
- the logarithm  $y$  is the exponent to which  $b$  must be raised to get  $x$ .

Also, since the logarithmic and exponential functions switch the  $x$  and  $y$  values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,

- the domain of the logarithm function with base  $b$  is  $(0, \infty)$ .
- the range of the logarithm function with base  $b$  is  $(-\infty, \infty)$ .

In our first example, we will convert logarithmic equations into exponential equations.

In the following video, we present more examples of rewriting logarithmic equations as exponential equations.

## Convert from Exponential to Logarithmic Form

## EXAMPLE

Write the following logarithmic equations in exponential form.

1.  $\log_6(\sqrt{6}) = \frac{1}{2}$
2.  $\log_3(9) = 2$

### Answer

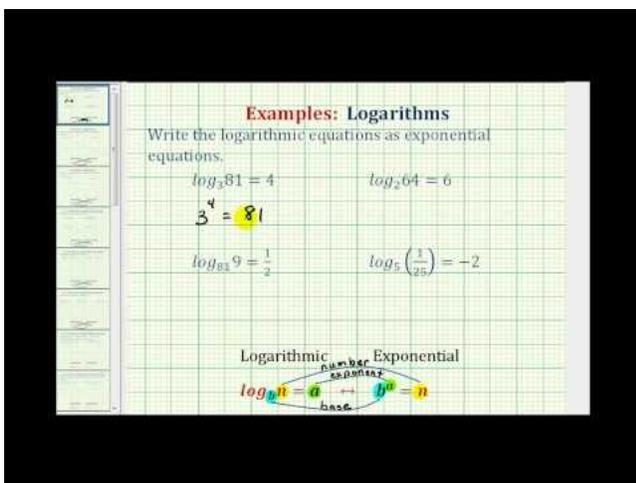
First, identify the values of  $b$ ,  $y$ , and  $x$ . Then, write the equation in the form  $b^y = x$ .

1.  $\log_6(\sqrt{6}) = \frac{1}{2}$

Here,  $b = 6$ ,  $y = \frac{1}{2}$ , and  $x = \sqrt{6}$ . Therefore, the equation  $\log_6(\sqrt{6}) = \frac{1}{2}$  can be written as  $6^{\frac{1}{2}} = \sqrt{6}$ .

2.  $\log_3(9) = 2$

Here,  $b = 3$ ,  $y = 2$ , and  $x = 9$ . Therefore, the equation  $\log_3(9) = 2$  can be written as  $3^2 = 9$ .



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## HOW TO: GIVEN AN EQUATION IN LOGARITHMIC FORM $\log_b(x) = y$ , CONVERT IT TO EXPONENTIAL FORM

1. Examine the equation  $y = \log_b x$  and identify  $b$ ,  $y$ , and  $x$ .
2. Rewrite  $\log_b x = y$  as  $b^y = x$ .

## THINK ABOUT IT

Can we take the logarithm of a negative number? Re-read the definition of a logarithm and formulate an answer. Think about the behavior of exponents. You can use the textbox below to formulate your ideas before you look at an answer.

### Answer

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of

zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

To convert from exponential form to logarithmic form, we follow the same steps in reverse. We identify the base  $b$ , exponent  $x$ , and output  $y$ . Then we write  $x = \log_b(y)$ .

## EXAMPLE

Write the following exponential equations in logarithmic form.

- $2^3 = 8$
- $5^2 = 25$
- $10^{-4} = \frac{1}{10,000}$

### Answer

First, identify the values of  $b$ ,  $y$ , and  $x$ . Then, write the equation in the form  $x = \log_b(y)$ .

1.  $2^3 = 8$

Here,  $b = 2$ ,  $x = 3$ , and  $y = 8$ . Therefore, the equation  $2^3 = 8$  can be written as  $\log_2(8) = 3$ .

2.  $5^2 = 25$

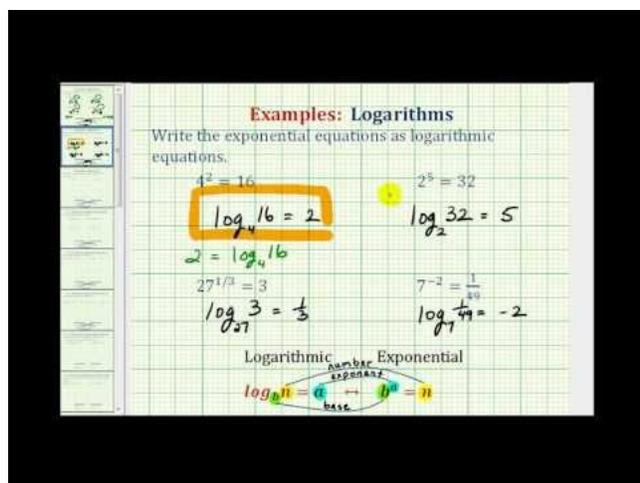
Here,  $b = 5$ ,  $x = 2$ , and  $y = 25$ . Therefore, the equation  $5^2 = 25$  can be written as  $\log_5(25) = 2$ .

3.  $10^{-4} = \frac{1}{10,000}$

Here,  $b = 10$ ,  $x = -4$ , and  $y = \frac{1}{10,000}$ . Therefore, the equation  $10^{-4} = \frac{1}{10,000}$  can be written as

$$\log_{10}\left(\frac{1}{10,000}\right) = -4.$$

In our last video, we show more examples of writing logarithmic equations as exponential equations.



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## Summary

The base  $b$  **logarithm** of a number is the exponent by which we must raise  $b$  to get that number. Logarithmic functions are the inverse of exponential functions, and it is often easier to understand them through this lens. We can never take the logarithm of a negative number, therefore  $\log_b(x) = y$  is defined for  $b > 0$

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# EVALUATE LOGARITHMS

## LEARNING OUTCOME

- Evaluate logarithms

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider  $\log_2 8$ . We ask, “To what exponent must 2 be raised in order to get 8?” Because we already know  $2^3 = 8$ , it follows that  $\log_2 8 = 3$ .

Now consider solving  $\log_7 49$  and  $\log_3 27$  mentally.

- We ask, “To what exponent must 7 be raised in order to get 49?” We know  $7^2 = 49$ . Therefore,  $\log_7 49 = 2$
- We ask, “To what exponent must 3 be raised in order to get 27?” We know  $3^3 = 27$ . Therefore,  $\log_3 27 = 3$

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let us evaluate  $\log_{\frac{2}{3}} \frac{4}{9}$  mentally.

- We ask, “To what exponent must  $\frac{2}{3}$  be raised in order to get  $\frac{4}{9}$ ?” We know  $2^2 = 4$  and  $3^2 = 9$ , so  $(\frac{2}{3})^2 = \frac{4}{9}$ . Therefore,  $\log_{\frac{2}{3}} (\frac{4}{9}) = 2$ .

In our first example, we will evaluate logarithms mentally.

## EXAMPLE

Solve  $y = \log_4 (64)$  without using a calculator.

### Answer

First we rewrite the logarithm in exponential form:  $4^y = 64$ . Next, we ask, “To what exponent must 4 be raised in order to get 64?”

We know

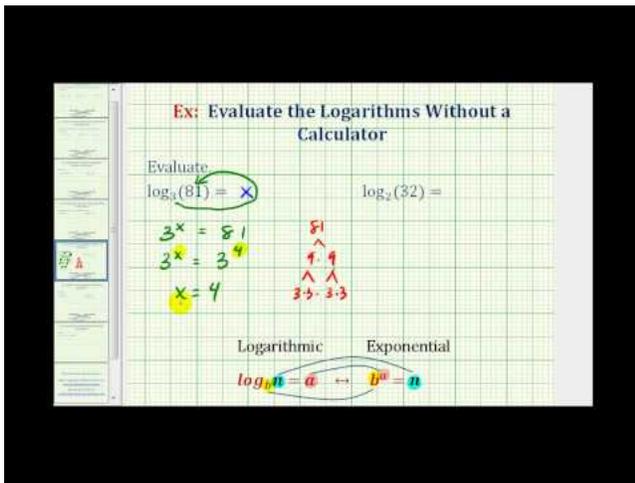
$$4^3 = 64$$

Therefore,

$$\log_4 (64) = 3$$

In our first video, we will show more examples of evaluating logarithms mentally; this helps you get familiar with what a logarithm represents.

In our next example, we will evaluate the logarithm of a reciprocal.



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## EXAMPLE

Evaluate  $y = \log_3\left(\frac{1}{27}\right)$  without using a calculator.

### Answer

First we rewrite the logarithm in exponential form:  $3^y = \frac{1}{27}$ . Next, we ask, “To what exponent must 3 be raised in order to get  $\frac{1}{27}$ ?”

We know  $3^3 = 27$ , but what must we do to get the reciprocal,  $\frac{1}{27}$ ? Recall from working with exponents that  $b^{-a} = \frac{1}{b^a}$ . We use this information to write:

$$\begin{aligned} 3^{-3} &= \frac{1}{3^3} \\ &= \frac{1}{27} \end{aligned}$$

Therefore,  $\log_3\left(\frac{1}{27}\right) = -3$ .

## HOW TO: GIVEN A LOGARITHM OF THE FORM $y = \log_b(x)$ , EVALUATE IT MENTALLY

1. Rewrite the argument  $x$  as a power of  $b$ :  $b^y = x$ .
2. Use previous knowledge of powers of  $b$  to identify  $y$  by asking, “To what exponent should  $b$  be raised in order to get  $x$ ?”

## Natural Logarithms

The most frequently used base for logarithms is  $e$ . Base  $e$  logarithms are important in calculus and some scientific applications; they are called **natural logarithms**. The base  $e$  logarithm,  $\log_e(x)$ , has its own notation,  $\ln(x)$ .

Most values of  $\ln(x)$  can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base,  $\ln 1 = 0$ . For other natural logarithms, we can use the  $\ln$  key that can be found on most scientific calculators. We can also find the natural logarithm of any power of  $e$  using the inverse property of logarithms.

## A GENERAL NOTE: DEFINITION OF THE NATURAL LOGARITHM

A **natural logarithm** is a logarithm with base  $e$ . We write  $\log_e(x)$  simply as  $\ln(x)$ . The natural logarithm of a positive number  $x$  satisfies the following definition.

For  $x > 0$ ,

$$y = \ln(x) \text{ can be written as } e^y = x$$

We read  $\ln(x)$  as, “the logarithm with base  $e$  of  $x$ ” or “the natural logarithm of  $x$ .”

The logarithm  $y$  is the exponent to which  $e$  must be raised to get  $x$ .

Since the functions  $y = e^x$  and  $y = \ln(x)$  are inverse functions,  $\ln(e^x) = x$  for all  $x$  and  $e^{\ln(x)} = x$  for  $x > 0$ .

In the next example, we will evaluate a natural logarithm using a calculator.

## EXAMPLE

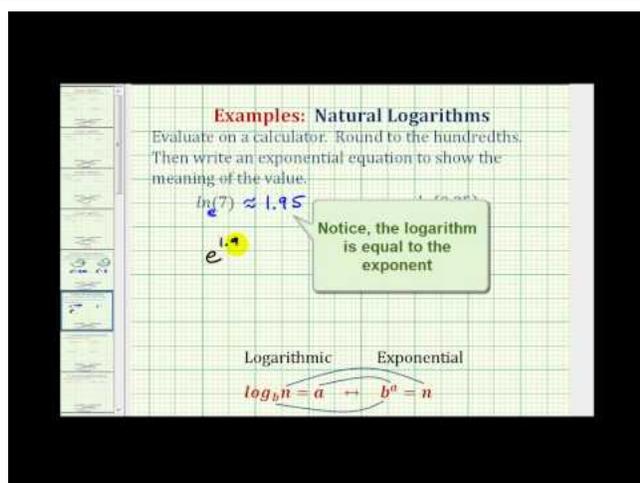
Evaluate  $y = \ln(500)$  to four decimal places using a calculator.

**Answer**

- Press **[LN]**.
- Enter 500, followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places,  $\ln(500) \approx 6.2146$

In our next video, we show more examples of how to evaluate natural logarithms using a calculator.



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## Common Logarithms

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10. In other words, the expression  $\log$  means  $\log_{10}$ . We call a base-10 logarithm a **common logarithm**. Common

logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

## DEFINITION OF COMMON LOGARITHM: LOG IS AN EXPONENT

A common logarithm is a logarithm with base 10. We write  $\log_{10}(x)$  simply as  $\log(x)$ . The common logarithm of a positive number,  $x$ , satisfies the following definition:

For  $x > 0$ ,

$y = \log(x)$  can be written as  $10^y = x$

We read  $\log(x)$  as "the logarithm with base 10 of  $x$ " or "log base 10 of  $x$ ".

The logarithm  $y$  is the exponent to which 10 must be raised to get  $x$ .

## EXAMPLE

Evaluate  $\log(1000)$  without using a calculator.

**Answer**

We know  $10^3 = 1000$ , therefore

$$\log(1000) = 3$$

## EXAMPLE

Evaluate  $y = \log(321)$  to four decimal places using a calculator.

**Answer**

- Press **[LOG]**.
- Enter 321, followed by **[ ) ]**.
- Press **[ENTER]**.

Rounding to four decimal places,  $\log(321) \approx 2.5065$ .

In our last example, we will use a logarithm to find the difference in magnitude of two different earthquakes.

## EXAMPLE

The amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. The equation  $10^x = 500$  represents this situation, where  $x$  is the difference in magnitudes on the Richter Scale. To the nearest thousandth, what was the difference in magnitudes?

**Answer**

We begin by rewriting the exponential equation in logarithmic form.

$$10^x = 500$$

$$\log(500) = x$$

Next we evaluate the logarithm using a calculator:

- Press **[LOG]**.
- Enter 500 followed by **[ ) ]**.
- Press **[ENTER]**.
- To the nearest thousandth,  $\log(500) \approx 2.699$

## Summary

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally because the logarithm is an exponent. Logarithms most commonly use base 10 and natural logarithms use base  $e$ . Logarithms can also be evaluated with most kinds of calculator.

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# GRAPHS OF LOGARITHMIC FUNCTIONS

## LEARNING Outcomes

- Identify the domain of a logarithmic function
- Graph logarithmic functions

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as  $y = b^x$  for any real number  $x$  and constant  $b > 0$ ,  $b \neq 1$ , where

- The domain of is  $(-\infty, \infty)$ .
- The range of is  $(0, \infty)$ .

In the last section we learned that the logarithmic function  $y = \log_b(x)$  is the inverse of the exponential function  $y = b^x$ . So, as inverse functions:

- The domain of  $y = \log_b(x)$  is the range of  $y = b^x$ :  $(0, \infty)$ .
- The range of  $y = \log_b(x)$  is the domain of  $y = b^x$ :  $(-\infty, \infty)$ .

## HOW TO: GIVEN A LOGARITHMIC FUNCTION, IDENTIFY THE DOMAIN

1. Set up an inequality showing the argument greater than zero.
2. Solve for  $x$ .
3. Write the domain in interval notation.

In our first example, we will show how to identify the domain of a logarithmic function.

Here is another example of how to identify the domain of a logarithmic function.

## Graph Logarithmic Functions

Creating a graphical representation of most functions gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the *cause* for an *effect*.

To illustrate, suppose we invest \$2500 in an account that offers an annual interest rate of 5%, compounded

## EXAMPLE

What is the domain of  $f(x) = \log_2(x + 3)$ ?

### Answer

The logarithmic function is defined only when the input is positive, so this function is defined when  $x + 3 > 0$ . Solving this inequality,

$$x + 3 > 0 \quad \text{The input must be positive.}$$

$$x > -3 \quad \text{Subtract 3.}$$

The domain of  $f(x) = \log_2(x + 3)$  is  $(-3, \infty)$ .

## EXAMPLE

What is the domain of  $f(x) = \log(5 - 2x)$ ?

### Answer

The logarithmic function is defined only when the input is positive, so this function is defined when  $5 - 2x > 0$ . Solving this inequality,

$$5 - 2x > 0 \quad \text{The input must be positive.}$$

$$-2x > -5 \quad \text{Subtract 5.}$$

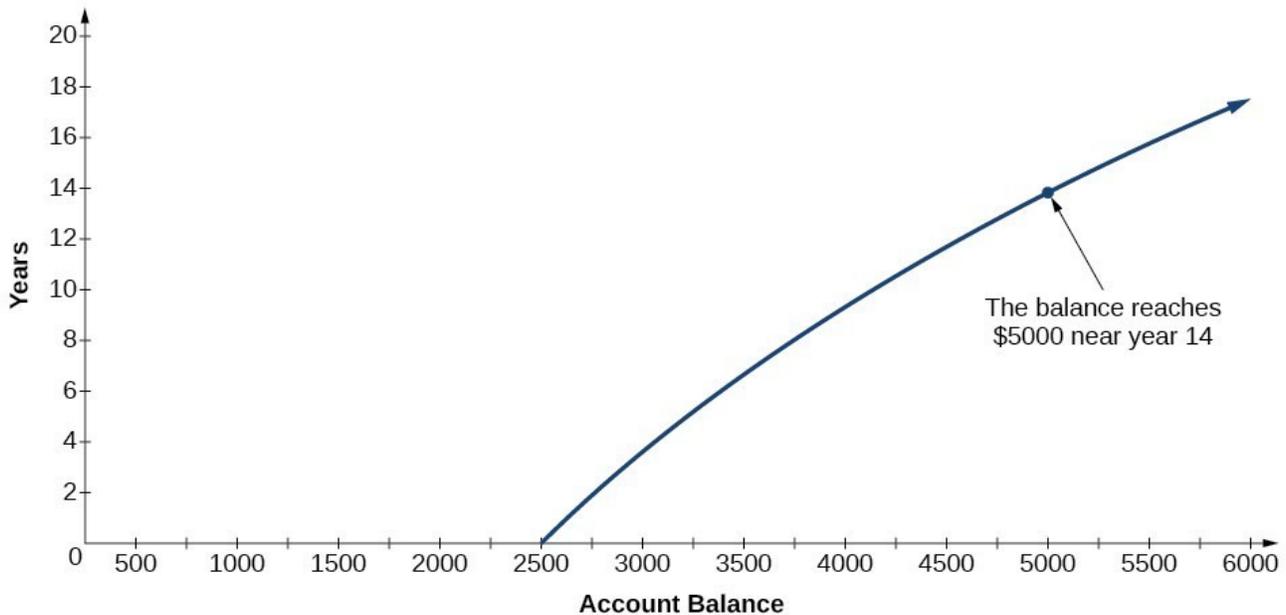
$$x < \frac{5}{2} \quad \text{Divide by } -2 \text{ and switch the inequality.}$$

The domain of  $f(x) = \log(5 - 2x)$  is  $(-\infty, \frac{5}{2})$ .

continuously. We already know that the balance in our account for any year  $t$  can be found with the equation  $A = 2500e^{0.05t}$ .

But what if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? The image below shows this point on the logarithmic graph.

## Logarithmic Model Showing Years as a Function of the Balance in the Account



A logarithmic graph showing how many years it will take for an account balance to double.

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function  $y = \log_b(x)$  along with all its transformations: shifts, stretches, compressions, and reflections.

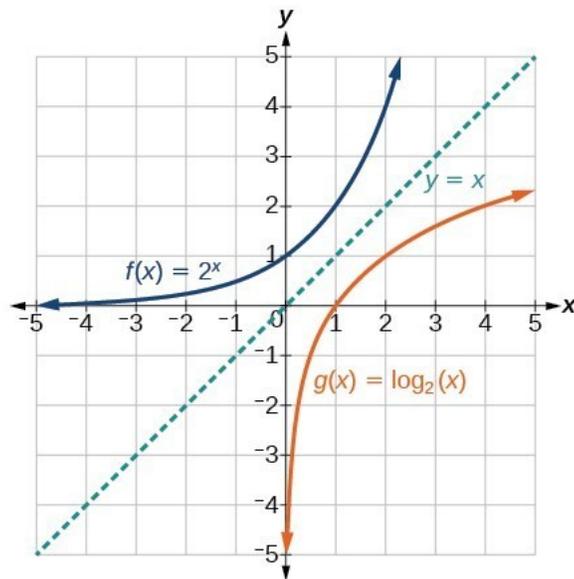
We begin with the parent function  $y = \log_b(x)$ . Because every logarithmic function of this form is the inverse of an exponential function of the form  $y = b^x$ , their graphs will be reflections of each other across the line  $y = x$ . To illustrate this, we can observe the relationship between the input and output values of  $y = 2^x$  and its equivalent  $x = \log_2(y)$  in the table below.

$x$	-3	-2	-1	0	1	2	3
$2^x = y$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$\log_2(y) = x$	-3	-2	-1	0	1	2	3

Using the inputs and outputs from the table above, we can build another table to observe the relationship between points on the graphs of the inverse functions  $f(x) = 2^x$  and  $g(x) = \log_2(x)$ .

$f(x)$ $= 2^x$	$(-3, \frac{1}{8})$	$(-2, \frac{1}{4})$	$(-1, \frac{1}{2})$	(0, 1)	(1, 2)	(2, 4)	(3, 8)
$g(x)$ $= \log_2(x)$	$(\frac{1}{8}, -3)$	$(\frac{1}{4}, -2)$	$(\frac{1}{2}, -1)$	(1, 0)	(2, 1)	(4, 2)	(8, 3)

As we would expect, the  $x$ - and  $y$ -coordinates are reversed for the inverse functions. The figure below shows the graph of  $f$  and  $g$ .



Notice that the graphs of  $f(x) = 2^x$  and  $g(x) = \log_2(x)$  are reflections about the line  $y = x$ .

Observe the following from the graph:

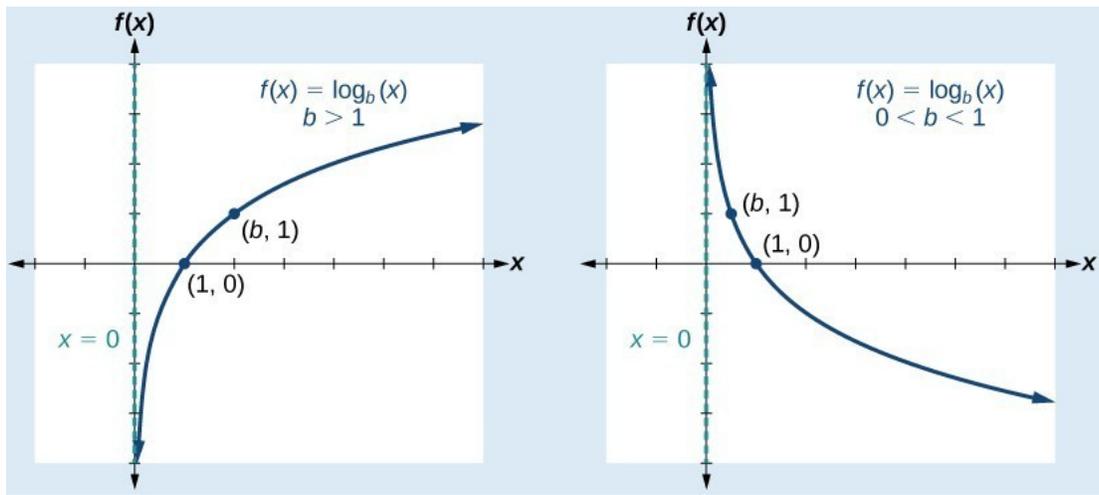
- $f(x) = 2^x$  has a  $y$ -intercept at  $(0, 1)$  and  $g(x) = \log_2(x)$  has an  $x$ -intercept at  $(1, 0)$ .
- The domain of  $f(x) = 2^x$ ,  $(-\infty, \infty)$ , is the same as the range of  $g(x) = \log_2(x)$ .
- The range of  $f(x) = 2^x$ ,  $(0, \infty)$ , is the same as the domain of  $g(x) = \log_2(x)$ .

## A General Note: Characteristics of the Graph of the Parent Function, $f(x) = \log_b(x)$

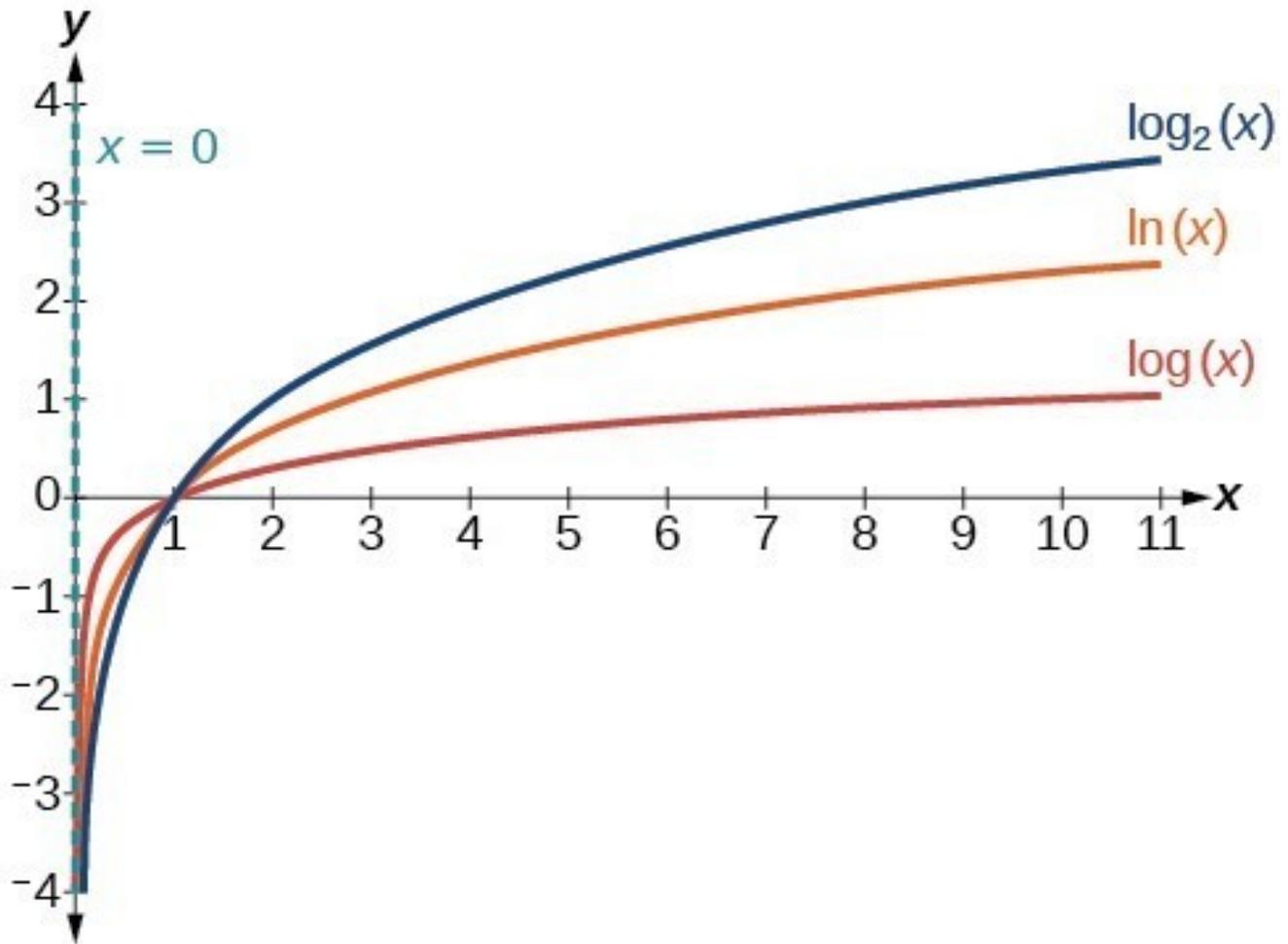
For any real number  $x$  and constant  $b > 0$ ,  $b \neq 1$ , we can see the following characteristics in the graph of  $f(x) = \log_b(x)$ :

- one-to-one function
- vertical asymptote:  $x = 0$
- domain:  $(0, \infty)$
- range:  $(-\infty, \infty)$
- $x$ -intercept:  $(1, 0)$  and key point  $(b, 1)$
- $y$ -intercept: none
- increasing if  $b > 1$
- decreasing if  $0 < b < 1$

The figure below shows how changing the base  $b$  in  $f(x) = \log_b(x)$  can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (*Note*: recall that the function  $\ln(x)$  has base  $e \approx 2.718$ .)



The graph of  $f(x) = \log_b(x)$  when  $b > 1$  and  $0 < b < 1$ .



The graphs of three logarithmic functions with different bases, all greater than 1.

In our next example, we will graph a logarithmic function of the form  $f(x) = \log_b(x)$ .

## Summary

To define the domain of a logarithmic function algebraically, set the argument greater than zero and solve. To plot a logarithmic function, it is easiest to find and plot the x-intercept and the key point  $(b, 1)$ .

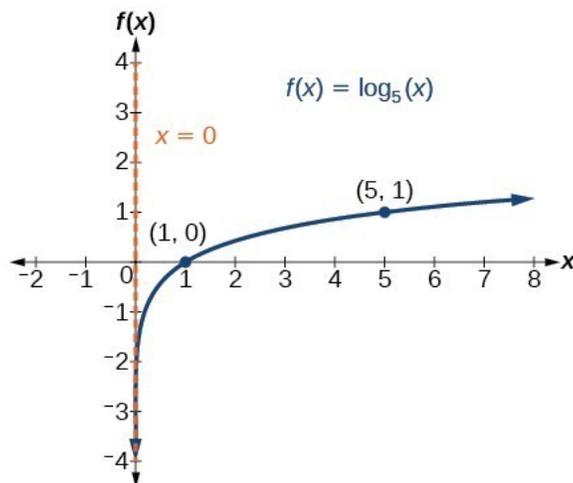
## EXAMPLE

Graph  $f(x) = \log_5(x)$ . State the domain and range.

### Answer

Before graphing, identify the behavior and key points for the graph.

- Since  $b = 5$  is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical line  $x = 0$ , and the right tail will increase slowly without bound.
- The  $x$ -intercept is  $(1, 0)$ .
- The key point  $(5, 1)$  is on the graph.
- We plot and label the points, and draw a smooth curve through the points.
- The domain is  $(0, \infty)$ , and the range is  $(-\infty, \infty)$ .



## HOW TO: GIVEN A LOGARITHMIC FUNCTION OF THE FORM $f(x) = \log_b(x)$ , GRAPH THE FUNCTION

1. Plot the  $x$ -intercept,  $(1, 0)$ .
2. Plot the key point  $(b, 1)$ .
3. Draw a smooth curve through the points.
4. State the domain,  $(0, \infty)$ , and the range,  $(-\infty, \infty)$ .

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# PUTTING IT TOGETHER: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

# Earthquake!

Joan's grandpa comes over for dinner every Sunday. Grandpa loves to talk, and it is really hard to understand him, and as he gets older, his hearing has been getting worse. Usually Joan tries to find an excuse to be out after dinner on Sundays, but since she had food poisoning on Saturday night, this week she was not feeling like leaving the house.

Grandpa was feeling extra chatty this week and since Joan was a captive audience, she got the full force of his attention. His stories started with the neighbor's dog who would not stop barking and then on to his arthritis.

"Joanie, do you know that I think I have arthritis in my ankles because of an earthquake injury?" says grandpa.

Grandpa went on to tell Joan the story of when he lived in Anchorage, Alaska in the 60's and lived through the Good Friday earthquake.



*A 2x6 plank driven through a ten-ply tire by the tsunami in Whittier, AK.*

The 1964 Alaskan earthquake, also known as the **Great Alaskan earthquake**, occurred at 5 : 36 p.m. AST on Good Friday, March 27. Across south-central Alaska, ground fissures, collapsing structures, and earthquake-created tsunamis caused about 139 deaths, some of them being friends of Joan's grandpa.

Lasting four minutes and thirty-eight seconds, the magnitude 8.5 (Richter scale) (Note: "Facts About the 1964 Alaska Earthquake." LiveScience. Accessed August 18, 2016. <http://www.livescience.com/44412-1964-alaska-earthquake-facts.html>.) megathrust earthquake was the most powerful recorded in North American history and the second most powerful recorded in world history. Soil liquefaction, fissures, landslides, and other ground failures caused major structural damage in several communities and much damage to property. Post-quake tsunamis severely affected numerous Alaskan communities as well as people and property in British Columbia, Washington, Oregon, and California. Tsunamis also caused damage in Hawaii and Japan. Evidence of motion directly related to the earthquake was reported from all over the world.

Surviving the quake inspired Joan's grandpa to study Geology in college. Joan had heard of the Richter Scale before, but did not really understand what it meant, so she asked her grandpa.

He explained that the Richter Scale magnitude is expressed in whole numbers and decimal fractions. For example, a magnitude 5.3 might be computed for a moderate earthquake, and a strong earthquake might be rated as magnitude 6.3. Because of the logarithmic basis of the scale, he said, each whole number increase in magnitude represents a tenfold increase in measured amplitude. For example, grandpa explained, a magnitude 6.3 earthquake corresponds to the release of about 31 times more energy than a magnitude 5.3. (Note: "Measuring the Size of an Earthquake." Measuring the Size of an Earthquake. Accessed August 05, 2016. <http://earthquake.usgs.gov/learn/topics/measure.php>.)



*Charles Richter*

Joan really enjoyed the conversation she had with her grandpa; she had never asked him about his work as a geologist, and she could tell it made him really happy that she did. At least something good came from Joan's disastrous date and food poisoning fiasco from the night before.

"1964 Quake: The Great Alaska Earthquake" is an eleven minute video created as part of USGS activities acknowledging the fifty year anniversary of the quake on March 27, 2014. The video features USGS geologist George Plafker, who, in the 1960s, correctly interpreted the quake as a subduction zone event. This was a great leap forward in resolving key mechanisms of the developing theory of plate tectonics. The extreme loss of life and destruction from this earthquake and accompanying tsunamis inspired the NOAA Tsunami Warning Centers and the USGS Earthquake Hazards Program.



A YouTube element has been excluded from this version of the text. You can view it online here:  
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# MODULE 14: EXPONENTIAL AND LOGARITHMIC EQUATIONS

## WHY IT MATTERS: EXPONENTIAL AND LOGARITHMIC EQUATIONS

### Why Learn About Exponential and Logarithmic Equations?

Joan enjoyed talking to her grandfather so much, that the next Sunday, she made sure she she would be there when he came to dinner. This time, though, she prepared a question for her grandfather. She had been learning about logarithmic and exponential functions in her math class and wanted to see if she could stump her grandfather with a math question.

One of the questions in Joan's homework on exponential and logarithmic functions had been about how to calculate the Richter scale measure of the magnitude of an earthquake. The following formula was given in her book:

$$R = \log \left( \frac{A}{A_0} \right)$$

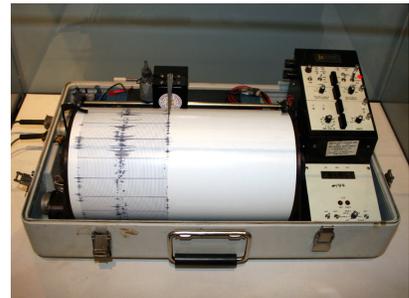
The question provided her with the quantity  $\frac{A}{A_0}$  and asked her to solve for R.  $A_0$  is a baseline measure of ground movement as detected by a seismometer. When there is an earthquake, wave amplitudes recorded by seismometers are expressed relative to this baseline. For example, Joan's book asked the following question:

An earthquake is measured with a wave amplitude 392 times as great as  $A_0$ . What is the magnitude of this earthquake, to the nearest tenth, using the Richter scale?

Joan hoped to give her grandfather the Richter scale magnitude for the Alaska quake, 8.5, and see if he could find how much greater the wave amplitude of that quake was than the baseline,  $A_0$ .

In this module, you will learn how to solve problems such as the one Joan is planning to use to try to stump her grandfather. We will come back to Joan and her grandfather at the end of this module to see if she was able to ask him a question he did not know how to answer.

You will learn about the properties of exponential and logarithmic functions in this module the same way that you learned about the properties of exponents. You will use the properties of logarithms and exponentials to solve equations that involve them.



*A Kinematics Seismograph formerly used by the United States Department of the Interior.*

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# INTRODUCTION TO LOGARITHMIC PROPERTIES

When you learned how to solve linear equations, you were likely introduced to the properties of real numbers. These properties help us know what the rules are for isolating and combining numbers and variables. For example, it is advantageous to know that multiplication and division “undo” each other when you want to solve an equation for a variable that is multiplied by a number. You may have also been introduced to properties and rules for writing and using exponents. In this section, we will show you properties of logarithms and how they can help you better understand what a logarithm means and eventually how to solve equations that contain them.

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# PRODUCT RULE FOR LOGARITHMS

## LEARNING OUTCOMES

- Define properties of logarithms, and use them to solve equations
- Define and use the product rule for logarithms

Recall that we can express the relationship between logarithmic form and its corresponding exponential form as follows:

$$\log_b(x) = y \Leftrightarrow b^y = x, b > 0, b \neq 1$$

Note that the base  $b$  is always positive and that the logarithmic and exponential functions “undo” each other. This means that logarithms have properties similar to exponents. Some important properties of logarithms are given in this section.

First, we will introduce some basic properties of logarithms followed by examples with integer arguments to help you get familiar with the relationship between exponents and logarithms.

Exponential and logarithmic functions are inverses of each other, and we can take advantage of this to evaluate and solve expressions and equations involving logarithms and exponentials. The inverse property

## ZERO AND IDENTITY EXPONENT RULE FOR LOGARITHMS AND EXPONENTIALS

$$\log_b 1 = 0, b > 0$$

$$\log_b b = 1, b > 0$$

### EXAMPLE

Use the the fact that exponentials and logarithms are inverses to prove the zero and identity exponent rule for the following:

1.  $\log_5 1 = 0$

2.  $\log_5 5 = 1$

**Answer**

1.  $\log_5 1 = 0$  since  $5^0 = 1$

2.  $\log_5 5 = 1$  since  $5^1 = 5$

of logarithms and exponentials gives us an explicit way to rewrite an exponential as a logarithm or a logarithm as an exponential.

## INVERSE PROPERTY OF LOGARITHMS AND EXPONENTIALS

$$\log_b (b^x) = x$$

$$b^{\log_b x} = x, x > 0, b > 0, b \neq 1$$

### EXAMPLE

Evaluate:

1.  $\log (100)$

2.  $e^{\ln(7)}$

**Answer**

1. Rewrite the logarithm as  $\log_{10} (10^2)$ , and then apply the inverse property  $\log_b (b^x) = x$  to get  $\log_{10} (10^2) = 2$ .

2. Rewrite the logarithm as  $e^{\log_e 7}$ , and then apply the inverse property  $b^{\log_b x} = x$  to get  $e^{\log_e 7} = 7$

Another property that can help us simplify logarithms is the one-to-one property. Essentially, this property states that if two logarithms have the same base, then their arguments – the stuff inside – are also equal to each other.

## THE ONE-TO-ONE PROPERTY OF LOGARITHMS

$$\log_b M = \log_b N \text{ if and only if } M = N$$

What if we had started with the equation  $\log_3 (3x) + \log_3 (2x + 5) = 2$ ? The one-to-one property does not

## EXAMPLE

Solve the equation  $\log_3(3x) = \log_3(2x + 5)$  for  $x$ .

### Answer

In order for this equation to be true, we must find a value for  $x$  such that  $3x = 2x + 5$

$3x = 2x + 5$  Set the arguments equal to each other.

$x = 5$  Subtract  $2x$ .

Check your answer by substituting 5 for  $x$ .

$$\log_3(3 \cdot 5) = \log_3(2 \cdot 5 + 5)$$

$$\log_3(15) = \log_3(15)$$

This is a true statement, so we must have found the correct value for  $x$ .

help us in this instance. Before we can solve an equation like this, we need a method for combining terms on the left side of the equation.

To recap, the properties of logarithms and exponentials that can help us understand, simplify, and solve these types of functions more easily include:

- Zero and Identity Exponent Rule:  $\log_b 1 = 0$ ,  $b > 0$  and  $\log_b b = 1$ ,  $b > 0$
- Inverse Property:

$$\log_b(b^x) = x$$

$$b^{\log_b x} = x, x > 0$$

- One-To-One Property:  $\log_b M = \log_b N$  if and only if  $M = N$

## The Product Rule for Logarithms

Recall that we use the *product rule of exponents* to combine the product of like bases raised to exponents by adding the exponents:  $x^a x^b = x^{a+b}$ . We have a similar property for logarithms, called the **product rule for logarithms**, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

### THE PRODUCT RULE FOR LOGARITHMS

The **product rule for logarithms** can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N) \text{ for } b > 0$$

Let  $m = \log_b M$  and  $n = \log_b N$ . In exponential form, these equations are  $b^m = M$  and  $b^n = N$ . It follows that

$$\begin{aligned} \log_b(MN) &= \log_b(b^m b^n) && \text{Substitute for } M \text{ and } N. \\ &= \log_b(b^{m+n}) && \text{Apply the product rule for exponents.} \\ &= m + n && \text{Apply the inverse property of logs.} \end{aligned}$$

$$= \log_b(M) + \log_b(N) \quad \text{Substitute for } m \text{ and } n.$$

Repeated applications of the product rule for logarithms allow us to simplify the logarithm of the product of any number of factors. Consider the following example:

### EXAMPLE

Using the product rule for logarithms, rewrite the logarithm of a product as the sum of logarithms of its factors.

$$\log_b(wxyz)$$

**Answer**

$$\log_b(wxyz) = \log_b w + \log_b x + \log_b y + \log_b z$$

In our next example, we will first factor the argument of a logarithm before expanding it with the product rule.

### EXAMPLE

Expand  $\log_3(30x(3x + 4))$ .

**Answer**

We begin by factoring the argument completely, expressing 30 as a product of primes.

$$\log_3(30x(3x + 4)) = \log_3(2 \cdot 3 \cdot 5 \cdot x \cdot (3x + 4))$$

Next we write the equivalent equation by summing the logarithm of each factor.

$$\log_3(30x(3x + 4)) = \log_3(2) + \log_3(3) + \log_3(5) + \log_3(x) + \log_3(3x + 4)$$

## Analysis of the Solution

It is tempting to use the distributive property when you see an expression like  $(30x(3x + 4))$ , but in this case, it is better to leave the argument of this logarithm as a product since you can then use the product rule for logarithms to simplify the expression.

The following video provides more examples of using the product rule to expand logarithms.

**EXAMPLE**

Use the product rule for logarithms to rewrite the logarithm of a product as the sum of logarithms of its factors.

- $\log_b(2xyz)$
- $\log_2(15x(7x + 2))$

Recall:  $\log_b(x) = y \iff b^y = x, b > 0, b \neq 1$

**THE PRODUCT RULE FOR LOGARITHMS**

$$\log_b(MN) = \log_b(M) + \log_b(N), b > 0$$

1.  $\log_b(2xyz) = \log_b(2 \cdot x \cdot y \cdot z)$   
 $= \log_b(2) + \log_b(x) + \log_b(y) + \log_b(z)$

2.  $\log_2(15x(7x+2)) = \log_2(3 \cdot 5 \cdot x \cdot (7x+2))$

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## Summary

Logarithms have properties that can help us simplify and solve expressions and equations that contain logarithms. Exponentials and logarithms are inverses of each other; therefore, we can define the product rule for logarithms. We can use this to simplify or solve expressions with logarithms. Given the logarithm of a product, use the product rule of logarithms to write an equivalent sum of logarithms as follows:

1. Factor the argument completely, expressing each whole number factor as a product of primes.
2. Write the equivalent expression by summing the logarithms of each factor.

## QUOTIENT AND POWER RULES FOR LOGARITHMS

### LEARNING OUTCOME

- Define and use the quotient and power rules for logarithms

For quotients, we have a similar rule for logarithms. Recall that we use the *quotient rule of exponents* to simplify division of like bases raised to powers by subtracting the exponents:  $\frac{x^a}{x^b} = x^{a-b}$ . The **quotient rule for logarithms** says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

### THE QUOTIENT RULE FOR LOGARITHMS

The **quotient rule for logarithms** can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b \left( \frac{M}{N} \right) = \log_b M - \log_b N$$

We can show  $\log_b \left( \frac{M}{N} \right) = \log_b (M) - \log_b (N)$ .

Given positive real numbers  $M$ ,  $N$ , and  $b$ , where  $b > 0$  we will show

$$\log_b \left( \frac{M}{N} \right) = \log_b (M) - \log_b (N).$$

Let  $m = \log_b M$  and  $n = \log_b N$ . In exponential form, these equations are  $b^m = M$  and  $b^n = N$ . It follows that

$$\begin{aligned} \log_b \left( \frac{M}{N} \right) &= \log_b \left( \frac{b^m}{b^n} \right) && \text{Substitute for } M \text{ and } N. \\ &= \log_b (b^{m-n}) && \text{Apply the quotient rule for exponents.} \\ &= m - n && \text{Apply the inverse property of logs.} \\ &= \log_b (M) - \log_b (N) && \text{Substitute for } m \text{ and } n. \end{aligned}$$

In the previous example, it was helpful to first factor the numerator and denominator and divide common terms. This gave us a simpler expression to use to write an equivalent expression. It is important to remember to subtract the logarithm of the denominator from the logarithm of the numerator. Always check to see if you need to expand further with the product rule.

### Analysis of the Solution

## EXAMPLE

Expand the following expression using the quotient rule for logarithms.

$$\log\left(\frac{2x^2+6x}{3x+9}\right)$$

### Answer

Factoring and canceling, we get:

$$\begin{aligned}\log\left(\frac{2x^2+6x}{3x+9}\right) &= \log\left(\frac{2x(x+3)}{3(x+3)}\right) && \text{Factor the numerator and denominator.} \\ &= \log\left(\frac{2x}{3}\right) && \text{Cancel the common factors.}\end{aligned}$$

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.

$$\begin{aligned}\log\left(\frac{2x}{3}\right) &= \log(2x) - \log(3) \\ &= \log(2) + \log(x) - \log(3)\end{aligned}$$

## EXAMPLE

Expand  $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$ .

### Answer

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.

$$\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right) = \log_2(15x(x-1)) - \log_2((3x+4)(2-x))$$

Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule, noting that the prime factors of the factor 15 are 3 and 5

$$\begin{aligned}\log_2(15x(x-1)) - \log_2((3x+4)(2-x)) &= [\log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1)] \\ &\quad - [\log_2(3x+4) + \log_2(2-x)] \\ &= \log_2(3) + \log_2(5) + \log_2(x) + \log_2(x-1) \\ &\quad - \log_2(3x+4) - \log_2(2-x)\end{aligned}$$

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for  $x = -\frac{4}{3}$  and  $x = 2$ . Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm, that  $x > 0$ ,  $x > 1$ ,  $x > -\frac{4}{3}$ , and  $x < 2$ . Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

In the following video, we show more examples of using the quotient rule for logarithms.

## Using the Power Rule for Logarithms

We have explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as  $x^2$ ? One method is as follows:

**EXAMPLE**

Expand the following expression using properties of logarithms.

$$\log \left( \frac{3x^2 + 9x}{2x + 6} \right)$$

**PROPERTIES OF LOGARITHMS**  
 $\log_b(MN) = \log_b(M) + \log_b(N)$ ,  $b > 0$   
 $\log_b \left( \frac{M}{N} \right) = \log_b(M) - \log_b(N)$ ,  $b > 0$

$\log \left( \frac{3x^2 + 9x}{2x + 6} \right) = \log \left( \frac{3x(x+3)}{2(x+3)} \right) = \log \left( \frac{3x}{2} \right)$   
 $= \log(3x) - \log 2$

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$$\begin{aligned} \log_b(x^2) &= \log_b(x \cdot x) \\ &= \log_b x + \log_b x \\ &= 2\log_b x \end{aligned}$$

Notice that we used the **product rule for logarithms** to simplify the example above. By doing so, we have derived the **power rule for logarithms** which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that, although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,

$$100 = 10^2 \quad \sqrt{3} = 3^{\frac{1}{2}} \quad \frac{1}{e} = e^{-1}$$

## THE POWER RULE FOR LOGARITHMS

The **power rule for logarithms** can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b M$$

### EXAMPLE

Expand  $\log_2 x^5$ .

**Answer**

The argument is already written as a power, so we identify the exponent, 5, and the base,  $x$ , and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_2(x^5) = 5\log_2 x$$

The power rule for logarithms is possible because we can use the product rule and combine like terms. In the next example, you will see that we can also rewrite an expression as a power in order to use the power rule.

Now let us use the power rule in reverse.

## Summary

## EXAMPLE

Expand  $\log_3(25)$  using the power rule for logs.

### Answer

Expressing the argument as a power, we get  $\log_3(25) = \log_3(5^2)$ .

Next we identify the exponent, 2, and the base, 5, and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.

$$\log_3(5^2) = 2\log_3(5)$$

## EXAMPLE

Rewrite  $4\ln(x)$  using the power rule for logs to a single logarithm with a leading coefficient of 1.

### Answer

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression  $4\ln(x)$ , we identify the factor, 4, as the exponent and the argument,  $x$ , as the base, and rewrite the product as a logarithm of a power:

$$4\ln(x) = \ln(x^4).$$

You can use the quotient rule of logarithms to write an equivalent difference of logarithms in the following way:

1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

To use the power rule of logarithms to write an equivalent product of a factor and a logarithm, consider the following:

1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

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# EXPAND AND CONDENSE LOGARITHMS

## LEARNING OUTCOMES

- Combine the product, power, and quotient rules to expand logarithmic expressions
- Combine the product, power, and quotient rules to condense logarithmic expressions

Taken together, the product rule, quotient rule, and power rule are often called “laws of logs.” Sometimes we apply more than one rule in order to simplify an expression. For example:

$$\begin{aligned}\log_b \left( \frac{6x}{y} \right) &= \log_b (6x) - \log_b y \\ &= \log_b 6 + \log_b x - \log_b y\end{aligned}$$

We can also use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal (fraction) has a negative power:

$$\begin{aligned}\log_b \left( \frac{A}{C} \right) &= \log_b (AC^{-1}) \\ &= \log_b (A) + \log_b (C^{-1}) \\ &= \log_b A + (-1) \log_b C \\ &= \log_b A - \log_b C\end{aligned}$$

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.



Remember that we can only apply laws of logs to products, quotients, powers, and roots—never with addition or subtraction inside the argument of the logarithm. Consider the following example

$$\begin{aligned}\log (10 + 100) &\stackrel{?}{=} \log (10) + \log (100) \\ \log (110) &\stackrel{?}{=} 1 + 2 \\ 2.04 &\neq 3\end{aligned}$$

Be careful to only apply the product rule when a logarithm has an argument that is a product or when you have a sum of logarithms.

In our first example, we will show that a logarithmic expression can be expanded by combining several of the rules of logarithms.

### EXAMPLE

Rewrite  $\ln \left( \frac{x^4 y}{7} \right)$  as a sum or difference of logs.

**Answer**

First, because we have a quotient of two expressions, we can use the quotient rule:

$$\ln \left( \frac{x^4 y}{7} \right) = \ln (x^4 y) - \ln (7)$$

Then seeing the product in the first term, we use the product rule:

$$\ln (x^4 y) - \ln (7) = \ln (x^4) + \ln (y) - \ln (7)$$

Finally, we use the power rule on the first term:

$$\ln (x^4) + \ln (y) - \ln (7) = 4 \ln (x) + \ln (y) - \ln (7)$$

We can also use the rules for logarithms to simplify the logarithm of a radical expression.

Let us do one more example with an expression that contains several different mathematical operations.

In the following video, we show another example of expanding logarithms.

## Condense Logarithms

## EXAMPLE

Expand  $\log(\sqrt{x})$ .

**Answer**

$$\begin{aligned}\log(\sqrt{x}) &= \log x^{\left(\frac{1}{2}\right)} \\ &= \frac{1}{2}\log x\end{aligned}$$

## THINK ABOUT IT

Can we expand  $\ln(x^2 + y^2)$ ? Use the space below to develop an argument one way or the other before you look at the solution.

**Answer**

No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm. Rewrite the expression as an equation and express it as an exponential to give yourself some proof.

$$m = \ln(x^2 + y^2)$$

If you rewrite this as an exponential, you get:

$$e^m = x^2 + y^2$$

From here, there is not much more you can do to make this expression more simple.

## EXAMPLE

Expand  $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$ .

**Answer**

We can expand by applying the Product and Quotient Rules.

$$\begin{aligned}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) &= \log_6 64 + \log_6 x^3 && \text{Apply the Quotient and Product Rules.} \\ &+ \log_6(4x+1) - \log_6(2x-1) \\ &= \log_6 2^6 + \log_6 x^3 + \log_6(4x+1) && \text{Simplify by writing 64 as } 2^6. \\ &- \log_6(2x-1) \\ &= 6\log_6 2 + 3\log_6 x && \text{Apply the Power Rule.} \\ &+ \log_6(4x+1) - \log_6(2x-1)\end{aligned}$$

We can use the rules of logarithms we just learned to condense sums and differences with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

In our next example, we show how to simplify a more complex logarithm by condensing it.

More examples of condensing logarithms are in the following video.

## Summary

Given a sum or difference of logarithms with the same base, we can write an equivalent expression as a

**EXAMPLE**

Expand the following expression using properties of logarithms.

$$\log_3 \left( \frac{2x(x-3)}{(2x-1)(x+4)} \right)$$

PROPERTIES OF LOGARITHMS ( $b > 0$ )

$$\log_b(MN) = \log_b(M) + \log_b(N)$$

$$\log_b \left( \frac{M}{N} \right) = \log_b(M) - \log_b(N)$$

$$\log_b(M^a) = a \log_b(M)$$

Handwritten solution:

$$\log_3 \left( \frac{2x(x-3)}{(2x-1)(x+4)} \right) = \log_3(2x(x-3)) - \log_3((2x-1)(x+4))$$

$$= \log_3(2) + \log_3(x) + \log_3(x-3) - (\log_3(2x-1) + \log_3(x+4))$$

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### EXAMPLE

Write  $\log_3(5) + \log_3(8) - \log_3(2)$  as a single logarithm.

**Answer**

Using the product and quotient rules:

$$\log_3(5) + \log_3(8) = \log_3(5 \cdot 8) = \log_3(40)$$

This simplifies our original expression to:

$$\log_3(40) - \log_3(2)$$

Then, using the quotient rule:

$$\log_3(40) - \log_3(2) = \log_3\left(\frac{40}{2}\right) = \log_3(20)$$

### EXAMPLE

Condense  $\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2)$ .

**Answer**

We apply the power rule first:

$$\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2) = \log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6)$$

Next we apply the product rule to the sum:

$$\log_2(x^2) + \log_2(\sqrt{x-1}) - \log_2((x+3)^6) = \log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6)$$

Finally, we apply the quotient rule to the difference:

$$\log_2(x^2\sqrt{x-1}) - \log_2((x+3)^6) = \log_2\frac{x^2\sqrt{x-1}}{(x+3)^6}$$

single logarithm.

1. Apply the power property first. Identify terms that are products of factors and a logarithm, and rewrite each as the logarithm of a power.
2. Next apply the product property. Rewrite sums of logarithms as the logarithm of a product.
3. Apply the quotient property last. Rewrite differences of logarithms as the logarithm of a quotient.

**EXAMPLE**

Write each logarithmic expression as a single logarithm.

1.  $\log_7(6) + \log_7(8) - \log_7(12)$

2.  $\log_2(2x) + \frac{1}{2}\log_2(x+1) - 2\log_2((x+2)^2)$

1.  $\log_7(6) + \log_7(8) - \log_7(12)$   
 $= \log_7(6 \cdot 8) - \log_7(12)$   
 $= \log_7(48) - \log_7(12)$   
 $= \log_7\left(\frac{48}{12}\right)$   
 $= \log_7(4)$

PROPERTIES OF LOGARITHMS ( $b > 0$ )  
 $\log_b(MN) = \log_b(M) + \log_b(N)$   
 $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$   
 $\log_b(M^a) = a\log_b(M)$

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## CHANGE OF BASE

### LEARNING OUTCOME

- Change the base of logarithmic expressions into base 10 or base  $e$

Most calculators can evaluate only common and natural logs. In order to evaluate logarithms with a base other than 10 or  $e$ , we use the **change-of-base formula** to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the **one-to-one** property and **power rule for logarithms**.

Given any positive real numbers  $M$ ,  $b$ , and  $n$ , where  $n \neq 1$  and  $b \neq 1$ , we show  $\log_b M = \frac{\log_n M}{\log_n b}$ .

Let  $y = \log_b M$ . Rewriting in exponential form, we have  $b^y = M$ . We can then take the log base  $n$  of both sides of the equation. It follows that:

$$\log_n(b^y) = \log_n M \quad \text{Apply the one-to-one property.}$$

$$y \log_n b = \log_n M \quad \text{Apply the power rule for logarithms.}$$

$$y = \frac{\log_n M}{\log_n b} \quad \text{Isolate } y.$$

$$\log_b M = \frac{\log_n M}{\log_n b} \quad \text{Substitute for } y.$$

For example, to evaluate  $\log_5 36$  using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.

$$\log_5 36 = \frac{\log(36)}{\log(5)} \quad \text{Apply the change of base formula using base 10.}$$
$$\approx 2.2266 \quad \text{Use a calculator to evaluate to 4 decimal places.}$$

Let us practice changing the base of a logarithmic expression from 5 to base  $e$ .

### EXAMPLE

Change  $\log_5 3$  to a quotient of natural logarithms.

**Answer**

Because we will be expressing  $\log_5 3$  as a quotient of natural logarithms, the new base  $n = e$ .

We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3. The denominator of the quotient will be the natural log with argument 5.

$$\log_b M = \frac{\ln M}{\ln b}$$

$$\log_5 3 = \frac{\ln 3}{\ln 5}$$

We can generalize the change of base formula in the following way:

### THE CHANGE-OF-BASE FORMULA

The **change-of-base formula** can be used to evaluate a logarithm with any base.

For any positive real numbers  $M$ ,  $b$ , and  $n$ , where  $n \neq 1$  and  $b \neq 1$ ,

$$\log_b M = \frac{\log_n M}{\log_n b}.$$

It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b M = \frac{\ln M}{\ln b}$$

and

$$\log_b M = \frac{\log M}{\log b}$$

As we stated earlier, the main reason for changing the base of a logarithm is to be able to evaluate it with a calculator. In the following example, we will use the change of base formula on a logarithmic expression, and then evaluate the result with a calculator.

In the following video, we show more examples using the change-of-base formula to evaluate logarithms.

## Summary

For practical purposes found in many different sciences or finance applications, you may want to evaluate a logarithm with a calculator. The change of base formula will allow you to change the base of any logarithm to either 10 or  $e$  so you can evaluate it with a calculator. Here we have summarized the steps for using the change of base formula to evaluate a logarithm with the form  $\log_b M$ .

1. Determine the new base  $n$ , remembering that the common log,  $\log(x)$ , has base 10, and the natural log,  $\ln(x)$ , has base  $e$ .
2. Rewrite the log as a quotient using the change-of-base formula:
  - The numerator of the quotient will be a logarithm with base  $n$  and argument  $M$ .

## EXAMPLE

Evaluate  $\log_2(10)$  using the change-of-base formula with a calculator.

### Answer

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm, which is the log base  $e$ .

$$\log_2 10 = \frac{\ln 10}{\ln 2} \quad \text{Apply the change of base formula using base } e.$$

$$\approx 3.3219 \quad \text{Use a calculator to evaluate to 4 decimal places.}$$

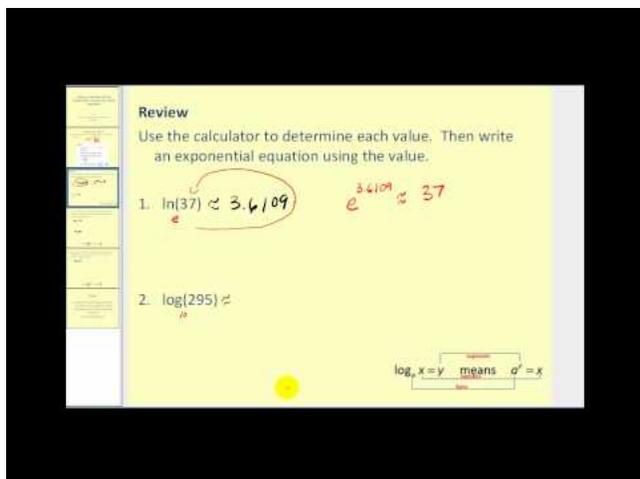
## THINK ABOUT IT

Can we change common logarithms to natural logarithms?

Write your ideas in the textbox below before looking at the solution.

### Answer

Yes. Remember that  $\log 9$  means  $\log_{10} 9$ . So,  $\log 9 = \frac{\ln 9}{\ln 10}$ .



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- The denominator of the quotient will be a logarithm with base  $n$  and argument  $b$ .

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# INTRODUCTION TO EXPONENTIAL AND LOGARITHMIC EQUATIONS

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.



*Wild rabbits in Australia. The rabbit population grew so quickly in Australia that the event became known as the "rabbit plague." (credit: Richard Taylor, Flickr)*

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section, we will learn techniques for solving exponential functions.

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## EXPONENTIAL EQUATIONS WITH LIKE BASES

### LEARNING OUTCOMES

- Identify an exponential equation whose terms all have the same base
- Identify cases where equations can be rewritten so all terms have the same base
- Apply the one-to-one property of exponents to solve an exponential equation

When an **exponential equation** has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then, we can set the exponents equal to one another and solve for the unknown.

For example, consider the equation  $3^{4x-7} = \frac{3^{2x}}{3}$ . To solve for  $x$ , we use the division property of exponents to rewrite the right side so that both sides have the common base, 3. Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for  $x$ :

$$3^{4x-7} = \frac{3^{2x}}{3}$$

$$3^{4x-7} = \frac{3^{2x}}{3^1} \quad \text{Rewrite 3 as } 3^1.$$

$$3^{4x-7} = 3^{2x-1} \quad \text{Use the division property of exponents.}$$

$$4x - 7 = 2x - 1 \quad \text{Apply the one-to-one property of exponents.}$$

$$2x = 6 \quad \text{Subtract } 2x \text{ and add 7 to both sides.}$$

$$x = 3 \quad \text{Divide by 2.}$$

In our first example, we solve an exponential equation whose terms all have a common base.

In general, we can summarize solving exponential equations whose terms all have the same base in this

## EXAMPLE

Solve  $2^{x-1} = 2^{2x-4}$ .

**Answer**

$2^{x-1} = 2^{2x-4}$  The common base is 2.

$x - 1 = 2x - 4$  By the one-to-one property the exponents must be equal.

$x = 3$  Solve for  $x$ .

way:

For any algebraic expressions  $S$  and  $T$ , and any positive real number  $b \neq 1$

$b^S = b^T$  if and only if  $S = T$

- Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form  $b^S = b^T$ .
- Use the one-to-one property to set the exponents equal to each other.
- Solve the resulting equation,  $S = T$ , for the unknown.

## Rewriting Equations So All Powers Have the Same Base

Sometimes we can rewrite the terms in an equation as powers with a common base and solve using the one-to-one property. This takes a keen eye for recognizing common powers. For example, you can rewrite 8 as  $2^3$  or 36 as  $6^2$  or  $\frac{1}{4}$  as  $(\frac{1}{2})^2$

Consider the equation  $256 = 4^{x-5}$ . We can rewrite both sides of this equation as a power of 2. Then we apply the rules of exponents, along with the one-to-one property, to solve for  $x$ :

$$256 = 4^{x-5}$$

$$2^8 = (2^2)^{x-5} \quad \text{Rewrite each side as a power with base 2.}$$

$$2^8 = 2^{2x-10} \quad \text{Use the power to a power property of exponents.}$$

$$8 = 2x - 10 \quad \text{Apply the one-to-one property of exponents.}$$

$$18 = 2x \quad \text{Add 10 to both sides.}$$

$$x = 9 \quad \text{Divide by 2.}$$

In the next example, we show how to find a common base for two expressions whose bases are 8 and 16. We can then solve the resulting equation using the one-to-one property of exponents.

In our next example, we are given an exponential equation that contains a square root. Remember that you can write roots as rational exponents, so you may be able to find like bases when it is not completely obvious at first.

By changing  $\sqrt{2}$  to  $2^{\frac{1}{2}}$ , we were able to solve the equation in the previous example. In general, here are some steps to consider when you are solving exponential equations. A good first step is always to determine whether you can rewrite the terms with a common base.

1. Rewrite each side of the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form  $b^S = b^T$ .
3. Use the one-to-one property to set the exponents equal to each other.
4. Solve the resulting equation,  $S = T$ , for the unknown.

In the following video, we show more examples of how to solve exponential equations by finding a common

## EXAMPLE

Solve  $8^{x+2} = 16^{x+1}$ .

**Answer**

$$8^{x+2} = 16^{x+1}$$

$$(2^3)^{x+2} = (2^4)^{x+1} \quad \text{Write 8 and 16 as powers of 2.}$$

$$2^{3x+6} = 2^{4x+4} \quad \text{To take a power of a power, multiply exponents.}$$

$$3x + 6 = 4x + 4 \quad \text{Use the one-to-one property to set the exponents equal to each other.}$$

$$x = 2 \quad \text{Solve for } x.$$

## EXAMPLE

Solve  $2^{5x} = \sqrt{2}$ .

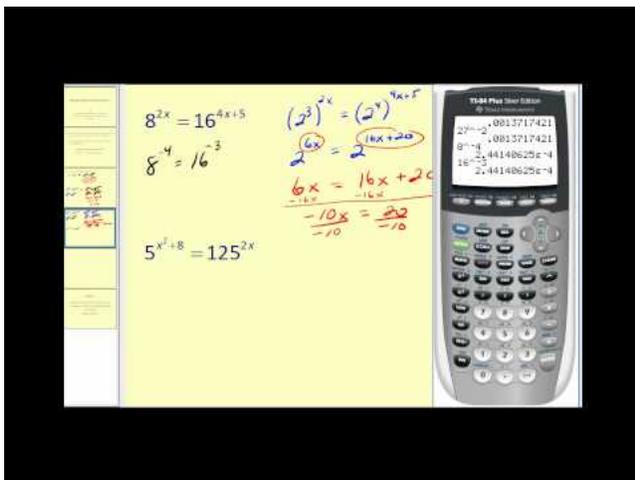
**Answer**

$$2^{5x} = 2^{\frac{1}{2}} \quad \text{Write the square root of 2 as a power of 2.}$$

$$5x = \frac{1}{2} \quad \text{Use the one-to-one property.}$$

$$x = \frac{1}{10} \quad \text{Solve for } x.$$

base.



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## THINK ABOUT IT

Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process? Write your thoughts in the textbox below before you check our proposed answer.

### Answer

No. Recall that the range of an exponential function is always positive. While solving the equation, we may obtain an expression that is undefined.

In the next example, we show you a case where there is no solution to an exponential equation. Remember how exponential functions are defined and ask yourself, “does this make sense”, before diving into solving exponential equations.

## EXAMPLE

Solve  $3^{x+1} = -2$ .

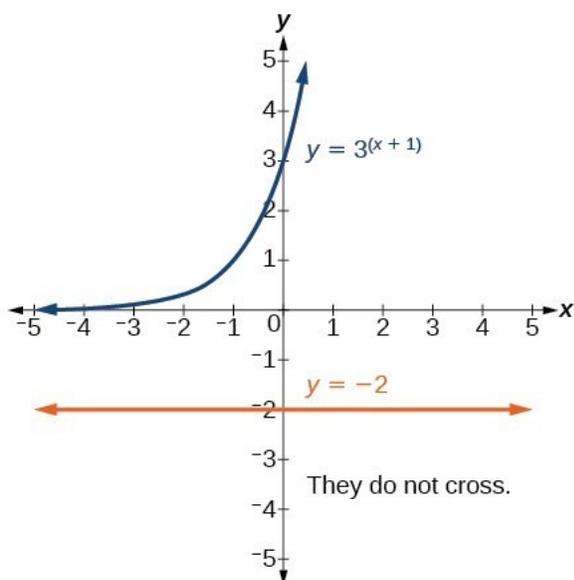
### Answer

This equation has no solution. There is no real value of  $x$  that will make the equation a true statement because any power of a positive number is positive.

For example  $3^2 = 9$  and  $2^4 = 16$ . Remember that we have defined exponential functions as having a base that is greater than 0.

## Analysis of the Solution

The figure below shows the graphs of the two separate expressions in the equation  $3^{x+1} = -2$  as  $y = 3^{x+1}$  and  $y = -2$ . The two graphs do not cross showing us that the left side is never equal to the right side. Thus the equation has no solution.



## Summary

We can use the one-to-one property of exponents to solve exponential equations whose bases are the same. The terms in some exponential equations can be rewritten with the same base, allowing us to use the same principle. There are exponential equations that do not have solutions because we define exponential functions as having a positive base. When restrictions are placed on the inputs of a function, it is natural that there will be restrictions on the output as well.

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# EXPONENTIAL EQUATIONS WITH UNLIKE BASES

## LEARNING OUTCOMES

- Use logarithms to solve exponential equations whose terms cannot be rewritten with the same base
- Solve exponential equations of the form  $y = Ae^{kt}$  for  $t$
- Recognize when there may be extraneous solutions or no solutions for exponential equations

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall, since  $\log(a) = \log(b)$  can be rewritten as  $a = b$ , we may apply logarithms with the same base on both sides of an exponential equation.

In our first example we will use the laws of logs combined with factoring to solve an exponential equation whose terms do not have the same base. Note how we rewrite the exponential terms as logarithms first.

## EXAMPLE

Solve  $5^{x+2} = 4^x$ .

**Answer**

There is no easy way to get the powers to have the same base for this equation.

$$\ln 5^{x+2} = \ln 4^x \quad \text{Take } \ln \text{ of both sides.}$$

$$(x+2)\ln 5 = x\ln 4 \quad \text{Use laws of logs.}$$

$$x\ln 5 + 2\ln 5 = x\ln 4 \quad \text{Use the distributive property.}$$

$$x\ln 5 - x\ln 4 = -2\ln 5 \quad \text{Get terms containing } x \text{ on one side and terms without } x \text{ on the other.}$$

$$x(\ln 5 - \ln 4) = -2\ln 5 \quad \text{On the left hand side, factor out an } x.$$

$$x\ln\left(\frac{5}{4}\right) = \ln\left(\frac{1}{25}\right) \quad \text{Use the laws of logs.}$$

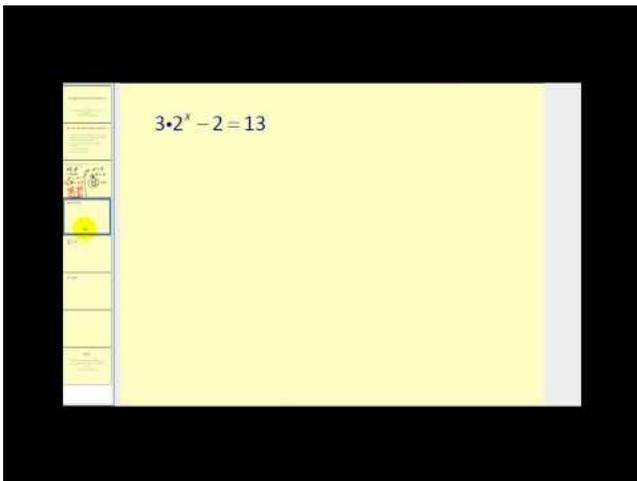
$$x = \frac{\ln\left(\frac{1}{25}\right)}{\ln\left(\frac{5}{4}\right)} \quad \text{Divide by the coefficient of } x.$$

In general we can solve exponential equations whose terms do not have like bases in the following way:

1. Apply the logarithm to both sides of the equation.
  - If one of the terms in the equation has base 10, use the common logarithm.
  - If none of the terms in the equation has base 10, use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

The following video provides more examples of solving exponential equations.

## Equations Containing $e$



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## THINK ABOUT IT

Is there any way to solve  $2^x = 3^x$ ?

Use the text box below to formulate an answer or example before you look at the solution.

### Answer

Yes. The solution is  $x = 0$ .

Base  $e$  is a very common base found in science, finance, and engineering applications. When we have an equation with a base  $e$  on either side, we can use the **natural logarithm** to solve it. Earlier, we introduced a formula that models continuous growth,  $y = Ae^{kt}$ . This formula is found in business, finance, and many biological and physical science applications. In our next example, we will show how to solve this equation for  $t$ , the elapsed time for the behavior in question.

## EXAMPLE

Solve  $100 = 20e^{2t}$ .

### Answer

$$100 = 20e^{2t}$$

$$5 = e^{2t} \quad \text{Divide by the coefficient of the base raised to a power.}$$

$$\ln 5 = 2t \quad \text{Take ln of both sides. Use the fact that ln} \\ (x) \text{ and } e^x \text{ are inverse functions.}$$

$$t = \frac{\ln 5}{2} \quad \text{Divide by the coefficient of } t.$$

## Analysis of the Solution

Using laws of logs, we can also write this answer in the form  $t = \ln \sqrt{5}$ . If we want a decimal approximation of the answer, we use a calculator.

## THINK ABOUT IT

Does every equation of the form  $y = Ae^{kt}$  have a solution? Write your thoughts or an example in the textbox below before you check the answer.

### Answer

No. There is a solution when  $k \neq 0$  and when  $y$  and  $A$  are either both 0 or neither 0 and they have the same sign. An example of an equation of this form that has no solution is  $2 = -3e^t$ .

We will provide one more example using the continuous growth formula, but this time we have to do a couple steps of algebra to get it in a form that can be solved.

## EXAMPLE

Solve  $4e^{2x} + 5 = 12$ .

### Answer

$$4e^{2x} + 5 = 12$$

$$4e^{2x} = 7 \quad \text{Combine like terms.}$$

$$e^{2x} = \frac{7}{4} \quad \text{Divide by the coefficient of the base raised to a power.}$$

$$2x = \ln\left(\frac{7}{4}\right) \quad \text{Take ln of both sides.}$$

$$x = \frac{1}{2}\ln\left(\frac{7}{4}\right) \quad \text{Solve for } x.$$

## Extraneous Solutions

Sometimes the methods used to solve an equation introduce an **extraneous solution**, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises when the logarithm is taken on both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is negative, there is no output.

In the next example, we will solve an exponential equation that is quadratic in form. We will factor first and then use the zero product principle. Note how we find two solutions but reject one that does not satisfy the original equation.

## Analysis of the Solution

When we plan to use factoring to solve a problem, we always get zero on one side of the equation, because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation  $e^x = -7$  because a positive number never equals a negative number. The solution  $x = \ln(-7)$  is not a real number, and in the real number system, this solution is rejected as an extraneous solution.

## Summary

The inverse operation of exponentiation is the logarithm, so we can use logarithms to solve exponential equations whose terms do not have the same bases. This is similar to using multiplication to “undo” division or addition to “undo” subtraction. It is important to check exponential equations for extraneous solutions or no solutions.

## EXAMPLE

Solve  $e^{2x} - e^x = 56$ .

**Answer**

$$e^{2x} - e^x = 56$$

$$e^{2x} - e^x - 56 = 0 \quad \text{Get one side of the equation equal to zero.}$$

$$(e^x + 7)(e^x - 8) = 0 \quad \text{Factor by the FOIL method.}$$

$$e^x + 7 = 0 \text{ or } e^x - 8 = 0 \quad \text{If a product is zero, then one factor must be zero.}$$

$$e^x = -7 \text{ or } e^x = 8 \quad \text{Isolate the exponentials.}$$

$$e^x = 8 \quad \text{Reject the equation in which the power equals a negative number.}$$

$$x = \ln 8 \quad \text{Solve the equation by taking the natural log of both sides.}$$

## THINK ABOUT IT

Does every logarithmic equation have a solution? Write your ideas, or a counter example, in the box below.

**Answer**

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.

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# LOGARITHMIC EQUATIONS

## LEARNING OUTCOMES

- Use the definition of a logarithm to solve logarithmic equations
- Use the rules of logarithms and one-to-one property of logarithms to solve a logarithmic equation

We have already seen that every **logarithmic equation**  $\log_b(x) = y$  can be written as the exponential equation  $b^y = x$ . We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation  $\log_2(2) + \log_2(3x - 5) = 3$ . To solve this equation, we can use rules of logarithms to rewrite the left side in compact form and then apply the definition of logs to solve for  $x$ :

$$\log_2(2) + \log_2(3x - 5) = 3$$

$\log_2(2(3x - 5)) = 3$	Apply the product rule of logarithms.
$\log_2(6x - 10) = 3$	Distribute.
$2^3 = 6x - 10$	Apply the definition of a logarithm.
$8 = 6x - 10$	Calculate $2^3$ .
$18 = 6x$	Add 10 to both sides.
$x = 3$	Divide by 6.

In our first example, we will show how to use techniques from solving linear equations to solve a logarithmic equation.

### EXAMPLE

Solve  $2\ln x + 3 = 7$ .

**Answer**

$$2\ln x + 3 = 7$$

$$2\ln x = 4 \quad \text{Subtract 3.}$$

$$\ln x = 2 \quad \text{Divide by 2.}$$

$$x = e^2 \quad \text{Rewrite in exponential form.}$$

## Analysis of the Solution

The solution for the previous equation was  $x = e^2$ ; this is often referred to as the exact solution. Sometimes, you may be asked to give an approximation, which would be more useful to someone using the result for a financial, scientific, or engineering application. The approximation for  $x = e^2$  can be found with a calculator.  $x = e^2 \approx 7.4$

In general, we can describe using the definition of a logarithm to solve logarithmic equations as follows:

For any algebraic expression  $S$  and real numbers  $b$  and  $c$ , where  $b > 0$ ,  $b \neq 1$ ,

$$\log_b(S) = c \text{ if and only if } b^c = S$$

Here is another example of what you may encounter when solving logarithmic equations.

In our next example, we will show you the power of using graphs to analyze solutions to logarithmic equations. By turning each side of the equation into a function and plotting them on the same set of axes, we can see how they interact with each other. In this case, we get a logarithmic function and a horizontal line and find that the solution is the point where the two intersect.

In the same way that a story can help you understand a complex concept, graphs can help us understand a wide variety of concepts in mathematics. Visual representations of the parts of an equation can be created by turning each side into a function and plotting the functions on the same set of axes.

## Use the One-to-One Property of Logarithms to Solve Logarithmic Equations

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers  $x > 0$ ,  $S > 0$ ,  $T > 0$  and any

## EXAMPLE

Solve  $2\ln(6x) = 7$ .

**Answer**

$$2\ln(6x) = 7$$

$$\ln(6x) = \frac{7}{2} \quad \text{Divide by 2.}$$

$$6x = e^{\left(\frac{7}{2}\right)} \quad \text{Use the definition of ln.}$$

$$x = \frac{1}{6}e^{\left(\frac{7}{2}\right)} \quad \text{Divide by 6.}$$

Exact answer:  $x = \frac{1}{6}e^{\left(\frac{7}{2}\right)}$

Approximate answer:  $x \approx 5.5$

## EXAMPLE

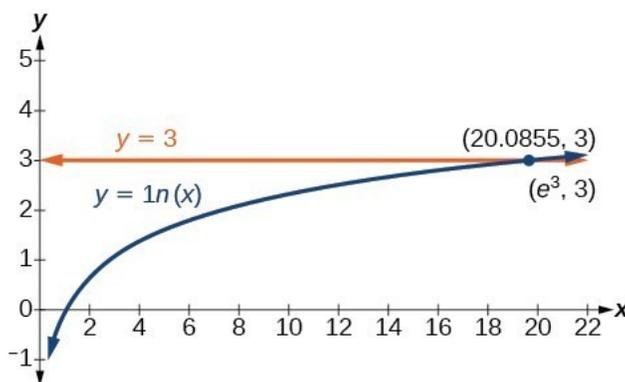
Solve  $\ln x = 3$ .

**Answer**

$$\ln x = 3$$

$$x = e^3 \quad \text{Use the definition of the natural logarithm.}$$

The graph below represents the graphs of each side of the equation. On the graph, the  $x$ -coordinate of the point at which the two graphs intersect is close to 20. In other words,  $e^3 \approx 20$ . A calculator gives a better approximation:  $e^3 \approx 20.0855$ .



The graphs of  $y = \ln x$  and  $y = 3$  cross at the point  $(e^3, 3)$ , which is approximately  $(20.0855, 3)$ .

positive real number  $b$ , where  $b \neq 1$ ,

$$\log_b S = \log_b T \text{ if and only if } S = T.$$

For example,

$$\text{If } \log_2(x - 1) = \log_2(8), \text{ then } x - 1 = 8.$$

So, if  $x - 1 = 8$ , then we can solve for  $x$ , and we get  $x = 9$ . To check, we can substitute  $x = 9$  into the original equation:  $\log_2(9 - 1) = \log_2(8) = 3$ . In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules

of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

## EXAMPLE

Solve  $\log(3x - 2) - \log(2) = \log(x + 4)$ .

Check your results.

**Answer**

$$\log(3x - 2) - \log(2) = \log(x + 4)$$

$$\log\left(\frac{3x-2}{2}\right) = \log(x + 4)$$

Apply the quotient rule of logarithms.

$$\frac{3x-2}{2} = x + 4$$

Apply the one to one property of a logarithm.

$$3x - 2 = 2x + 8$$

Multiply both sides of the equation by 2.

$$x = 10$$

Subtract  $2x$  and add 2.

To check the result, substitute  $x = 10$  into  $\log(3x - 2) - \log(2) = \log(x + 4)$ .

$$\log(3(10) - 2) - \log(2) = \log((10) + 4)$$

$$\log(28) - \log(2) = \log(14)$$

$$\log\left(\frac{28}{2}\right) = \log(14)$$

The solution checks.

The answer is  $x = 10$ .

Note, when solving an equation involving logarithms, always check to see if the answer is correct or if there is an extraneous solution.

In general, we can summarize solving logarithmic equations as follows:

For  $\log_b S = \log_b T$  if and only if  $S = T$

1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation has the form  $\log_b S = \log_b T$ .
2. Use the one-to-one property to set the arguments equal to each other.
3. Solve the resulting equation,  $S = T$ , for the unknown.

## Analysis of the Solution

There are two solutions:  $x = 3$  or  $x = -1$ . The solution  $x = -1$  is negative, but it checks when substituted into the original equation because the argument of the logarithm function is still positive.

In the following video, we show more examples of solving logarithmic equations.

## Summary

We can combine the definition of logarithms and algebraic tools used for solving linear equations to solve logarithmic equations. Graphing each side of a logarithmic equation helps you analyze the solution.

## EXAMPLE

Solve  $\ln(x^2) = \ln(2x + 3)$ .

**Answer**

$$\ln(x^2) = \ln(2x + 3)$$

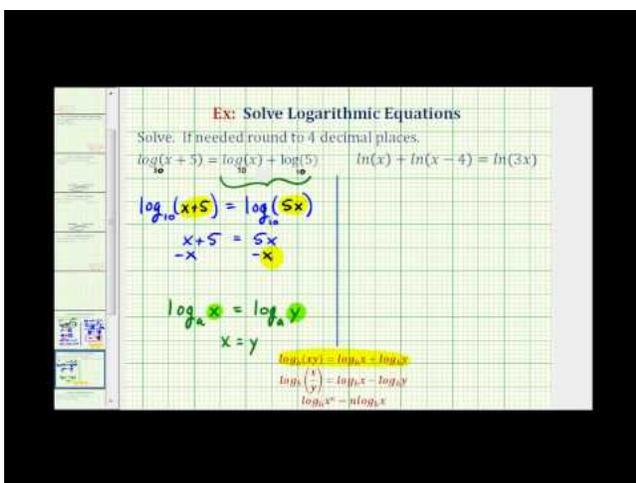
$$x^2 = 2x + 3 \quad \text{Use the one-to-one property of the logarithm.}$$

$$x^2 - 2x - 3 = 0 \quad \text{Get zero on one side before factoring.}$$

$$(x - 3)(x + 1) = 0 \quad \text{Factor using FOIL.}$$

$$x - 3 = 0 \text{ or } x + 1 = 0 \quad \text{If a product is zero, one of the factors must be zero.}$$

$$x = 3 \text{ or } x = -1 \quad \text{Solve for } x.$$



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# APPLIED EXPONENTIAL AND LOGARITHMIC EQUATIONS

## LEARNING OUTCOMES

- Solve half-life problems
- Solve pH problems

- Solve problems involving Richter scale readings
- Solve problems involving decibels

In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.



One such application is called **half-life**, which refers to the amount of time it takes for half a given quantity of radioactive material to decay. The table below lists the half-life for several of the more common radioactive substances.

Substance	Use	Half-life
gallium-67	nuclear medicine	80 hours
cobalt-60	manufacturing	5.3 years
technetium-99m	nuclear medicine	6 hours
americium-241	construction	432 years
carbon-14	archeological dating	5,715 years
uranium-235	atomic power	703,800,000 years

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:

$$A(t) = A_0 e^{\frac{\ln(0.5)}{T} t}$$

$$A(t) = A_0 e^{\ln(0.5) \frac{t}{T}}$$

$$A(t) = A_0 (e^{\ln(0.5)})^{\frac{t}{T}}$$

$$A(t) = A_0 \left(\frac{1}{2}\right)^{\frac{t}{T}}$$

where

- $A_0$  is the amount initially present
- $T$  is the half-life of the substance
- $t$  is the time period over which the substance is studied
- $A(t)$  is the amount of the substance present after time  $t$

## Analysis of the Solution

Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium-235 remaining is 900 grams.

## pH

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0

## EXAMPLE

How long will it take for ten percent of a 1000-gram sample of uranium-235 to decay?

**Answer**

$$y = 1000e^{\frac{\ln(0.5)}{703,800,000}t}$$

$$900 = 1000e^{\frac{\ln(0.5)}{703,800,000}t} \quad \text{After 10\% decays, 900 grams are left.}$$

$$0.9 = e^{\frac{\ln(0.5)}{703,800,000}t} \quad \text{Divide by 1000.}$$

$$\ln(0.9) = \ln\left(e^{\frac{\ln(0.5)}{703,800,000}t}\right) \quad \text{Take ln of both sides.}$$

$$\ln(0.9) = \frac{\ln(0.5)}{703,800,000}t \quad \ln(e^M) = M$$

$$t = 703,800,000 \times \frac{\ln(0.9)}{\ln(0.5)} \text{ years} \quad \text{Solve for } t.$$

$$t \approx 106,979,777 \text{ years}$$



to 14. Substances with a pH less than 7 are considered acidic, and substances with a pH greater than 7 are said to be alkaline. In our next example, we will find how doubling the concentration of hydrogen ions in a liquid affects pH.

## EXAMPLE

In chemistry,  $\text{pH} = -\log [H^+]$ . If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH?

**Answer**

Suppose  $C$  is the original concentration of hydrogen ions, and  $P$  is the original pH of the liquid. Then  $P = -\log(C)$ . If the concentration is doubled, the new concentration is  $2C$ . Then the pH of the new liquid is  $\text{pH} = -\log(2C)$ .

Using the product rule of logs:

$$\text{pH} = -\log(2C) = -(\log(2) + \log(C)) = -\log(2) - \log(C)$$

Since  $P = -\log(C)$ , the new pH is:

$$\text{pH} = P - \log(2) \approx P - 0.301$$

When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301.

## Richter Scale

The Richter scale is a logarithmic function that is used to measure the magnitude of earthquakes. The magnitude of an earthquake is related to how much energy is released by the quake. Instruments called seismographs detect movement in the earth; the smallest movement that can be detected shows on a seismograph as a wave with amplitude  $A_0$ .

A – the measure of the amplitude of the earthquake wave  
 $A_0$  – the amplitude of the smallest detectable wave (or standard wave)

From this you can find R, the Richter scale measure of the magnitude of the earthquake using the formula:

$$R = \log \left( \frac{A}{A_0} \right)$$

The intensity of an earthquake will typically measure between 2 and 10 on the Richter scale. Any earthquakes registering below a 5 are fairly minor; they may shake the ground a bit but are seldom strong enough to cause much damage. Earthquakes with a Richter rating between 5 and 7.9 are much more severe, and any quake above an 8 is likely to cause massive damage. (The highest rating ever recorded for an earthquake is 9.5 during the 1960 Valdivia earthquake in Chile.)

**RICHTER SCALE** of earthquake energy:

Each level is **10** times stronger than the previous level

	Description	Occurrence	In Population	Movement
1	SMALL	DAILY	every minute	small
2	SMALL	DAILY	every hour	small
3	SMALL	DAILY	every day	small
4	SMALL	DAILY	every week	moderate sudden
5	MODERATE	MONTHLY	every 10 years	strong sudden
6	MODERATE	MONTHLY	every 30 years	strong sudden
7	MAJOR	MONTHLY	every 50 years	severe sudden
8	GREAT	YEARLY	every 100 years	very severe
9	GREAT	YEARLY	every 300 years	very severe
10	SUPER	RARELY	every 1000 years	extreme

The Richter Scale of earthquake energy.

## EXAMPLE

An earthquake is measured with a wave amplitude 392 times as great as  $A_0$ . What is the magnitude of this earthquake, to the nearest tenth, using the Richter scale?

### Answer

Use the Richter scale equation.

$$R = \log \left( \frac{A}{A_0} \right)$$

Since A is 392 times as large as  $A_0$ ,  $A = 392A_0$ . Substitute this expression in for A.

$$R = \log \left( \frac{392A_0}{A_0} \right)$$

Simplify the argument of the logarithm, and then use a calculator to evaluate.

$$\begin{aligned} R &= \log \left( \frac{392A_0}{A_0} \right) \\ &= \log (392) \\ &= 2.5932 \\ &\approx 2.6 \end{aligned}$$

The earthquake registered 2.6 on the Richter scale.

A difference of 1 point on the Richter scale equates to a 10-fold difference in the amplitude of the earthquake (which is related to the wave strength). This means that an earthquake that measures 3.6 on the Richter scale has 10 times the amplitude of one that measures 2.6.

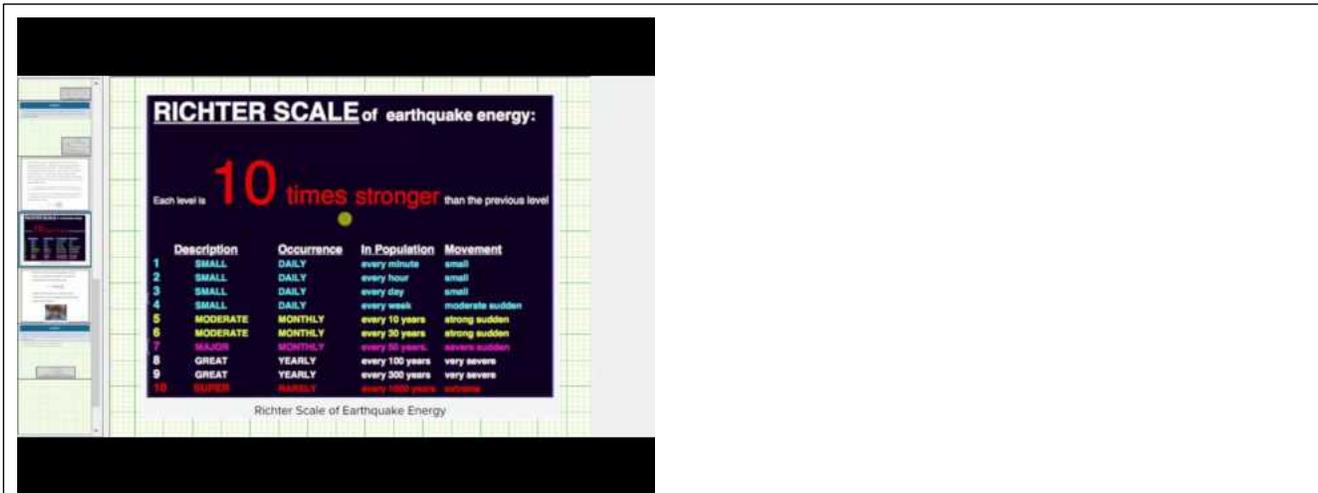
In the Richter scale example, the wave amplitude of the earthquake was 392 times the smallest detectable wave. What if it were 10 times that or 3,920 times normal? To find the measurement of that size earthquake on the Richter scale, you find  $\log 3920$ . A calculator gives a value of 3.5932 . . . or 3.6 when rounded to the nearest tenth. One extra point on the Richter scale can mean a lot more shaking!

In the following video, we show another example of how to calculate the magnitude of an earthquake.

## Decibels

Sound is measured in a logarithmic scale using a unit called a decibel. The formula looks similar to the Richter scale:

$$d = 10 \log \left( \frac{P}{P_0} \right)$$



A YouTube element has been excluded from this version of the text. You can view it online here: <https://courses.lumenlearning.com/intermediatealgebra/?p=3550>

where  $P$  is the power or intensity of the sound and  $P_0$  is the weakest sound that the human ear can hear. In the next example we will find how much more intense the noise from a dishwasher is than the noise from a hot water pump.



*Logarithms can be used to measure how loud sound is.*

The following video provides another example of comparing the intensity of two sounds.

Applications of logarithms and exponentials are everywhere in science. We hope the examples here have given you an idea of how useful they can be.

## EXAMPLE

One hot water pump has a noise rating of 50 decibels. One dishwasher, however, has a noise rating of 62 decibels. The dishwasher noise is how many times more intense than the hot water pump noise?

### Answer

You cannot easily compare the two noises using the formula, but you can compare them to  $P_0$ . Start by finding the intensity of noise for the hot water pump. Use  $h$  for the intensity of the hot water pump's noise.

$$50 = 10 \log \left( \frac{h}{P_0} \right)$$

Divide both sides of the equation by 10 to get the log by itself.

$$5 = \log \left( \frac{h}{P_0} \right)$$

Rewrite the equation as an exponential equation.

$$10^5 = \frac{h}{P_0}$$

Solve for  $h$ , the intensity of the water pump.

$$h = P_0 10^5$$

Repeat the same process to find the intensity of the noise for the dishwasher; use  $d$  to represent the intensity of the sound of the dishwasher. Remember that  $P_0$  is a baseline for the most faint sound the human ear can hear.

$$62 = 10 \log \left( \frac{d}{P_0} \right)$$

$$6.2 = \log \left( \frac{d}{P_0} \right)$$

$$10^{6.2} = \frac{d}{P_0}$$

$$d = P_0 10^{6.2}$$

To compare  $d$  to  $h$ , you can divide. (Think: if the dishwasher's noise is twice as intense as the pump's, then  $d$  should be  $2h$ —that is,  $\frac{d}{h}$  should be 2.)

Use the laws of exponents to simplify the quotient.

$$\frac{d}{h} = \frac{P_0 10^{6.2}}{P_0 10^5} = \frac{10^{6.2}}{10^5} = 10^{6.2-5} = 10^{1.2}$$

The dishwasher's noise is  $10^{1.2}$  (or about 15.85) times as intense as the hot water pump.

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# PUTTING IT TOGETHER: EXPONENTIAL AND LOGARITHMIC EQUATIONS

# WRPS Records Virginia Earthquake

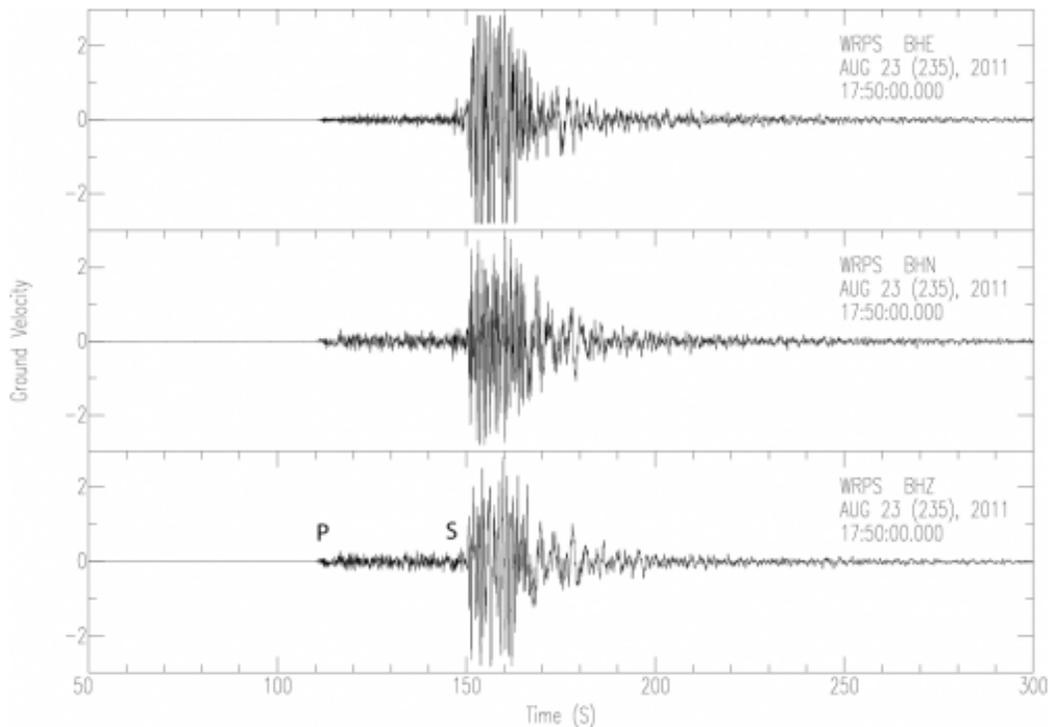
8/23/2011

Virginia Magnitude 5.9 Earthquake, Origin time 17:51:03 UTC, August 23, 2011

Ground Motion recorded on seismic station at Penn State, University Park Campus (Deike Bldg)

BHZ=vertical ground motion; BHN=ground motion N-S direction; BHE=ground motion E-W direction

Note that siesmogram starts at 17:50, about 1 minute before the earthquake.



**P and S waves are labeled. The largest amplitude S waves lasted for about 15 seconds.**

*Ground Motion recorded on seismic station at Penn State, University Park Campus (Deike Bldg).*

In the beginning of this module, we left Joan with a plan to try and stump her grandfather, the geologist, with the following math problem.

The Alaska quake of 1964 had a Richter scale value of 8.5. How many times greater than the baseline was the measured wave amplitude recorded by a seismograph?

$$8.5 = \log \left( \frac{A}{A_0} \right)$$

He was so excited to have found something he and Joan could talk about; he beamed when he saw that she had a problem for him to solve.

“I probably knew how to solve that at one time, but it has been a long time,” said grandpa. “Will you show me?”

First, Joan explained to her grandpa that the goal of the problem was to get the argument out of the logarithm. She remembered that a logarithm is an exponent, so her first step was to rewrite the logarithm as an exponential:

$$8.5 = \log\left(\frac{A}{A_0}\right)$$

$$10^{8.5} = \frac{A}{A_0}$$

$$10^{8.5} \cdot A_0 = A$$

Joan entered  $10^{8.5}$  into a calculator and got the following number:

316, 227, 766.017

Wow, the amplitude of the waves that caused the Alaska earthquake of 1964 was 3 million times greater than the baseline reading!

When working with values that are very large or very small, it is very helpful to work in logarithmic or exponential scales. It helps scientists be able to avoid having to do calculations with massive numbers. Additionally, it makes comparisons between different measurements easier to understand.

We hope you have enjoyed learning with Joan and her many friends and family members.

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