## College Algebra Corequisite

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# MODULE 1: ALGEBRA ESSENTIALS 

## WHY IT MATTERS: ALGEBRA ESSENTIALS

## Why learn about the essentials of algebra?

It is often said that mathematics is the language of science. If this is true, then numbers must be an essential part of the language of mathematics. The earliest use of numbers occurred 100 centuries ago in the Middle East to count, or enumerate items. Farmers, cattlemen, and tradesmen used tokens, stones, or markers to signify a single quantity - a sheaf of grain, a head of livestock, or a fixed length of cloth, for example. Doing so made commerce possible, leading to improved communications and the spread of civilization.

Three to four thousand years ago, Egyptians introduced fractions. They first used them to show reciprocals. Later, they used them to represent the amount when a quantity was divided into equal parts.

But what if there were no cattle to trade or an entire crop of grain was lost in a flood? How could someone indicate the existence of nothing?


Yuan Dynasty iron magic square showing Persian/ Arabic numbers (ca. 1271 1368) From earliest times, people had thought of a "base state" while counting and used various symbols to represent this null condition. However, it was not until about the fifth century A.D. in India that zero was added to the number system and used as a numeral in calculations.

Clearly there was also a need for numbers to represent loss or debt. In India, in the seventh century A.D., negative numbers were used as solutions to mathematical equations and commercial debts. The opposites of the counting numbers expanded the number system even further.

## LEARNING OUTCOMES

## Review Topics for Success

- Write a composite number as a product of its prime factors.
- Find the least common multiple (LCM) of a list of numbers.
- Perform operations on fractions.
- Use the order of operations to simplify expressions.
- Recognize and combine like terms in an expression.

Real Numbers

- Classify a real number.
- Perform calculations using order of operations.
- Use the properties of real numbers.
- Evaluate and simplify algebraic expressions.


## Exponents and Scientific Notation

- Use the rules of exponents to simplify exponential expressions.
- Use scientific notation.


## Radicals and Rational Exponents

- Evaluate and simplify square roots.
- Rationalize a denominator that contains a square root.
- Rewrite a radical expression using rational exponents.


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## REVIEW TOPICS FOR SUCCESS

## Learning Outcomes

By the end of this section, you will be able to:

- Find the prime factorization of a number
- Find the least common multiple of a number
- Perform operations on fractions.
- Use the order of operations to simplify expressions.
- Recognize and combine like terms in an expression.

Before embarking upon a study of the concepts, techniques, and applications of College Algebra, it is important to warm up by practicing math concepts you've probably seen before. This practice will help refresh some of the topics you may not have used recently but which will be extended into new topics in this course. A good review will help make new math less challenging and will enable deeper thought about familiar ideas.

Each module in this course will begin with a review of the material you'll need to use, and timely reminders will appear throughout the text exactly where you'll need them. Of course, you can return to Algebra Essentials any time you need to refresh the basics. Even if your instructor doesn't specifically assign material from the review features, it's a good idea to make sure you are familiar with the vocabulary, concepts, and processes involved.

We'll begin by reviewing important skills that you'll use over and over again throughout the course: prime factorizations, fractions, and order of operations.

## RECALL FOR SUCCESS

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

# PRIME FACTORIZATION AND THE LEAST COMMON MULTIPLE 

## LEARNING OUTCOMES

- Find the prime factorization of a number
- Find the least common multiple of a list of numbers

The word factor can be both a noun and a verb. To factor a number is to rewrite it by breaking it up into a product of smaller numbers. For example, we can factor 24 by writing it as $6 * 4$. We say that 6 and 4 are factors of 24 . But so are 2 and 12 . There are many ways to write this number.

A useful skill to have when doing algebra is the ability to rewrite numbers and expressions in helpful forms. For example, consider some different forms of the number 24.
$24 \quad \frac{72}{3} \quad \sqrt{576} \quad 6 * 4 \quad 2 \cdot 2 \cdot 2 \cdot 3$.
Composite numbers, like 24, are natural numbers that can be written as products of other natural numbers. Prime numbers are natural numbers that have only two possible factors, themselves and the number 1. The final form of 24 in the list above, $2 \cdot 2 \cdot 2 \cdot 3$, is called its prime factorization. When we write the prime factorization of a number, we are writing it as a product of only its prime factors. Being able to find the prime factorization of a composite number is an especially useful skill to have when doing algebra.

## Prime Factorization

The prime factorization of a number is the product of prime numbers that equals the number.
You may want to refer to the following list of prime numbers less than 50 as you work through this section.
$2,3,5,7,11,13,17,19,23,29,31,37,41,43,47$
Tip: Knowing the first five prime numbers will come in handy when reducing fractions.

## Prime Factorization Using the Factor Tree Method

One way to find the prime factorization of a number is to make a factor tree. We start by writing the number, and then writing it as the product of two factors. We write the factors below the number and connect them to the number with a small line segment-a "branch" of the factor tree.

If a factor is prime, we circle it (like a bud on a tree), and do not factor that "branch" any further. If a factor is not prime, we repeat this process, writing it as the product of two factors and adding new branches to the tree.

We continue until all the branches end with a prime. When the factor tree is complete, the circled primes give us the prime factorization.

For example, let's find the prime factorization of 36 . We can start with any factor pair such as 3 and 12 . We write 3 and 12 below 36 with branches connecting them.


The factor 3 is prime, so we circle it. The factor 12 is composite, so we need to find its factors. Let's use 3 and 4 . We write these factors on the tree under the 12 .


The factor 3 is prime, so we circle it. The factor 4 is composite, and it factors into $2 \cdot 2$. We write these factors under the 4 . Since 2 is prime, we circle both 2 s.


The prime factorization is the product of the circled primes. We generally write the prime factorization in order from least to greatest.
$2 \cdot 2 \cdot 3 \cdot 3$
In cases like this, where some of the prime factors are repeated, we can write prime factorization in exponential form.
$2 \cdot 2 \cdot 3 \cdot 3$
$2^{2} \cdot 3^{2}$
Note that we could have started our factor tree with any factor pair of 36 . We chose 12 and 3 , but the result would have been the same if we had started with 2 and 18,4 and 9 , or 6 and 6.

## Find the prime factorization of a composite number using the tree method

1. Find any factor pair of the given number, and use these numbers to create two branches.
2. If a factor is prime, that branch is complete. Circle the prime.
3. If a factor is not prime, write it as the product of a factor pair and continue the process.
4. Write the composite number as the product of all the circled primes.

## EXAMPLE

Find the prime factorization of 48 using the factor tree method.
Solution:

We can start our tree using any factor pair of 48. Let's use 2 and 24. We circle the 2 because it is prime and so that branch is complete.


Now we will factor 24 . Let's use 4 and 6 .


Neither factor is prime, so we do not circle either.We factor the 4 , using 2 and 2 .We factor 6 , using 2 and 3.
We circle the 2 s and the 3 since they are prime. Now all of the branches end in a prime.

$2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$
$2^{4} \cdot 3$

Check this on your own by multiplying all the factors together. The result should be 48 .

## TRY IT

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The following video shows how to find the prime factorization of 60 using the factor tree method.

There are other methods that work well to find the prime factorization of a number. Any method that factors out small primes repeatedly until there are only prime factors remaining is acceptable. See the video below for a demonstration of using stacked division to find a prime factorization.

## Finding Least Common Multiples

Consider the multiples of two numbers, 3 and 4 . What is the least common multiple between them?
Certainly 3 and 4 have many multiples in common: 12, 24, 36, etc. But 12 is the smallest among them.


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## EXAMPLE

Find the prime factorization of 84 using the factor tree method.
Answer
Solution:

We start with the factor pair 4 and 21.
Neither factor is prime so we factor them further.

Now the factors are all prime, so we circle them.

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The smallest number that is a multiple of two numbers is called the least common multiple (LCM). So the LCM of 3 and 4 is 12 .

One of the reasons we find prime factorizations is to use them to find the least common multiple of two or more numbers. This will be useful when we add and subtract fractions with different denominators.


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## Prime Factors Method

We can find the least common multiple of two numbers by inspecting their prime factors. We'll use this method to find the LCM of 12 and 18.

We start by finding the prime factorization of each number.

$$
12=2 \cdot 2 \cdot 3 \quad \text { and } \quad 18=2 \cdot 3 \cdot 3
$$

Then we find the largest instance of each prime appearing in any one factorization. Here, we see that the number 2 appears twice in the factorization of 12, but only once in 18. And the number 3 appears twice in 18, but only once in 12 . So, we select $2 \cdot 2$ and $3 \cdot 3$ and multiply them together to find the LCM.

$$
\begin{array}{ccc}
12=2 \cdot 2 \cdot 3 & \text { and } & 18=2 \cdot 3 \cdot 3 \\
2 \cdot 2 & * & 3 \cdot 3 .
\end{array}
$$

$$
2 \cdot 2 \cdot 3 \cdot 3=36
$$

The LCM of 12 and 18 is 36 .
If a prime appears the same number of times in any factorization, just select one instance. Let's find the LCM of a list of numbers as an example.

Find the LCM of $12,18,20$, and 60 . First we'll write the numbers as products of primes.

$$
12=2 \cdot 2 \cdot 3 \quad 18=2 \cdot 3 \cdot 3 \quad 20=2 \cdot 2 \cdot 5 \quad 60=2 \cdot 2 \cdot 3 \cdot 5
$$

We see that the largest instance of the number 2 appears in the numbers 12,20 , and 60 . But we just select one largest instance. The largest instance of the number 3 appears in 18, where there are two factors of 3. And the number 5 appears once in 20 and once in 60, so we'll select one 5.

The LCM of $12,18,20$, and 60 is

$$
2 \cdot 2 \cdot 3 \cdot 3 \cdot 5=180 .
$$

So, 180 is the smallest number that $12,18,20$, and 60 all divide evenly into.

1. Find the prime factorization of each number.
2. Inspect each factorization for the largest number of instances of each prime number appearing in each factorization
3. Select one set of each largest instance of each prime factor appearing.
4. Multiply the selected factors together to obtain the LCM.

It can be helpful to write the prime factors next to each number, matching them vertically in columns, then bringing down the primes in each column to collect the factors whose product is the LCM. The following examples illustrate this technique.

## EXAMPLE

Find the LCM of 15 and 18 using the prime factors method.
Answer
Solution:

Write each number as a product of primes.

| $15=3 \cdot 5$ | $18=2 \cdot 3 \cdot 3$ |
| :--- | :--- |
| $15=3 \cdot$ | 5 |
| $18=2 \cdot 3 \cdot 3$ |  | vertically when possible.

$18=2 \cdot 3 \cdot 3$

Bring down the primes in each column.

| 15 | $=3 \cdot$ |
| ---: | :--- |
| 18 | $=2 \cdot 2 \cdot 3 \cdot 3$ |
| LCM | $=2 \cdot 3 \cdot 3 \cdot 5$ |

LCM $=2 \cdot 3 \cdot 3 \cdot 5$ The LCM of 15 and 18 is 90 .

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In the next video we see how to find the Least Common Multiple by using prime factorization.

Find the LCM of 50 and 100 using the prime factors method.
Answer
Solution:
Write the prime factorization of each number.

$$
50=2 \cdot 5 \cdot 5 \quad 100=2 \cdot 2 \cdot 5 \cdot 5
$$

Write each number as a product of primes, matching primes vertically when possible.

$$
50=2 \cdot 5 \cdot 5100=2 \cdot 2 \cdot 5 \cdot 5
$$

Bring down the primes in each column.

Multiply the factors to get the LCM.
$50=2 \cdot 5 \cdot 5$
$100=2 \cdot 2 \cdot 5 \cdot 5$

| LCM $=2 \cdot 2 \cdot 5 \cdot 5$ |
| :--- |

LCM $=2 \cdot 2 \cdot 2 \cdot 5 \cdot 5$ The LCM of
50 and 100 is 100.

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## OPERATIONS ON FRACTIONS

## Learning Outcomes

- Add or Subtract fractions.
- Simplify fractions.
- Multiply fractions.
- Divide fractions.

Math students and working adults often find their knowledge of how to add, subtract, multiply, and divide fractions has grown rusty with disuse. We tend to rely on calculators to do a lot of the work of fractions for us. College Algebra, though, builds up some important techniques for handling expressions and equations that are based on operations on fractions. So it is important to refamiliarize yourself with these skills. This section will remind you how to do operations on fractions. As you work through the rest of the course, you can return this section as needed for a quick reminder of operations on fractions.

Before we get started, let's define some terminology.

- product: the result of multiplication
- factor: something being multiplied - for $3 \cdot 2=6$, both 3 and 2 are factors of 6
- numerator: the top part of a fraction - the numerator in the fraction $\frac{2}{3}$ is 2
- denominator: the bottom part of a fraction - the denominator in the fraction $\frac{2}{3}$ is 3


## A Note About Instructions

Certain words are used in math textbooks and by teachers to provide students with instructions on what to do with a given problem. For example, you may see instructions such as find or simplify. It is important to understand what these words mean so you can successfully work through the problems in this course. Here is a short list of some problem instructions along with their descriptions as they will be used in this module.

| Instruction | Interpretation |
| :---: | :---: |
| Find | Perform the indicated mathematical operations which may include addition, subtraction, multiplication, division (later, use of the word find will be expanded to solving equations, as in find the value of variable). |
| Simplify | 1) Perform the indicated mathematical operations including addition, subtraction, multiplication, division <br> 2) Write a mathematical statement in smallest terms so there are no other mathematical operations that can be performed-often found in problems related to fractions and the order of operations |
| Evaluate | 1) Perform the indicated mathematical operations including addition, subtraction, multiplication, division <br> 2) Substitute a given value for a variable in an expression and then perform the indicated mathematical operations |
| Reduce | Write a mathematical statement in smallest or lowest terms so there are no other mathematical operations that can be performed-often found in problems related to fractions or division |

## Add Fractions

When you need to add or subtract fractions, you will need to first make sure that the fractions have the same denominator. The denominator tells you how many pieces the whole has been broken into, and the numerator tells you how many of those pieces you are using.

The "parts of a whole" concept can be modeled with pizzas and pizza slices. For example, imagine a pizza is cut into 4 pieces, and someone takes 1 piece. Now, $\frac{1}{4}$ of the pizza is gone and $\frac{3}{4}$ remains. Note that both of these fractions have a denominator of 4 , which refers to the number of slices the whole pizza has been cut into. What if you have another pizza that had been cut into 8 equal parts and 3 of those parts were gone, leaving $\frac{5}{8}$ ?


How can you describe the total amount of pizza that is left with one number rather than two different fractions? You need a common denominator, technically called the least common multiple. Remember that if a number is a multiple of another, you can divide them and have no remainder.

One way to find the least common multiple of two or more numbers is to first multiply each by $1,2,3,4$, etc. For example, find the least common multiple of 2 and 5 .

First, list all the multiples of 2 :

$$
\begin{array}{l|l}
2 \cdot 1=2 & 5 \cdot 1=5 \\
2 \cdot 2=4 & 5 \cdot 2=10
\end{array}
$$

Then list all the multiples of 5 :

| $2 \cdot 3=6$ | $5 \cdot 3=15$ |
| :--- | :--- |
| $2 \cdot 4=8$ | $5 \cdot 4=20$ |
| $2 \cdot 5=10$ | $5 \cdot 5=25$ |

The smallest multiple they have in common will be the common denominator to use to rewrite each fraction into equivalent fractions. See the example below for a demonstration of our pizza problem.

## EXAMPLE

One pizza, cut into four slices, has one missing. Another pizza of the same size has been cut into eight pieces, of which three have been removed. Describe the total amount of pizza left in the two pizzas using common terms.

## Answer

Rewrite the fractions $\frac{3}{4}$ and $\frac{5}{8}$ as fractions with a least common denominator.
Find the least common multiple of the denominators. This is the least common denominator.
Multiples of $4: 4, \mathbf{8}, 12,16,24$
Multiples of 8 : 8, 16, $\mathbf{2 4}$
The least common denominator is 8 -the smallest multiple they have in common.
Rewrite $\frac{3}{4}$ with a denominator of 8 . You have to multiply both the top and bottom by 2 so you don't change the relationship between them.
$\frac{3}{4} \cdot \frac{2}{2}=\frac{6}{8}$
We don't need to rewrite $\frac{5}{8}$ since it already has the common denominator.

## Answer

Both $\frac{6}{8}$ and $\frac{5}{8}$ have the same denominator, and you can describe how much pizza is left with common terms. Add the numerators and put them over the common denominator.
We have $\frac{6}{8}$ of the first pizza and $\frac{5}{8}$ of the second pizza left. That's $\frac{11}{8}$ of an identically sized pizza, or 1 and $\frac{3}{8}$, pizza still on the table.

To add fractions with unlike denominators, first rewrite them with like denominators. Then add or subtract the numerators over the common denominator.

## ADDING FRACTIONS WITH UNLIKE DENOMINATORS

1. Find a common denominator.
2. Rewrite each fraction as an equivalent fraction using the common denominator.
3. Now that the fractions have a common denominator, you can add the numerators.
4. Simplify by canceling out all common factors in the numerator and denominator.

## Simplify a Fraction

It is a common convention in mathematics to present fractions in lowest terms. We call this
practice simplifying or reducing the fraction, and it can be accomplished by canceling (dividing) the common factors in a fraction's numerator and denominator. We can do this because a fraction represents division, and for any number $a, \frac{a}{a}=1$.

For example, to simplify $\frac{6}{9}$ you can rewrite 6and 9 using the smallest factors possible as follows:
$\frac{6}{9}=\frac{2 \cdot 3}{3 \cdot 3}$
Since there is a 3 in both the numerator and denominator, and fractions can be considered division, we can divide the 3 in the top by the 3 in the bottom to reduce to 1 .
$\frac{6}{9}=\frac{2 \cdot \not 2}{3 \cdot \not 2}=\frac{2 \cdot 1}{3}=\frac{2}{3}$
In the next example you are shown how to add two fractions with different denominators, then simplify the answer.

## EXAMPLE

Add $\frac{2}{3}+\frac{1}{5}$. Simplify the answer.

## Answer

Since the denominators are not alike, find a common denominator by multiplying the denominators.
$3 \cdot 5=15$
Rewrite each fraction with a denominator of 15 .

$$
\begin{aligned}
& \frac{2}{3} \cdot \frac{5}{5}=\frac{10}{15} \\
& \frac{1}{5} \cdot \frac{3}{3}=\frac{3}{15}
\end{aligned}
$$

Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified.
$\frac{10}{15}+\frac{3}{15}=\frac{13}{15}$

## Answer

$$
\frac{2}{3}+\frac{1}{5}=\frac{13}{15}
$$

You can find a common denominator by finding the common multiples of the denominators. The least common multiple is the easiest to use.

In the following video you will see an example of how to add two fractions with different denominators.
You can also add more than two fractions as long as you first find a common denominator for all of them. An example of a sum of three fractions is shown below. In this example, you will use the prime factorization method to find the LCM.

## Subtract Fractions

## EXAMPLE

Add $\frac{3}{7}+\frac{2}{21}$. Simplify the answer.
Answer
Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 7 and 21.
Multiples of $7: 7,14, \mathbf{2 1}$
Multiples of 21:21
Rewrite each fraction with a denominator of 21 .

$$
\begin{aligned}
& \frac{3}{7} \cdot \frac{3}{3}=\frac{9}{21} \\
& \frac{2}{21}
\end{aligned}
$$

Add the fractions by adding the numerators and keeping the denominator the same. Make sure the fraction cannot be simplified.
$\frac{9}{21}+\frac{2}{21}=\frac{11}{21}$

Answer
$\frac{3}{7}+\frac{2}{21}=\frac{11}{21}$


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## THINK ABOUT IT

Add $\frac{3}{4}+\frac{1}{6}+\frac{5}{8}$. Simplify the answer and write as a mixed number.
What makes this example different than the previous ones? Use the box below to write down a few thoughts about how you would add three fractions with different denominators together.
Answer

Since the denominators are not alike, find the least common denominator by finding the least common multiple (LCM) of 4,6 , and 8 .
$4=2 \cdot 2$
$6=3 \cdot 2$
$8=2 \cdot 2 \cdot 2$
LCM : $2 \cdot 2 \cdot 2 \cdot 3=24$
Rewrite each fraction with a denominator of 24 .
$\frac{3}{4} \cdot \frac{6}{6}=\frac{18}{24}$
$\frac{1}{6} \cdot \frac{4}{4}=\frac{4}{24}$
$\frac{5}{8} \cdot \frac{3}{3}=\frac{15}{24}$
Add the fractions by adding the numerators and keeping the denominator the same.
$\frac{18}{24}+\frac{4}{24}+\frac{15}{24}=\frac{37}{24}$
Write the improper fraction as a mixed number and simplify the fraction.
$\frac{37}{24}=1 \frac{13}{24}$

Answer
$\frac{3}{4}+\frac{1}{6}+\frac{5}{8}=1 \frac{13}{24}$
Subtracting fractions follows the same technique as adding them. First, determine whether or not the denominators are alike. If not, rewrite each fraction as an equivalent fraction, all having the same denominator. Below are some examples of subtracting fractions whose denominators are not alike.

## Example

Subtract $\frac{1}{5}-\frac{1}{6}$. Simplify the answer.
Answer
The fractions have unlike denominators, so you need to find a common denominator. Recall that a common denominator can be found by multiplying the two denominators together.
$5 \cdot 6=30$
Rewrite each fraction as an equivalent fraction with a denominator of 30 .
$\frac{1}{5} \cdot \frac{6}{6}=\frac{6}{30}$
$\frac{1}{6} \cdot \frac{5}{5}=\frac{5}{30}$
Subtract the numerators. Simplify the answer if needed.
$\frac{6}{30}-\frac{5}{30}=\frac{1}{30}$
Answer

$$
\frac{1}{5}-\frac{1}{6}=\frac{1}{30}
$$

The example below shows how to use multiples to find the least common multiple, which will be the least common denominator.

## EXAMPLE

Subtract $\frac{5}{6}-\frac{1}{4}$. Simplify the answer.
Answer
Find the least common multiple of the denominators-this is the least common denominator.
Multiples of $6: 6, \mathbf{1 2}, 18,24$
Multiples of $4: 4,8, \mathbf{1 2}, 16,20$
12 is the least common multiple of 6 and 4 .
Rewrite each fraction with a denominator of 12 .

$$
\begin{aligned}
& \frac{5}{6} \cdot \frac{2}{2}=\frac{10}{12} \\
& \frac{1}{4} \cdot \frac{3}{3}=\frac{3}{12}
\end{aligned}
$$

Subtract the fractions. Simplify the answer if needed.
$\frac{10}{12}-\frac{3}{12}=\frac{7}{12}$
Answer

$$
\frac{5}{6}-\frac{1}{4}=\frac{7}{12}
$$

In the following video you will see an example of how to subtract fractions with unlike denominators.


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Just as you add, subtract, multiply, and divide when working with whole numbers, you also use these operations when working with fractions. There are many times when it is necessary to multiply fractions. A model may help you understand multiplication of fractions.

When you multiply a fraction by a fraction, you are finding a "fraction of a fraction." Suppose you have $\frac{3}{4}$ of a candy bar and you want to find $\frac{1}{2}$ of the $\frac{3}{4}$ :
${ }^{\frac{3}{4}}$


By dividing each fourth in half, you can divide the candy bar into eighths.


Then, choose half of those to get $\frac{3}{8}$.


In both of the above cases, to find the answer, you can multiply the numerators together and the denominators together.

## MULTIPLYING TWO FRACTIONS

$\frac{a}{b} \cdot \frac{c}{d}=\frac{a \cdot c}{b \cdot d}=\frac{\text { product of the numerators }}{\text { product of the denominators }}$

# MULTIPLYING MORE THAN TWO FRACTIONS <br> $\frac{a}{b} \cdot \frac{c}{d} \cdot \frac{e}{f}=\frac{a \cdot c \cdot e}{b \cdot d \cdot f}$ 

## EXAMPLE

Multiply $\frac{2}{3} \cdot \frac{4}{5}$

## Answer

Multiply the numerators and multiply the denominators.
$\frac{2 \cdot 4}{3 \cdot 5}$
Simplify, if possible. This fraction is already in lowest terms.
$\frac{8}{15}$

Answer
$\frac{8}{15}$

To review: if a fraction has common factors in the numerator and denominator, we can reduce the fraction to its simplified form by removing the common factors.

For example,

- Given $\frac{8}{15}$, the factors of 8 are: $1,2,4,8$ and the factors of 15 are: $1,3,5,15 . \frac{8}{15}$ is simplified because there are no common factors of 8 and 15 .
- Given $\frac{10}{15}$, the factors of 10 are: $1,2,5,10$ and the factors of 15 are: $1,3,5,15 . \frac{10}{15}$ is not simplified because 5 is a common factor of 10 and 15 .

You can simplify first, before you multiply two fractions, to make your work easier. This allows you to work with smaller numbers when you multiply.

In the following video you will see an example of how to multiply two fractions, then simplify the answer.


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## THINK ABOUT IT

Multiply $\frac{2}{3} \cdot \frac{1}{4} \cdot \frac{3}{5}$. Simplify the answer.
What makes this example different than the previous ones? Use the box below to write down a few thoughts about how you would multiply three fractions together.
Answer
Multiply the numerators and multiply the denominators.
$\frac{2 \cdot 1 \cdot 3}{3 \cdot 4}$
Simplify first by canceling (dividing) the common factors of 3 and 2.3 divided by 3 is 1 , and 2 divided by 2 is 1 .
$\frac{2 \cdot 1 \cdot 3}{3 \cdot(2 \cdot 2) \cdot 5}$
$\frac{\not 2 \cdot 1 \cdot \not 2}{\not Z \cdot(\not 2 \cdot 2) \cdot 5}$
$\frac{1}{10}$

## Divide Fractions

There are times when you need to use division to solve a problem. For example, if painting one coat of paint on the walls of a room requires 3 quarts of paint and you have a bucket that contains 6 quarts of paint, how many coats of paint can you paint on the walls? You divide 6 by 3 for an answer of 2 coats. There will also be times when you need to divide by a fraction. Suppose painting a closet with one coat only required $\frac{1}{2}$ quart of paint. How many coats could be painted with the 6 quarts of paint? To find the answer, you need to divide 2 by the fraction, $\frac{1}{2}$.

Before we begin dividing fractions, let's cover some important terminology.

- reciprocal: two fractions are reciprocals if their product is 1 (Don't worry; we will show you examples of what this means.)
- quotient: the result of division

Dividing fractions requires using the reciprocal of a number or fraction. If you multiply two numbers together and get 1 as a result, then the two numbers are reciprocals. Here are some examples of reciprocals:

| Original number | Reciprocal | Product |
| :--- | :--- | :--- |
| $\frac{3}{4}$ | $\frac{4}{3}$ | $\frac{3}{4} \cdot \frac{4}{3}=\frac{3 \cdot 4}{4 \cdot 3}=\frac{12}{12}=1$ |
| $\frac{1}{2}$ | $\frac{2}{1}$ | $\frac{1}{2} \cdot \frac{2}{1}=\frac{1 \cdot 2}{2 \cdot 1}=\frac{2}{2}=1$ |
| $3=\frac{3}{1}$ | $\frac{1}{3}$ | $\frac{3}{1} \cdot \frac{1}{3}=\frac{3 \cdot 1}{1 \cdot 3}=\frac{3}{3}=1$ |
| $2 \frac{1}{3}=\frac{7}{3}$ | $\frac{3}{7}$ | $\frac{7}{3} \cdot \frac{3}{7}=\frac{7 \cdot 3}{3 \cdot 7}=\frac{21}{21}=1$ |

Sometimes we call the reciprocal the "flip" of the other number: flip $\frac{2}{5}$ to get the reciprocal $\frac{5}{2}$.

## Division by Zero

You know what it means to divide by 2 or divide by 10 , but what does it mean to divide a quantity by 0 ? Is this even possible? Can you divide 0 by a number? Consider the fraction
$\frac{0}{8}$
We can read it as, "zero divided by eight." Since multiplication is the inverse of division, we could rewrite this as a multiplication problem.
$? \cdot 8=0$.
We can infer that the unknown must be 0 since that is the only number that will give a result of 0 when it is multiplied by 8 .

Now let's consider the reciprocal of $\frac{0}{8}$ which would be $\frac{8}{0}$. If we rewrite this as a multiplication problem, we will have
$? \cdot 0=8$.
This doesn't make any sense. There are no numbers that you can multiply by zero to get a result of 8 . The reciprocal of $\frac{8}{0}$ is undefined, and in fact, all division by zero is undefined.

Caution! Division by zero is undefined and so is the reciprocal of any fraction that has a zero in the numerator. For any real number a, $\frac{a}{0}$ is undefined. Additionally, the reciprocal of $\frac{0}{a}$ will always be undefined.

## Divide a Fraction by a Whole Number

When you divide by a whole number, you are multiplying by the reciprocal. In the painting example where you need 3 quarts of paint for a coat and have 6 quarts of paint, you can find the total number of coats that can be painted by dividing 6 by $3,6 \div 3=2$. You can also multiply 6 by the reciprocal of 3 , which is $\frac{1}{3}$, so the multiplication problem becomes
$\frac{6}{1} \cdot \frac{1}{3}=\frac{6}{3}=2$

## DIVIDING IS MULTIPLYING BY THE RECIPROCAL

For all division, you can turn the operation into multiplication by using the reciprocal. Dividing is the same as multiplying by the reciprocal.

The same idea will work when the divisor (the thing being divided) is a fraction. If you have $\frac{3}{4}$ of a candy bar and need to divide it among 5 people, each person gets $\frac{1}{5}$ of the available candy:
$\frac{1}{5}$ of $\frac{3}{4}=\frac{1}{5} \cdot \frac{3}{4}=\frac{3}{20}$
Each person gets $\frac{3}{20}$ of a whole candy bar.
If you have a recipe that needs to be divided in half, you can divide each ingredient by 2 , or you can multiply each ingredient by $\frac{1}{2}$ to find the new amount.

For example, dividing by 6 is the same as multiplying by the reciprocal of 6 , which is $\frac{1}{6}$. Look at the diagram of two pizzas below. How can you divide what is left (the red shaded region) among 6 people fairly?


Each person gets one piece, so each person gets $\frac{1}{4}$ of a pizza.

Dividing a fraction by a whole number is the same as multiplying by the reciprocal, so you can always use multiplication of fractions to solve division problems.

## EXAMPLE

Find $\frac{2}{3} \div 4$

## Answer

Write your answer in lowest terms.
Dividing by 4 or $\frac{4}{1}$ is the same as multiplying by the reciprocal of 4 , which is $\frac{1}{4}$.
$\frac{2}{3} \div 4=\frac{2}{3} \cdot \frac{1}{4}$
Multiply numerators and multiply denominators.
$\frac{2 \cdot 1}{3 \cdot 4}=\frac{2}{12}$
Simplify to lowest terms by dividing numerator and denominator by the common factor 4 .
$\frac{1}{6}$

Answer
$\frac{2}{3} \div 4=\frac{1}{6}$

## EXAMPLE

Divide. $9 \div \frac{1}{2}$
Answer
Write your answer in lowest terms.
Dividing by $\frac{1}{2}$ is the same as multiplying by the reciprocal of $\frac{1}{2}$, which is $\frac{2}{1}$.
$9 \div \frac{1}{2}=\frac{9}{1} \cdot \frac{2}{1}$
Multiply numerators and multiply denominators.
$\frac{9 \cdot 2}{1 \cdot 1}=\frac{18}{1}=18$
This answer is already simplified to lowest terms.

Answer
$9 \div \frac{1}{2}=18$

## Divide a Fraction by a Fraction

Sometimes you need to solve a problem that requires dividing by a fraction. Suppose you have a pizza that is already cut into 4 slices. How many $\frac{1}{2}$ slices are there?


There are 8 slices. You can see that dividing 4 by $\frac{1}{2}$ gives the same result as multiplying 4 by 2 .
What would happen if you needed to divide each slice into thirds?


You would have 12 slices, which is the same as multiplying 4 by 3 .

## DIVIDING WITH FRACTIONS

1. Find the reciprocal of the number that follows the division symbol.
2. Multiply the first number (the one before the division symbol) by the reciprocal of the second number (the one after the division symbol).

Any easy way to remember how to divide fractions is the phrase "keep, change, flip." This means to KEEP the first number, CHANGE the division sign to multiplication, and then FLIP (use the reciprocal) of the second number.

When solving a division problem by multiplying by the reciprocal, remember to write all whole numbers and mixed numbers as improper fractions. The final answer should be simplified and written as a mixed number.

In the following video you will see an example of how to divide an integer by a fraction, as well as an example of how to divide a fraction by another fraction.

## EXAMPLE

Divide $\frac{2}{3} \div \frac{1}{6}$
Answer
Multiply by the reciprocal.
KEEP $\frac{2}{3}$
CHANGE $\div$ to .
FLIP $\frac{1}{6}$
$\frac{2}{3} \cdot \frac{6}{1}$
Multiply numerators and multiply denominators.
$\frac{2 \cdot 6}{3 \cdot 1}=\frac{12}{3}$
Simplify.
$\frac{12}{3}=4$

Answer
$\frac{2}{3} \div \frac{1}{6}=4$

## EXAMPLE

Divide $\frac{3}{5} \div \frac{2}{3}$
Answer
Multiply by the reciprocal. Keep $\frac{3}{5}$, change $\div$ to $\cdot$, and flip $\frac{2}{3}$.
$\frac{3}{5} \cdot \frac{3}{2}$
Multiply numerators and multiply denominators.
$\frac{3 \cdot 3}{5 \cdot 2}=\frac{9}{10}$

Answer
$\frac{3}{5} \div \frac{2}{3}=\frac{9}{10}$


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## COMBINING LIKE TERMS AND ORDER OF OPERATIONS

## Learning Outcomes

- Recognize and combine like terms in an expression
- Use the order of operations to simplify expressions

Some important terminology before we begin:

- operations/operators: In mathematics we call things like multiplication, division, addition, and subtraction operations. They are the verbs of the math world, doing work on numbers and variables. The symbols used to denote operations are called operators, such as,,$+- \times, \div$. As you learn more math, you will learn more operators.
- term: Examples of terms would be $2 x$ and $-\frac{3}{2}$ or $a^{3}$. Even lone integers can be a term, like 0 .
- expression: A mathematical expression is one that connects terms with mathematical operators. For example $\frac{1}{2}+\left(2^{2}\right)-9 \div \frac{6}{7}$ is an expression.

One way we can simplify expressions is to combine like terms. Like terms are terms where the variables match exactly (exponents included). Examples of like terms would be $5 x y$ and $-3 x y$, or $8 a^{2} b$ and $a^{2} b$, or -3 and 8 . If we have like terms we are allowed to add (or subtract) the numbers in front of the variables, then keep the variables the same. As we combine like terms we need to interpret subtraction signs as part of the following term. This means if we see a subtraction sign, we treat the following term like a negative term. The sign always stays with the term.

This is shown in the following examples:

## EXAMPLE

Combine like terms: $5 x-2 y-8 x+7 y$
Answer
The like terms in this expression are:
$5 x$ and $-8 x$
$-2 y$ and $7 y$
Note how we kept the sign in front of each term.
Combine like terms:
$5 x-8 x=-3 x$
$-2 y+7 y=5 y$
Note how signs become operations when you combine like terms.
Simplified Expression:
$5 x-2 y-8 x+7 y=-3 x+5 y$
In the following video you will be shown how to combine like terms using the idea of the distributive property. Note that this is a different method than is shown in the written examples on this page, but it obtains the same result.


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In the video that follows, you will be shown another example of combining like terms. Pay attention to why you are not able to combine all three terms in the example.

## Order of Operations

You may or may not recall the order of operations for applying several mathematical operations to one expression. Just as it is a social convention for us to drive on the right-hand side of the road, the order of operations is a set of conventions used to provide order when you are required to use several mathematical

Combine like terms: $x^{2}-3 x+9-5 x^{2}+3 x-1$
Answer
The like terms in this expression are:
$x^{2}$ and $-5 x^{2}$
$-3 x$ and $3 x$
9 and -1
Combine like terms:
$x^{2}-5 x^{2}=-4 x^{2}$
$-3 x+3 x=0$
$9-1=8$
Simplified Expression:
$-4 x^{2}+8$


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operations for one expression.

## THE ORDER OF OPERATIONS

- Perform all operations within grouping symbols first. Grouping symbols include parentheses ( ), brackets [ ], braces \{ \}, and fraction bars.
- Evaluate exponents or square roots.
- Multiply or divide, from left to right.
- Add or subtract, from left to right.

This order of operations is true for all real numbers.

## Example

Simplify $7-5+3 \cdot 8$
Answer
According to the order of operations, multiplication comes before addition and subtraction.

Multiply $3 \cdot 8$
$7-5+3 \cdot 8$
$7-5+24$
Now, add and subtract from left to right. 7-5 comes first.
$2+24$
Finally, add.
$2+24=26$

Answer
$7-5+3 \cdot 8=26$
In the following example, you will be shown how to simplify an expression that contains both multiplication and subtraction using the order of operations.


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When you are applying the order of operations to expressions that contain fractions, decimals, and negative numbers, you will need to recall how to do these computations as well.

## Example

Simplify $3 \cdot \frac{1}{3}-8 \div \frac{1}{4}$

## Answer

According to the order of operations, multiplication and division come before addition and subtraction. Sometimes it helps to add parentheses to help you know what comes first, so let's put parentheses around the multiplication and division since it will come before the subtraction.
$\left(3 \cdot \frac{1}{3}\right)-\left(8 \div \frac{1}{4}\right)$
Multiply $3 \cdot \frac{1}{3}$ first.
$\left(3 \cdot \frac{1}{3}\right)-\left(8 \div \frac{1}{4}\right)$
(1) $-\left(8 \div \frac{1}{4}\right)$

Now, divide $8 \div \frac{1}{4}$.
$8 \div \frac{1}{4}=\frac{8}{1} \cdot \frac{4}{1}=32$
Subtract.
$(1)-(32)=-31$

## Answer

$3 \cdot \frac{1}{3}-8 \div \frac{1}{4}=-31$
In the following video you are shown how to use the order of operations to simplify an expression that contains multiplication, division, and subtraction with terms that contain fractions.


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## Exponents and Square Roots

In this section, we expand our skills with applying the order of operation rules to expressions with exponents and square roots. If the expression has exponents or square roots, they are to be performed after parentheses and other grouping symbols have been simplified and before any multiplication, division, subtraction, and addition that are outside the parentheses or other grouping symbols.

Recall that an expression such as $7^{2}$ is exponential notation for $7 \cdot 7$. (Exponential notation has two parts: the base and the exponent or the power. $\ln 7^{2}, 7$ is the base and 2 is the exponent; the exponent determines how many times the base is multiplied by itself.)

Exponents are a way to represent repeated multiplication; the order of operations places it before any other multiplication, division, subtraction, and addition is performed.

## Example

Simplify $3^{2} \cdot 2^{3}$.
Answer

This problem has exponents and multiplication in it. According to the order of operations, simplifying $3^{2}$ and $2^{3}$ comes before multiplication.
$3^{2} \cdot 2^{3}$
$3^{2}$ is $3 \cdot 3$, which equals 9 .
$9 \cdot 2^{3}$
$2^{3}$ is $2 \cdot 2 \cdot 2$, which equals 8 .
9 - 8
Multiply.
$9 \cdot 8=72$

## Answer

$3^{2} \cdot 2^{3}=72$

In the video that follows, an expression with exponents on its terms is simplified using the order of operations.


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When there are grouping symbols within grouping symbols, calculate from the inside to the outside. That is, begin simplifying within the innermost grouping symbols first.

Remember that parentheses can also be used to show multiplication. In the example that follows, both uses of parentheses-as a way to represent a group, as well as a way to express multiplication-are shown.

## Example

Simplify $(3+4)^{2}+(8)(4)$

## Answer

This problem has parentheses, exponents, multiplication, and addition in it. The first set of parentheses is a grouping symbol. The second set indicates multiplication.

Grouping symbols are handled first. Add numbers in parentheses.

$$
(3+4)^{2}+(8)(4)
$$

$(7)^{2}+(8)(4)$
Simplify $7^{2}$.

$$
\begin{aligned}
& 7^{2}+(8)(4) \\
& 49+(8)(4)
\end{aligned}
$$

Multiply.

$$
\begin{aligned}
& 49+(8)(4) \\
& 49+(32)
\end{aligned}
$$

Add.
$49+32=81$

Answer

$$
(3+4)^{2}+(8)(4)=81
$$

In the following video, you are shown how to use the order of operations to simplify an expression with grouping symbols, exponents, multiplication, and addition.


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In the next example we will simplify an expression that has a square root.

## Example

Simplify $\frac{\sqrt{7+2}+2^{2}}{(8)(4)-11}$

## Answer

This problem has all the operations to consider with the order of operations.
Grouping symbols are handled first, in this case the fraction bar. We will simplify the top and bottom separately.
To simplify the top:
$\sqrt{7+2}+2^{2}$
Add the numbers inside the square root, simplify the result and $2^{2}$

$$
\sqrt{7+2}+2^{2}
$$

$=\sqrt{9}+4$
$=3+4=7$
To simplify the bottom:
(8)(4) - 11

Multiply 8 and 4 first, then subtract 11 .
(8)(4) $-11=32-11=21$

Now put the fraction back together to see if any more simplifying needs to be done.
$\frac{7}{21}$, this can be reduced to $\frac{1}{3}$

## Answer

$$
\frac{\sqrt{7+2}+2^{2}}{(8)(4)-11}=\frac{1}{3}
$$



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## THINK ABOUT IT

These problems are very similar to the examples given above. How are they different and what tools do you need to simplify them?
a) Simplify $(1.5+3.5)-2(0.5 \cdot 6)^{2}$. This problem has parentheses, exponents, multiplication, subtraction, and addition in it, as well as decimals instead of integers.
Use the box below to write down a few thoughts about how you would simplify this expression with decimals and grouping symbols.

Answer
Grouping symbols are handled first. Add numbers in the first set of parentheses.
$(1.5+3.5)-2(0.5 \cdot 6)^{2}$
$5-2(0.5 \cdot 6)^{2}$
Multiply numbers in the second set of parentheses.
$5-2(0.5 \cdot 6)^{2}$
$5-2(3)^{2}$
Evaluate exponents.
$5-2(3)^{2}$
5-2.9
Multiply.
5-2.9
5-18
Subtract.
$5-18=-13$

## Answer

$(1.5+3.5)-2(0.5 \cdot 6)^{2}=-13$
b) Simplify $\left(\frac{1}{2}\right)^{2}+\left(\frac{1}{4}\right)^{3} \cdot 32$

Use the box below to write down a few thoughts about how you would simplify this expression with fractions and grouping symbols.

## Answer

This problem has exponents, multiplication, and addition in it, as well as fractions instead of integers.
According to the order of operations, simplify the terms with the exponents first, then multiply, then add.
$\left(\frac{1}{2}\right)^{2}+\left(\frac{1}{4}\right)^{3} \cdot 32$
Evaluate: $\left(\frac{1}{2}\right)^{2}=\frac{1}{2} \cdot \frac{1}{2}=\frac{1}{4}$
$\frac{1}{4}+\left(\frac{1}{4}\right)^{3} \cdot 32$
Evaluate: $\left(\frac{1}{4}\right)^{3}=\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4}=\frac{1}{64}$
$\frac{1}{4}+\frac{1}{64} \cdot 32$
Multiply.
$\frac{1}{4}+\frac{32}{64}$
Simplify. $\frac{32}{64}=\frac{1}{2}$, so you can add $\frac{1}{4}+\frac{1}{2}$.
$\frac{1}{4}+\frac{1}{2}=\frac{3}{4}$

Answer
$\left(\frac{1}{2}\right)^{2}+\left(\frac{1}{4}\right)^{3} \cdot 32=\frac{3}{4}$
Some people use a saying to help them remember the order of operations. This saying is called PEMDAS or Please Excuse My Dear Aunt Sally. The first letter of each word begins with the same letter of an arithmetic operation.

Please $\Rightarrow$ Parentheses (and other grouping symbols)
Excuse $\Rightarrow$ Exponents
My Dear $\Rightarrow$ Multiplication and Division (from left to right)
Aunt Sally $\Rightarrow$ Addition and Subtraction (from left to right)
Note: Even though multiplication comes before division in the saying, division could be performed first. Which is performed first, between multiplication and division, is determined by which comes first when reading from left to right. The same is true of addition and subtraction. Don't let the saying confuse you about this!

The order of operations gives us a consistent sequence to use in computation. Without the order of operations, you could come up with different answers to the same computation problem. (Some of the early calculators, and some inexpensive ones, do NOT use the order of operations. In order to use these calculators, the user has to input the numbers in the correct order.)

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## SUMMARY: REVIEW

## Key Concepts

- To find the prime factorization of a number, use the factor tree method to divide away small prime numbers until only prime factors remain. The product of those prime factors is the prime factorization of the number.
- To find the least common multiple (LCM) of a list of numbers, write each number as a product of its prime factors, select the largest instance of each prime that appears in any one number, then multiply the selections together to obtain the LCM.
- To add or subtract fractions, first rewrite each fraction as an equivalent fraction, all having the same denominator, then add or subtract the numerators and place the result over the common denominator.
- $\frac{a}{b} \pm \frac{c}{d}=\frac{a d \pm b c}{b d}$
- To multiply fractions, place the product of the numerators over the product of the denominator.
- $\frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}$
- To divide fractions, multiply the first fraction by the reciprocal of the second
- $\frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}=\frac{a d}{b c}$.
- To simplify a fraction, rewrite the numerator and denominator as products of their prime factors, then cancel ratios of common factors until there are no more common factors between the top and the bottom.
- Division by zero is undefined.
- When applying the order of operations, first simplify inside grouping symbols, then evaluate exponents or radicals, then multiply or divide from left to right in the order each appears, and finally add or subtract from left to right in the order each appears.


## Glossary

composite number a natural number that can be written as the product of other natural numbers. For example, the number 10 can be written as the product of 2 and 5 .
denominator the bottom part of a fraction - the denominator in the fraction $\frac{2}{3}$ is 3
evaluate to substitute a given value for a variable in an expression and then perform the indicated mathematical operation
exponent, exponential notation a superscript, called the exponent or power, over a number, called the base, that tells how many times to use the base as a factor
expression several terms connected together by the operation of addition or subtraction
factor a number being multiplied, for example, for $2 \cdot 5=10$, the numbers 2 and 5 are factors of 10
least common multiple, LCM Given two or more numbers, the least common multiple between them is the smallest number that each of the given numbers divides evenly into
like terms terms in which the variables match exactly (exponents included)
numerator the top part of a fraction - the numerator in the fraction $\frac{2}{3}$ is 2
operations/operators multiplication, division, addition, subtraction
prime number a number that is divisible only by itself and 1
product the result of multiplication
radical a symbol indicating the square root, cube root, etc. of a number
simplify, reduce to write a mathematical statement in smallest terms
term a single number or variable, or the product of a number and a variable, which may also contain one or more exponents or radicals

## Learning Outcomes

By the end of this section, you will be able to:

- Classify a real number as a natural, whole, integer, rational, or irrational number.
- Perform calculations using order of operations.
- Use the following properties of real numbers: commutative, associative, distributive, inverse, and identity.
- Evaluate algebraic expressions.
- Simplify algebraic expressions.

Because of the evolution of the number system, we can now perform complex calculations using several categories of real numbers. In this section we will explore sets of numbers, perform calculations with different kinds of numbers, and begin to learn about the use of numbers in algebraic expressions.

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## CLASSIFY A REAL NUMBER

## LEARNING OUTCOMES

- Write integers as rational numbers
- Identify rational numbers
- Classify real numbers into sets


## REVIEW FOR SUCCESS

About the notation for the sets of numbers we'll explore in this section:

- Natural numbers: The most familiar numbers are the natural numbers (sometimes called counting numbers): $1,2,3$, and so on. The mathematical symbol for the set of all natural numbers is written as $\mathbb{N}$.
- Whole numbers: The set of whole numbers includes all natural numbers as well as 0 . There is no special notation for this set.
- Integers: The set of all natural numbers together with their opposites (negative), including 0 , is called the set of integers, notated $\mathbb{Z}$.
- Rational numbers: A rational number, $\mathbb{Q}$, is a number that can be expressed as a ratio of integers (a fraction with an integer numerator and a positive, non-zero integer denominator).
- Real numbers: The real numbers include all the numbers above. The symbol for the real numbers is $\mathbb{R}$.
- Irrational numbers: All the real numbers that are not rational are called irrational numbers. These numbers cannot be expressed as a fraction of integers. Irrational numbers can be notated by the
symbol $\mathbb{R} \backslash \mathbb{Q}$, that is, the set of all real numbers minus the set of all rational numbers.

The numbers we use for counting, or enumerating items, are the natural numbers: $1,2,3,4,5$, and so on. We describe them in set notation as $\{1,2,3, \ldots\}$ where the ellipsis (...) indicates that the numbers continue to infinity. The natural numbers are, of course, also called the counting numbers. Any time we enumerate the members of a team, count the coins in a collection, or tally the trees in a grove, we are using the set of natural numbers. The set of whole numbers is the set of natural numbers plus zero: $\{0,1,2,3, \ldots\}$.The set of integers adds the opposites of the natural numbers to the set of whole numbers: $\{\ldots,-3,-2,-1,0,1,2,3$, ...\}. It is useful to note that the set of integers is made up of three distinct subsets: negative integers, zero, and positive integers. In this sense, the positive integers are just the natural numbers. Another way to think about it is that the natural numbers are a subset of the integers.

$$
\begin{array}{lll}
\text { negative integers } & \text { zero } & \text { positive integers } \\
\ldots,-3,-2,-1, & 0, & 1,2,3, \ldots
\end{array}
$$

The set of rational numbers is written as $\left\{\left.\frac{m}{n} \right\rvert\, m\right.$ and $n$ are integers and $\left.n \neq 0\right\}$. Notice from the definition that rational numbers are fractions (or quotients) containing integers in both the numerator and the denominator, and the denominator is never 0 . We can also see that every natural number, whole number, and integer is a rational number with a denominator of 1.

## WHY MUST THE DENOMINATOR OF A QUOTIENT NEVER BE 0?

Recall that division by zero is undefined.

- For any number $a, \frac{0}{a}=0$
- For any number $a, \frac{\stackrel{a}{0}}{0}=$ undefined

Because they are fractions, any rational number can also be expressed in decimal form. Any rational number can be represented as either:

1. a terminating decimal: $\frac{15}{8}=1.875$, or
2. a repeating decimal: $\frac{4}{11}=0.36363636 \cdots=0 . \overline{36}$

We use a line drawn over the repeating block of numbers instead of writing the group multiple times.

## EXAMPLE: WRITING INTEGERS AS RATIONAL NUMBERS

Write each of the following as a rational number.

1. 7
2. 0
3. -8

Answer
Write a fraction with the integer in the numerator and 1 in the denominator.

1. $7=\frac{7}{1}$
2. $0=\frac{0}{1}$
3. $-8=-\frac{8}{1}$

## TRY IT

Write each of the following as a rational number.

1. 11
2. 3
3. -4

Answer

1. $\frac{11}{1}$
2. $\frac{3}{1}$
3. $-\frac{4}{1}$

## EXAMPLE: IDENTIFYING RATIONAL NUMBERS

Write each of the following rational numbers as either a terminating or repeating decimal.

1. $-\frac{5}{7}$
2. $\frac{15}{5}$
3. $\frac{13}{25}$

## Answer

Write each fraction as a decimal by dividing the numerator by the denominator.

1. $-\frac{5}{7}=-0 . \overline{714285}$, a repeating decimal
2. $\frac{15}{5}=3$ (or 3.0 ), a terminating decimal
3. $\frac{13}{25}=0.52$, a terminating decimal

## Irrational Numbers

At some point in the ancient past, someone discovered that not all numbers are rational numbers. A builder, for instance, may have found that the diagonal of a square with unit sides was not 2 or even $\frac{3}{2}$, but was something else. Or a garment maker might have observed that the ratio of the circumference to the diameter of a roll of cloth was a little bit more than 3 , but still not a rational number. Such numbers are said to be irrational because they cannot be written as fractions. These numbers make up the set of irrational numbers. Irrational numbers cannot be expressed as a fraction of two integers. It is impossible to describe this set of numbers by a single rule except to say that a number is irrational if it is not rational. We can express this symbolically using a particular kind of notation called set notation as $\{h \mid h$ is not a rational number $\}$, that is, "the set of all numbers $h$ such that $h$ is not a rational number."

## Real Numbers

Given any number $n$, we know that $n$ is either rational or irrational. It cannot be both. The sets of rational and

## EXAMPLE: DIFFERENTIATING RATIONAL AND IRRATIONAL NUMBERS

Determine whether each of the following numbers is rational or irrational. If it is rational, determine whether it is a terminating or repeating decimal.

1. $\sqrt{25}$
2. $\frac{33}{9}$
3. $\sqrt{11}$
4. $\frac{17}{34}$
5. $0.3033033303333 \ldots$

## Answer

1. $\sqrt{25}$ : This can be simplified as $\sqrt{25}=5$. Therefore, $\sqrt{25}$ is rational.
2. $\frac{33}{9}$ : Because it is a fraction, $\frac{33}{9}$ is a rational number. Next, simplify and divide.

$$
\frac{33}{9}=\frac{11 \cdot 3}{3 \cdot 3}=\frac{11}{3}=3 . \overline{6}
$$

So, $\frac{33}{9}$ is rational and a repeating decimal.
3. $\sqrt{11}$ : This cannot be simplified any further. Therefore, $\sqrt{11}$ is an irrational number.
4. $\frac{17}{34}$ : Because it is a fraction, $\frac{17}{34}$ is a rational number. Simplify and divide.

$$
\frac{17}{34}=\frac{1 \cdot 17}{2 \cdot 17}=\frac{1}{2}=0.5
$$

So, $\frac{17}{34}$ is rational and a terminating decimal.
5. $0.3033033303333 \ldots$ is not a terminating decimal. Also note that there is no repeating pattern because the group of 3 s increases each time. Therefore it is neither a terminating nor a repeating decimal and, hence, not a rational number. It is an irrational number.

## TRY IT

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irrational numbers together make up the set of real numbers. As we saw with integers, the real numbers can be divided into three subsets: negative real numbers, zero, and positive real numbers. Each subset includes fractions, decimals, and irrational numbers according to their algebraic sign (+ or - ). Zero is considered neither positive nor negative.

The real numbers can be visualized on a horizontal number line with an arbitrary point chosen as 0 , with negative numbers to the left of 0 and positive numbers to the right of 0 . A fixed unit distance is then used to mark off each integer (or other basic value) on either side of 0 . Any real number corresponds to a unique position on the number line. The converse is also true: each location on the number line corresponds to exactly one real number. This is known as a one-to-one correspondence. We refer to this as the real number line.


The real number line

## EXAMPLE: CLASSIFYING REAL NUMBERS

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

1. $-\frac{10}{3}$
2. $\sqrt{5}$
3. $-\sqrt{289}$
4. $-6 \pi$
5. $0.615384615384 \ldots$

## Answer

1. $-\frac{10}{3}$ is negative and rational. It lies to the left of 0 on the number line.
2. $\sqrt{5}$ is positive and irrational. It lies to the right of 0 .
3. $-\sqrt{289}=-\sqrt{17^{2}}=-17$ is negative and rational. It lies to the left of 0 .
4. $-6 \pi$ is negative and irrational. It lies to the left of 0 .
5. $0.615384615384 \ldots$ is a repeating decimal so it is rational and positive. It lies to the right of 0 .

## TRY IT

Classify each number as either positive or negative and as either rational or irrational. Does the number lie to the left or the right of 0 on the number line?

1. $\sqrt{73}$
2. $-11.411411411 \ldots$
3. $\frac{47}{19}$
4. $-\frac{\sqrt{5}}{2}$
5. 6.210735

## Answer

1. positive, irrational; right
2. negative, rational; left
3. positive, rational; right
4. negative, irrational; left
5. positive, rational; right

## Sets of Numbers as Subsets

Beginning with the natural numbers, we have expanded each set to form a larger set, meaning that there is a subset relationship between the sets of numbers we have encountered so far. These relationships become more obvious when seen as a diagram.


Sets of numbers. $N$ : the set of natural numbers $W$ : the set of whole numbers $I$ : the set of integers $Q$ : the set of rational numbers $Q$ : the set of irrational numbers

## A GENERAL NOTE: SETS OF NUMBERS

The set of natural numbers includes the numbers used for counting: $\{1,2,3, \ldots\}$.
The set of whole numbers is the set of natural numbers plus zero: $\{0,1,2,3, \ldots\}$.
The set of integers adds the negative natural numbers to the set of whole numbers:
$\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}$.
The set of rational numbers includes fractions written as $\left\{\left.\frac{m}{n} \right\rvert\, m\right.$ and $n$ are integers and $\left.n \neq 0\right\}$.
The set of irrational numbers is the set of numbers that are not rational, are nonrepeating, and are nonterminating: $\{h \mid h$ is not a rational number $\}$.

Watch this video for an overview of the sets of numbers, and how to identify which set a number belongs to.

## EXAMPLE: DIFFERENTIATING THE SETS OF NUMBERS

Classify each number as being a natural number ( $N$ ), whole number ( $W$ ), integer ( $($ ), rational number $(Q)$, and/or irrational number ( $Q$ ).

1. $\sqrt{36}$
2. $\frac{8}{3}$
3. $\sqrt{73}$
4. -6
5. $3.2121121112 \ldots$

Answer

| N | W | I | Q | Q' |
| :--- | :--- | :--- | :--- | :--- |
| 1. $\sqrt{36}=6$ | X | X | X | X |
| 2. $\frac{8}{3}=2 . \overline{6}$ |  |  | X |  |
| 3. $\sqrt{73}$ |  |  |  |  |
| 4. -6 |  |  |  | X |
| 5. $3.2121121112 \ldots$ | X | X |  |  |

## TRY IT

Classify each number as being a natural number ( $N$ ), whole number ( $W$ ), integer ( $($ ), rational number $(Q)$, and/or irrational number ( $Q$ ).
a. $-\frac{35}{7}$
b. 0
c. $\sqrt{169}$
d. $\sqrt{24}$
e. $4.763763763 \ldots$

Answer

| $\boldsymbol{N}$ | W | $\boldsymbol{I}$ | Q | Q' |
| :--- | :--- | :--- | :--- | :--- |
| a. $-\frac{35}{7}$ | X | X |  |  |
| b. 0 | X | X | X |  |
| c. $\sqrt{169}$ | X | X | X | X |
| d. $\sqrt{24}$ | X |  |  |  |
| e. $4.763763763 \ldots$ | X |  |  |  |

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## PROPERTIES OF REAL NUMBERS

## LEARNING OUTCOMES

- Use the order of operations to simplify an algebraic expression.
- Use properties of real numbers to simplify algebraic expressions.

When we multiply a number by itself, we square it or raise it to a power of 2 . For example, $4^{2}=4 \cdot 4=16$. We can raise any number to any power. In general, the exponential notation $a^{n}$ means that the number or variable $a$ is used as a factor $n$ times.

$$
a^{n}=a \cdot a \cdot a \cdots \cdots a
$$

In this notation, $a^{n}$ is read as the $n$th power of $a$, where $a$ is called the base and $n$ is called the exponent. A term in exponential notation may be part of a mathematical expression, which is a combination of numbers and operations. For example, $24+6 \cdot \frac{2}{3}-4^{2}$ is a mathematical expression.

To evaluate a mathematical expression, we perform the various operations. However, we do not perform them in any random order. We use the order of operations. This is a sequence of rules for evaluating such expressions.

Recall that in mathematics we use parentheses ( ), brackets [ ], and braces \{ \} to group numbers and expressions so that anything appearing within the symbols is treated as a unit. Additionally, fraction bars, radicals, and absolute value bars are treated as grouping symbols. When evaluating a mathematical expression, begin by simplifying expressions within grouping symbols.

The next step is to address any exponents or radicals. Afterward, perform multiplication and division from left to right and finally addition and subtraction from left to right.

Let's take a look at the expression provided.

$$
24+6 \cdot \frac{2}{3}-4^{2}
$$

There are no grouping symbols, so we move on to exponents or radicals. The number 4 is raised to a power of 2 , so simplify $4^{2}$ as 16 .

$$
\begin{aligned}
& 24+6 \cdot \frac{2}{3}-4^{2} \\
& 24+6 \cdot \frac{2}{3}-16
\end{aligned}
$$

Next, perform multiplication or division, left to right.

$$
\begin{gathered}
24+6 \cdot \frac{2}{3}-16 \\
24+4-16
\end{gathered}
$$

Lastly, perform addition or subtraction, left to right.

$$
\begin{gathered}
24+4-16 \\
28-16
\end{gathered}
$$

Therefore, $24+6 \cdot \frac{2}{3}-4^{2}=12$.

## HOW WAS THE FRACTION ABOVE SIMPLIFIED?

Recall, when mulitplying fractions, we multiply the numerators together and place the result over the product of the denominators.

- $\frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}$
- $6 \cdot \frac{2}{3}=\frac{6}{1} \cdot \frac{2}{3}=\frac{12}{3}=4$

For some complicated expressions, several passes through the order of operations will be needed. For instance, there may be a radical expression inside parentheses that must be simplified before the parentheses are evaluated. Following the order of operations ensures that anyone simplifying the same mathematical expression will get the same result.

## A GENERAL NOTE: ORDER OF OPERATIONS

Operations in mathematical expressions must be evaluated in a systematic order, which can be simplified using the acronym PEMDAS:
$\mathbf{P}$ (arentheses)
E(xponents)

## HOW TO: GIVEN A MATHEMATICAL EXPRESSION, SIMPLIFY IT USING THE ORDER OF OPERATIONS.

1. Simplify any expressions within grouping symbols.
2. Simplify any expressions containing exponents or radicals.
3. Perform any multiplication and division in order, from left to right.
4. Perform any addition and subtraction in order, from left to right.

Watch the following video for more examples of using the order of operations to simplify an expression.

## Using Properties of Real Numbers

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

## Commutative Properties

The commutative property of addition states that numbers may be added in any order without affecting the sum.

$$
a+b=b+a
$$

We can better see this relationship when using real numbers.

$$
(-2)+7=5 \text { and } 7+(-2)=5
$$

Similarly, the commutative property of multiplication states that numbers may be multiplied in any order without affecting the product.

$$
a \cdot b=b \cdot a
$$

Again, consider an example with real numbers.

$$
(-11) \cdot(-4)=44 \text { and }(-4) \cdot(-11)=44
$$

It is important to note that neither subtraction nor division is commutative. For example, $17-5$ is not the same as $5-17$. Similarly, $20 \div 5 \neq 5 \div 20$.

## Associative Properties

The associative property of multiplication tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.

$$
a(b c)=(a b) c
$$

Consider this example.

$$
(3 \cdot 4) \cdot 5=60 \text { and } 3 \cdot(4 \cdot 5)=60
$$

The associative property of addition tells us that numbers may be grouped differently without affecting the sum.

$$
a+(b+c)=(a+b)+c
$$

## EXAMPLE: USING THE ORDER OF OPERATIONS

Use the order of operations to evaluate each of the following expressions.

1. $(3 \cdot 2)^{2}-4(6+2)$
2. $\frac{5^{2}-4}{7}-\sqrt{11-2}$
3. $6-|5-8|+3(4-1)$
4. $\frac{14-3 \cdot 2}{2 \cdot 5-3^{2}}$
5. $7(5 \cdot 3)-2\left[(6-3)-4^{2}\right]+1$

## Answer

1. 

$$
\begin{aligned}
(3 \cdot 2)^{2} & =(6)^{2}-4(8) & & \text { Simplify parentheses } \\
& =36-4(8) & & \text { Simplify exponent } \\
& =36-32 & & \text { Simplify multiplication } \\
& =4 & & \text { Simplify subtraction }
\end{aligned}
$$

2. 

$$
\begin{aligned}
\frac{5^{2}-4}{7}-\sqrt{11-2} & =\frac{5^{2}-4}{7}-\sqrt{9} & & \text { Simplify grouping systems (radical) } \\
& =\frac{5^{2}-4}{7}-3 & & \text { Simplify radical } \\
& =\frac{25-4}{7}-3 & & \text { Simplify exponent } \\
& =\frac{21}{7}-3 & & \text { Simplify subtraction in numerator } \\
& =3-3 & & \text { Simplify division } \\
& =0 & & \text { Simplify subtraction }
\end{aligned}
$$

Note that in the first step, the radical is treated as a grouping symbol, like parentheses. Also, in the third step, the fraction bar is considered a grouping symbol so the numerator is considered to be grouped.
3.
$6-|5-8|+3(4-1)=6-|-3|+3(3) \quad$ Simplify inside grouping system
$=6-3+3(3) \quad$ Simplify absolute value
$=6-3+9 \quad$ Simplify multiplication
$=3+9 \quad$ Simplify subtraction
$=12 \quad$ Simplify addition
4.

$$
\begin{aligned}
\frac{14-3 \cdot 2}{2 \cdot 5-3^{2}} & =\frac{14-3 \cdot 2}{2 \cdot 5-9} & & \text { Simplify exponent } \\
& =\frac{14-6}{10-9} & & \text { Simplify products } \\
& =\frac{8}{1} & & \text { Simplify quotient } \\
& =8 & & \text { Simplify quotient }
\end{aligned}
$$

In this example, the fraction bar separates the numerator and denominator, which we simplify separately until the last step.
5.
$7(5 \cdot 3)-2\left[(6-3)-4^{2}\right]+1=7(15)-2\left[(3)-4^{2}\right]+1 \quad$ Simplify inside parentheses

| $7(15)-2(3-16)+1$ |  |
| :--- | :--- |
| $=7(15)-2(-13)+1$ |  |
| $=105+26+1$ |  |
| $=132$ |  |
| $=1$ | MultiplyAdd |

## TRY IT

Use the order of operations to evaluate the following expression:
$\sqrt{(21-5)}-2\left(7+9^{2}\right)+\frac{23-5}{3^{2}}$
Answer

$$
\begin{aligned}
& =\sqrt{(4)}-2(7+81)+\frac{23-5}{3^{2}} \\
& =\sqrt{(4)}-2(88)+\frac{18}{9} \\
& =2-176+2 \\
& =-172
\end{aligned}
$$

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## RECALL: SIMPLIFYING FRACTIONS

To simplify a fraction, look for common factors in the numerator and the denominator. In the video below, a fraction, $\frac{14}{26}$ must be simplified. Since $\frac{14}{26}=\frac{2 \cdot 7}{2 \cdot 13}$, and since $\frac{2}{2}=1$, this fraction simplifies to $\frac{7}{13}$.


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This property can be especially helpful when dealing with negative integers. Consider this example.

$$
[15+(-9)]+23=29 \text { and } 15+[(-9)+23]=29
$$

Are subtraction and division associative? Review these examples.

$$
\begin{aligned}
8-(3-15) & \stackrel{?}{=}(8-3)-15 \\
8-(-12) & \stackrel{?}{=} 5-15 \\
20 & \neq 20-10 \\
64 \div(8 \div 4) & \stackrel{?}{=}(64 \div 8) \div 4 \\
64 \div 2 & \stackrel{?}{=} 8 \div 4 \\
32 & \neq 2
\end{aligned}
$$

As we can see, neither subtraction nor division is associative.

## Distributive Property

The distributive property states that the product of a factor times a sum is the sum of the factor times each term in the sum.

$$
a \cdot(b+c)=a \cdot b+a \cdot c
$$

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.


Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12 , multiplying it by 7 , and adding the products.

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.

$$
\begin{aligned}
6+(3 \cdot 5) & \stackrel{?}{=}(6+3) \cdot(6+5) \\
6+(15) & \stackrel{?}{=}(9) \cdot(11) \\
21 & \neq 99
\end{aligned}
$$

Multiplication does not distribute over subtraction, and division distributes over neither addition nor subtraction.

A special case of the distributive property occurs when a sum of terms is subtracted.

$$
a-b=a+(-b)
$$

For example, consider the difference $12-(5+3)$. We can rewrite the difference of the two terms 12 and $(5+3)$ by turning the subtraction expression into addition of the opposite. So instead of subtracting $(5+3)$, we add the opposite.

$$
12+(-1) \cdot(5+3)
$$

Now, distribute -1 and simplify the result.

$$
\begin{aligned}
12+(-1) \cdot(5+3) & =12+[(-1) \cdot 5+(-1) \cdot 3] \\
& =12+(-5-3) \\
& =12+(-8) \\
& =4
\end{aligned}
$$

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.

$$
\begin{aligned}
12-(5+3) & =12+(-5-3) \\
& =12+(-8) \\
& =4
\end{aligned}
$$

## Identity Properties

The identity property of addition states that there is a unique number, called the additive identity (0) that, when added to a number, results in the original number.

$$
a+0=a
$$

The identity property of multiplication states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.

$$
a \cdot 1=a
$$

For example, we have $(-6)+0=-6$ and $23 \cdot 1=23$. There are no exceptions for these properties; they work for every real number, including 0 and 1.

## Inverse Properties

The inverse property of addition states that, for every real number $a$, there is a unique number, called the additive inverse (or opposite), denoted-a, that, when added to the original number, results in the additive identity, 0.

$$
a+(-a)=0
$$

For example, if $a=-8$, the additive inverse is 8 , since $(-8)+8=0$.
The inverse property of multiplication holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number a, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, that, when multiplied by the original number, results in the multiplicative identity, 1.

$$
a \cdot \frac{1}{a}=1
$$

For example, if $a=-\frac{2}{3}$, the reciprocal, denoted $\frac{1}{a}$, is $-\frac{3}{2}$ because

$$
a \cdot \frac{1}{a}=\left(-\frac{2}{3}\right) \cdot\left(-\frac{3}{2}\right)=1
$$

## A GENERAL NOTE: PROPERTIES OF REAL NUMBERS

The following properties hold for real numbers $a, b$, and $c$.

|  | Addition | Multiplication |
| :---: | :---: | :---: |
| Commutative Property | $a+b=b+a$ | $a \cdot b=b \cdot a$ |
| Associative Property | $a+(b+c)=(a+b)+c$ | $a(b c)=(a b) c$ |
| Distributive Property | $a \cdot(b+c)=a \cdot b+a \cdot c$ |  |
| Identity Property | There exists a unique real number called the additive identity, 0 , such that, for any real number $a$ $a+0=a$ | There exists a unique real number called the multiplicative identity, 1 , such that, for any real number a $a \cdot 1=a$ |
| Inverse Property | Every real number a has an additive inverse, or opposite, denoted $-a$, such that $a+(-a)=0$ | Every nonzero real number a has a multiplicative inverse, or reciprocal, denoted $\frac{1}{a}$, such that $a \cdot\left(\frac{1}{a}\right)=1$ |

## USING THE INVERSE PROPERTY OF NUMBERS: SIMPLIFYING FRACTIONS

Recall when simplifying fractions, we look for common factors in the numerator and denominator to "cancel out." What we mean by that is that common factors, $\frac{a}{a}$, divide to the number 1 . This is based on the inverse property of multiplication.
Because the inverse property states that
$a \cdot\left(\frac{1}{a}\right)=1$,
and because fraction multiplication gives that
$\frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}$,
We have
$\frac{a}{1} \cdot\left(\frac{1}{a}\right)=\frac{a}{a}=1$

## EXAMPLE: USING PROPERTIES OF REAL NUMBERS

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

1. $3 \cdot 6+3 \cdot 4$
2. $(5+8)+(-8)$
3. $6-(15+9)$
4. $\frac{4}{7} \cdot\left(\frac{2}{3} \cdot \frac{7}{4}\right)$
5. $100 \cdot[0.75+(-2.38)]$

## Answer

1. 

$$
\begin{aligned}
3 \cdot 6+3 \cdot 4 & =3 \cdot(6+4) & & \text { Distributive property } \\
& =3 \cdot 10 & & \text { Simplify } \\
& =30 & & \text { Simplify }
\end{aligned}
$$

2. 

$$
\begin{aligned}
(5+8)+(-8) & =5+[8+(-8)] & & \text { Associative property of addition } \\
& =5+0 & & \text { Inverse property of addition } \\
& =5 & & \text { Identity property of addition }
\end{aligned}
$$

3. 

$6-(15+9)=6+(-15-9) \quad$ Distributive property

$$
\begin{array}{ll}
=6+(-24) & \text { Simplify } \\
=-18 & \text { Simplify }
\end{array}
$$

4. 

$\frac{4}{7} \cdot\left(\frac{2}{3} \cdot \frac{7}{4}\right)=\frac{4}{7} \cdot\left(\frac{7}{4} \cdot \frac{2}{3}\right) \quad$ Commutative property of multiplication

$$
=\left(\frac{4}{7} \cdot \frac{7}{4}\right) \cdot \frac{2}{3} \quad \text { Associative property of multiplication }
$$

$$
=1 \cdot \frac{2}{3} \quad \text { Inverse property of multiplication }
$$

$$
=\frac{2}{3} \quad \text { Identity property of multiplication }
$$

5. 

$$
\begin{aligned}
100 \cdot[0.75+(-2.38)] & =100 \cdot 0.75+100 \cdot(-2.38) & & \text { Distributive property } \\
& =75+(-238) & & \text { Simplify } \\
& =-163 & & \text { Simplify }
\end{aligned}
$$

## TRY IT

Use the properties of real numbers to rewrite and simplify each expression. State which properties apply.

1. $\left(-\frac{23}{5}\right) \cdot\left[11 \cdot\left(-\frac{5}{23}\right)\right]$
2. $5 \cdot(6.2+0.4)$
3. $18-(7-15)$
4. $\frac{17}{18}+\cdot\left[\frac{4}{9}+\left(-\frac{17}{18}\right)\right]$
5. $6 \cdot(-3)+6 \cdot 3$

## Answer

1. 11, commutative property of multiplication, associative property of multiplication, inverse property of multiplication, identity property of multiplication;
2. 33, distributive property;
3. 26, distributive property;
4. $\frac{4}{9}$, commutative property of addition, associative property of addition, inverse property of addition, identity property of addition;
5. 0 , distributive property, inverse property of addition, identity property of addition

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## EVALUATE AND SIMPLIFY ALGEBRAIC EXPRESSIONS

## LEARNING OUTCOMES

- List the constants and variables in an algebraic expression.
- Evaluate an algebraic expression.
- Use an algebraic formula.

So far, the mathematical expressions we have seen have involved real numbers only. In mathematics, we may see expressions such as $x+5, \frac{4}{3} \pi r^{3}$, or $\sqrt{2 m^{3} n^{2}}$. In the expression $x+5,5$ is called a constant because it does not vary and $x$ is called a variable because it does. (In naming the variable, ignore any exponents or radicals containing the variable.) An algebraic expression is a collection of constants and variables joined together by the algebraic operations of addition, subtraction, multiplication, and division.

We have already seen some real number examples of exponential notation, a shorthand method of writing products of the same factor. When variables are used, the constants and variables are treated the same way.

$$
\begin{array}{ll}
(-3)^{5}=(-3) \cdot(-3) \cdot(-3) \cdot(-3) \cdot(-3) & x^{5}=x \cdot x \cdot x \cdot x \cdot x \\
(2 \cdot 7)^{3}=(2 \cdot 7) \cdot(2 \cdot 7) \cdot(2 \cdot 7) & (y z)^{3}=(y z) \cdot(y z) \cdot(y z)
\end{array}
$$

In each case, the exponent tells us how many factors of the base to use, whether the base consists of constants or variables.

Any variable in an algebraic expression may take on or be assigned different values. When that happens, the value of the algebraic expression changes. To evaluate an algebraic expression means to determine the value of the expression for a given value of each variable in the expression. Replace each variable in the expression with the given value, then simplify the resulting expression using the order of operations. If the algebraic expression contains more than one variable, replace each variable with its assigned value and simplify the expression as before.

## EXAMPLE: DESCRIBING ALGEBRAIC EXPRESSIONS

List the constants and variables for each algebraic expression.

1. $x+5$
2. $\frac{4}{3} \pi r^{3}$
3. $\sqrt{2 m^{3} n^{2}}$

Answer

|  | Constants | Variables |
| :--- | :--- | :--- |
| 1. $x+5$ | 5 | $x$ |
| 2. $\frac{4}{3} \pi r^{3}$ | $\frac{4}{3}, \pi$ | $r$ |
| 3. $\sqrt{2 m^{3} n^{2}}$ | 2 | $m, n$ |

## TRY IT

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In the following video we present more examples of how to evaluate an expression for a given value.

## Formulas

An equation is a mathematical statement indicating that two expressions are equal. The expressions can be numerical or algebraic. The equation is not inherently true or false, but only a proposition. The values that make the equation true, the solutions, are found using the properties of real numbers and other results. For example, the equation $2 x+1=7$ has the unique solution $x=3$ because when we substitute 3 for $x$ in the equation, we obtain the true statement $2(3)+1=7$.

A formula is an equation expressing a relationship between constant and variable quantities. Very often the equation is a means of finding the value of one quantity (often a single variable) in terms of another or other

## EXAMPLE: EVALUATING AN ALGEBRAIC EXPRESSION AT DIFFERENT VALUES

Evaluate the expression $2 x-7$ for each value for $x$.

1. $x=0$
2. $x=1$
3. $x=\frac{1}{2}$
4. $x=-4$

Answer

1. Substitute 0 for $x$.

$$
\begin{aligned}
2 x-7 & =2(0)-7 \\
& =0-7 \\
& =-7
\end{aligned}
$$

2. Substitute 1 for $x$.

$$
\begin{aligned}
2 x-7 & =2(1)-7 \\
& =2-7 \\
& =-5
\end{aligned}
$$

3. Substitute $\frac{1}{2}$ for $x$.

$$
\begin{aligned}
2 x-7 & =2\left(\frac{1}{2}\right)-7 \\
& =1-7 \\
& =-6
\end{aligned}
$$

4. Substitute -4 for $x$.

$$
\begin{aligned}
2 x-7 & =2(-4)-7 \\
& =-8-7 \\
& =-15
\end{aligned}
$$

## TRY IT

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quantities. One of the most common examples is the formula for finding the area $A$ of a circle in terms of the radius $r$ of the circle: $A=\pi r^{2}$. For any value of $r$, the area $A$ can be found by evaluating the expression $\pi r^{2}$.

## Simplify Algebraic Expressions

Sometimes we can simplify an algebraic expression to make it easier to evaluate or to use in some other way. To do so, we use the properties of real numbers. We can use the same properties in formulas because they contain algebraic expressions.

## EXAMPLE: EVALUATING ALGEBRAIC EXPRESSIONS

Evaluate each expression for the given values.

1. $x+5$ for $x=-5$
2. $\frac{t}{2 t-1}$ for $t=10$
3. $\frac{4}{3} \pi r^{3}$ for $r=5$
4. $a+a b+b$ for $a=11, b=-8$
5. $\sqrt{2 m^{3} n^{2}}$ for $m=2, n=3$

## Answer

1. Substitute -5 for $x$.

$$
\begin{aligned}
x+5 & =(-5)+5 \\
& =0
\end{aligned}
$$

2. Substitute 10 for $t$.

$$
\begin{aligned}
\frac{t}{2 t-1} & =\frac{(10)}{2(10)-1} \\
& =\frac{10}{20-1} \\
& =\frac{10}{19}
\end{aligned}
$$

3. Substitute 5 for $r$.

$$
\begin{aligned}
\frac{4}{3} \pi r^{3} & =\frac{4}{3} \pi(5)^{3} \\
& =\frac{4}{3} \pi(125) \\
& =\frac{500}{3} \pi
\end{aligned}
$$

4. Substitute 11 for $a$ and -8 for $b$.

$$
\begin{aligned}
a+a b+b & =(11)+(11)(-8)+(-8) \\
& =11-8-8 \\
& =-85
\end{aligned}
$$

5. Substitute 2 for $m$ and 3 for $n$.

$$
\begin{aligned}
\sqrt{2 m^{3} n^{2}} & =\sqrt{2(2)^{3}(3)^{2}} \\
& =\sqrt{2(8)(9)} \\
& =\sqrt{144} \\
& =12
\end{aligned}
$$

## BE CAREFUL WHEN SIMPLIFYING FRACTIONS!

Why does the fraction $\frac{(25)}{3(25)-1}$ not simplify to $\frac{(25)}{3(25)-1}=\frac{1}{3-1}=\frac{1}{2}$ ?
Using the inverse property of multiplication, we are permitted to "cancel out" common factors in the numerator and denominator such that $\frac{a}{a}=1$.

But be careful! We have no rule that allows us to cancel numbers in the top and bottom of a fractions that are contained in sums or differences. You'll see this idea reappear frequently throughout the course.

## TRY IT

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## EXAMPLE: USING A FORMULA

A right circular cylinder with radius $r$ and height $h$ has the surface area $S$ (in square units) given by the formula $S=2 \pi r(r+h)$. Find the surface area of a cylinder with radius 6 in . and height 9 in . Leave the answer in terms of $\pi$.


Right circular cylinder

## Answer

Evaluate the expression $2 \pi r(r+h)$ for $r=6$ and $h=9$.

$$
\begin{aligned}
S & =2 \pi r(r+h) \\
& =2 \pi(6)[(6)+(9)] \\
& =2 \pi(6)(15) \\
& =180 \pi
\end{aligned}
$$

The surface area is $180 \pi$ square inches.

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Figure 4
A photograph with length $L$ and width $W$ is placed in a matte of width 8 centimeters (cm). The area of the matte (in square centimeters, or $\mathrm{cm}^{2}$ ) is found to be $A=(L+16)(W+16)-L \cdot W$. Find the area of a matte for a photograph with length 32 cm and width 24 cm .
Answer
$1,152 \mathrm{~cm}^{2}$
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## RECALL: OPERATIONS ON FRACTIONS

When simplifying algebraic expressions, we may sometimes need to add, subtract, simplify, multiply, or divide fractions. It is important to be able to do these operations on the fractions without converting them to decimals.
To multiply fractions, multiply the numerators and place them over the product of the denominators.
$\frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}$
To divide fractions, multiply the first by the reciprocal of the second.
$\frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}=\frac{a d}{b c}$
To simplify fractions, find common factors in the numerator and denominator that cancel.
$\frac{24}{32}=\frac{2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}=\frac{3}{2 \cdot 2}=\frac{3}{4}$
To add or subtract fractions, first rewrite each fraction as an equivalent fraction such that each has a common denominator, then add or subtract the numerators and place the result over the common denominator.
$\frac{a}{b} \pm \frac{c}{d}=\frac{a d \pm b c}{b d}$

## EXAMPLE: SIMPLIFYING ALGEBRAIC EXPRESSIONS

Simplify each algebraic expression.

1. $3 x-2 y+x-3 y-7$
2. $2 r-5(3-r)+4$
3. $\left(4 t-\frac{5}{4} s\right)-\left(\frac{2}{3} t+2 s\right)$
4. $2 m n-5 m+3 m n+n$

## Answer

$3 x-2 y+x-3 y-7=3 x+x-2 y-3 y-7 \quad$ Commutative property of addition
1.

$$
=4 x-5 y-7
$$

Simplify

$$
\begin{aligned}
2 r-5(3-r)+4 & =2 r-15+5 r+4 & & \text { Distributive property } \\
& =2 r+5 r-15+4 & & \text { Commutative property of addition } \\
& =7 r-11 & & \text { Simplify }
\end{aligned}
$$

2. 

$$
\begin{aligned}
4 t-\frac{5}{4} s-\left(\frac{2}{3} t+2 s\right) & =4 t-\frac{5}{4} s-\frac{2}{3} t-2 s & & \text { Distributive property } \\
& =4 t-\frac{2}{3} t-\frac{5}{4} s-2 s & & \text { Commutative property of addition }
\end{aligned}
$$

$$
\text { 3. } \quad=\frac{12}{3} t-\frac{2}{3} t-\frac{5}{4} s-\frac{8}{4} s \quad \text { Common Denominators }
$$

$$
=\frac{10}{3} t-\frac{13}{4} s \quad \text { Simplify }
$$

$$
\text { 4. } \begin{array}{rlrl}
m n-5 m+3 m n+n & =2 m n+3 m n-5 m+n \\
& =5 m n-5 m+n & & \\
\text { Commutative property of addition } \\
\text { Simplify }
\end{array}
$$

## TRY IT

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## EXAMPLE: SIMPLIFYING A FORMULA

A rectangle with length $L$ and width $W$ has a perimeter $P$ given by $P=L+W+L+W$. Simplify this expression.
Answer

$$
\begin{array}{ll}
P=L+W+L+W & \\
P=L+L+W+W & \text { Commutative property of addition } \\
P=2 L+2 W & \text { Simplify } \\
P=2(L+W) & \text { Distributive property }
\end{array}
$$

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## SUMMARY: REAL NUMBERS

## Key Concepts

- Rational numbers may be written as fractions or terminating or repeating decimals.
- Determine whether a number is rational or irrational by writing it as a decimal.
- The rational numbers and irrational numbers make up the set of real numbers. A number can be classified as natural, whole, integer, rational, or irrational.
- The order of operations is used to evaluate expressions.
- The real numbers under the operations of addition and multiplication obey basic rules, known as the properties of real numbers. These are the commutative properties, the associative properties, the distributive property, the identity properties, and the inverse properties.
- Algebraic expressions are composed of constants and variables that are combined using addition, subtraction, multiplication, and division. They take on a numerical value when evaluated by replacing variables with constants.
- Formulas are equations in which one quantity is represented in terms of other quantities. They may be simplified or evaluated as any mathematical expression.


## Glossary

algebraic expression constants and variables combined using addition, subtraction, multiplication, and division
associative property of addition the sum of three numbers may be grouped differently without affecting the result; in symbols, $a+(b+c)=(a+b)+c$
associative property of multiplication the product of three numbers may be grouped differently without affecting the result; in symbols, $a \cdot(b \cdot c)=(a \cdot b) \cdot c$
base in exponential notation, the expression that is being multiplied
commutative property of addition two numbers may be added in either order without affecting the result; in symbols, $a+b=b+a$
commutative property of multiplication two numbers may be multiplied in any order without affecting the result; in symbols, $a \cdot b=b \cdot a$
constant a quantity that does not change value
distributive property the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols, $a \cdot(b+c)=a \cdot b+a \cdot c$
equation a mathematical statement indicating that two expressions are equal
exponent in exponential notation, the raised number or variable that indicates how many times the base is being multiplied
exponential notation a shorthand method of writing products of the same factor
formula an equation expressing a relationship between constant and variable quantities
identity property of addition there is a unique number, called the additive identity, 0 , which, when added to a number, results in the original number; in symbols, $a+0=a$
identity property of multiplication there is a unique number, called the multiplicative identity, 1 , which, when multiplied by a number, results in the original number; in symbols, $a \cdot 1=a$
integers the set consisting of the natural numbers, their opposites, and 0 : $\{\ldots,-3,-2,-1,0,1,2,3, \ldots\}$
inverse property of addition for every real number $a$, there is a unique number, called the additive inverse (or opposite), denoted $-a$, which, when added to the original number, results in the additive identity, 0 ; in symbols, $a+(-a)=0$
inverse property of multiplication for every non-zero real number $a$, there is a unique number, called the multiplicative inverse (or reciprocal), denoted $\frac{1}{a}$, which, when multiplied by the original number, results in the multiplicative identity, 1 ; in symbols, $a \cdot \frac{1}{a}=1$
irrational numbers the set of all numbers that are not rational; they cannot be written as either a terminating or repeating decimal; they cannot be expressed as a fraction of two integers
natural numbers the set of counting numbers: $\{1,2,3, \ldots\}$
order of operations a set of rules governing how mathematical expressions are to be evaluated, assigning priorities to operations
rational numbers the set of all numbers of the form $\frac{m}{n}$, where $m$ and $n$ are integers and $n \neq 0$. Any rational number may be written as a fraction or a terminating or repeating decimal.
real number line a horizontal line used to represent the real numbers. An arbitrary fixed point is chosen to represent 0 ; positive numbers lie to the right of 0 and negative numbers to the left.
real numbers the sets of rational numbers and irrational numbers taken together
variable a quantity that may change value
whole numbers the set consisting of 0 plus the natural numbers: $\{0,1,2,3, \ldots\}$

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# INTRODUCTION TO EXPONENTS AND SCIENTIFIC NOTATION 

## Learning Outcomes

By the end of this section, you will be able to:

- Use the product rule of exponents.
- Use the quotient rule of exponents.
- Use the power rule of exponents.
- Use the zero exponent rule of exponents.
- Use the negative rule of exponents.
- Find the power of a product and a quotient.
- Simplify exponential expressions.
- Use scientific notation.

Mathematicians, scientists, and economists commonly encounter very large and very small numbers. But it may not be obvious how common such figures are in everyday life. For instance, a pixel is the smallest unit of light that can be perceived and recorded by a digital camera. A particular camera might record an image that is 2,048 pixels by 1,536 pixels, which is a very high resolution picture. It can also perceive a color depth (gradations in colors) of up to 48 bits per frame, and can shoot the equivalent of 24 frames per second. The maximum possible number of bits of information used to film a one-hour (3,600-second) digital film is then an extremely large number.

Using a calculator, we enter $2,048 \times 1,536 \times 48 \times 24 \times 3,600$ and press ENTER. The calculator displays 1.304596316 E 13 . What does this mean? The "E13" portion of the result represents the exponent 13 of ten, so there are a maximum of approximately $1.3 \times 10^{13}$ bits of data in that one-hour film. In this section, we review rules of exponents first and then apply them to calculations involving very large or small numbers.

## RULES FOR EXPONENTS

## LEARNING OUTCOMES

- Use the product rule for exponents.
- Use the quotient rule for exponents.
- Use the power rule for exponents.

Consider the product $x^{3} \cdot x^{4}$. Both terms have the same base, $x$, but they are raised to different exponents. Expand each expression, and then rewrite the resulting expression.

$$
\begin{aligned}
& x^{3} \cdot x^{4}=(x \cdot x \cdot x)(x \cdot x \cdot x \cdot x) \\
&=x \cdot x \cdot x^{3 \text { factors factors }} \quad \begin{array}{l}
4 \text { factors } \\
\\
\end{array} \\
&=x^{7}
\end{aligned}
$$

The result is that $x^{3} \cdot x^{4}=x^{3+4}=x^{7}$.
Notice that the exponent of the product is the sum of the exponents of the terms. In other words, when multiplying exponential expressions with the same base, we write the result with the common base and add the exponents. This is the product rule of exponents.

$$
a^{m} \cdot a^{n}=a^{m+n}
$$

Now consider an example with real numbers.

$$
2^{3} \cdot 2^{4}=2^{3+4}=2^{7}
$$

We can always check that this is true by simplifying each exponential expression. We find that $2^{3}$ is $8,2^{4}$ is 16 , and $2^{7}$ is 128 . The product $8 \cdot 16$ equals 128 , so the relationship is true. We can use the product rule of exponents to simplify expressions that are a product of two numbers or expressions with the same base but different exponents.

## A GENERAL NOTE: THE PRODUCT RULE OF EXPONENTS

For any real number $a$ and natural numbers $m$ and $n$, the product rule of exponents states that

$$
a^{m} \cdot a^{n}=a^{m+n}
$$

In the following video we show more examples of how to use the product rule to simplify an expression with exponents.

## EXAMPLE: USING THE PRODUCT RULE

Write each of the following products with a single base. Do not simplify further.

1. $t^{5} \cdot t^{3}$
2. $(-3)^{5} \cdot(-3)$
3. $x^{2} \cdot x^{5} \cdot x^{3}$

## Answer

Use the product rule to simplify each expression.

1. $t^{5} \cdot t^{3}=t^{5+3}=t^{8}$
2. $(-3)^{5} \cdot(-3)=(-3)^{5} \cdot(-3)^{1}=(-3)^{5+1}=(-3)^{6}$
3. $x^{2} \cdot x^{5} \cdot x^{3}$

At first, it may appear that we cannot simplify a product of three factors. However, using the associative property of multiplication, begin by simplifying the first two.

$$
x^{2} \cdot x^{5} \cdot x^{3}=\left(x^{2} \cdot x^{5}\right) \cdot x^{3}=\left(x^{2+5}\right) \cdot x^{3}=x^{7} \cdot x^{3}=x^{7+3}=x^{10}
$$

Notice we get the same result by adding the three exponents in one step.

$$
x^{2} \cdot x^{5} \cdot x^{3}=x^{2+5+3}=x^{10}
$$

## TRY IT

Write each of the following products with a single base. Do not simplify further.

1. $k^{6} \cdot k^{9}$
2. $\left(\frac{2}{y}\right)^{4} \cdot\left(\frac{2}{y}\right)$
3. $t^{3} \cdot t^{6} \cdot t^{5}$

## Answer

1. $k^{15}$
2. $\left(\frac{2}{y}\right)^{5}$
3. $t^{14}$

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## Using the Quotient Rule of Exponents

The quotient rule of exponents allows us to simplify an expression that divides two numbers with the same base but different exponents. In a similar way to the product rule, we can simplify an expression such as $\frac{y^{m}}{y^{n}}$ , where $m>n$. Consider the example $\frac{y^{9}}{y^{5}}$. Perform the division by canceling common factors.


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$$
\begin{aligned}
\frac{y^{9}}{y^{5}} & =\frac{y \cdot y \cdot y \cdot y \cdot y \cdot y \cdot y}{y \cdot y \cdot y \cdot y \cdot y} \\
& =\frac{\not y \cdot \not y \cdot \not y \cdot \not y \cdot \not y \cdot y \cdot y \cdot y \cdot y}{\not y \cdot \not y \cdot \not y \cdot \not y \cdot \not y} \\
& =\frac{y \cdot y \cdot y \cdot y}{1} \\
& =y^{4}
\end{aligned}
$$

## RECALL

When simplifying fractions, "canceling out" always leaves a 1 behind because $\frac{a}{a}=1$.
Notice that the exponent of the quotient is the difference between the exponents of the divisor and dividend.

$$
\frac{a^{m}}{a^{n}}=a^{m-n}
$$

In other words, when dividing exponential expressions with the same base, we write the result with the common base and subtract the exponents.

$$
\frac{y^{9}}{y^{5}}=y^{9-5}=y^{4}
$$

For the time being, we must be aware of the condition $m>n$. Otherwise, the difference $m-n$ could be zero or negative. Those possibilities will be explored shortly. Also, instead of qualifying variables as nonzero each time, we will simplify matters and assume from here on that all variables represent nonzero real numbers.

## A GENERAL NOTE: THE QUOTIENT RULE OF EXPONENTS

For any real number $a$ and natural numbers $m$ and $n$, such that $m>n$, the quotient rule of exponents states that

$$
\frac{a^{m}}{a^{n}}=a^{m-n}
$$

## EXAMPLE: USING THE QUOTIENT RULE

Write each of the following products with a single base. Do not simplify further.

1. $\frac{(-2)^{14}}{(-2)^{9}}$
2. $\frac{t^{23}}{t^{15}}$
3. $\frac{(z \sqrt{2})^{5}}{z \sqrt{2}}$

Answer
Use the quotient rule to simplify each expression.

1. $\frac{(-2)^{14}}{(-2)^{9}}=(-2)^{14-9}=(-2)^{5}$
2. $\frac{t^{23}}{t^{15}}=t^{23-15}=t^{8}$
3. $\frac{(z \sqrt{2})^{5}}{z \sqrt{2}}=(z \sqrt{2})^{5-1}=(z \sqrt{2})^{4}$

## TRY IT

Write each of the following products with a single base. Do not simplify further.

1. $\frac{s^{75}}{s^{68}}$
2. $\frac{(-3)^{6}}{-3}$
3. $\frac{\left(e f^{2}\right)^{5}}{\left(e f^{2}\right)^{3}}$

Answer

1. $s^{7}$
2. $(-3)^{5}$
3. $\left(e f^{2}\right)^{2}$

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Watch this video to see more examples of how to use the quotient rule for exponents.


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Suppose an exponential expression is raised to some power. Can we simplify the result? Yes. To do this, we use the power rule of exponents. Consider the expression $\left(x^{2}\right)^{3}$. The expression inside the parentheses is multiplied twice because it has an exponent of 2 . Then the result is multiplied three times because the entire expression has an exponent of 3 .

$$
\begin{aligned}
\left(x^{2}\right)^{3} & =\left(x^{2}\right) \cdot\left(x^{2}\right) \cdot\left(x^{2}\right) \\
& =\overbrace{(\overbrace{x \cdot x}^{2 \text { factors }}) \cdot(\overbrace{x \cdot x}^{2 \text { factors }}) \cdot(\overbrace{x \cdot x}^{2 \text { factors }})}^{3 \text { factors }}) \\
& =x \cdot x \cdot x \cdot x \cdot x \cdot x \\
& =x^{6}
\end{aligned}
$$

The exponent of the answer is the product of the exponents: $\left(x^{2}\right)^{3}=x^{2 \cdot 3}=x^{6}$. In other words, when raising an exponential expression to a power, we write the result with the common base and the product of the exponents.

$$
\left(a^{m}\right)^{n}=a^{m \cdot n}
$$

Be careful to distinguish between uses of the product rule and the power rule. When using the product rule, different terms with the same bases are raised to exponents. In this case, you add the exponents. When using the power rule, a term in exponential notation is raised to a power. In this case, you multiply the exponents.

| Product Rule |  |  |  |  | Power Rule |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $5^{3} \cdot 5^{4}$ | = | $5^{3+4}$ | $=$ | $5^{7}$ | but | $\left(5^{3}\right)^{4}$ | $=$ | $5^{3 \cdot 4}$ | = | $5^{12}$ |
| $x^{5} \cdot x^{2}$ | = | $x^{5+2}$ | $=$ | $x^{7}$ | but | $\left(x^{5}\right)^{2}$ | = | $x^{5 \cdot 2}$ | $=$ | $x^{10}$ |
| $(3 a)^{7} \cdot(3 a)^{10}$ | = | $(3 a)^{7+1-}$ | $=$ | $(3 a)^{17}$ | but | $\left((3 a)^{7}\right)^{10}$ | $=$ | $(3 a)^{7 \cdot 10}$ | $=$ | $(3 a)^{70}$ |

## A GENERAL NOTE: THE POWER RULE OF EXPONENTS

For any real number $a$ and positive integers $m$ and $n$, the power rule of exponents states that

$$
\left(a^{m}\right)^{n}=a^{m \cdot n}
$$

## EXAMPLE: USING THE POWER RULE

Write each of the following products with a single base. Do not simplify further.

1. $\left(x^{2}\right)^{7}$
2. $\left((2 t)^{5}\right)^{3}$
3. $\left((-3)^{5}\right)^{11}$

Answer
Use the power rule to simplify each expression.

1. $\left(x^{2}\right)^{7}=x^{2.7}=x^{14}$
2. $\left((2 t)^{5}\right)^{3}=(2 t)^{5 \cdot 3}=(2 t)^{15}$
3. $\left((-3)^{5}\right)^{11}=(-3)^{5 \cdot 11}=(-3)^{55}$

## TRY IT

Write each of the following products with a single base. Do not simplify further.

1. $\left((3 y)^{8}\right)^{3}$
2. $\left(t^{5}\right)^{7}$
3. $\left((-g)^{4}\right)^{4}$

Answer

1. $(3 y)^{24}$
2. $t^{35}$
3. $(-g)^{16}$

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The following video gives more examples of using the power rule to simplify expressions with exponents.


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## ZERO AND NEGATIVE EXPONENTS

## LEARNING OUTCOMES

- Simplify expressions with exponents equal to zero.
- Simplify expressions with negative exponents.
- Simplify exponential expressions.

Return to the quotient rule. We made the condition that $m>n$ so that the difference $m-n$ would never be zero or negative. What would happen if $m=n$ ? In this case, we would use the zero exponent rule of exponents to simplify the expression to 1 . To see how this is done, let us begin with an example.

$$
\frac{t^{8}}{t^{8}}=\frac{t^{\phi}}{t^{8}}=1
$$

If we were to simplify the original expression using the quotient rule, we would have

$$
\frac{t^{8}}{t^{8}}=t^{8-8}=t^{0}
$$

If we equate the two answers, the result is $t^{0}=1$. This is true for any nonzero real number, or any variable representing a real number.

$$
a^{0}=1
$$

The sole exception is the expression $0^{0}$. This appears later in more advanced courses, but for now, we will consider the value to be undefined.

## A GENERAL NOTE: THE ZERO EXPONENT RULE OF EXPONENTS

For any nonzero real number $a$, the zero exponent rule of exponents states that

$$
a^{0}=1
$$

## USING ORDER OF OPERATIONS WITH FRACTIONS

When simplifying expressions with exponents, it is sometimes helpful to rely on the rule for multiplying fractions to separate the factors before doing work on them. For example, to simplify the expression $\frac{5 a^{m} z^{2}}{a^{m} z}$ using exponent rules, you may find it helpful to break the fraction up into a product of fractions, then simplify.

$$
\frac{5 a^{m} z^{2}}{a^{m} z}=5 \cdot \frac{a^{m}}{a^{m}} \cdot \frac{z^{2}}{z}=5 \cdot a^{m-m} \cdot z^{2-1}=5 \cdot a^{0} \cdot z^{1}=5 z
$$

In this video we show more examples of how to simplify expressions with zero exponents.

## Using the Negative Rule of Exponents

Another useful result occurs if we relax the condition that $m>n$ in the quotient rule even further. For example, can we simplify $\frac{h^{3}}{h^{5}}$ ? When $m<n$-that is, where the difference $m-n$ is negative-we can use the negative rule of exponents to simplify the expression to its reciprocal.

Divide one exponential expression by another with a larger exponent. Use our example, $\frac{h^{3}}{h^{5}}$.

$$
\begin{aligned}
\frac{h^{3}}{h^{5}} & =\frac{h \cdot h \cdot h}{h \cdot h \cdot h \cdot h \cdot h} \\
& =\frac{\not K \cdot \not h \cdot \not K}{\not K \cdot \not K \cdot \not K \cdot h \cdot h} \\
& =\frac{1}{h \cdot h} \\
& =\frac{1}{h^{2}}
\end{aligned}
$$

If we were to simplify the original expression using the quotient rule, we would have

$$
\begin{aligned}
\frac{h^{3}}{h^{5}} & =h^{3-5} \\
& =h^{-2}
\end{aligned}
$$

## EXAMPLE: USING THE ZERO EXPONENT RULE

Simplify each expression using the zero exponent rule of exponents.

1. $\frac{c^{3}}{c^{3}}$
2. $\frac{-3 x^{5}}{x^{5}}$
3. $\frac{\left(j^{2} k\right)^{4}}{\left(j^{2} k\right) \cdot\left(j^{2} k\right)^{3}}$
4. $\frac{5\left(r s^{2}\right)^{2}}{\left(r s^{2}\right)^{2}}$

Answer
Use the zero exponent and other rules to simplify each expression.

1. $\frac{c^{3}}{c^{3}}=c^{3-3}$
2. 

$$
=c^{0}
$$

$$
=1
$$

$$
\frac{-3 x^{5}}{x^{5}}=-3 \cdot \frac{x^{5}}{x^{5}}
$$

2. 

$$
\begin{aligned}
& =-3 \cdot x^{5-5} \\
& =-3 \cdot x^{0}
\end{aligned}
$$

$$
=-3 \cdot 1
$$

$$
=-3
$$

$$
\frac{\left(j^{2} k\right)^{4}}{\left(j^{2} k\right) \cdot\left(j^{2} k\right)^{3}}=\frac{\left(j^{2} k\right)^{4}}{\left(j^{2} k\right)^{1+3}} \quad \text { Use the product rule in the denominator. }
$$

3. 

$$
=\frac{\left(j^{2} k\right)^{4}}{\left(j^{2} k\right)^{4}} \quad \text { Simplify }
$$

$$
=\left(j^{2} k\right)^{4-4} \quad \text { Use the quotient rule. }
$$

$$
=\left(j^{2} k\right)^{0} \quad \text { Simplify }
$$

$$
\begin{gathered}
\quad=1 \\
\frac{5\left(r s^{2}\right)^{2}}{\left(r s^{2}\right)^{2}}=5\left(r s^{2}\right)^{2-2} \quad \text { Use the quotient rule. }
\end{gathered}
$$

4. $\quad=5\left(r s^{2}\right)^{0} \quad$ Simplify.
$=5 \cdot 1 \quad$ Use the zero exponent rule.
$=5 \quad$ Simplify.

## TRY IT

Simplify each expression using the zero exponent rule of exponents.

1. $\frac{t^{7}}{t^{7}}$
2. $\frac{\left(d e^{2}\right)^{11}}{2\left(d e^{2}\right)^{11}}$
3. $\frac{w^{4} \cdot w^{2}}{w^{6}}$
4. $\frac{t^{3} \cdot t^{4}}{t^{2} \cdot t^{5}}$

## Answer

1. 1
2. $\frac{1}{2}$
3. 1
4. 1

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Putting the answers together, we have $h^{-2}=\frac{1}{h^{2}}$. This is true for any nonzero real number, or any variable representing a nonzero real number.

A factor with a negative exponent becomes the same factor with a positive exponent if it is moved across the fraction bar-from numerator to denominator or vice versa.

$$
a^{-n}=\frac{1}{a^{n}} \text { and } a^{n}=\frac{1}{a^{-n}}
$$

We have shown that the exponential expression $a^{n}$ is defined when $n$ is a natural number, 0 , or the negative of a natural number. That means that $a^{n}$ is defined for any integer $n$. Also, the product and quotient rules and all of the rules we will look at soon hold for any integer $n$.

For any nonzero real number $a$ and natural number $n$, the negative rule of exponents states that $a^{-n}=\frac{1}{a^{n}}$ and $a^{n}=\frac{1}{a^{-n}}$

## EXAMPLE: USING THE NEGATIVE EXPONENT RULE

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

1. $\frac{\theta^{3}}{\theta^{10}}$
2. $\frac{z^{2} \cdot z}{z^{4}}$
3. $\frac{\left(-5 t^{3}\right)^{4}}{\left(-5 t^{3}\right)^{8}}$

## Answer

1. $\frac{\theta^{3}}{\theta^{10}}=\theta^{3-10}=\theta^{-7}=\frac{1}{\theta^{7}}$
2. $\frac{z^{2} \cdot z}{z^{4}}=\frac{z^{2+1}}{z^{4}}=\frac{z^{3}}{z^{4}}=z^{3-4}=z^{-1}=\frac{1}{z}$
3. $\frac{\left(-5 t^{3}\right)^{4}}{\left(-5 t^{3}\right)^{8}}=\left(-5 t^{3}\right)^{4-8}=\left(-5 t^{3}\right)^{-4}=\frac{1}{\left(-5 t^{3}\right)^{4}}$

## TRY IT

Write each of the following quotients with a single base. Do not simplify further. Write answers with positive exponents.

1. $\frac{(-3 t)^{2}}{(-3 t)^{8}}$
2. $\frac{f^{47}}{f^{49} \cdot f}$
3. $\frac{2 k^{4}}{5 k^{7}}$

## Answer

1. $\frac{1}{(-3 t)^{6}}$
2. $\frac{1}{f^{3}}$
3. $\frac{2}{5 k^{3}}$

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Watch this video to see more examples of simplifying expressions with negative exponents.


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## EXAMPLE: USING THE PRODUCT AND QUOTIENT RULES

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

1. $b^{2} \cdot b^{-8}$
2. $(-x)^{5} \cdot(-x)^{-5}$
3. $\frac{-7 z}{(-7 z)^{5}}$

Answer

1. $b^{2} \cdot b^{-8}=b^{2-8}=b^{-6}=\frac{1}{b^{6}}$
2. $(-x)^{5} \cdot(-x)^{-5}=(-x)^{5-5}=(-x)^{0}=1$
3. $\frac{-7 z}{(-7 z)^{5}}=\frac{(-7 z)^{1}}{(-7 z)^{5}}=(-7 z)^{1-5}=(-7 z)^{-4}=\frac{1}{(-7 z)^{4}}$

## TRY IT

Write each of the following products with a single base. Do not simplify further. Write answers with positive exponents.

1. $t^{-11} \cdot t^{6}$
2. $\frac{25^{12}}{25^{13}}$

## Answer

1. $t^{-5}=\frac{1}{t^{5}}$
2. $\frac{1}{25}$

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## Finding the Power of a Product

To simplify the power of a product of two exponential expressions, we can use the power of a product rule of exponents, which breaks up the power of a product of factors into the product of the powers of the factors.
For instance, consider $(p q)^{3}$. We begin by using the associative and commutative properties of multiplication to regroup the factors.

$$
\begin{aligned}
(p q)^{3} & =(p q) \cdot(p q) \cdot(p q) \\
& =p \cdot q \cdot p \cdot q \cdot p \cdot q \\
& =\stackrel{3 \text { factors }}{p} \cdot \stackrel{p}{p} p \cdot \stackrel{3 \text { factors }}{q} \cdot q \cdot q \\
& =p^{3} \cdot q^{3}
\end{aligned}
$$

In other words, $(p q)^{3}=p^{3} \cdot q^{3}$.

## A GENERAL NOTE: THE POWER OF A PRODUCT RULE OF EXPONENTS

For any real numbers $a$ and $b$ and any integer $n$, the power of a product rule of exponents states that

$$
(a b)^{n}=a^{n} b^{n}
$$

In the following video we show more examples of how to find hte power of a product.

## Finding the Power of a Quotient

To simplify the power of a quotient of two expressions, we can use the power of a quotient rule, which states that the power of a quotient of factors is the quotient of the powers of the factors. For example, let's look at the following example.

$$
\left(e^{-2} f^{2}\right)^{7}=\frac{f^{14}}{e^{14}}
$$

Let's rewrite the original problem differently and look at the result.

$$
\begin{aligned}
\left(e^{-2} f^{2}\right)^{7} & =\left(\frac{f^{2}}{e^{2}}\right)^{7} \\
& =\frac{f^{14}}{e^{14}}
\end{aligned}
$$

It appears from the last two steps that we can use the power of a product rule as a power of a quotient rule.

## EXAMPLE: USING THE POWER OF A PRODUCT RULE

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

1. $\left(a b^{2}\right)^{3}$
2. $(2 t)^{15}$
3. $\left(-2 w^{3}\right)^{3}$
4. $\frac{1}{(-7 z)^{4}}$
5. $\left(e^{-2} f^{2}\right)^{7}$

Answer
Use the product and quotient rules and the new definitions to simplify each expression.

1. $\left(a b^{2}\right)^{3}=(a)^{3} \cdot\left(b^{2}\right)^{3}=a^{1 \cdot 3} \cdot b^{2 \cdot 3}=a^{3} b^{6}$
2. $2 t^{15}=(2)^{15} \cdot(t)^{15}=2^{15} t^{15}=32,768 t^{15}$
3. $\left(-2 w^{3}\right)^{3}=(-2)^{3} \cdot\left(w^{3}\right)^{3}=-8 \cdot w^{3 \cdot 3}=-8 w^{9}$
4. $\frac{1}{(-7 z)^{4}}=\frac{1}{(-7)^{4} \cdot(z)^{4}}=\frac{1}{2,401 z^{4}}$
5. $\left(e^{-2} f^{2}\right)^{7}=\left(e^{-2}\right)^{7} \cdot\left(f^{2}\right)^{7}=e^{-2 \cdot 7} \cdot f^{2 \cdot 7}=e^{-14} f^{14}=\frac{f^{14}}{e^{14}}$

## TRY IT

Simplify each of the following products as much as possible using the power of a product rule. Write answers with positive exponents.

1. $\left(g^{2} h^{3}\right)^{5}$
2. $(5 t)^{3}$
3. $\left(-3 y^{5}\right)^{3}$
4. $\frac{1}{\left(a^{6} b^{7}\right)^{3}}$
5. $\left(r^{3} s^{-2}\right)^{4}$

Answer

1. $g^{10} h^{15}$
2. $125 t^{3}$
3. $-27 y^{15}$
4. $\frac{1}{a^{18} b^{21}}$
5. $\frac{r^{12}}{s^{8}}$

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$$
\begin{aligned}
\left(e^{-2} f^{2}\right)^{7} & =\left(\frac{f^{2}}{e^{2}}\right)^{7} \\
& =\frac{\left(f^{2}\right)^{7}}{\left(e^{2}\right)^{7}} \\
& =\frac{f^{2 \cdot 7}}{e^{2 \cdot 7}} \\
& =\frac{f^{14}}{e^{14}}
\end{aligned}
$$

## A GENERAL NOTE: THE POWER OF A QUOTIENT RULE OF EXPONENTS

For any real numbers $a$ and $b$ and any integer $n$, the power of a quotient rule of exponents states that

$$
\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}
$$

## Simplifying Exponential Expressions

Recall that to simplify an expression means to rewrite it by combing terms or exponents; in other words, to write the expression more simply with fewer terms. The rules for exponents may be combined to simplify expressions.

In the following video we show more examples of how to find the power of a quotient.

## EXAMPLE: USING THE POWER OF A QUOTIENT RULE

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

1. $\left(\frac{4}{z^{11}}\right)^{3}$
2. $\left(\frac{p}{q^{3}}\right)^{6}$
3. $\left(\frac{-1}{t^{2}}\right)^{27}$
4. $\left(j^{3} k^{-2}\right)^{4}$
5. $\left(m^{-2} n^{-2}\right)^{3}$

Answer

1. $\left(\frac{4}{z^{11}}\right)^{3}=\frac{(4)^{3}}{\left(z^{11}\right)^{3}}=\frac{64}{z^{11 \cdot 3}}=\frac{64}{z^{33}}$
2. $\left(\frac{p}{q^{3}}\right)^{6}=\frac{(p)^{6}}{\left(q^{3}\right)^{6}}=\frac{p^{1 \cdot 6}}{q^{3 \cdot 6}}=\frac{p^{6}}{q^{18}}$
3. $\left(\frac{-1}{t^{2}}\right)^{27}=\frac{(-1)^{27}}{\left(t^{2}\right)^{27}}=\frac{-1}{t^{2 \cdot 27}}=\frac{-1}{t^{54}}=-\frac{1}{t^{54}}$
4. $\left(j^{3} k^{-2}\right)^{4}=\left(\frac{j^{3}}{k^{2}}\right)^{4}=\frac{\left(j^{3}\right)^{4}}{\left(k^{2}\right)^{4}}=\frac{j^{3 \cdot 4}}{k^{2 \cdot 4}}=\frac{j^{12}}{k^{8}}$
5. $\left(m^{-2} n^{-2}\right)^{3}=\left(\frac{1}{m^{2} n^{2}}\right)^{3}=\frac{(1)^{3}}{\left(m^{2} n^{2}\right)^{3}}=\frac{1}{\left(m^{2}\right)^{3}\left(n^{2}\right)^{3}}=\frac{1}{m^{2 \cdot 3} \cdot n^{2 \cdot 3}}=\frac{1}{m^{6} n^{6}}$

## TRY IT

Simplify each of the following quotients as much as possible using the power of a quotient rule. Write answers with positive exponents.

1. $\left(\frac{b^{5}}{c}\right)^{3}$
2. $\left(\frac{5}{u^{8}}\right)^{4}$
3. $\left(\frac{-1}{w^{3}}\right)^{35}$
4. $\left(p^{-4} q^{3}\right)^{8}$
5. $\left(c^{-5} d^{-3}\right)^{4}$

Answer

1. $\frac{b^{15}}{c^{3}}$
2. $\frac{625}{u^{32}}$
3. $\frac{-1}{w^{105}}$
4. $\frac{q^{24}}{p^{32}}$
5. $\frac{1}{c^{20} d^{12}}$

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## EXAMPLE: SIMPLIFYING EXPONENTIAL EXPRESSIONS

Simplify each expression and write the answer with positive exponents only.

1. $\left(6 m^{2} n^{-1}\right)^{3}$
2. $17^{5} \cdot 17^{-4} \cdot 17^{-3}$
3. $\left(\frac{u^{-1} v}{v^{-1}}\right)^{2}$
4. $\left(-2 a^{3} b^{-1}\right)\left(5 a^{-2} b^{2}\right)$
5. $\left(x^{2} \sqrt{2}\right)^{4}\left(x^{2} \sqrt{2}\right)^{-4}$
6. $\frac{\left(3 w^{2}\right)^{5}}{\left(6 w^{-2}\right)^{2}}$

## Answer

$$
\begin{aligned}
& \left(6 m^{2} n^{-1}\right)^{3}=(6)^{3}\left(m^{2}\right)^{3}\left(n^{-1}\right)^{3} \quad \text { The power of a product rule } \\
& =6^{3} m^{2 \cdot 3} n^{-1 \cdot 3} \quad \text { The power rule } \\
& =216 m^{6} n^{-3} \quad \text { Simplify. } \\
& \begin{aligned}
& =216 m^{6} n^{-3} \\
& =\frac{216 m^{6}}{n^{3}}
\end{aligned} \\
& 17^{5} \cdot 17^{-4} \cdot 17^{-3}=\quad 17^{5-4-3} \\
& 17^{5-4-3} \quad \text { The product rule } \\
& 2 . \\
& 1 . \\
& \text { The negative exponent rule } \\
& \text { Simplify. } \\
& \text { The negative exponent rule } \\
& \left(\frac{u^{-1} v}{v^{-1}}\right)^{2}=\frac{\left(u^{-1} v\right)^{2}}{\left(v^{-1}\right)^{2}} \quad \text { The power of a quotient rule } \\
& =\frac{u^{-2} v^{2}}{v^{-2}} \quad \text { The power of a product rule } \\
& =u^{-2} v^{2-(-2)} \quad \text { The quotient rule } \\
& =u^{-2} v^{4} \quad \text { Simplify. } \\
& =\frac{v^{4}}{u^{2}} \quad \text { The negative exponent rule } \\
& 3 . \\
& =\frac{1}{17^{2}} \text { or } \frac{1}{289} \quad \text { The negative exponent rule } \\
& \begin{aligned}
& \left(v^{-1}\right)^{2} & & \\
= & \frac{u^{-2} v^{2}}{v^{-2}} & & \text { The power of a product rule } \\
= & u^{-2} v^{2-(-2)} & & \text { The quotient rule } \\
= & u^{-2} v^{4} & & \text { Simplify. } \\
= & \frac{v^{4}}{u^{2}} & & \text { The negative exponent rule }
\end{aligned}
\end{aligned}
$$

4. 

$$
\begin{array}{rlrl}
\left(-2 a^{3} b^{-1}\right)\left(5 a^{-2} b^{2}\right) & = & & -2 \cdot 5 \cdot a^{3} \cdot a^{-2} \cdot b^{-1} \cdot b^{2} \\
& =-10 \cdot a^{3-2} \cdot b^{-1+2} & \\
& =-10 a b & & \\
\left(x^{2} \sqrt{2}\right)^{4}\left(x^{2} \sqrt{2}\right)^{-4} & =\left(x^{2} \sqrt{2}\right)^{4-4} & & \text { The product rule } \\
& =\left(x^{2} \sqrt{2}\right)^{0} & & \text { Simplify. } \\
& =1 & & \text { The zero exponent rule }
\end{array}
$$

5. 

The product rule Simplify.

$$
\begin{aligned}
\frac{\left(3 w^{2}\right)^{5}}{\left(6 w^{-2}\right)^{2}} & =\frac{(3)^{5} \cdot\left(w^{2}\right)^{5}}{(6)^{2} \cdot\left(w^{-2}\right)^{2}} & & \text { The power of a product rule } \\
& =\frac{3^{5} w^{2 \cdot 5}}{6^{2} w^{-2 \cdot 2}} & & \text { The power rule } \\
& =\frac{243 w^{10}}{36 w^{-4}} & & \text { Simplify. } \\
& =\frac{27 w^{10-(-4)}}{4} & & \text { The quotient rule and reduce fraction } \\
& =\frac{27 w^{14}}{4} & & \text { Simplify. }
\end{aligned}
$$

## TRY IT

Simplify each expression and write the answer with positive exponents only.

1. $\left(2 u v^{-2}\right)^{-3}$
2. $x^{8} \cdot x^{-12} \cdot x$
3. $\left(\frac{e^{2} f^{-3}}{f^{-1}}\right)^{2}$
4. $\left(9 r^{-5} s^{3}\right)\left(3 r^{6} s^{-4}\right)$
5. $\left(\frac{4}{9} t w^{-2}\right)^{-3}\left(\frac{4}{9} t w^{-2}\right)^{3}$
6. $\frac{\left(2 h^{2} k\right)^{4}}{\left(7 h^{-1} k^{2}\right)^{2}}$

## Answer

1. $\frac{v^{6}}{8 u^{3}}$
2. $\frac{1}{x^{3}}$
3. $\frac{e^{4}}{f^{4}}$
4. $\frac{27 r}{s}$
5. 1
6. $\frac{16 h^{10}}{49}$

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## SCIENTIFIC NOTATION

## LEARNING OUTCOMES

- Convert standard notation to scientific notation.
- Convert from scientific to standard notation.
- Apply scientific notation in an application.

Recall at the beginning of the section that we found the number $1.3 \times 10^{13}$ when describing bits of information in digital images. Other extreme numbers include the width of a human hair, which is about 0.00005 m , and the radius of an electron, which is about 0.00000000000047 m . How can we effectively work read, compare, and calculate with numbers such as these?

A shorthand method of writing very small and very large numbers is called scientific notation, in which we express numbers in terms of exponents of 10 . To write a number in scientific notation, move the decimal point to the right of the first digit in the number. Write the digits as a decimal number between 1 and 10. Count the number of places $n$ that you moved the decimal point. Multiply the decimal number by 10 raised to a power of $n$. If you moved the decimal left as in a very large number, $n$ is positive. If you moved the decimal right as in a small large number, $n$ is negative.

For example, consider the number $2,780,418$. Move the decimal left until it is to the right of the first nonzero digit, which is 2.


We obtain 2.780418 by moving the decimal point 6 places to the left. Therefore, the exponent of 10 is 6 , and it is positive because we moved the decimal point to the left. This is what we should expect for a large number.

$$
2.780418 \times 10^{6}
$$

Working with small numbers is similar. Take, for example, the radius of an electron, 0.00000000000047 m . Perform the same series of steps as above, except move the decimal point to the right.


Be careful not to include the leading 0 in your count. We move the decimal point 13 places to the right, so the exponent of 10 is 13 . The exponent is negative because we moved the decimal point to the right. This is what we should expect for a small number.

$$
4.7 \times 10^{-13}
$$

## A GENERAL NOTE: SCIENTIFIC NOTATION

A number is written in scientific notation if it is written in the form $a \times 10^{n}$, where $1 \leq|a|<10$ and $n$ is an integer.

## Converting from Scientific to Standard Notation

To convert a number in scientific notation to standard notation, simply reverse the process. Move the decimal $n$ places to the right if $n$ is positive or $n$ places to the left if $n$ is negative and add zeros as needed. Remember, if $n$ is positive, the value of the number is greater than 1 , and if $n$ is negative, the value of the number is less than one.

## Using Scientific Notation in Applications

Scientific notation, used with the rules of exponents, makes calculating with large or small numbers much easier than doing so using standard notation. For example, suppose we are asked to calculate the number of atoms in 1 L of water. Each water molecule contains 3 atoms ( 2 hydrogen and 1 oxygen). The average drop of water contains around $1.32 \times 10^{21}$ molecules of water and 1 L of water holds about $1.22 \times 10^{4}$ average drops. Therefore, there are approximately $3 \cdot\left(1.32 \times 10^{21}\right) \cdot\left(1.22 \times 10^{4}\right) \approx 4.83 \times 10^{25}$ atoms in 1 L of water. We simply multiply the decimal terms and add the exponents. Imagine having to perform the calculation without using scientific notation!

When performing calculations with scientific notation, be sure to write the answer in proper scientific notation. For example, consider the product $\left(7 \times 10^{4}\right) \cdot\left(5 \times 10^{6}\right)=35 \times 10^{10}$. The answer is not in proper

## EXAMPLE: CONVERTING STANDARD NOTATION TO SCIENTIFIC NOTATION

Write each number in scientific notation.

1. Distance to Andromeda Galaxy from Earth: $24,000,000,000,000,000,000,000 \mathrm{~m}$
2. Diameter of Andromeda Galaxy: 1,300,000,000,000,000,000,000 m
3. Number of stars in Andromeda Galaxy: 1,000,000,000,000
4. Diameter of electron: 0.00000000000094 m
5. Probability of being struck by lightning in any single year: 0.00000143

## Answer

1. 

$24,000,000,000,000,000,000,000 \mathrm{~m}$
$\leftarrow 22$ places
$2.4 \times 10^{22} \mathrm{~m}$
2.
$1,300,000,000,000,000,000,000 \mathrm{~m}$ $\leftarrow 21$ places
$1.3 \times 10^{21} \mathrm{~m}$
3.
$1,000,000,000,000$
$\leftarrow 12$ places
$1 \times 10^{12}$
4.
0.00000000000094 m
$\rightarrow \begin{array}{r}\rightarrow 6 \text { places } \\ 9.4 \times 10^{-13} \mathrm{~m}\end{array}$
5.
0.00000143
$\rightarrow 6$ places
$1.43 \times 10^{-6}$

## Analysis of the Solution

Observe that, if the given number is greater than 1 , as in examples a-c, the exponent of 10 is positive; and if the number is less than 1 , as in examples d-e, the exponent is negative.

## TRY IT

Write each number in scientific notation.

1. U.S. national debt per taxpayer (April 2014): \$152,000
2. World population (April 2014): 7,158,000,000
3. World gross national income (April 2014): $\$ 85,500,000,000,000$
4. Time for light to travel $1 \mathrm{~m}: 0.00000000334 \mathrm{~s}$
5. Probability of winning lottery (match 6 of 49 possible numbers): 0.0000000715

Answer

1. $\$ 1.52 \times 10^{5}$
2. $7.158 \times 10^{9}$
3. $\$ 8.55 \times 10^{13}$
4. $3.34 \times 10^{-9}$
5. $7.15 \times 10^{-8}$

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## EXAMPLE: CONVERTING SCIENTIFIC NOTATION TO STANDARD NOTATION

Convert each number in scientific notation to standard notation.

1. $3.547 \times 10^{14}$
2. $-2 \times 10^{6}$
3. $7.91 \times 10^{-7}$
4. $-8.05 \times 10^{-12}$

## Answer

1. 

$3.547 \times 10^{14}$
3.54700000000000
$\rightarrow 14$ places
$354,700,000,000,000$
2.
$-2 \times 10^{6}$
$-2.000000$
$\rightarrow 6$ places
$-2,000,000$
3.
$7.91 \times 10^{-7}$
0000007.91
$\rightarrow 7$ places
0.000000791
4.
$-8.05 \times 10^{-12}$
$-000000000008.05$
$\rightarrow 12$ places
$-0.00000000000805$

## TRY IT

Convert each number in scientific notation to standard notation.

1. $7.03 \times 10^{5}$
2. $-8.16 \times 10^{11}$
3. $-3.9 \times 10^{-13}$
4. $8 \times 10^{-6}$

Answer

1. 703,000
2. $-816,000,000,000$
3. -0.00000000000039
4. 0.000008

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## WHAT PROPERTIES OF NUMBERS ENABLE OPERATIONS ON SCIENTIFIC NOTATION?

In the example above, $3 \cdot\left(1.32 \times 10^{21}\right) \cdot\left(1.22 \times 10^{4}\right) \approx 4.83 \times 10^{25}$. How are we are able to simply multiply the decimal terms and add the exponents? What properties of numbers enable this? Recall that multiplication is both commutative and associative. That means, as long as multiplication is the only operation being performed, we can move the factors around to suit our needs. Lastly, the product rule for exponents, $a^{m} \cdot a^{n}=a^{m+n}$, allows us to add the exponents on the base of 10 .
$(3) \cdot\left(1.32 \times 10^{21}\right) \cdot\left(1.22 \times 10^{4}\right)=(3 \cdot 1.32 \cdot 1.22) \times\left(10^{4} \cdot 10^{25}\right) \approx 4.83 \times 10^{25}$.
scientific notation because 35 is greater than 10 . Consider 35 as $3.5 \times 10$. That adds a ten to the exponent of the answer.

$$
(35) \times 10^{10}=(3.5 \times 10) \times 10^{10}=3.5 \times\left(10 \times 10^{10}\right)=3.5 \times 10^{11}
$$

Watch the following video to see more examples of writing numbers in scientific notation.

## EXAMPLE: USING SCIENTIFIC NOTATION

Perform the operations and write the answer in scientific notation.

1. $\left(8.14 \times 10^{-7}\right)\left(6.5 \times 10^{10}\right)$
2. $\left(4 \times 10^{5}\right) \div\left(-1.52 \times 10^{9}\right)$
3. $\left(2.7 \times 10^{5}\right)\left(6.04 \times 10^{13}\right)$
4. $\left(1.2 \times 10^{8}\right) \div\left(9.6 \times 10^{5}\right)$
5. $\left(3.33 \times 10^{4}\right)\left(-1.05 \times 10^{7}\right)\left(5.62 \times 10^{5}\right)$

Answer
1.

| $\left(8.14 \times 10^{-7}\right)\left(6.5 \times 10^{10}\right)$ | $=(8.14 \times 6.5)\left(10^{-7} \times 10^{10}\right)$ |  |  |
| ---: | :--- | ---: | :--- |
|  | $=(52.91)\left(10^{3}\right)$ |  | Commutative and associative properties 0 |
|  | $=5.291 \times 10^{4}$ |  | Product rule of exponents |

2. 

$\left(4 \times 10^{5}\right) \div\left(-1.52 \times 10^{9}\right)=\left(\frac{4}{-1.52}\right)\left(\frac{10^{5}}{10^{9}}\right) \quad$ Commutative and associative properties of multily $=(-2.63)\left(10^{-4}\right) \quad$ Quotient rule of exponents $=-2.63 \times 10^{-4} \quad$ Scientific notation
3.
$\left(2.7 \times 10^{5}\right)\left(6.04 \times 10^{13}\right)=(2.7 \times 6.04)\left(10^{5} \times 10^{13}\right) \quad$ Commutative and associative properties of $n$ $=(16.308)\left(10^{18}\right) \quad$ Product rule of exponents $=1.6308 \times 10^{19} \quad$ Scientific notation
4.
$\left(1.2 \times 10^{8}\right) \div\left(9.6 \times 10^{5}\right)=\left(\frac{1.2}{9.6}\right)\left(\frac{10^{8}}{10^{5}}\right) \quad$ Commutative and associative properties of multiplica $=(0.125)\left(10^{3}\right) \quad$ Quotient rule of exponents $=1.25 \times 10^{2} \quad$ Scientific notation
5.
$\left(3.33 \times 10^{4}\right)\left(-1.05 \times 10^{7}\right)\left(5.62 \times 10^{5}\right)=[3.33 \times(-1.05) \times 5.62]\left(10^{4} \times 10^{7} \times 10^{5}\right)$

$$
\begin{aligned}
& \approx(-19.65)\left(10^{16}\right) \\
& =-1.965 \times 10^{17}
\end{aligned}
$$



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## TRY IT

Perform the operations and write the answer in scientific notation.

1. $\left(-7.5 \times 10^{8}\right)\left(1.13 \times 10^{-2}\right)$
2. $\left(1.24 \times 10^{11}\right) \div\left(1.55 \times 10^{18}\right)$
3. $\left(3.72 \times 10^{9}\right)\left(8 \times 10^{3}\right)$
4. $\left(9.933 \times 10^{23}\right) \div\left(-2.31 \times 10^{17}\right)$
5. $\left(-6.04 \times 10^{9}\right)\left(7.3 \times 10^{2}\right)\left(-2.81 \times 10^{2}\right)$

## Answer

1. $-8.475 \times 10^{6}$
2. $8 \times 10^{-8}$
3. $2.976 \times 10^{13}$
4. $-4.3 \times 10^{6}$
5. $\approx 1.24 \times 10^{15}$

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## EXAMPLE: APPLYING SCIENTIFIC NOTATION TO SOLVE PROBLEMS

In April 2014, the population of the United States was about 308,000,000 people. The national debt was about $\$ 17,547,000,000,000$. Write each number in scientific notation, rounding figures to two decimal places, and find the amount of the debt per U.S. citizen. Write the answer in both scientific and standard notations.

## Answer

The population was $308,000,000=3.08 \times 10^{8}$.
The national debt was $\$ 17,547,000,000,000 \approx \$ 1.75 \times 10^{13}$.
To find the amount of debt per citizen, divide the national debt by the number of citizens.

$$
\begin{aligned}
\left(1.75 \times 10^{13}\right) \div\left(3.08 \times 10^{8}\right) & =\left(\frac{1.75}{3.08}\right) \cdot\left(\frac{10^{13}}{10^{8}}\right) \\
& \approx 0.57 \times 10^{5} \\
& =5.7 \times 10^{4}
\end{aligned}
$$

The debt per citizen at the time was about $\$ 5.7 \times 10^{4}$, or $\$ 57,000$.

## TRY IT

An average human body contains around $30,000,000,000,000$ red blood cells. Each cell measures approximately 0.000008 m long. Write each number in scientific notation and find the total length if the cells were laid end-to-end. Write the answer in both scientific and standard notations.
Answer
Number of cells: $3 \times 10^{13}$; length of a cell: $8 \times 10^{-6} \mathrm{~m}$; total length: $2.4 \times 10^{8} \mathrm{~m}$ or $240,000,000 \mathrm{~m}$.
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## SUMMARY: EXPONENTS AND SCIENTIFIC NOTATION

## Key Equations

## Rules of Exponents

For nonzero real numbers $a$ and $b$ and integers $m$ and $n$

| Product rule | $a^{m} \cdot a^{n}=a^{m+n}$ |
| :--- | :--- |
| Quotient rule | $\frac{a^{m}}{a^{n}}=a^{m-n}$ |
| Power rule | $\left(a^{m}\right)^{n}=a^{m \cdot n}$ |
| Zero exponent rule | $a^{0}=1$ |
| Negative rule | $a^{-n}=\frac{1}{a^{n}}$ |
| Power of a product rule | $(a \cdot b)^{n}=a^{n} \cdot b^{n}$ |
| Power of a quotient rule | $\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}$ |

## Key Concepts

- Products of exponential expressions with the same base can be simplified by adding exponents.
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents.
- Powers of exponential expressions with the same base can be simplified by multiplying exponents.
- An expression with exponent zero is defined as 1 .
- An expression with a negative exponent is defined as a reciprocal.
- The power of a product of factors is the same as the product of the powers of the same factors.
- The power of a quotient of factors is the same as the quotient of the powers of the same factors.
- The rules for exponential expressions can be combined to simplify more complicated expressions.
- Scientific notation uses powers of 10 to simplify very large or very small numbers.
- Scientific notation may be used to simplify calculations with very large or very small numbers.


## Glossary

scientific notation a shorthand notation for writing very large or very small numbers in the form $a \times 10^{n}$ where $1 \leq|a|<10$ and $n$ is an integer

# INTRODUCTION TO RADICALS AND RATIONAL EXPONENTS 

## Learning Outcomes

By the end of this section, you will be able to:

- Evaluate square roots.
- Use the product rule to simplify square roots.
- Use the quotient rule to simplify square roots.
- Add and subtract square roots.
- Rationalize denominators.
- Use rational roots.

A hardware store sells 16 -ft ladders and 24 -ft ladders. A window is located 12 feet above the ground. A ladder needs to be purchased that will reach the window from a point on the ground 5 feet from the building. To find out the length of ladder needed, we can draw a right triangle as shown in Figure 1, and use the Pythagorean Theorem.


Figure 1

$$
\begin{aligned}
a^{2}+b^{2} & =c^{2} \\
5^{2}+12^{2} & =c^{2} \\
169 & =c^{2}
\end{aligned}
$$

Now we need to find out the length that, when squared, is 169, to determine which ladder to choose. In other word we need to find a square root. In this section we will investigate methods of finding solutions to problems such as this one.

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## EVALUATE AND SIMPLIFY SQUARE ROOTS

## LEARNING OUTCOMES

- Evaluate principal square roots.
- Use the product rule to simplify square roots.

When the square root of a number is squared, the result is the original number. Since $4^{2}=16$, the square root of 16 is 4 . The square root function is the inverse of the squaring function just as subtraction is the inverse of addition. To undo squaring, we take the square root.

In general terms, if $a$ is a positive real number, then the square root of $a$ is a number that, when multiplied by itself, gives $a$. The square root could be positive or negative because multiplying two negative numbers gives a positive number. The principal square root is the nonnegative number that when multiplied by itself equals $a$. The square root obtained using a calculator is the principal square root.

The principal square root of $a$ is written as $\sqrt{a}$. The symbol is called a radical, the term under the symbol is called the radicand, and the entire expression is called a radical expression.


## A GENERAL NOTE: PRINCIPAL SQUARE ROOT

The principal square root of $a$ is the nonnegative number that, when multiplied by itself, equals $a$. It is written as a radical expression, with a symbol called a radical over the term called the radicand: $\sqrt{a}$.

$$
\begin{gathered}
\mathrm{Q} \& \mathrm{~A} \\
\text { DOES } \sqrt{25}= \pm 5 ?
\end{gathered}
$$

No. Although both $5^{2}$ and $(-5)^{2}$ are 25 , the radical symbol implies only a nonnegative root, the principal square root. The principal square root of 25 is $\sqrt{25}=5$.

## EXAMPLE: EVALUATING SQUARE ROOTS

Evaluate each expression.

1. $\sqrt{100}$
2. $\sqrt{\sqrt{16}}$
3. $\sqrt{25+144}$
4. $\sqrt{49}-\sqrt{81}$

## Answer

1. $\sqrt{100}=10$ because $10^{2}=100$
2. $\sqrt{\sqrt{16}}=\sqrt{4}=2$ because $4^{2}=16$ and $2^{2}=4$
3. $\sqrt{25+144}=\sqrt{169}=13$ because $13^{2}=169$
4. $\sqrt{49}-\sqrt{81}=7-9=-2$ because $7^{2}=49$ and $9^{2}=81$

No. $\sqrt{25}+\sqrt{144}=5+12=17$. This is not equivalent to $\sqrt{25+144}=13$. The order of operations requires us to add the terms in the radicand before finding the square root.

## TRY IT

Evaluate each expression.

1. $\sqrt{225}$
2. $\sqrt{\sqrt{81}}$
3. $\sqrt{25-9}$
4. $\sqrt{36}+\sqrt{121}$

Answer

1. 15
2. 3
3. 4
4. 17

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## Use the Product Rule to Simplify Square Roots

To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. There are several properties of square roots that allow us to simplify complicated radical expressions. The first rule we will look at is the product rule for simplifying square roots, which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite $\sqrt{15}$ as $\sqrt{3} \cdot \sqrt{5}$. We can also use the product rule to express the product of multiple radical expressions as a single radical expression.

## A GENERAL NOTE: THE PRODUCT RULE FOR SIMPLIFYING SQUARE ROOTS

If $a$ and $b$ are nonnegative, the square root of the product $a b$ is equal to the product of the square roots of $a$ and $b$.

$$
\sqrt{a b}=\sqrt{a} \cdot \sqrt{b}
$$

## HOW TO: GIVEN A SQUARE ROOT RADICAL EXPRESSION, USE THE PRODUCT RULE TO SIMPLIFY IT.

1. Factor any perfect squares from the radicand.
2. Write the radical expression as a product of radical expressions.
3. Simplify.

## RECALL PRIME FACTORIZATION

It can be helpful, when simplifying square roots, to write the radicand as a product of primes in order to find perfect squares under the radical.
Example. $\sqrt{288}=\sqrt{2 \cdot 3^{2} \cdot 4^{2}}=3 \cdot 4 \cdot \sqrt{2}=12 \sqrt{2}$.

## EXAMPLE: USING THE PRODUCT RULE TO SIMPLIFY SQUARE ROOTS

Simplify the radical expression.

1. $\sqrt{300}$
2. $\sqrt{162 a^{5} b^{4}}$

## Answer

1. 

$\sqrt{100 \cdot 3} \quad$ Factor perfect square from radicand.
$\sqrt{100} \cdot \sqrt{3} \quad$ Write radical expression as product of radical expressions.
$10 \sqrt{3} \quad$ Simplify.
2.
$\sqrt{81 a^{4} b^{4} \cdot 2 a} \quad$ Factor perfect square from radicand.
$\sqrt{81 a^{4} b^{4}} \cdot \sqrt{2 a} \quad$ Write radical expression as product of radical expressions.
$9 a^{2} b^{2} \sqrt{2 a} \quad$ Simplify.

## TRY IT

Simplify $\sqrt{50 x^{2} y^{3} z}$.

## Answer

$5|x||y| \sqrt{2 y z}$. Notice the absolute value signs around $x$ and $y$ ? That's because their value must be positive!

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## HOW TO: GIVEN THE PRODUCT OF MULTIPLE RADICAL EXPRESSIONS, USE THE PRODUCT RULE TO COMBINE THEM INTO ONE RADICAL EXPRESSION.

1. Express the product of multiple radical expressions as a single radical expression.
2. Simplify.

## Using the Quotient Rule to Simplify Square Roots

Just as we can rewrite the square root of a product as a product of square roots, so too can we rewrite the square root of a quotient as a quotient of square roots, using the quotient rule for simplifying square roots. It

## EXAMPLE: USING THE PRODUCT RULE TO SIMPLIFY THE PRODUCT OF MULTIPLE SQUARE ROOTS

Simplify the radical expression. $\sqrt{12} \cdot \sqrt{3}$
Answer
$\sqrt{12 \cdot 3}$ Express the product as a single radical expression.
$\sqrt{36} \quad$ Simplify.
6

## TRY IT

Simplify $\sqrt{50 x} \cdot \sqrt{2 x}$ assuming $x>0$.

## Answer

$10 x$
Because $x>0$, we do not need an absolute values.
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can be helpful to separate the numerator and denominator of a fraction under a radical so that we can take their square roots separately. We can rewrite $\sqrt{\frac{5}{2}}$ as $\frac{\sqrt{5}}{\sqrt{2}}$.

## A GENERAL NOTE: THE QUOTIENT RULE FOR SIMPLIFYING SQUARE ROOTS

The square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of $a$ and $b$, where $b \neq 0$.
$\sqrt{\frac{a}{b}}=\frac{\sqrt{ } \bar{a}}{\sqrt{b}}$

## HOW TO: GIVEN A RADICAL EXPRESSION, USE THE QUOTIENT RULE TO SIMPLIFY IT.

1. Write the radical expression as the quotient of two radical expressions.
2. Simplify the numerator and denominator.

## RECALL SIMPLIFYING FRACTIONS

To simplify fractions, find common factors in the numerator and denominator that cancel.

$$
\text { Example: } \quad \frac{24}{32}=\frac{\not 2 \cdot \not 2 \cdot \not 2 \cdot 3}{\not 2 \cdot \not 2 \cdot \not 2 \cdot 2 \cdot 2}=\frac{3}{2 \cdot 2}=\frac{3}{4}
$$

## EXAMPLE: USING THE QUOTIENT RULE TO SIMPLIFY SQUARE ROOTS

Simplify the radical expression.
$\sqrt{\frac{5}{36}}$

Answer
$\frac{\sqrt{5}}{\sqrt{36}} \quad$ Write as quotient of two radical expressions.
$\frac{\sqrt{5}}{6} \quad$ Simplify denominator.

## TRY IT

Simplify $\sqrt{\frac{2 x^{2}}{9 y^{4}}}$.
Answer
$\frac{x \sqrt{2}}{3 y^{2}}$. We do not need the absolute value signs for $y^{2}$ because that term will always be nonnegative.
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## EXAMPLE: USING THE QUOTIENT RULE TO SIMPLIFY AN EXPRESSION WITH TWO SQUARE ROOTS

Simplify the radical expression.
$\frac{\sqrt{234 x^{11} y}}{\sqrt{26 x^{7} y}}$
Answer
$\sqrt{\underline{234 x^{11} y}}$
$\sqrt{26 x^{7} y}$ Combine numerator and denominator into one radical expression.
$\sqrt{9 x^{4}} \quad$ Simplify fraction.
$3 x^{2} \quad$ Simplify square root.

## TRY IT

Simplify $\frac{\sqrt{9 a^{5} b^{14}}}{\sqrt{3 a^{4} b^{5}}}$.
Answer
$b^{4} \sqrt{3 a b}$

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In the following video you will see more examples of how to simplify radical expressions with variables.


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## OPERATIONS ON SQUARE ROOTS

## LEARNING OUTCOMES

- Add and subtract square roots.
- Rationalize denominators.

We can add or subtract radical expressions only when they have the same radicand and when they have the same radical type such as square roots. For example, the sum of $\sqrt{2}$ and $3 \sqrt{2}$ is $4 \sqrt{2}$. However, it is often possible to simplify radical expressions, and that may change the radicand. The radical expression $\sqrt{18}$ can be written with a 2 in the radicand, as $3 \sqrt{2}$, so $\sqrt{2}+\sqrt{18}=\sqrt{2}+3 \sqrt{2}=4 \sqrt{2}$.

## HOW TO: GIVEN A RADICAL EXPRESSION REQUIRING ADDITION OR SUBTRACTION OF SQUARE ROOTS, SOLVE.

1. Simplify each radical expression.
2. Add or subtract expressions with equal radicands.

## EXAMPLE: ADDING SQUARE ROOTS

Add $5 \sqrt{12}+2 \sqrt{3}$.

## Answer

We can rewrite $5 \sqrt{12}$ as $5 \sqrt{4 \cdot 3}$. According the product rule, this becomes $5 \sqrt{4} \sqrt{3}$. The square root of $\sqrt{4}$ is 2 , so the expression becomes 5 (2) $\sqrt{3}$, which is $10 \sqrt{3}$. Now we can the terms have the same radicand so we can add.
$10 \sqrt{3}+2 \sqrt{3}=12 \sqrt{3}$

## TRY IT

Add $\sqrt{5}+6 \sqrt{20}$.

## Answer

$13 \sqrt{5}$
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Watch this video to see more examples of adding roots.


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in the next video we show more examples of how to subtract radicals.

## Rationalize Denominators

When an expression involving square root radicals is written in simplest form, it will not contain a radical in the denominator. We can remove radicals from the denominators of fractions using a process called rationalizing the denominator.

We know that multiplying by 1 does not change the value of an expression. We use this property of multiplication to change expressions that contain radicals in the denominator. To remove radicals from the

## EXAMPLE: SUBTRACTING SQUARE ROOTS

Subtract $20 \sqrt{72 a^{3} b^{4} c}-14 \sqrt{8 a^{3} b^{4} c}$.

## Answer

Rewrite each term so they have equal radicands.

$$
\begin{aligned}
20 \sqrt{72 a^{3} b^{4} c} & =20 \sqrt{9} \sqrt{4} \sqrt{2} \sqrt{a} \sqrt{a^{2}} \sqrt{\left(b^{2}\right)^{2}} \sqrt{c} \\
& =20(3)(2) a b^{2} \sqrt{2 a c} \\
& =120 a b^{2} \sqrt{2 a c} \\
14 \sqrt{8 a^{3} b^{4} c} & =14 \sqrt{2} \sqrt{4} \sqrt{a} \sqrt{a^{2}} \sqrt{\left(b^{2}\right)^{2}} \sqrt{c} \\
& =14(2) a b^{2} \sqrt{2 a c} \\
& =28 a b^{2} \sqrt{2 a c}
\end{aligned}
$$

Now the terms have the same radicand so we can subtract.

$$
120 a b^{2} \sqrt{2 a c}-28 a b^{2} \sqrt{2 a c}=92 a b^{2} \sqrt{2 a c}
$$

Note that we do not need an absolute value around the a because the $a^{3}$ under the radical means that $a$ can't be negative.

## TRY IT

Subtract $3 \sqrt{80 x}-4 \sqrt{45 x}$.

## Answer

0

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We leverage an important and useful identity in this section in a technique commonly used in college algebra:
rewriting an expression by multiplying it by a well-chosen form of the number 1 .
Because the additive identity states that $a \cdot 1=a$, we are able to multiply the top and bottom of any fraction by the same number without changing its value. We use this idea when we rationalize the denominator.
denominators of fractions, multiply by the form of 1 that will eliminate the radical.
For a denominator containing a single term, multiply by the radical in the denominator over itself. In other words, if the denominator is $b \sqrt{c}$, multiply by $\frac{\sqrt{c}}{\sqrt{c}}$.

For a denominator containing the sum or difference of a rational and an irrational term, multiply the numerator and denominator by the conjugate of the denominator, which is found by changing the sign of the radical portion of the denominator. If the denominator is $a+b \sqrt{c}$, then the conjugate is $a-b \sqrt{c}$.

## HOW TO: GIVEN AN EXPRESSION WITH A SINGLE SQUARE ROOT RADICAL TERM IN THE DENOMINATOR, RATIONALIZE THE DENOMINATOR.

1. Multiply the numerator and denominator by the radical in the denominator.
2. Simplify.

## EXAMPLE: RATIONALIZING A DENOMINATOR CONTAINING A SINGLE TERM

Write $\frac{2 \sqrt{3}}{3 \sqrt{10}}$ in simplest form.
Answer
The radical in the denominator is $\sqrt{10}$. So multiply the fraction by $\frac{\sqrt{10}}{\sqrt{10}}$. Then simplify.

$$
\begin{aligned}
\frac{2 \sqrt{3}}{3 \sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} & =\frac{2 \sqrt{30}}{30} \\
& =\frac{\sqrt{30}}{15}
\end{aligned}
$$

## TRY IT

Write $\frac{12 \sqrt{3}}{\sqrt{2}}$ in simplest form.
Answer
$6 \sqrt{6}$
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## HOW TO: GIVEN AN EXPRESSION WITH A RADICAL TERM AND A CONSTANT IN THE DENOMINATOR, RATIONALIZE THE DENOMINATOR.

1. Find the conjugate of the denominator.
2. Multiply the numerator and denominator by the conjugate.
3. Use the distributive property.
4. Simplify.

## EXAMPLE: RATIONALIZING A DENOMINATOR CONTAINING TWO TERMS

Write $\frac{4}{1+\sqrt{5}}$ in simplest form.
Answer
Begin by finding the conjugate of the denominator by writing the denominator and changing the sign. So the conjugate of $1+\sqrt{5}$ is $1-\sqrt{5}$. Then multiply the fraction by $\frac{1-\sqrt{5}}{1-\sqrt{5}}$.

$$
\begin{aligned}
\frac{4}{1+\sqrt{5}} \cdot \frac{1-\sqrt{5}}{1-\sqrt{5}} & =\frac{4-4 \sqrt{5}}{-4} & & \text { Use the distributive property. } \\
& =\sqrt{5}-1 & & \text { Simplify. }
\end{aligned}
$$

## TRY IT

Write $\frac{7}{2+\sqrt{3}}$ in simplest form.
Answer
$14-7 \sqrt{3}$
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## NTH ROOTS AND RATIONAL EXPONENTS

## LEARNING OUTCOMES

- Simplify Nth roots.
- Write radicals as rational exponents.


## RECALL: OPERATIONS ON FRACTIONS

When simplifying handling nth roots and rational exponents, we often need to perform operations on fractions. It's important to be able to do these operations on the fractions without converting them to decimals. Recall the rules for operations on fractions.

- To multiply fractions, multiply the numerators and place them over the product of the denominators.

$$
\text { - } \frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}
$$

- To divide fractions, multiply the first by the reciprocal of the second.
- $\frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}=\frac{a d}{b c}$
- To simplify fractions, find common factors in the numerator and denominator that cancel.
- $\frac{24}{32}=\frac{2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}=\frac{3}{2 \cdot 2}=\frac{3}{4}$
- To add or subtract fractions, first rewrite each fraction as an equivalent fraction such that each has a common denominator, then add or subtract the numerators and place the result over the common denominator.
- $\frac{a}{b} \pm \frac{c}{d}=\frac{a d \pm b c}{b d}$


## Using Rational Roots

Although square roots are the most common rational roots, we can also find cube roots, 4th roots, 5th roots, and more. Just as the square root function is the inverse of the squaring function, these roots are the inverse of their respective power functions. These functions can be useful when we need to determine the number
that, when raised to a certain power, gives a certain number.
Suppose we know that $a^{3}=8$. We want to find what number raised to the 3rd power is equal to 8 . Since $2^{3}=8$, we say that 2 is the cube root of 8 .

The $n$th root of $a$ is a number that, when raised to the $n$th power, gives $a$. For example, -3 is the 5 th root of -243 because $(-3)^{5}=-243$. If $a$ is a real number with at least one $n$th root, then the principal $n$th root of $a$ is the number with the same sign as $a$ that, when raised to the $n$th power, equals $a$.

The principal $n$th root of $a$ is written as $\sqrt[n]{a}$, where $n$ is a positive integer greater than or equal to 2 . In the radical expression, $n$ is called the index of the radical.

## A GENERAL NOTE: PRINCIPAL NTH ROOT

If $a$ is a real number with at least one $n$th root, then the principal $n$th root of $a$, written as $\sqrt[n]{a}$, is the number with the same sign as $a$ that, when raised to the $n$th power, equals $a$. The index of the radical is $n$.

## EXAMPLE: SIMPLIFYING NTH ROOTS

Simplify each of the following:

1. $\sqrt[5]{-32}$
2. $\sqrt[4]{4} \cdot \sqrt[4]{1,024}$
3. $-\sqrt[3]{\frac{8 x^{6}}{125}}$
4. $8 \sqrt[4]{3}-\sqrt[4]{48}$

## Answer

1. $\sqrt[5]{-32}=-2$ because $(-2)^{5}=-32$
2. First, express the product as a single radical expression. $\sqrt[4]{4,096}=8$ because $8^{4}=4,096$
3. $\frac{-\sqrt[3]{8 x^{6}}}{\sqrt[3]{125}} \quad$ Write as quotient of two radical expressions.
$\frac{-2 x^{2}}{5} \quad$ Simplify.
4. $8 \sqrt[4]{3}-2 \sqrt[4]{3} \quad$ Simplify to get equal radicands.
$6 \sqrt[4]{3} \quad$ Add.

## TRY IT

Simplify.

1. $\sqrt[3]{-216}$
2. $\frac{3 \sqrt[4]{80}}{\sqrt[4]{5}}$
3. $6 \sqrt[3]{9,000}+7 \sqrt[3]{576}$

## Answer

1. -6
2. 6
3. $88 \sqrt[3]{9}$

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## Using Rational Exponents

Radical expressions can also be written without using the radical symbol. We can use rational (fractional) exponents. The index must be a positive integer. If the index $n$ is even, then $a$ cannot be negative.

$$
a^{\frac{1}{n}}=\sqrt[n]{a}
$$

We can also have rational exponents with numerators other than 1. In these cases, the exponent must be a fraction in lowest terms. We raise the base to a power and take an nth root. The numerator tells us the power and the denominator tells us the root.

$$
a^{\frac{m}{n}}=(\sqrt[n]{a})^{m}=\sqrt[n]{a^{m}}
$$

All of the properties of exponents that we learned for integer exponents also hold for rational exponents.

## RATIONAL EXPONENTS

Rational exponents are another way to express principal $n$th roots. The general form for converting between a radical expression with a radical symbol and one with a rational exponent is

$$
a^{\frac{m}{n}}=(\sqrt[n]{a})^{m}=\sqrt[n]{a^{m}}
$$

## HOW TO: GIVEN AN EXPRESSION WITH A RATIONAL EXPONENT, WRITE THE EXPRESSION AS A RADICAL.

1. Determine the power by looking at the numerator of the exponent.
2. Determine the root by looking at the denominator of the exponent.
3. Using the base as the radicand, raise the radicand to the power and use the root as the index.

Watch this video to see more examples of how to write a radical with a fractional exponent.

## EXAMPLE: WRITING RATIONAL EXPONENTS AS RADICALS

Write $343^{\frac{2}{3}}$ as a radical. Simplify.
Answer
The 2 tells us the power and the 3 tells us the root.
$343^{\frac{2}{3}}=(\sqrt[3]{343})^{2}=\sqrt[3]{343^{2}}$
We know that $\sqrt[3]{343}=7$ because $7^{3}=343$. Because the cube root is easy to find, it is easiest to find the cube root before squaring for this problem. In general, it is easier to find the root first and then raise it to a power.
$343^{\frac{2}{3}}=(\sqrt[3]{343})^{2}=7^{2}=49$

## TRY IT

Write $9^{\frac{5}{2}}$ as a radical. Simplify.
Answer
$(\sqrt{9})^{5}=3^{5}=243$
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## EXAMPLE: WRITING RADICALS AS RATIONAL EXPONENTS

Write $\frac{4}{\sqrt[7]{a^{2}}}$ using a rational exponent.
Answer
The power is 2 and the root is 7 , so the rational exponent will be $\frac{2}{7}$. We get $\frac{4}{a^{\frac{2}{7}}}$. Using properties of exponents, we get $\frac{4}{\sqrt[7]{a^{2}}}=4 a^{\frac{-2}{7}}$.

## TRY IT

Write $x \sqrt{(5 y)^{9}}$ using a rational exponent.
Answer
$x(5 y)^{\frac{9}{2}}$


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## EXAMPLE: SIMPLIFYING RATIONAL EXPONENTS

Simplify:

1. $5\left(2 x^{\frac{3}{4}}\right)\left(3 x^{\frac{1}{5}}\right)$
2. $\left(\frac{16}{9}\right)^{-\frac{1}{2}}$

## Answer

1. 

$30 x^{\frac{3}{4}} x^{\frac{1}{5}} \quad$ Multiply the coefficients.
$30 x^{\frac{3}{4}+\frac{1}{5}} \quad$ Use properties of exponents.
$30 x^{\frac{19}{20}} \quad$ Simplify.
2.
$\left(\frac{9}{16}\right)^{\frac{1}{2}} \quad$ Use definition of negative exponents.
$\sqrt{\frac{9}{16}} \quad$ Rewrite as a radical.

| $\frac{\sqrt{9}}{\sqrt{16}}$ | Use the q |
| :--- | :--- |
| $\frac{3}{4}$ | Simplify. |

Simplify $(8 x)^{\frac{1}{3}}\left(14 x^{\frac{6}{5}}\right)$
Answer
$28 x^{\frac{23}{15}}$

Simplify $\frac{\sqrt{y}}{y^{\frac{2}{5}}}$
Answer
$y^{\frac{1}{2}}-y^{\frac{2}{5}}$
$y^{\frac{5}{10}}-y^{\frac{4}{10}}$
$y^{\frac{1}{10}}$ or $\sqrt[10]{y}$
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## SUMMARY: RADICALS AND RATIONAL EXPONENTS

## Key Concepts

- The principal square root of a number $a$ is the nonnegative number that when multiplied by itself equals $a$.
- If $a$ and $b$ are nonnegative, the square root of the product $a b$ is equal to the product of the square roots of $a$ and $b$
- If $a$ and $b$ are nonnegative, the square root of the quotient $\frac{a}{b}$ is equal to the quotient of the square roots of $a$ and $b$
- We can add and subtract radical expressions if they have the same radicand and the same index.
- Radical expressions written in simplest form do not contain a radical in the denominator. To eliminate the square root radical from the denominator, multiply both the numerator and the denominator by the conjugate of the denominator.
- The principal $n$th root of $a$ is the number with the same sign as $a$ that when raised to the $n$th power equals $a$. These roots have the same properties as square roots.
- Radicals can be rewritten as rational exponents and rational exponents can be rewritten as radicals.
- The properties of exponents apply to rational exponents.


## Glossary

index the number above the radical sign indicating the $n$th root
principal $n$th root the number with the same sign as $a$ that when raised to the $n$th power equals $a$
principal square root the nonnegative square root of a number $a$ that, when multiplied by itself, equals $a$
radical the symbol used to indicate a root
radical expression an expression containing a radical symbol
radicand the number under the radical symbol

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## PUTTING IT TOGETHER: ALGEBRA ESSENTIALS

At the beginning of this module, you read that mathematics is the language of science. You also discovered that additional numbers were developed to deal with real-world applications, including the existence of nothing and debt. Fortunately, the sets and the formats of numbers has continued to develop so that scientists and other researchers have the mathematical tools they need to investigate the natural world.

It just so happens that you need a specific number format to describe the light you are using to read these words-scientific notation. Light is a form of energy transferred as an electromagnetic (EM) wave. X-rays, microwaves, and radio waves are other forms of EM waves. They are differentiated by their properties of wavelength, which is the distance from one part of a wave to the same part of the next wave, and frequency, which is the number of waves that pass a point in a given amount of time. The arrangement of EM waves in order of wavelength and frequency is known as the electromagnetic spectrum. You can see from this example, that visible light makes up only a very small portion of the spectrum.

So what is the difference between blue light and red light? If you look at the diagram, you see that blue light is labeled at about 420 nanometers $(\mathrm{nm})$. Writing this out as meters gives $4.20 \times 10^{-7} \mathrm{~m}$-that's really


The electromagnetic spectrum with the visible light range enlarged. small. Red light, meanwhile, is about 700 nm or $\left(7.0 \times 10^{-7}\right) \mathrm{m}$. That small difference has a huge impact on what you see.

The interesting thing about EM waves is that they all travel at the same speed in the vacuum of space$3.0 \times 10^{8} \mathrm{~m} / \mathrm{s}$. So you can calculate either wavelength or frequency if you know the other.

$$
v=f \lambda
$$

where,

$$
\begin{aligned}
& v=\text { speed (velocity) } \\
& f=\text { frequency } \\
& \lambda=\text { wavelength }
\end{aligned}
$$

For example, let's find the frequency of red light with a wavelength of $7.0 \times 10^{-7} \mathrm{~m}$. First, rearrange to formula to solve for frequency. Then substitute in the given information.

$$
\begin{aligned}
f & =\frac{v}{\lambda} \\
& =\frac{3.0 \times 10^{8} \mathrm{~m} / \mathrm{s}}{7.0 \times 10^{-7} \mathrm{~m}} \text { Hertz } \\
& =4.0 \times 10^{14} \mathrm{~s}^{-1}
\end{aligned}
$$

You can see that wavelength and frequency are inversely related. So blue light, which has a smaller wavelength than red light, will have a greater frequency if traveling at the same speed.

So what does it matter if you can calculate the frequency of light? It just so happens that you may be able to predict the future of the universe! If a wave is produced by one object and observed by another, the frequency of the wave changes if either the source of the wave or the observer move relative to the other. This is true for sound waves as well, and you may already be familiar with this phenomenon if you have listened to a siren or train whistle as it approaches and passes you. When it comes to sound, frequency determines pitch so the sound gets higher as it approaches and lower as it moves away. This is known as the Doppler effect.

Now back to the universe. Objects in the universe, such as stars, give off light. Suppose you are an astronomer on Earth. If the source of light is moving toward Earth, the frequency of the light will increase. An increase in frequency means that it shifts toward the blue end of the spectrum, and is appropriately known as a blueshift. If the source of light is moving away from Earth, the frequency will decrease and shifts toward the red end of the spectrum. This shift, as you may have guessed, is a redshift.

Astronomers have used their understanding of the Doppler effect to make measurements regarding light from distant galaxies. As a result, they have learned that galaxies are moving apart from one another and the universe is expanding. Furthermore, the speed at which an object moves away is proportional to its distance. Because of the tremendous distances in space and the fast speed of light, none of this would be possible without being able to use formulas involving scientific notation.

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# MODULE 2: POLYNOMIAL AND RATIONAL EXPRESSIONS 

## WHY IT MATTERS: POLYNOMIALS AND RATIONAL EXPRESSIONS

## What are Polynomials and Rational Expressions Used For?

In economics we learn that profit is the difference between revenue (money coming in) and costs (money going out). Positive profit means that there is a net inflow of money, while negative profit means that money is being lost.

Suppose that you own a company that manufactures and sells furniture. A particular chair sells for $\$ 40$. If your company builds and sells 500 of


Arrangement of US currency; money serves as a medium of financial exchange in economics. these chairs in one month, then your monthly revenue would be $\$ 20,000$. In general, if you manufacture and sell $x$ chairs, then the revenue $R$, in dollars, would be given by the formula:
$R=40 x$
Now let's talk about costs. Let's say it costs $\$ 10,000$ each month just to keep the factory open, regardless of how many chairs are produced. This amount might include the electric and gas bills as well as the monthly paychecks for each of your employees. These costs, which do not depend on the number of items produced, are called the overhead or fixed costs. Furthermore, suppose the materials cost $\$ 25$ per chair. That means that if we produced $x$ chairs, then the total cost for materials would be $\$ 25 x$. The costs which vary based on the number of items produced are called variable costs. Together, with the overhead, we can derive a total estimated cost formula.
$C=($ overhead $)+($ variable costs $)=10,000+25 x$
Both formulas for revenue and cost are examples of polynomials. In fact both polynomials are what we call degree 1, or linear polynomials, because the variable $x$ only occurs with exponent 1 . What is the profit formula? Simply subtract the two expressions (Profit = Revenue - Costs), to find:

$$
\begin{aligned}
& P=R-C \\
& P=(40 x)-(10,000+25 x) \\
& P=40 x-10,000-25 x
\end{aligned}
$$

$P=15 x-10,000$
Will your company make money if it manufactures and sells 500 chairs? What about 800 chairs?
Now imagine that your small company has received more and more business over the past few months. You're producing so many chairs that you had to hire new employees, purchase new trucks and even plan for a larger factory. Those costs are adding up!

After carefully analyzing monthly expenses, you estimate the additional costs (above and beyond the overhead and variable costs) to be modeled by a degree 2 (or quadratic) term, $0.001 x^{2}$. Here are your new cost and profit formulas:
$C=10,000+15 x+0.001 x^{2}$
$P=40 x-\left(10,000+25 x+0.001 x^{2}\right)=-0.001 x^{2}+15 x-10,000$
These new formulas are also polynomials, but they behave differently than the

previous formulas for large values of $x$.
Finally, let's suppose you want to find out whether it is still cost-effective to produce your $\$ 40$ chairs in mass quantities. Perhaps the best way to find out is to determine the average profit when $x$ chairs are manufactured and sold. As always, an average is equal to the total amount ( $P$, for profit in this case) divided by the number of items $(x)$.

$$
\frac{P}{x}=\frac{-0.001 x^{2}+15 x-10,000}{x}
$$

This time, the formula for average profit is no longer a polynomial, but a rational expression, which is a ratio of two polynomials.

In this module you'll learn how to classify and perform algebraic operations on polynomials and rational expressions. We will check back in on our company's profits at the end of the module.

## LEARNING OUTCOMES

## Review Topics for Success

- Identify and evaluate polynomials.
- Find the greatest common factor of a list of expressions.
- Find the greatest common factor of a polynomial.
- Define and simplify rational expressions.


## Polynomial Basics

- Identify the degree, leading coefficient, and leading term of a polynomial.
- Add and subtract polynomials.
- Multiply polynomials.
- Square a binomial.
- Find a difference of squares.
- Perform operations on polynomials with several variables.


## Factoring Polynomials

- Identify and factor the greatest common factor of a polynomial.
- Factor a trinomial with leading coefficient 1.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor a sum and difference of cubes.
- Factor an expression with negative or fractional exponents.


## Rational Expressions

- Simplify rational expressions.
- Multiply and divide rational expressions.
- Find the least common denominator of two rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.


## REVIEW TOPICS FOR SUCCESS

## Learning Outcomes

By the end of this section, you will be able to:

- Identify polynomial expressions
- Evaluate polynomial expressions
- Find the greatest common factor of a list of expressions
- Find the greatest common factor of a polynomial
- Recognize and define a rational expression
- Determine the domain of a rational expression
- Simplify a rational expression

Before we can perform algebraic operations on polynomials and rational expressions, we need to be able to recognize and define these objects. We'll also need to review some skills that make our work on these expressions possible.

We've already seen how finding the least common multiple makes it possible to do addition and subtraction on fractions. The greatest common factor is another object we can find that makes it possible to work with polynomial expressions in useful ways. Finally, we'll extend our knowledge of fractions as ratios of expressions by defining and simplifying rational expressions.

Warm up for this module by refreshing these important concepts and skills. As you study these review topics, recall that you can also return to Module 1 Algebra Essentials any time you need to refresh the basics.

## RECALL FOR SUCCESS

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

## IDENTIFY AND EVALUATE POLYNOMIALS

## Learning Outcomes

- Identify a polynomial
- Evaluate a polynomial for given values

A polynomial is an expression consisting of a sum or difference of terms in which each term consists of a real number, a variable, or the product of a real number and one or more variables with non-negative integer exponents. Non negative integers are $0,1,2,3,4, \ldots$

## Identify a polynomial

The following table is intended to help you tell the difference between what is a polynomial and what is not.

| IS a <br> Polynomial | Is NOT a <br> Polynomial | Because |
| :--- | :--- | :--- |
| $2 x^{2}-\frac{1}{2} x-9$ | $\frac{2}{x^{2}}+x$ | Polynomials only have variables in the numerator |
| $\frac{y}{4}-y^{3}$ | $\frac{2}{y}+4$ | Polynomials only have variables in the numerator |
| $\sqrt{12}(a)+9$ | $\sqrt{a}+7$ | Roots are equivalent to rational exponents, and polynomials only have <br> integer exponents |

The basic building block of a polynomial is a monomial. A monomial is one term and can be a number, a variable, or the product of a number and variables with an exponent. The number part of the term is called the coefficient.


A polynomial containing two terms, such as $2 x-9$, is called a binomial. A polynomial containing three terms, such as $-3 x^{2}+8 x-7$, is called a trinomial.

We can find the degree of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the leading term because it is usually written first. The coefficient of the leading term is called the leading coefficient. When a polynomial is written so that the powers are descending, we say that it is in standard form. It is important to note that polynomials only have integer exponents.


## EXAMPLE

Which of the following expressions is a polynomial? Select all that apply.

1. $-\frac{1}{12} x^{3}+5+2 x^{2}$
2. $5 x^{\frac{1}{2}}-2 x^{3}+7 x$
3. $7 p-p^{11}-1$
4. $x^{-1}+x^{3}-9$

## Answer

1. $-\frac{1}{12} x^{3}+5+2 x^{2}$ is a polynomial.
2. $5 x^{\frac{1}{2}}-2 x^{3}+7 x$ is not a polynomial because it contains a non-integer exponent.
3. $7 p-p^{11}-1$ is a polynomial.
4. $x^{-1}+x^{3}-9$ is not a polynomial because it contains a negative exponent.

The table below illustrates some examples of monomials, binomials, trinomials, and other polynomials. They are all written in standard form.

| Monomials | Binomials | Trinomials | Other Polynomials |
| :--- | :--- | :--- | :--- |
| 15 | $3 y+13$ | $x^{3}-x^{2}+1$ | $5 x^{4}+3 x^{3}-6 x^{2}+2 x$ |
| $\frac{1}{2} x$ | $4 p-7$ | $3 x^{2}+2 x-9$ | $\frac{1}{3} x^{5}-2 x^{4}+\frac{2}{9} x^{3}-x^{2}+4 x-\frac{5}{6}$ |
| $-4 y^{3}$ | $3 x^{2}+\frac{5}{8} x$ | $3 y^{3}+y^{2}-2$ | $3 t^{3}-3 t^{2}-3 t-3$ |
| $16 n^{4}$ | $14 y^{3}+3 y$ | $a^{7}+2 a^{5}-3 a^{3}$ | $q^{7}+2 q^{5}-3 q^{3}+q$ |

When the coefficient of a polynomial term is 0 , you usually do not write the term at all (because 0 times anything is 0 , and adding 0 doesn't change the value). The last binomial above could be written as a trinomial, $14 y^{3}+0 y^{2}+3 y$.

A term without a variable is called a constant term, and the degree of that term is 0 . For example 13 is the constant term in $3 y+13$. You would usually say that $14 y^{3}+3 y$ has no constant term or that the constant term is 0 .

The following video illustrates how to identify which expressions are polynomials.

## Evaluate a polynomial

You can evaluate polynomials just as you can other kinds of expressions. To evaluate an expression for a value of the variable, you substitute the value for the variable every time it appears. Then use the order of
categorize it as a monomial, binomial, or trinomial.

1. $5 x^{3}+\frac{1}{8} x^{2}+\sqrt{5}$
$2 \frac{-9}{x-3}+7 y^{3}$
$3 \frac{t}{4}+\sqrt{2} t$
2. $3 \sqrt[3]{n}-2 n+5$
3. $\frac{1}{z^{1}}+2 z-1$

6 $2 a^{-4}+3 a^{3}-4 a+2$
Trinomid

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operations to find the resulting value for the expression.

## Example

Evaluate $3 x^{2}-2 x+1$ for $x=-1$.

## Answer

Substitute -1 for each $x$ in the polynomial.
$3(-1)^{2}-2(-1)+1$
Following the order of operations, evaluate exponents first.
$3(1)-2(-1)+1$
Multiply 3 times 1 , and then multiply -2 times -1 .
$3+(-2)(-1)+1$
Change the subtraction to addition of the opposite.
$3+2+1$
Find the sum.

Answer
$3 x^{2}-2 x+1=6$, for $x=-1$

## Example

Evaluate $-\frac{2}{3} p^{4}+2^{3}-p$ for $p=3$.

## Answer

Substitute 3 for each $p$ in the polynomial.
$-\frac{2}{3}(3)^{4}+2(3)^{3}-3$
Following the order of operations, evaluate exponents first and then multiply.
$-\frac{2}{3}(81)+2(27)-3$
Add and then subtract to get -3 .
$-54+54-3$

Answer
$-\frac{2}{3} p^{4}+2 p^{3}-p=-3$, for $p=3$
In the following video we show more examples of evaluating polynomials for given values of the variable.


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## GREATEST COMMON FACTOR

## LEARNING OUTCOMES

- Find the greatest common factor of a list of expressions
- Find the greatest common factor of a polynomial

Factors are the building blocks of multiplication. They are the numbers that you can multiply together to produce another number. For example, 2 and 10 are factors of 20 , as are $4,5,1,20$. To factor a number is to
rewrite it as a product. $20=4 \cdot 5$ or $20=1 \cdot 20$. In algebra, we use the word factor as both a noun something being multiplied - and as a verb - the action of rewriting a sum or difference as a product. Factoring is very helpful in simplifying expressions and solving equations involving polynomials.

The greatest common factor (GCF) of two numbers is the largest number that divides evenly into both numbers: 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 . The GCF of two or more algebraic expressions works the same way: $4 x$ is the GCF of $16 x$ and $20 x^{2}$ because it is the largest algebraic expression that divides evenly into both $16 x$ and $20 x^{2}$.

## Find the GCF of a list of algebraic expressions

We begin by finding the GCF of a list of numbers, then we'll extend the technique to monomial expressions containing variables.

A good technique for finding the GCF of a list of numbers is to write each number as a product of its prime factors. Then, match all the common factors between each prime factorization. The product of all the common factors will build the greatest common factor.

## EXAMPLE

Find the greatest common factor of 24 and 36 .

## Answer

Step 1: Factor each coefficient into primes. Write all variables with exponents in expanded form.

Step 2: List all factorsmatching common factors in a column.

In each column, circle the common factors.

Step 3: Bring down the common factors that all expressions share.

Factor 24 and 36.

Circle the 2 , 2 , and 3 that are shared by both numbers.

$$
\begin{aligned}
& 24=2 \cdot 2 \cdot 2 \cdot 3 \\
& 36=2 \cdot 2 \cdot 3 \cdot 3 \\
& \hline 24=2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \\
& 36=2 \cdot\left(\begin{array}{l}
3 \\
2 \cdot \\
3 \cdot 3
\end{array}\right. \\
& \mathrm{GCF}=2 \cdot 2 \cdot 3 \\
& \mathrm{GCF}=12
\end{aligned}
$$

Bring down the 2, 2, 3 and then multiply.


Step 4: Multiply the factors.

The GCF of 24 and 36 is 12 .
Notice that since the GCF is a factor of both numbers, 24 and 36 can be written as multiples of 12 .

$$
\begin{aligned}
24 & =12 \cdot 2 \\
36 & =12 \cdot 3
\end{aligned}
$$

Here's a summary of the technique.

## FIND THE GREATEST COMMON FACTOR

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors-matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all expressions share.
4. Multiply the factors.

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In the previous example, we found the greatest common factor of a list of constants. The greatest common factor of an algebraic expression can contain variables raised to powers along with coefficients.

To find the GCF of an expression containing variable terms, first find the GCF of the coefficients, then find the GCF of the variables. The GCF of the variables will be the smallest degree of each variable that appears in each term. Here's an example using the matching method from the example above.

| EXAMPLE |  |
| :--- | :--- |
| Find the greatest common factor of $5 x$ and 15. <br> Answer <br> Solution |  |
| Factor each number into primes.  <br> Circle the common factors in each column.  <br> Bring down the common factors.  <br>   | GCF $=3$ <br>  |

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In the examples so far, the GCF was a constant. In the next two examples we will get variables in the greatest common factor.

Here are some examples of finding the GCF of a list of more than two expressions.

The common-factor matching method works well for finding the GCF of the coefficients, but when finding the GCF of the variables, you may have noticed that we can simply select the smallest power of a variable that appears in each term.

## EXAMPLE

Find the greatest common factor of $12 x^{2}$ and $18 x^{3}$.
Answer
Solution

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors. Multiply the factors.

$$
\begin{aligned}
& 12 x^{2}=2 \cdot 2 \cdot 3 \cdot \quad x \cdot x \\
& 18 x^{3}=2 \cdot \quad 3 \cdot 3 \cdot x \cdot x \cdot x \\
& \mathrm{GCF}=2 \cdot 3 \cdot x \cdot x \\
& \mathrm{GCF}=6 x^{2}
\end{aligned}
$$

The GCF of $12 x^{2}$ and $18 x^{3}$ is $6 x^{2}$

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## EXAMPLE

Find the greatest common factor of $14 x^{3}, 8 x^{2}, 10 x$.
Answer
Solution

Factor each coefficient into primes and write
the variables with exponents in expanded form.
Circle the common factors in each column.
Bring down the common factors.
Multiply the factors.


The GCF of $14 x^{3}$ and $8 x^{2}$ and $10 x$ is $2 x$

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In fact, the GCF of a set of expressions in the form $x^{n}$ will always be the exponent of lowest degree.
Watch the following video to see another example of how to find the GCF of two monomials that have one variable.

Sometimes you may encounter a polynomial with more than one variable, so it is important to check whether both variables are part of the GCF. In the next example we find the GCF of two terms which both contain two


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## variables.

## Example

Find the greatest common factor of $81 c^{3} d$ and $45 c^{2} d^{2}$.

## Answer

$$
\begin{aligned}
81 c^{3} d & =3 \cdot 3 \cdot 3 \cdot 3 \cdot c \cdot c \cdot c \cdot d \\
45 c^{2} d^{2} & =3 \cdot 3 \cdot 5 \cdot c \cdot c \cdot d \cdot d \\
\mathrm{GCF} & =3 \cdot 3 \cdot c \cdot c \cdot d
\end{aligned}
$$

## Answer

$$
\mathrm{GCF}=9 c^{2} d
$$

The video that follows shows another example of finding the greatest common factor of two monomials with more than one variable.


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## Find the GCF of a polynomial

Now that you have practiced finding the GCF of a term with one and two variables, the next step is to find the GCF of a polynomial. Later in this module we will apply this idea to factoring the GCF out of a polynomial. That is, doing the distributive property "backwards" to divide the GCF away from each of the terms in the polynomial. In preparation, practice finding the GCF of a given polynomial.

Recall that a polynomial is an expression consisting of a sum or difference of terms. To find the GCF of a polynomial, inspect each term for common factors just as you previously did with a list of expressions.

No matter how large the polynomial, you can use the same technique described below to identify its GCF.

## HOW TO: GIVEN A POLYNOMIAL EXPRESSION, FIND THE GREATEST COMMON FACTOR.

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.

## EXAMPLE

Find the GCF of $6 x^{3} y^{3}+45 x^{2} y^{2}+21 x y$.

## Answer

The GCF of 6,45 , and 21 is 3 .
The GCF of $x^{3}, x^{2}$, and $x$ is $x$.
And the GCF of $y^{3}, y^{2}$, and $y$ is $y$.
Combine these to find the GCF of the polynomial, $3 x y$.

## TRY IT

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[^0]
## DEFINE AND SIMPLIFY RATIONAL EXPRESSIONS

## LEARNING OUTCOMES

- Recognize and define a rational expression
- Determine the domain of a rational expression
- Simplify a rational expression

Rational expressions are fractions that have a polynomial in the numerator, denominator, or both. Although rational expressions can seem complicated because they contain variables, they can be simplified using the techniques used to simplify expressions such as $\frac{4 x^{3}}{12 x^{2}}$. Because rational expressions often contain a variable in the denominator, though, there are certain values of the variable that will would cause the denominator to equal zero. When the denominator of a fraction is zero, the fraction is said to be undefined.

When working with rational expressions, it is important to identify values that would cause the denominator to equal zero and state them explicitly as numbers that can't be included in the set of values for which the expression may be evaluated. We call the set of values for which the expression is defined the domain of the expression.

## Determine the domain of a rational expression

Consider the following rational expression evaluated at $x=2$ :
Evaluate $\frac{x}{x-2}$ for $x=2$
Substitute $x=2$

$$
\begin{aligned}
& \frac{2}{2-2} \\
& =\frac{2}{0}
\end{aligned}
$$

But we have stated that a fraction with a zero in the denominator is not defined. This means that for the expression $\frac{x}{x-2}, x$ cannot be 2 because it will result in an undefined fraction. In general, finding values for a variable that will not result in division by zero allows us to identify all the values for which the expression is defined. We call the set of all such values the domain of the expression (later we'll apply them to equations and functions as well).

## Domain of a rational expression

The domain of a rational expression is a collection of the values for the variable that will not result in an undefined mathematical operation such as division by zero. For a = any real number, we can notate the
domain in the following way:
$x$ is all real numbers where $x \neq a$
We cannot divide a number $c$ by zero $\left(\frac{c}{0}=?\right)$ because if we could, there would have to exist some number that, when multiplied by 0 , would recover $c(? \cdot 0=c)$. There are no numbers that can do this, so we say that division by zero is undefined. In simplifying rational expressions, pay attention to what values of the variable(s) in the expression would make the denominator equal zero. These values cannot be included in the domain. We call them excluded values.
(Note that although the denominator cannot be equivalent to 0 , the numerator can-this is why you only look for excluded values in the denominator of a rational expression.)

For rational expressions, the domain will exclude values for which the value of the denominator is 0 . The following example illustrates finding the domain of an expression.

## Example

Identify the domain of the expression. $\frac{x+7}{x+9}$
Answer

We can see that if $x=-9$, the denominator would equal zero because $-9+9=0$.
So we exclude -9 from the set of possible values in the domain of $\frac{x+7}{x+9}$.
The domain is all real numbers except -9 .

## Simplify Rational Expressions

Before we begin to discuss simplifying rational expressions, let's review the difference between a factor, a term, and an expression.

Factors are the building blocks of multiplication. They are the numbers that you can multiply together to produce another number: 2 and 10 are factors of 20 , as are $4,5,1,20$.

Terms are single numbers, or variables and numbers connected by multiplication. $-4,6 x$ and $x^{2}$ are all terms.

Expressions are groups of terms connected by addition and subtraction. $2 x^{2}-5$ is an expression.
Simplify: $\frac{2 x^{2}}{12 x}$
Just as we did when we simplified fractions containing only numbers, we'll write the numerator and denominator each as a product of its smallest factors. Then, we can cancel common factors in the numerator and denominator by dividing them.

$$
\begin{aligned}
& \frac{2 x^{2}}{12 x} \\
& =\frac{2 \cdot x \cdot x}{2 \cdot 3 \cdot 2 \cdot x} \\
& =\frac{\not 2 \cdot \not x \cdot x}{\not 2 \cdot 3 \cdot 2 \cdot \not x}=\frac{x}{6}
\end{aligned}
$$

We can do this because $\frac{2}{2}=1$ and $\frac{x}{x}=1$, so our expression simplifies to $\frac{x}{6}$
Compare that to the expression $\frac{2 x^{2}+x}{2 x+2}$. Notice the numerator and denominator in this rational expression consist of polynomials (sums or differences of terms). We would have to write those polynomials as products, a process called factoring, before canceling any terms or variables. We'll see how to do that later in this module.

The polynomial above, written in factored form is $\frac{2 x(x+1)}{2(x+1)}$. Note the common factors of 2 and $(x+1)$ in the numerator and denominator. Since these terms are being multiplied together, we can cancel the common factors.

$$
\begin{aligned}
& \frac{2 x(x+1)}{2(x+1)} \\
& =\frac{2 \cdot x \cdot(x+1)}{2 \cdot(x+1)} \\
& =\frac{\not \partial \cdot x \cdot(x+1)}{\not 2 \cdot(x+1)} \\
& =\frac{x}{1}=x
\end{aligned}
$$

We'll discover how to write the numerator and denominator of a rational expression as products later in the module. For now, consider the steps to simplify a rational expression.

## HOW TO: GIVEN A RATIONAL EXPRESSION, SIMPLIFY IT

1. Write the numerator and denominator as products of factors.
2. Cancel any common factors.

## EXAMPLE

Simplify the rational expression $\frac{x(x-7)(x+2)}{x(x+3)(x+2)}$ and state its domain.

## Answer

$\frac{x(x-7)(x+2)}{x(x+3)(x+2)}=\frac{\not x(x-7)(x+2)}{\not x(x+3)(x+2)}=\frac{(x-7)}{(x+3)}$
Since the unsimplified expression would have been undefined for $x=0,-3$, or -2 , the domain is all real numbers except $-3,-2$ and 0 .

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## SUMMARY: REVIEW

## Key Concepts

- To evaluate a polynomial expression, substitute a particular value for each variable and simplify.
- To find the GCF of a polynomial, first find the GCF of the coefficients, then find the GCF of the variables, then multiply them together to form the GCF of the polynomial.
- To simplify a rational expression, first factor the numerator and denominator, exclude any possible values of the variable that could cause the denominator to equal zero, then find and cancel common factors in the numerator and denominator.
- Division by zero is undefined.


## Glossary

coefficient a real number multiplying a variable in a polynomial
constant a term in a polynomial with no variable
degree the highest exponent of a variable term in a polynomial
domain of a rational expresion all numbers for which the expression is defined (for which the denominator does not equal zero)
excluded valuesvalues excluded from possible values of the variable of a rational expression because they would cause the denominator to equal zero
factor as a verb, to write a number or, in the case of polynomials, a sum or difference, as a product
factors as a noun, numbers multiplied together to form another number
greatest common factor (GCF) the largest number or expression that divides evenly into a list of expressions or into each term of a polynomial
leading coefficient the coefficient of the highest degree term of a polynomial
monomial, binomial, trinomial a polynomial made of one, two, or three terms, respectively
polynomial an expression consisting of a sum or difference of terms in which each term consists of a real number, a variable, or the product of a real number and one or more variables with non-negative integer exponents
rational expression an expression with a polynomial in the numerator and a polynomial in the denominator

## INTRODUCTION TO POLYNOMIAL BASICS

## Learning Outcomes

By the end of this section, you will be able to:

- Identify the degree, leading coefficient, and leading term of a polynomial.
- Add and subtract polynomials.
- Multiply polynomials.
- Square a binomial.
- Find a difference of squares.
- Perform operations on polynomials with several variables.

Earl is building a doghouse whose front is in the shape of a square topped with a triangle. There will be a rectangular door through which the dog can enter and exit the house. Earl wants to find the area of the front of the doghouse so that he can purchase the correct amount of paint. Using the measurements of the front of the house shown in Figure 1 below, we can create an expression that combines several variable terms which allows us to solve this problem and others like it.


Figure 1
First, find the area of the square in square feet.

$$
\begin{aligned}
A & =s^{2} \\
& =(2 x)^{2} \\
& =4 x^{2}
\end{aligned}
$$

Then, find the area of the triangle in square feet.

$$
\begin{aligned}
A & =\frac{1}{2} b h \\
& =\frac{1}{2}(2 x)\left(\frac{3}{2}\right) \\
& =\frac{3}{2} x
\end{aligned}
$$

Next, find the area of the rectangular door in square feet.

$$
\begin{aligned}
A & =l w \\
& =x \cdot 1 \\
& =x
\end{aligned}
$$

The area of the front of the doghouse can be found by adding the areas of the square and the triangle and then subtracting the area of the rectangle. When we do this, we get
$4 x^{2}+\frac{3}{2} x-x \mathrm{ft}^{2}$, or $4 x^{2}+\frac{1}{2} x \mathrm{ft}^{2}$.
In this module, we will examine expressions such as this one which combine several variable terms.

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## OPERATIONS ON POLYNOMIALS

## LEARNING OUTCOMES

- Identify the degree, leading coefficient, and leading term of a polynomial.
- Add and subtract polynomials.
- Multiply polynomials.

The area of the front of the doghouse described in the introduction was $4 x^{2}+\frac{1}{2} x \mathrm{ft}^{2}$.
This is an example of a polynomial which is a sum of or difference of terms each consisting of a variable raised to a nonnegative integer power. A number multiplied by a variable raised to an exponent, such as $384 \pi$, is known as a coefficient. Coefficients can be positive, negative, or zero, and can be whole numbers, decimals, or fractions. Each product $a_{i} x^{i}$, such as $384 \pi w$, is a term of a polynomial. If a term does not contain a variable, it is called a constant.

A polynomial containing only one term, such as $5 x^{4}$, is called a monomial. A polynomial containing two terms, such as $2 x-9$, is called a binomial. A polynomial containing three terms, such as $-3 x^{2}+8 x-7$, is called a trinomial.

We can find the degree of a polynomial by identifying the highest power of the variable that occurs in the polynomial. The term with the highest degree is called the leading term because it is usually written first. The coefficient of the leading term is called the leading coefficient. When a polynomial is written so that the powers are descending, we say that it is in standard form.


## A GENERAL NOTE: POLYNOMIALS

A polynomial is an expression that can be written in the form

$$
a_{n} x^{n}+\cdots+a_{2} x^{2}+a_{1} x+a_{0}
$$

Each real number $a_{i}$ is called a coefficient. The number $a_{0}$ that is not multiplied by a variable is called a constant. Each product $a_{i} x^{i}$ is a term of a polynomial. The highest power of the variable that occurs in the polynomial is called the degree of a polynomial. The leading term is the term with the highest power, and its coefficient is called the leading coefficient.

## HOW TO: GIVEN A POLYNOMIAL EXPRESSION, IDENTIFY THE DEGREE AND LEADING COEFFICIENT

1. Find the highest power of $x$ to determine the degree.
2. Identify the term containing the highest power of $x$ to find the leading term.

3 . Identify the coefficient of the leading term.

## EXAMPLE: IDENTIFYING THE DEGREE AND LEADING COEFFICIENT OF A POLYNOMIAL

For the following polynomials, identify the degree, the leading term, and the leading coefficient.

1. $3+2 x^{2}-4 x^{3}$
2. $5 t^{5}-2 t^{3}+7 t$
3. $6 p-p^{3}-2$

Answer

1. The highest power of $x$ is 3 , so the degree is 3 . The leading term is the term containing that degree, $-4 x^{3}$. The leading coefficient is the coefficient of that term, -4 .
2. The highest power of $t$ is 5 , so the degree is 5 . The leading term is the term containing that degree, $5 t^{5}$ . The leading coefficient is the coefficient of that term, 5 .
3 . The highest power of $p$ is 3 , so the degree is 3 . The leading term is the term containing that degree, $-p^{3}$, The leading coefficient is the coefficient of that term, -1 .

## TRY IT

Identify the degree, leading term, and leading coefficient of the polynomial $4 x^{2}-x^{6}+2 x-6$.

Answer
The degree is 6 , the leading term is $-x^{6}$, and the leading coefficient is -1 .

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In the following video example, we will identify the terms, leading coefficient, and degree of a polynomial.


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## Adding and Subtracting Polynomials

We can add and subtract polynomials by combining like terms, which are terms that contain the same variables raised to the same exponents. For example, $5 x^{2}$ and $-2 x^{2}$ are like terms and can be added to get $3 x^{2}$, but $3 x$ and $3 x^{2}$ are not like terms and therefore cannot be added.

## HOW TO: GIVEN MULTIPLE POLYNOMIALS, ADD OR SUBTRACT THEM TO SIMPLIFY THE EXPRESSIONS

1. Combine like terms.
2. Simplify and write in standard form. Standard form means you start with the leading term, and write the rest of the terms in descending order by degree.

Watch this video to see more examples of adding and subtracting polynomials.

## Multiplying Polynomials

Multiplying polynomials is a bit more challenging than adding and subtracting polynomials. We must use the distributive property to multiply each term in the first polynomial by each term in the second polynomial. We then combine like terms. We can also use a shortcut called the FOIL method when multiplying binomials. Certain special products follow patterns that we can memorize and use instead of multiplying the polynomials by hand each time. We will look at a variety of ways to multiply polynomials.

## Multiplying Polynomials Using the Distributive Property

To multiply a number by a polynomial we use the distributive property. The number must be distributed to

## EXAMPLE: ADDING POLYNOMIALS

Find the sum.
$\left(12 x^{2}+9 x-21\right)+\left(4 x^{3}+8 x^{2}-5 x+20\right)$
Answer
$4 x^{3}+\left(12 x^{2}+8 x^{2}\right)+(9 x-5 x)+(-21+20)$ Combine like terms.
$4 x^{3}+20 x^{2}+4 x-1 \quad$ Simplify.

## Analysis of the Solution

We can check our answers to these types of problems using a graphing calculator. To check, graph the original problem as given along with the simplified answer. The two graphs should be the same. Be sure to use the same window to compare the graphs. Using different windows can make the expressions seem equivalent when they are not.

## TRY IT

Find the sum.
$\left(2 x^{3}+5 x^{2}-x+1\right)+\left(2 x^{2}-3 x-4\right)$

## Answer

$2 x^{3}+7 x^{2}-4 x-3$
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## BE CAREFUL WHEN SUBTRACTING POLYNOMIALS

When subtracting a polynomial from another, be careful to subtract each term in the second from the first. That is, use the distributive property to distribute the minus sign through the second polynomial.

$$
\begin{array}{ll}
\left(3 x^{2}-2 x+9\right)-\left(x^{2}-4 x+5\right) & \text { Distribute the negative in front of the parenthesis } \\
3 x^{2}-2 x+9-x^{2}-(-4 x)-5 & \text { Be careful when subtracting a negative. } \\
3 x^{2}-x^{2}-2 x+4 x+9-5 & \text { Rearrange terms in descending order of degree } \\
2 x^{2}+2 x+4 & \text { Combine like terms. }
\end{array}
$$

each term of the polynomial. In $2(x+7)$ we can distribute 2 to obtain the expression $2 x+14$. When multiplying polynomials, the distributive property allows us to multiply each term of the first polynomial by each term of the second. We then add the products together and combine like terms to simplify.

Watch this video to see more examples of how to use the distributive property to multiply polynomials.

## Using FOIL to Multiply Binomials

A shortcut called FOIL is sometimes used to find the product of two binomials. It is called FOIL because we

## EXAMPLE: SUBTRACTING POLYNOMIALS

Find the difference.
$\left(7 x^{4}-x^{2}+6 x+1\right)-\left(5 x^{3}-2 x^{2}+3 x+2\right)$
Answer

$$
\begin{array}{ll}
7 x^{4}-5 x^{3}+\left(-x^{2}+2 x^{2}\right)+(6 x-3 x)+(1-2) & \text { Combine like terms. } \\
7 x^{4}-5 x^{3}+x^{2}+3 x-1 & \text { Simplify }
\end{array}
$$

Analysis of the Solution
Note that finding the difference between two polynomials is the same as adding the opposite of the second polynomial to the first.

## TRY IT

Find the difference.
$\left(-7 x^{3}-7 x^{2}+6 x-2\right)-\left(4 x^{3}-6 x^{2}-x+7\right)$
Answer
$-11 x^{3}-x^{2}+7 x-9$
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## HOW TO: GIVEN THE MULTIPLICATION OF TWO POLYNOMIALS, USE THE DISTRIBUTIVE PROPERTY TO SIMPLIFY THE EXPRESSION

1. Multiply each term of the first polynomial by each term of the second.
2. Combine like terms.

## EXAMPLE: MULTIPLYING POLYNOMIALS USING THE DISTRIBUTIVE PROPERTY

Find the product.
$(2 x+1)\left(3 x^{2}-x+4\right)$
Answer
$2 x\left(3 x^{2}-x+4\right)+1\left(3 x^{2}-x+4\right) \quad$ Use the distributive property.
$\left(6 x^{3}-2 x^{2}+8 x\right)+\left(3 x^{2}-x+4\right) \quad$ Multiply.
$6 x^{3}+\left(-2 x^{2}+3 x^{2}\right)+(8 x-x)+4$ Combine like terms.
$6 x^{3}+x^{2}+7 x+4 \quad$ Simplify.

## Analysis of the Solution

We can use a table to keep track of our work, as shown in the table below. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.

|  | $3 x^{2}$ | $-x$ | +4 |
| :--- | :--- | :--- | :--- |
| $2 x$ | $6 x^{3}$ | $-2 x^{2}$ | $8 x$ |
| +1 | $3 x^{2}$ | $-x$ | 4 |

## USING THE DISTRIBUTIVE PROPERTY

When multiplying a binomial and another polynomial with two or more terms, be sure to multiply each term in the first to each term in the second.

## TRY IT

Find the product.
$(3 x+2)\left(x^{3}-4 x^{2}+7\right)$
Answer
$3 x^{4}-10 x^{3}-8 x^{2}+21 x+14$

Analysis of the Solution
We can use a table to keep track of our work as shown below. Write one polynomial across the top and the other down the side. For each box in the table, multiply the term for that row by the term for that column. Then add all of the terms together, combine like terms, and simplify.

|  | $3 x^{2}$ | $-x$ | +4 |
| :--- | :--- | :--- | :--- |
| $2 x$ | $6 x^{3}$ | $-2 x^{2}$ | $8 x$ |
| +1 | $3 x^{2}$ | $-x$ | 4 |

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multiply the first terms, the outer terms, the inner terms, and then the last terms of each binomial.


The FOIL method is simply just the distributive property. We are multiplying each term of the first binomial by each term of the second binomial and then combining like terms.

## HOW TO: GIVEN TWO BINOMIALS, MULTIPLYING USING FOIL

1. Multiply the first terms of each binomial.
2. Multiply the outer terms of the binomials.
3. Multiply the inner terms of the binomials.
4. Multiply the last terms of each binomial.
5. Add the products.
6. Combine like terms and simplify.

## EXAMPLE: USING FOIL TO MULTIPLY BINOMIALS

Use FOIL to find the product.
$(2 x-18)(3 x+3)$

## Answer

Find the product of the first terms.


$$
2 x \cdot 3 x=6 x^{2}
$$

Find the product of the outer terms.

$2 x \cdot 3=6 x$
Find the product of the inner terms.


Find the product of the last terms.

$-18 \cdot 3=-54$
$6 x^{2}+6 x-54 x-54 \quad$ Add the products.
$6 x^{2}+(6 x-54 x)-54$ Combine like terms.
$6 x^{2}-48 x-54 \quad$ Simplify.

## TRY IT

Use FOIL to find the product.
$(x+7)(3 x-5)$
Answer
$3 x^{2}+16 x-35$
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## SPECIAL CASES OF POLYNOMIALS

## LEARNING OUTCOMES

- Square a binomial.
- Find a difference of squares.
- Perform operations on polynomials with several variables.


## Perfect Square Trinomials

Certain binomial products have special forms. When a binomial is squared, the result is called a perfect square trinomial. We can find the square by multiplying the binomial by itself. However, there is a special form that each of these perfect square trinomials takes, and memorizing the form makes squaring binomials much easier. Let's look at a few perfect square trinomials to familiarize ourselves with the form.

$$
\begin{aligned}
& (x+5)^{2}=x^{2}+10 x+25 \\
& (x-3)^{2}=x^{2}-6 x+9 \\
& (4 x-1)^{2}=4 x^{2}-8 x+1
\end{aligned}
$$

Notice that the first term of each trinomial is the square of the first term of the binomial and, similarly, the last term of each trinomial is the square of the last term of the binomial. The middle term is double the product of the two terms. Lastly, we see that the first sign of the trinomial is the same as the sign of the binomial.

## A GENERAL NOTE: PERFECT SQUARE TRINOMIALS

When a binomial is squared, the result is the first term squared added to double the product of both terms and the last term squared.

$$
(x+a)^{2}=(x+a)(x+a)=x^{2}+2 a x+a^{2}
$$

## HOW TO: GIVEN A BINOMIAL, SQUARE IT USING THE FORMULA FOR PERFECT SQUARE TRINOMIALS

1. Square the first term of the binomial.
2. Square the last term of the binomial.
3. For the middle term of the trinomial, double the product of the two terms.
4. Add and simplify.

## Difference of Squares

Another special product is called the difference of squares which occurs when we multiply a binomial by another binomial with the same terms but the opposite sign. Let's see what happens when we multiply $(x+1)(x-1)$ using the FOIL method.

## EXAMPLE: EXPANDING PERFECT SQUARES

Expand $(3 x-8)^{2}$.
Answer
Begin by squaring the first term and the last term. For the middle term of the trinomial, double the product of the two terms.

$$
(3 x)^{2}-2(3 x)(8)+(-8)^{2}
$$

$9 x^{2}-48 x+64$.

## TRY IT

Expand $(4 x-1)^{2}$.

## Answer

$16 x^{2}-8 x+1$
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$$
\begin{aligned}
(x+1)(x-1) & =x^{2}-x+x-1 \\
& =x^{2}-1
\end{aligned}
$$

The middle term drops out resulting in a difference of squares. Just as we did with the perfect squares, let's look at a few examples.
$(x+5)(x-5)=x^{2}-25$
$(x+11)(x-11)=x^{2}-121$
$(2 x+3)(2 x-3)=4 x^{2}-9$
Because the sign changes in the second binomial, the outer and inner terms cancel each other out, and we are left only with the square of the first term minus the square of the last term.

## Q \& A

Is there a special form for the sum of squares?
No. The difference of squares occurs because the opposite signs of the binomials cause the middle terms to disappear. There are no two binomials that multiply to equal a sum of squares.

## A GENERAL NOTE: DIFFERENCE OF SQUARES

When a binomial is multiplied by a binomial with the same terms separated by the opposite sign, the result is the square of the first term minus the square of the last term.

$$
(a+b)(a-b)=a^{2}-b^{2}
$$

## Performing Operations with Polynomials of Several Variables

We have looked at polynomials containing only one variable. However, a polynomial can contain several

## HOW TO: GIVEN A BINOMIAL MULTIPLIED BY A BINOMIAL WITH THE <br> SAME TERMS BUT THE OPPOSITE SIGN, FIND THE DIFFERENCE OF SQUARES

1. Square the first term of the binomials.
2. Square the last term of the binomials.
3. Subtract the square of the last term from the square of the first term.

## EXAMPLE: MULTIPLYING BINOMIALS RESULTING IN A DIFFERENCE OF SQUARES

Multiply $(9 x+4)(9 x-4)$.
Answer
Square the first term to get $(9 x)^{2}=81 x^{2}$. Square the last term to get $4^{2}=16$. Subtract the square of the last term from the square of the first term to find the product of $81 x^{2}-16$.

## TRY IT

Multiply $(2 x+7)(2 x-7)$.
Answer
$4 x^{2}-49$
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variables. All of the same rules apply when working with polynomials containing several variables. Consider an example:
$(a+2 b)(4 a-b-c)$
$a(4 a-b-c)+2 b(4 a-b-c) \quad$ Use the distributive property.
$4 a^{2}-a b-a c+8 a b-2 b^{2}-2 b c \quad$ Multiply.
$4 a^{2}+(-a b+8 a b)-a c-2 b^{2}-2 b c \quad$ Combine like terms.
$4 a^{2}+7 a b-a c-2 b c-2 b^{2} \quad$ Simplify.

## EXAMPLE: MULTIPLYING POLYNOMIALS CONTAINING SEVERAL VARIABLES

Multiply $(x+4)(3 x-2 y+5)$.

## Answer

Follow the same steps that we used to multiply polynomials containing only one variable.
$x(3 x-2 y+5)+4(3 x-2 y+5) \quad$ Use the distributive property.
$3 x^{2}-2 x y+5 x+12 x-8 y+20 \quad$ Multiply.
$3 x^{2}-2 x y+(5 x+12 x)-8 y+20$ Combine like terms.
$3 x^{2}-2 x y+17 x-8 y+20 \quad$ Simplify.

## TRY IT

$(3 x-1)(2 x+7 y-9)$.
Answer
$6 x^{2}+21 x y-29 x-7 y+9$
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## SUMMARY: POLYNOMIAL BASICS

## Key Equations

## perfect square trinomial <br> difference of squares

 $(x+a)^{2}=(x+a)(x+a)=x^{2}+2 a x+a^{2}$$(a+b)(a-b)=a^{2}-b^{2}$

## Key Concepts

- A polynomial is a sum of terms each consisting of a variable raised to a nonnegative integer power. The degree is the highest power of the variable that occurs in the polynomial. The leading term is the term containing the highest degree, and the leading coefficient is the coefficient of that term.
- We can add and subtract polynomials by combining like terms.
- To multiply polynomials, use the distributive property to multiply each term in the first polynomial by each term in the second. Then add the products.
- FOIL (First, Outer, Inner, Last) is a shortcut that can be used to multiply binomials.
- Perfect square trinomials and difference of squares are special products.
- Follow the same rules to work with polynomials containing several variables.


## Glossary

binomial a polynomial containing two terms
coefficient any real number $a_{i}$ in a polynomial of the form $a_{n} x^{n}+\cdots+a_{2} x^{2}+a_{1} x+a_{0}$ degree the highest power of the variable that occurs in a polynomial
difference of squares the binomial that results when a binomial is multiplied by a binomial with the same terms, but the opposite sign
leading coefficient the coefficient of the leading term
leading term the term containing the highest degree
monomial a polynomial containing one term
perfect square trinomial the trinomial that results when a binomial is squared
polynomial a sum of terms each consisting of a variable raised to a nonnegative integer power
term of a polynomial any $a_{i} x^{i}$ of a polynomial of the form $a_{n} x^{n}+\cdots+a_{2} x^{2}+a_{1} x+a_{0}$
trinomial a polynomial containing three terms

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## INTRODUCTION TO FACTORING POLYNOMIALS

## Learning Outcomes

By the end of this section, you will be able to:

- Identify and factor the greatest common factor of a polynomial.
- Factor a trinomial with leading coefficient 1.
- Factor by grouping.
- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor a sum and difference of cubes.
- Factor an expression with negative or fractional exponents.

Imagine that we are trying to find the area of a lawn so that we can determine how much grass seed to purchase. The lawn is the green portion in the figure below.


The area of the entire region can be found using the formula for the area of a rectangle.

$$
\begin{aligned}
A & =l w \\
& =10 x \cdot 6 x \\
& =60 x^{2} \text { units }^{2}
\end{aligned}
$$

The areas of the portions that do not require grass seed need to be subtracted from the area of the entire region. The two square regions each have an area of $A=s^{2}=4^{2}=16$ units $^{2}$. The other rectangular region has one side of length $10 x-8$ and one side of length 4 , giving an area of
$A=l w=4(10 x-8)=40 x-32$ units $^{2}$. So the region that must be subtracted has an area of $2(16)+40 x-32=40 x$ units $^{2}$.

The area of the region that requires grass seed is found by subtracting $60 x^{2}-40 x$ units $^{2}$. This area can also be expressed in factored form as $20 x(3 x-2)$ units $^{2}$. We can confirm that this is an equivalent expression by multiplying.

Many polynomial expressions can be written in simpler forms by factoring. In this section, we will look at a variety of methods that can be used to factor polynomial expressions.

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## LEARNING OUTCOMES

- Identify and factor the greatest common factor of a polynomial.
- Factor a trinomial with leading coefficient 1.
- Factor by grouping.

Recall that the greatest common factor (GCF) of two numbers is the largest number that divides evenly into both numbers. For example, 4 is the GCF of 16 and 20 because it is the largest number that divides evenly into both 16 and 20 . The GCF of polynomials works the same way: $4 x$ is the GCF of $16 x$ and $20 x^{2}$ because it is the largest polynomial that divides evenly into both $16 x$ and $20 x^{2}$.

Finding and factoring out a GCF from a polynomial is the first skill involved in factoring polynomials.

## Factoring a GCF out of a polynomial

When factoring a polynomial expression, our first step is to check to see if each term contains a common factor. If so, we factor out the greatest amount we can from each term. To make it less challenging to find this GCF of the polynomial terms, first look for the GCF of the coefficients, and then look for the GCF of the variables.

## A GENERAL NOTE: GREATEST COMMON FACTOR

The greatest common factor (GCF) of a polynomial is the largest polynomial that divides evenly into each term of the polynomial.

To factor out a GCF from a polynomial, first identify the greatest common factor of the terms. You can then use the distributive property "backwards" to rewrite the polynomial in a factored form. Recall that the distributive property of multiplication over addition states that a product of a number and a sum is the same as the sum of the products.

## Distributive Property Forward and Backward

Forward: We distribute $a$ over $b+c$.
$a(b+c)=a b+a c$.
Backward: We factor $a$ out of $a b+a c$.
$a b+a c=a(b+c)$.
We have seen that we can distribute a factor over a sum or difference. Now we see that we can "undo" the distributive property with factoring.

## Example

Factor $25 b^{3}+10 b^{2}$.

## Answer

Find the GCF.

$$
\begin{aligned}
& 25 b^{3}=5 \cdot 5 \cdot b \cdot b \cdot b \\
& 10 b^{2}=5 \cdot 2 \cdot b \cdot b
\end{aligned}
$$

$\mathrm{GCF}=5 \cdot b \cdot b=5 b^{2}$
Rewrite each term with the GCF as one factor.

$$
\begin{aligned}
& 25 b^{3}=5 b^{2} \cdot 5 b \\
& 10 b^{2}=5 b^{2} \cdot 2
\end{aligned}
$$

Rewrite the polynomial using the factored terms in place of the original terms.
$5 b^{2}(5 b)+5 b^{2}(2)$
Factor out the $5 b^{2}$.
$5 b^{2}(5 b+2)$

Answer
$5 b^{2}(5 b+2)$

## Analysis

The factored form of the polynomial $25 b^{3}+10 b^{2}$ is $5 b^{2}(5 b+2)$. You can check this by doing the multiplication. $5 b^{2}(5 b+2)=25 b^{3}+10 b^{2}$.

Note that if you do not factor the greatest common factor at first, you can continue factoring, rather than start all over.

For example:

$$
\begin{aligned}
25 b^{3}+10 b^{2} & =5\left(5 b^{3}+2 b^{2}\right) & & \text { Factor out } 5 . \\
& =5 b^{2}(5 b+2) & & \text { Factor out } b^{2} .
\end{aligned}
$$

Notice that you arrive at the same simplified form whether you factor out the GCF immediately or if you pull out factors individually.

In the following video we see two more examples of how to find and factor the GCF from binomials.


## HOW TO: GIVEN A POLYNOMIAL EXPRESSION, FACTOR OUT THE GREATEST COMMON FACTOR

1. Identify the GCF of the coefficients.
2. Identify the GCF of the variables.
3. Combine to find the GCF of the expression.
4. Determine what the GCF needs to be multiplied by to obtain each term in the expression.
5. Write the factored expression as the product of the GCF and the sum of the terms we need to multiply by.

## EXAMPLE: FACTORING THE GREATEST COMMON FACTOR

Factor $6 x^{3} y^{3}+45 x^{2} y^{2}+21 x y$.

## Answer

First find the GCF of the expression. The GCF of 6,45 , and 21 is 3 . The GCF of $x^{3}, x^{2}$, and $x$ is $x$. (Note that the GCF of a set of expressions of the form $x^{n}$ will always be the lowest exponent.) The GCF of $y^{3}, y^{2}$ , and $y$ is $y$. Combine these to find the GCF of the polynomial, $3 x y$.
Next, determine what the GCF needs to be multiplied by to obtain each term of the polynomial. We find that $3 x y\left(2 x^{2} y^{2}\right)=6 x^{3} y^{3}, 3 x y(15 x y)=45 x^{2} y^{2}$, and $3 x y(7)=21 x y$.
Finally, write the factored expression as the product of the GCF and the sum of the terms we needed to multiply by.

$$
(3 x y)\left(2 x^{2} y^{2}+15 x y+7\right)
$$

## Analysis of the Solution

After factoring, we can check our work by multiplying. Use the distributive property to confirm that $(3 x y)\left(2 x^{2} y^{2}+15 x y+7\right)=6 x^{3} y^{3}+45 x^{2} y^{2}+21 x y$.

The GCF may not always be a monomial. Here is an example of a GCF that is a binomial.

## TRY IT

Factor $x\left(b^{2}-a\right)+6\left(b^{2}-a\right)$ by pulling out the GCF.

## Answer

$\left(b^{2}-a\right)(x+6)$
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Watch this video to see more examples of how to factor the GCF from a trinomial.

## Factoring a Trinomial with Leading Coefficient 1

Although we should always begin by looking for a GCF, pulling out the GCF is not the only way that


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polynomial expressions can be factored. The polynomial $x^{2}+5 x+6$ has a GCF of 1 , but it can be written as the product of the factors $(x+2)$ and $(x+3)$.

Trinomials of the form $x^{2}+b x+c$ can be factored by finding two numbers with a product of $c$ and a sum of $b$ . The trinomial $x^{2}+10 x+16$, for example, can be factored using the numbers 2 and 8 because the product of these numbers is 16 and their sum is 10 . The trinomial can be rewritten as the product of $(x+2)$ and $(x+8)$.

## A GENERAL NOTE: FACTORING A TRINOMIAL WITH LEADING COEFFICIENT 1

A trinomial of the form $x^{2}+b x+c$ can be written in factored form as $(x+p)(x+q)$ where $p q=c$ and $p+q=b$.

## Q \& A

Can every trinomial be factored as a product of binomials?
No. Some polynomials cannot be factored. These polynomials are said to be prime.

## HOW TO: GIVEN A TRINOMIAL IN THE FORM $x^{2}+b x+c$, FACTOR IT

1. List factors of $c$.
2. Find $p$ and $q$, a pair of factors of $c$ with a sum of $b$.
3. Write the factored expression $(x+p)(x+q)$.

## Factoring by Grouping

Trinomials with leading coefficients other than 1 are slightly more complicated to factor. For these trinomials, we can factor by grouping by dividing the $x$ term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression. The trinomial $2 x^{2}+5 x+3$ can be rewritten as $(2 x+3)(x+1)$ using this process. We begin by rewriting the original expression as $2 x^{2}+2 x+3 x+3$ and then factor each portion of the expression to obtain $2 x(x+1)+3(x+1)$. We then

## EXAMPLE: FACTORING A TRINOMIAL WITH LEADING COEFFICIENT 1

Factor $x^{2}+2 x-15$.

## Answer

We have a trinomial with leading coefficient $1, b=2$, and $c=-15$. We need to find two numbers with a product of -15 and a sum of 2 . In the table, we list factors until we find a pair with the desired sum.

| Factors of -15 | Sum of Factors |
| :--- | :--- |
| $1,-15$ | -14 |
| $-1,15$ | 14 |
| $3,-5$ | -2 |
| $-3,5$ | 2 |

Now that we have identified $p$ and $q$ as -3 and 5 , write the factored form as $(x-3)(x+5)$.

Analysis of the Solution
We can check our work by multiplying. Use FOIL to confirm that $(x-3)(x+5)=x^{2}+2 x-15$.
Q \& A

Does the order of the factors matter?
No. Multiplication is commutative, so the order of the factors does not matter.

## TRY IT

Factor $x^{2}-7 x+6$.
Answer
$(x-6)(x-1)$
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pull out the GCF of $(x+1)$ to find the factored expression.

## A GENERAL NOTE: FACTORING BY GROUPING

To factor a trinomial of the form $a x^{2}+b x+c$ by grouping, we find two numbers with a product of $a c$ and a sum of $b$. We use these numbers to divide the $x$ term into the sum of two terms and factor each portion of the expression separately then factor out the GCF of the entire expression.

1. List factors of $a c$.
2. Find $p$ and $q$, a pair of factors of $a c$ with a sum of $b$.
3. Rewrite the original expression as $a x^{2}+p x+q x+c$.
4. Pull out the GCF of $a x^{2}+p x$.
5. Pull out the GCF of $q x+c$.
6. Factor out the GCF of the expression.

## EXAMPLE: FACTORING A TRINOMIAL BY GROUPING

Factor $5 x^{2}+7 x-6$ by grouping.

## Answer

We have a trinomial with $a=5, b=7$, and $c=-6$. First, determine $a c=-30$. We need to find two numbers with a product of -30 and a sum of 7 . In the table, we list factors until we find a pair with the desired sum.

| Factors of -30 | Sum of Factors |
| :--- | :--- |
| $1,-30$ | -29 |
| $-1,30$ | 29 |
| $2,-15$ | -13 |
| $-2,15$ | 13 |
| $3,-10$ | -7 |
| $-3,10$ | 7 |

So $p=-3$ and $q=10$.
$5 x^{2}-3 x+10 x-6 \quad$ Rewrite the original expression as $a x^{2}+p x+q x+c$.
$x(5 x-3)+2(5 x-3)$ Factor out the GCF of each part.
$(5 x-3)(x+2) \quad$ Factor out the GCF of the expression.

## Analysis of the Solution

We can check our work by multiplying. Use FOIL to confirm that $(5 x-3)(x+2)=5 x^{2}+7 x-6$.

## TRY IT

Factor the following.

1. $2 x^{2}+9 x+9$
2. $6 x^{2}+x-1$

## Answer

1. $(2 x+3)(x+3)$
2. $(3 x-1)(2 x+1)$

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In the next video we see another example of how to factor a trinomial by grouping.


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## FACTORING SPECIAL CASES

## LEARNING OUTCOMES

- Factor a perfect square trinomial.
- Factor a difference of squares.
- Factor a sum and difference of cubes.
- Factor an expression with negative or fractional exponents.


## Factoring a Perfect Square Trinomial

A perfect square trinomial is a trinomial that can be written as the square of a binomial. Recall that when a binomial is squared, the result is the square of the first term added to twice the product of the two terms and
the square of the last term.

$$
a^{2}+2 a b+b^{2}=(a+b)^{2}
$$

and
$a^{2}-2 a b+b^{2}=(a-b)^{2}$

We can use this equation to factor any perfect square trinomial.

## A GENERAL NOTE: PERFECT SQUARE TRINOMIALS

A perfect square trinomial can be written as the square of a binomial:

$$
a^{2}+2 a b+b^{2}=(a+b)^{2}
$$

## HOW TO: GIVEN A PERFECT SQUARE TRINOMIAL, FACTOR IT INTO THE SQUARE OF A BINOMIAL

1. Confirm that the first and last term are perfect squares.
2. Confirm that the middle term is twice the product of $a b$.
3. Write the factored form as $(a+b)^{2}$.

## EXAMPLE: FACTORING A PERFECT SQUARE TRINOMIAL

Factor $25 x^{2}+20 x+4$.

## Answer

Notice that $25 x^{2}$ and 4 are perfect squares because $25 x^{2}=(5 x)^{2}$ and $4=2^{2}$. Then check to see if the middle term is twice the product of $5 x$ and 2 . The middle term is, indeed, twice the product:
$2(5 x)(2)=20 x$. Therefore, the trinomial is a perfect square trinomial and can be written as $(5 x+2)^{2}$.

## TRY IT

Factor $49 x^{2}-14 x+1$.
Answer
$(7 x-1)^{2}$
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## Factoring a Difference of Squares

A difference of squares is a perfect square subtracted from a perfect square. Recall that a difference of squares can be rewritten as factors containing the same terms but opposite signs because the middle terms cancel each other out when the two factors are multiplied.

$$
a^{2}-b^{2}=(a+b)(a-b)
$$

We can use this equation to factor any differences of squares.

## A GENERAL NOTE: DIFFERENCES OF SQUARES

A difference of squares can be rewritten as two factors containing the same terms but opposite signs.

$$
a^{2}-b^{2}=(a+b)(a-b)
$$

## HOW TO: GIVEN A DIFFERENCE OF SQUARES, FACTOR IT INTO BINOMIALS

1. Confirm that the first and last term are perfect squares.
2. Write the factored form as $(a+b)(a-b)$.

## EXAMPLE: FACTORING A DIFFERENCE OF SQUARES

Factor $9 x^{2}-25$.

## Answer

Notice that $9 x^{2}$ and 25 are perfect squares because $9 x^{2}=(3 x)^{2}$ and $25=5^{2}$. The polynomial represents a difference of squares and can be rewritten as $(3 x+5)(3 x-5)$.

## TRY IT

Factor $81 y^{2}-100$.

## Answer

$(9 y+10)(9 y-10)$
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## Q \& A

Is there a formula to factor the sum of squares?
No. A sum of squares cannot be factored.
Watch this video to see another example of how to factor a difference of squares.

## Factoring the Sum and Difference of Cubes

Now we will look at two new special products: the sum and difference of cubes. Although the sum of squares cannot be factored, the sum of cubes can be factored into a binomial and a trinomial.

$$
a^{3}+b^{3}=(a+b)\left(a^{2}-a b+b^{2}\right)
$$

Similarly, the sum of cubes can be factored into a binomial and a trinomial but with different signs.


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$$
a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right)
$$

We can use the acronym SOAP to remember the signs when factoring the sum or difference of cubes. The first letter of each word relates to the signs: Same Opposite Always Positive. For example, consider the following example.

$$
x^{3}-2^{3}=(x-2)\left(x^{2}+2 x+4\right)
$$

The sign of the first 2 is the same as the sign between $x^{3}-2^{3}$. The sign of the $2 x$ term is opposite the sign between $x^{3}-2^{3}$. And the sign of the last term, 4, is always positive.

## A GENERAL NOTE: SUM AND DIFFERENCE OF CUBES

We can factor the sum of two cubes as

$$
a^{3}+b^{3}=(a+b)\left(a^{2}-a b+b^{2}\right)
$$

We can factor the difference of two cubes as

$$
a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right)
$$

## HOW TO: GIVEN A SUM OF CUBES OR DIFFERENCE OF CUBES, FACTOR IT

1. Confirm that the first and last term are cubes, $a^{3}+b^{3}$ or $a^{3}-b^{3}$.
2. For a sum of cubes, write the factored form as $(a+b)\left(a^{2}-a b+b^{2}\right)$. For a difference of cubes, write the factored form as $(a-b)\left(a^{2}+a b+b^{2}\right)$.

In the following two video examples we show more binomials that can be factored as a sum or difference of cubes.

## Factoring Expressions with Fractional or Negative Exponents

Expressions with fractional or negative exponents can be factored by pulling out a GCF. Look for the variable or exponent that is common to each term of the expression and pull out that variable or exponent

## EXAMPLE: FACTORING A SUM OF CUBES

Factor $x^{3}+512$.
Answer
Notice that $x^{3}$ and 512 are cubes because $8^{3}=512$. Rewrite the sum of cubes as $(x+8)\left(x^{2}-8 x+64\right)$

## Analysis of the Solution

After writing the sum of cubes this way, we might think we should check to see if the trinomial portion can be factored further. However, the trinomial portion cannot be factored, so we do not need to check.

## TRY IT

Factor the sum of cubes $216 a^{3}+b^{3}$.

## Answer

$(6 a+b)\left(36 a^{2}-6 a b+b^{2}\right)$

## EXAMPLE: FACTORING A DIFFERENCE OF CUBES

Factor $8 x^{3}-125$.

## Answer

Notice that $8 x^{3}$ and 125 are cubes because $8 x^{3}=(2 x)^{3}$ and $125=5^{3}$. Write the difference of cubes as $(2 x-5)\left(4 x^{2}+10 x+25\right)$.

## Analysis of the Solution

Just as with the sum of cubes, we will not be able to further factor the trinomial portion.

## TRY IT

Factor the difference of cubes: $1,000 x^{3}-1$.
Answer
$(10 x-1)\left(100 x^{2}+10 x+1\right)$
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raised to the lowest power. These expressions follow the same factoring rules as those with integer exponents. For instance, $2 x^{\frac{1}{4}}+5 x^{\frac{3}{4}}$ can be factored by pulling out $x^{\frac{1}{4}}$ and being rewritten as $x^{\frac{1}{4}}\left(2+5 x^{\frac{1}{2}}\right)$.

The following video provides more examples of how to factor expressions with negative exponents.


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## EXAMPLE: FACTORING AN EXPRESSION WITH FRACTIONAL OR NEGATIVE EXPONENTS

Factor $3 x(x+2)^{\frac{-1}{3}}+4(x+2)^{\frac{2}{3}}$.

## Answer

Factor out the term with the lowest value of the exponent. In this case, that would be $(x+2)^{-\frac{1}{3}}$.
$(x+2)^{-\frac{1}{3}}(3 x+4(x+2))$ Factor out the GCF.
$(x+2)^{-\frac{1}{3}}(3 x+4 x+8) \quad$ Simplify.
$(x+2)^{-\frac{1}{3}}(7 x+8)$

The following video provides more examples of how to factor expressions with fractional exponents.


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## TRY IT

Factor $2(5 a-1)^{\frac{3}{4}}+7 a(5 a-1)^{-\frac{1}{4}}$.
Answer
$(5 a-1)^{-\frac{1}{4}}(17 a-2)$
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## SUMMARY: FACTORING POLYNOMIALS

## Key Equations

| difference of squares |
| :--- |
| perfect square trinomial |
| sum of cubes |
| difference of cubes |

$$
\begin{aligned}
& a^{2}-b^{2}=(a+b)(a-b) \\
& a^{2}+2 a b+b^{2}=(a+b)^{2} \\
& a^{3}+b^{3}=(a+b)\left(a^{2}-a b+b^{2}\right) \\
& a^{3}-b^{3}=(a-b)\left(a^{2}+a b+b^{2}\right)
\end{aligned}
$$

## Key Concepts

- The greatest common factor, or GCF, can be factored out of a polynomial. Checking for a GCF should be the first step in any factoring problem.
- Trinomials with leading coefficient 1 can be factored by finding numbers that have a product of the third term and a sum of the second term.
- Trinomials can be factored using a process called factoring by grouping.
- Perfect square trinomials and the difference of squares are special products and can be factored using equations.
- The sum of cubes and the difference of cubes can be factored using equations.
- Polynomials containing fractional and negative exponents can be factored by pulling out a GCF.


## Glossary

factor by grouping a method for factoring a trinomial of the form $a x^{2}+b x+c$ by dividing the $x$ term into the sum of two terms, factoring each portion of the expression separately, and then factoring out the GCF of the entire expression
greatest common factor the largest polynomial that divides evenly into each polynomial

## INTRODUCTION TO RATIONAL EXPRESSIONS

## Learning Outcomes

By the end of this section, you will be able to:

- Simplify rational expressions.
- Multiply and divide rational expressions.
- Find the least common denominator of two rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

A pastry shop has fixed costs of $\$ 280$ per week and variable costs of $\$ 9$ per box of pastries. The shop's costs per week in terms of $x$, the number of boxes made, is $280+9 x$. We can divide the costs per week by the number of boxes made to determine the cost per box of pastries.

$$
\frac{280+9 x}{x}
$$

Notice that the result is a polynomial expression divided by a second polynomial expression. In this section, we will explore quotients of polynomial expressions.

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\section*{MULTIPLYING AND DIVIDING RATIONAL EXPRESSIONS}
- Simplify rational expressions.
- Multiply and divide rational expressions.

\section*{Simplifying Rational Expressions}

The quotient of two polynomial expressions is called a rational expression. We can apply the properties of fractions to rational expressions such as simplifying the expressions by canceling common factors from the numerator and the denominator. To do this, we first need to factor both the numerator and denominator. Let's start with the rational expression shown.
\(\frac{x^{2}+8 x+16}{x^{2}+11 x+28}\)
We can factor the numerator and denominator to rewrite the expression as \(\frac{(x+4)^{2}}{(x+4)(x+7)}\)
Then we can simplify the expression by canceling the common factor \((x+4)\) to get \(\frac{x+4}{x+7}\)

\section*{HOW TO: GIVEN A RATIONAL EXPRESSION, SIMPLIFY IT}
1. Factor the numerator and denominator.
2. Cancel any common factors.

\section*{EXAMPLE: SIMPLIFYING RATIONAL EXPRESSIONS}

Simplify \(\frac{x^{2}-9}{x^{2}+4 x+3}\).

\section*{Answer}
\(\frac{(x+3)(x-3)}{(x+3)(x+1)} \quad\) Factor the numerator and the denominator.
\(\frac{x-3}{x+1} \quad\) Cancel common factor \((x+3)\).

Analysis of the Solution
We can cancel the common factor because any expression divided by itself is equal to 1 .

\section*{Q \& A}

Can the \(x^{2}\) term be cancelled in the above example?
No. A factor is an expression that is multiplied by another expression. The \(x^{2}\) term is not a factor of the numerator or the denominator.

\section*{TRY IT}

Simplify \(\frac{x-6}{x^{2}-36}\).
Answer

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\section*{Multiplying Rational Expressions}

\section*{RECALL MULTIPLYING FRACTIONS}

To multiply fractions, multiply the numerators and place them over the product of the denominators.
\(\frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}\)
It is helpful to factor the numerator and denominator and cancel common factors before multiplying terms together.

Multiplication of rational expressions works the same way as multiplication of any other fractions. We multiply the numerators to find the numerator of the product, and then multiply the denominators to find the denominator of the product. Before multiplying, it is helpful to factor the numerators and denominators just as we did when simplifying rational expressions. We are often able to simplify the product of rational expressions.

\section*{HOW TO: GIVEN TWO RATIONAL EXPRESSIONS, MULTIPLY THEM}
1. Factor the numerator and denominator.
2. Multiply the numerators.
3. Multiply the denominators.
4. Simplify.

\section*{EXAMPLE: MULTIPLYING RATIONAL EXPRESSIONS}

Multiply the rational expressions and show the product in simplest form:
\[
\frac{x^{2}+4 x-5}{3 x+18} \cdot \frac{2 x-1}{x+5}
\]

\section*{Answer}
\(\frac{(x+5)(x-1)}{3(x+6)} \cdot \frac{(2 x-1)}{(x+5)}\) Factor the numerator and denominator.
\(\frac{(x+5)(x-1)(2 x-1)}{3(x+6)(x+5)} \quad\) Multiply numerators and denominators.
\(\frac{(x+5)(x-1)(2 x-1)}{3(x+6)(x+5)}\) Cancel common factors to simplify.
\(\frac{(x-1)(2 x-1)}{3(x+6)}\)

\section*{TRY IT}

Multiply the rational expressions and show the product in simplest form:
\[
\frac{x^{2}+11 x+30}{x^{2}+5 x+6} \cdot \frac{x^{2}+7 x+12}{x^{2}+8 x+16}
\]

Answer
\(\frac{(x+5)(x+6)}{(x+2)(x+4)}\)
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\section*{Dividing Rational Expressions}

\section*{RECALL DIVIDING FRACTIONS}

To divide fractions, multiply the first by the reciprocal of the second.
\(\frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}=\frac{a d}{b c}\)
Remember to factor first and cancel common factors in the numerator and denominator before multiplying individual terms together.

Division of rational expressions works the same way as division of other fractions. To divide a rational expression by another rational expression, multiply the first expression by the reciprocal of the second. Using this approach, we would rewrite \(\frac{1}{x} \div \frac{x^{2}}{3}\) as the product \(\frac{1}{x} \cdot \frac{3}{x^{2}}\). Once the division expression has been rewritten as a multiplication expression, we can multiply as we did before.
\[
\frac{1}{x} \cdot \frac{3}{x^{2}}=\frac{3}{x^{3}}
\]

\section*{HOW TO: GIVEN TWO RATIONAL EXPRESSIONS, DIVIDE THEM}
1. Rewrite as the first rational expression multiplied by the reciprocal of the second.
2. Factor the numerators and denominators.
3. Multiply the numerators.
4. Multiply the denominators.
5. Simplify.

\section*{EXAMPLE: DIVIDING RATIONAL EXPRESSIONS}

Divide the rational expressions and express the quotient in simplest form:
\[
\frac{2 x^{2}+x-6}{x^{2}-1} \div \frac{x^{2}-4}{x^{2}+2 x+1}
\]

\section*{Answer}
\(\frac{2 x^{2}+x-6}{x^{2}-1} \cdot \frac{x^{2}+2 x+1}{x^{2}-4}\) \(\frac{(2 x-3)(x+2)}{(x+1)(x-1)} \cdot \frac{(x+1)(x+1)}{(x+2)(x-2)}\)
\(\frac{(2 x+3)(x+1)}{(x-1)(x-2)} \quad\) Multiply numerators and denominators.
\(\frac{2 x^{2}+5 x+3}{x^{2}-3 x+2}\)
Factor and cancel common factors.

Simplify.

Rewrite as the first fraction multiplied by the reciprocal of the second fractio

\section*{TRY IT}

Divide the rational expressions and express the quotient in simplest form:
\[
\frac{9 x^{2}-16}{3 x^{2}+17 x-28} \div \frac{3 x^{2}-2 x-8}{x^{2}+5 x-14}
\]

\section*{Answer}

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\section*{ADDING AND SUBTRACTING RATIONAL EXPRESSIONS}

\section*{LEARNING OUTCOMES}
- Find the least common denominator of two rational expressions.
- Add and subtract rational expressions.
- Simplify complex rational expressions.

Adding and subtracting rational expressions works just like adding and subtracting numerical fractions. To add fractions, we need to find a common denominator. Let's look at an example of fraction addition.
\[
\begin{aligned}
\frac{5}{24}+\frac{1}{40} & =\frac{25}{120}+\frac{3}{120} \\
& =\frac{28}{120} \\
& =\frac{7}{30}
\end{aligned}
\]

\section*{HOW DID WE KNOW WHAT NUMBER TO USE FOR THE DENOMINATOR?}

In the example above, we rewrote the fractions as equivalent fractions with a common denominator of 120. Recall that we use the least common multiple of the original denominators.

To find the LCM of 24 and 40 , rewrite 24 and 40 as products of primes, then select the largest set of each prime appearing.
\(24=2^{3} \cdot 3\)
\(40=2^{3} \cdot 5\)
We choose \(2^{3} \cdot 3 \cdot 5=120\) as the LCM, since that's the largest number of factors of 2,3 , and 5 we see. The LCM is 120.
We multiply each numerator with just enough of the LCM to make each denominator 120 to get the equivalent fractions.
When referring to fractions, we call the LCM the least common denominator, or the LCD.

We have to rewrite the fractions so they share a common denominator before we are able to add. We must do the same thing when adding or subtracting rational expressions.

The easiest common denominator to use will be the least common denominator or LCD. The LCD is the smallest multiple that the denominators have in common. To find the LCD of two rational expressions, we factor the expressions and multiply all of the distinct factors. For instance, if the factored denominators were \((x+3)(x+4)\) and \((x+4)(x+5)\), then the LCD would be \((x+3)(x+4)(x+5)\).

Once we find the LCD, we need to multiply each expression by the form of 1 that will change the denominator to the LCD. We would need to multiply the expression with a denominator of \((x+3)(x+4)\) by \(\frac{x+5}{x+5}\) and the expression with a denominator of \((x+4)(x+5)\) by \(\frac{x+3}{x+3}\).

\section*{HOW TO: GIVEN TWO RATIONAL EXPRESSIONS, ADD OR SUBTRACT THEM}
1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators.
5. Simplify.

\section*{Simplifying Complex Rational Expressions}

A complex rational expression is a rational expression that contains additional rational expressions in the numerator, the denominator, or both. We can simplify complex rational expressions by rewriting the numerator and denominator as single rational expressions and dividing. The complex rational expression \(\frac{a}{\frac{1}{b}+c}\) can be simplified by rewriting the numerator as the fraction \(\frac{a}{1}\) and combining the expressions in the denominator as \(\frac{1+b c}{b}\). We can then rewrite the expression as a multiplication problem using the reciprocal of the denominator. We get \(\frac{a}{1} \cdot \frac{b}{1+b c}\) which is equal to \(\frac{a b}{1+b c}\).

\section*{EXAMPLE: ADDING RATIONAL EXPRESSIONS}

Add the rational expressions:
\[
\frac{5}{x}+\frac{6}{y}
\]

\section*{Answer}

First, we have to find the LCD. In this case, the LCD will be \(x y\). We then multiply each expression by the appropriate form of 1 to obtain \(x y\) as the denominator for each fraction.
\(\frac{5}{x} \cdot \frac{y}{y}+\frac{6}{y} \cdot \frac{x}{x}\)
\(\frac{5 y}{x y}+\frac{6 x}{x y}\)
Now that the expressions have the same denominator, we simply add the numerators to find the sum.
\[
\frac{6 x+5 y}{x y}
\]

Analysis of the Solution
Multiplying by \(\frac{y}{y}\) or \(\frac{x}{x}\) does not change the value of the original expression because any number divided by itself is 1 , and multiplying an expression by 1 gives the original expression.

\section*{TRY IT}

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\section*{EXAMPLE: SUBTRACTING RATIONAL EXPRESSIONS}

Subtract the rational expressions:
\[
\frac{6}{x^{2}+4 x+4}-\frac{2}{x^{2}-4}
\]

Answer
\[
\begin{array}{ll}
\frac{6}{(x+2)^{2}}-\frac{2}{(x+2)(x-2)} & \text { Factor. } \\
\frac{6}{(x+2)^{2}} \cdot \frac{x-2}{x-2}-\frac{2}{(x+2)(x-2)} \cdot \frac{x+2}{x+2} & \text { Multiply each fraction to get the LCD as the denominatc } \\
\frac{6(x-2)}{(x+2)^{2}(x-2)}-\frac{2(x+2)}{(x+2)^{2}(x-2)} & \text { Multiply. } \\
\frac{6 x-12-(2 x+4)}{(x+2)^{2}(x-2)} & \text { Apply distributive property. } \\
\frac{4 x-16}{(x+2)^{2}(x-2)} & \text { Subtract. } \\
\frac{4(x-4)}{(x+2)^{2}(x-2)} & \text { Simplify. }
\end{array}
\]

\section*{Q \& A}

Do we have to use the LCD to add or subtract rational expressions?
No. Any common denominator will work, but it is easiest to use the LCD.

\section*{TRY IT}

Subtract the rational expressions: \(\frac{3}{x+5}-\frac{1}{x-3}\).
Answer
\(\frac{2(x-7)}{(x+5)(x-3)}\)
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\section*{HOW TO: GIVEN A COMPLEX RATIONAL EXPRESSION, SIMPLIFY IT}
1. Combine the expressions in the numerator into a single rational expression by adding or subtracting.
2. Combine the expressions in the denominator into a single rational expression by adding or subtracting.
3. Rewrite as the numerator divided by the denominator.
4. Rewrite as multiplication.
5. Multiply.
6. Simplify.

\section*{EXAMPLE: SIMPLIFYING COMPLEX RATIONAL EXPRESSIONS}

Simplify: \(\frac{y+\frac{1}{x}}{\frac{x}{y}}\).
Answer
Begin by combining the expressions in the numerator into one expression.
\(y \cdot \frac{x}{x}+\frac{1}{x} \quad\) Multiply by \(\frac{x}{x}\) to get LCD as denominator.
\(\frac{x y}{x}+\frac{1}{x}\)
\(\frac{x y+1}{x}\) Add numerators.
Now the numerator is a single rational expression and the denominator is a single rational expression.
\[
\frac{\frac{x y+1}{x}}{\frac{x}{y}}
\]

We can rewrite this as division and then multiplication.
\(\frac{x y+1}{x} \div \frac{x}{y}\)
\(\frac{x y+1}{x} \cdot \frac{y}{x} \quad\) Rewrite as multiplication.
\(\frac{y(x y+1)}{x^{2}} \quad\) Multiply.

\section*{TRY IT}

Simplify: \(\frac{\frac{x}{y}-\frac{y}{x}}{y}\)
Answer
\(\frac{x^{2}-y^{2}}{x y^{2}}\)
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=64

\section*{Q \& A}

Can a complex rational expression always be simplified?
Yes. We can always rewrite a complex rational expression as a simplified rational expression.

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\section*{SUMMARY: RATIONAL EXPRESSIONS}

\section*{Key Concepts}
- Rational expressions can be simplified by canceling common factors in the numerator and denominator.
- We can multiply rational expressions by multiplying the numerators and multiplying the denominators.
- To divide rational expressions, multiply by the reciprocal of the second expression.
- Adding or subtracting rational expressions requires finding a common denominator.
- Complex rational expressions have fractions in the numerator or the denominator. These expressions can be simplified.

\section*{Glossary}
least common denominator the smallest multiple that two denominators have in common
rational expression the quotient of two polynomial expressions
```

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At the beginning of the module, we discussed a hypothetical furniture company that manufactures and sells a certain kind of chair at $\$ 40$. Let's recall the revenue $(R)$, cost $(C)$ and profit $(P)$ formulas in terms of the number of chairs sold $(x)$. We will write each one in standard form.
$R=40 x$
$C=0.001 x^{2}+25 x+10,000$
$P=-0.001 x^{2}+15 x-10,000$
Each of these three formulas is a polynomial. The revenue is degree 1, while the cost and profit are both degree 2.

Now suppose our company also makes tables to go with the chairs priced at
 $\$ 150$. The revenue for making and selling $x$ tables would then be $R=150 x$. Materials for the table cost $\$ 80$ per unit. If our factory is already making chairs, then there is no need to count the fixed costs again in table production costs, so our total cost formula for $x$ tables is:
$C=0.002 x^{2}+80 x$
The degree 2 term comes from some additional analysis of the costs. Don't worry about where this term comes from; instead, think of it as a given quantity. For simplicity, we're going to assume that tables and chairs are sold as a set (one table and one chair per set). What is the total profit generated when 100 sets are built and sold? What about 200?

Because each set consists of one table and one chair, the total revenue generated by selling $x$ sets can be found by adding the revenue polynomials.
$R=40 x+150 x=190 x$
Furthermore, the total costs of manufacturing $x$ sets is found by adding the cost polynomials.

$$
\begin{aligned}
& C=\left(0.001 x^{2}+25 x+10,000\right)+\left(0.002 x^{2}+80 x\right) \\
& C=0.003 x^{2}+105 x+10,000
\end{aligned}
$$

Finally, we find the profit by subtracting polynomials, $P=R-C$. Don't forget to distribute the minus sign and then collect like terms.

$$
\begin{aligned}
& P=190 x-\left(0.003 x^{2}+105 x+10,000\right) \\
& P=190 x-0.003 x^{2}-105 x-10,000
\end{aligned}
$$

$$
P=-0.003 x^{2}+85 x-10,000
$$

So how much profit results from making and selling 100 sets? Plug in $x=100$ to find out.

$$
P=-0.003(100)^{2}+85(100)-10,000=-1530
$$

That negative value means that we lost $\$ 1,530$ on this project! But if you think about it, this result probably means we just haven't made enough of our product to cover the fixed costs. Let's see if increasing the production helps our bottom line. What is the profit in making and selling 200 table-chair sets?
$P=-0.003(200)^{2}+85(200)-10,000=6880$

This time our company has made $\$ 6,880$ in profit. Not bad for one month of work. But let's dig a little deeper. What is the average profit per set? First set up the formula (which is an example of a rational function).
$\frac{P}{x}=\frac{-0.003 x^{2}+85 x-10,000}{x}$
For example, the average profit when $x=200$ is:
$\frac{-0.003(200)^{2}+85(200)-10,000}{200}=\frac{6880}{200}=34.4$
That means that we are making roughly $\$ 34.40$ in profit per set. Below you can see how the average profit changes when more sets are produced and sold. About how many sets would we have to make in order to get the most average profit?

| Number of sets $x$ | Average profit per set $\frac{P}{x}$ |
| :---: | :---: |
| 200 | $\$ 34.40$ |
| 400 | $\$ 58.80$ |
| 600 | $\$ 66.53$ |
| 800 | $\$ 70.10$ |
| 1000 | $\$ 72.00$ |
| 1200 | $\$ 73.07$ |
| 1400 | $\$ 73.66$ |
| 1600 | $\$ 73.95$ |
| 1800 | $\$ 74.04$ |
| 2000 | $\$ 74.00$ |
| 2200 | $\$ 73.85$ |

Table shows the correlation between the number of sets sold and the profit per set.
According to our table, the highest average profit of $\$ 74.04$ occurs when roughly 1800 sets are produced and sold. Looks like we need to increase production!

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# MODULE 3: THE RECTANGULAR COORDINATE SYSTEM AND EQUATIONS OF LINES 

## WHY IT MATTERS: THE RECTANGULAR COORDINATE SYSTEM AND EQUATIONS OF LINES

## Why Learn about the Rectangular Coordinate System?

Caroline is a full-time college student planning a spring break vacation. To earn enough money for the trip, she has taken a part-time job at the local bank that pays $\$ 15.00 / \mathrm{hr}$, and she opened a savings account with an initial deposit of $\$ 400$ on January 15 . She arranged for direct deposit of her payroll checks. If spring break begins March 20 and the trip will cost approximately $\$ 2,500$, how many hours will she have to work to earn enough to pay for her vacation? If she can only work 4 hours per day, how many days per week will she have to work? How many weeks will it take? In this section, we will investigate problems like this and others which generate graphs like the line in the figure below.


## Review Topics

- Use properties of equality to isolate variables and solve algebraic equations
- Solve equations containing absolute value
- Use the properties of equality and the distributive property to solve equations containing parentheses
- Clear fractions and decimals from equations to make them easier to solve
- Identify equations that have one solution, no solution, or an infinite number of solutions
- Recognize when a linear equation that contains absolute value does not have a solution


## Points and Lines in the Plane

- Define the components of the Cartesian coordinate system.
- Plot points on the Cartesian coordinate plane.
- Use the distance formula to find the distance between two points in the plane.
- Use the midpoint formula to find the midpoint between two points.
- Plot linear equations in two variables on the coordinate plane.
- Use intercepts to plot lines.
- Use a graphing utility to graph a linear equation on a coordinate plane.


## Equations of Lines

- Use slope-intercept form to plot and write equations of lines.
- Use point-slope form to write the equation of a line.
- Write the equation of a line in standard form.
- Recognize vertical and horizontal lines from their graphs and equations.
- Determine whether two lines are parallel or perpendicular.
- Find the equations of parallel and perpendicular lines.
- Write the equations of lines that are parallel or perpendicular to a given line.

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## REVIEW TOPICS FOR SUCCESS

## Learning Outcomes

By the end of this section, you will be able to:

- Use properties of equality to isolate variables and solve algebraic equations
- Solve equations containing absolute value
- Use the properties of equality and the distributive property to solve equations containing parentheses
- Clear fractions and decimals from equations to make them easier to solve
- Identify equations that have one solution, no solution, or an infinite number of solutions
- Recognize when a linear equation that contains absolute value does not have a solution

We begin our exploration of the coordinate plane and equations of lines with a review of the properties of equality that permit us to solve equations like $x+7=9$. We can just look at that equation and see that $x$ must be 2 , since 2 is the number we add to 7 to obtain 9 .

You can solve equations like these in your head quickly, but other equations are more complicated. Multistep equations, ones that take several steps to solve, can still be simplified and solved by applying basic algebraic rules such as the multiplication and addition properties of equality.

In this section we will explore methods for solving multi-step equations that contain grouping symbols and several mathematical operations. We will also learn techniques for solving multi-step equations that contain absolute values. Finally, we will learn that some equations have no solutions, while others have an infinite number of solutions.

Warm up for this module by refreshing these important concepts and skills. As you study these review topics, recall that you can also return to Module 1 Algebra Essentials any time you need to refresh the basics.

## RECALL FOR SUCCESS

Look for red boxes like this one throughout the module. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

## MULTI-STEP EQUATIONS

- Use properties of equality to isolate variables and solve algebraic equations
- Solve equations containing absolute value


## Use Properties of Equality to Isolate Variables and Solve Algebraic Equations

## REVIEW TERMINOLOGY: EXPRESSIONS

Recall that a mathematical expression consists of terms connected by addition or subtraction, each term of which consists of variables and numbers connected by multiplication or division.

- variables: variables are symbols that stand for an unknown quantity; they are often represented with letters, like $x, y$, or $z$.
- coefficient: Sometimes a variable is multiplied by a number. This number is called the coefficient of the variable. For example, the coefficient of $3 x$ is 3 .
- term: a single number, or variables and numbers connected by multiplication. $-4,6 x$ and $x^{2}$ are all terms.
- expression: groups of terms connected by addition and subtraction. $2 x^{2}-5$ is an expression.

An equation is a mathematical statement of the equivalency of two expressions.. An equation will always contain an equal sign with an expression on each side. Think of an equal sign as meaning "the same as." Some examples of equations are $y=m x+b, \frac{3}{4} r=v^{3}-r$, and $2(6-d)+f(3+k)=\frac{1}{4} d$.

The following figure shows how coefficients, variables, terms, and expressions all come together to make equations. In the equation $2 x-3^{2}=10 x$, the variable is $x$, a coefficient is 10 , a term is $10 x$, an expression is $2 x-3^{2}$.


There are some equations that you can solve in your head quickly. For example, what is the value of $y$ in the equation $2 y=6$ ? Chances are you didn't need to get out a pencil and paper to calculate that $y=3$. You only needed to do one thing to get the answer: divide 6 by 2 .

Other equations are more complicated. Solving $4\left(\frac{1}{3} t+\frac{1}{2}\right)=6$ without writing anything down is difficult! That is because this equation contains not just a variable but also fractions and terms inside parentheses. This is a multi-step equation, one that takes several steps to solve. Although multi-step equations take more time and more operations, they can still be simplified and solved by applying basic algebraic rules.

Remember that you can think of an equation as a balance scale, with the goal being to rewrite the equation so that it is easier to solve but still balanced. The addition property of equality and the multiplication property of equality explain how you can keep the scale, or the equation, balanced. Whenever you perform an operation to one side of the equation, if you perform the same exact operation to the other side, you will keep both sides of the equation equal.

If the equation is in the form $a x+b=c$, where $x$ is the variable, you can solve the equation as before. First "undo" the addition and subtraction and then "undo" the multiplication and division.

## Example

Solve $3 y+2=11$.
Answer
Subtract 2 from both sides of the equation to get the term with the variable by itself.
$3 y+2=11$

$$
\begin{array}{rr}
-2 & -2 \\
\hline 3 y=9
\end{array}
$$

Divide both sides of the equation by 3 to get a coefficient of 1 for the variable.

$$
\begin{gathered}
\underline{3 y}=\underline{9} \\
3
\end{gathered}
$$

$$
y=3
$$

## Example

Solve $3 x+5 x+4-x+7=88$.

## Answer

There are three like terms involving a variable: $3 x, 5 x$, and $-x$. Combine these like terms. 4 and 7 are also like terms and can be added.

$$
\begin{gathered}
3 x+5 x+4-x+7=88 \\
7 x+11=88
\end{gathered}
$$

The equation is now in the form $a x+b=c$, so we can solve as before.
Subtract 11 from both sides.

$$
7 x+11=88
$$

$$
-11 \quad-11
$$

$$
7 x=77
$$

Divide both sides by 7 .

$$
\underline{7 x}=\underline{77}
$$

7
7

$$
x=11
$$

Some equations may have the variable on both sides of the equal sign, as in this equation:
$4 x-6=2 x+10$.
To solve this equation, we need to "move" one of the variable terms. This can make it difficult to decide which side to work with. It does not matter which term gets moved, $4 x$ or $2 x$; however, to avoid negative coefficients, you can move the smaller term.

In this video, we show an example of solving equations that have variables on both sides of the equal sign.

## Solving Multi-Step Equations With Absolute Value

We can apply the same techniques we used for solving a one-step equation which contained absolute value to an equation that will take more than one step to solve. Let us start with an example where the first step is to write two equations, one equal to positive 26 and one equal to negative 26 .

## EXAMPLES

Solve: $4 x-6=2 x+10$

## Answer

Choose the variable term to move-to avoid negative terms choose $2 x$

$$
\begin{aligned}
& 4 x-6=2 x+10 \\
& -2 x-2 x \\
& \hline 4 x-6=10
\end{aligned}
$$

Now add 6 to both sides to isolate the term with the variable.
$4 x-6=10$

| $+6+6$ |
| :---: |

$2 x=16$

Now divide each side by 2 to isolate the variable x .

$$
\frac{2 x}{2}=\frac{16}{2}
$$

$$
x=8
$$



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## Example

Solve for $p$. $|2 p-4|=26$
Answer
Write the two equations that will give an absolute value of 26 .
$2 p-4=26 \quad$ or $\quad 2 p-4=-26$
Solve each equation for $p$ by isolating the variable.

| $2 p-4=26$ | $2 p-4=-26$ |
| :---: | :---: |
| +4 +4 | +4 +4 |
| $\underline{2 p}=\underline{30}$ | $\underline{2 p}=\underline{-22}$ |
| 2 | $2 \quad 2$ |
| $p=15$ | $p=-11$ |

Check the solutions in the original equation.

$$
\begin{aligned}
|2 p-4| & =26 & |2 p-4| & =26 \\
|2(15)-4| & =26 & |2(-11)-4| & =26 \\
|30-4| & =26 & |-22-4| & =26 \\
|26| & =26 & |-26| & =26
\end{aligned}
$$

Both solutions check!
In the next video, we show more examples of solving a simple absolute value equation.


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Now let us look at an example where you need to do an algebraic step or two before you can write your two equations. The goal here is to get the absolute value on one side of the equation by itself. Then we can proceed as we did in the previous example.

## Example

Solve for $w .3|4 w-1|-5=10$

## Answer

Isolate the term with the absolute value by adding 5 to both sides.
$3|4 w-1|-5=10$

$$
+5+5
$$

$3|4 w-1|=15$
Divide both sides by 3. Now the absolute value is isolated.

$$
\underline{3|4 w-1|}=\underline{15}
$$

$$
\begin{aligned}
& 3 \\
& |4 w-1|=5
\end{aligned}
$$

Write the two equations that will give an absolute value of 5 and solve them.

$$
\begin{aligned}
& 4 w-1=5 \quad \text { or } \quad 4 w-1=-5 \\
& \begin{array}{lll}
+1+1 \\
\frac{4 w}{\underline{4}}=\underline{6} \\
4
\end{array} \quad \frac{+1}{\underline{4 w}}=\frac{-4}{4} \\
& w=\frac{3}{2} \quad w=-1 \\
& w=\frac{3}{2} \quad \text { or } \quad-1
\end{aligned}
$$

Check the solutions in the original equation.

$$
\begin{array}{rlrl}
3|4 w-1|-5 & =10 & 3|4 w-1|-5 & =10 \\
3\left|4\left(\frac{3}{2}\right)-1\right|-5 & =10 & 3|4 w-1|-5 & =10 \\
3\left|\frac{12}{2}-1\right|-5 & =10 & 3|4(-1)-1|-5 & =10 \\
3|6-1|-5 & =10 & 3|-4-1|-5 & =10 \\
3(5)-5 & =10 & 3|-5|-5 & =10 \\
15-5 & =10 & 15-5 & =10 \\
10 & =10 & 10 & =10
\end{array}
$$

Both solutions check!
In the two videos that follow, we show examples of how to solve an absolute value equation that requires you to isolate the absolute value first using mathematical operations.


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## SOLVE EQUATIONS WITH THE DISTRIBUTIVE PROPERTY

## Learning Outcomes

- Use the properties of equality and the distributive property to solve equations containing parentheses
- Clear fractions and decimals from equations to make them easier to solve


## The Distributive Property

As we solve linear equations, we often need to do some work to write the linear equations in a form we are familiar with solving. This section will focus on manipulating an equation we are asked to solve in such a way that we can use the skills we learned for solving multi-step equations to ultimately arrive at the solution.

If an equation you encounter contains parentheses, you'll need to clear them from the equations before attempting to combine like terms. To clear parentheses from an equation, use the distributive property to multiply the number in front of the parentheses by each term inside of the parentheses. If the number in front of the parentheses is negative, multiply the negative against each term inside the parentheses.

For all real numbers $a, b$, and $c, a(b+c)=a b+a c$.
What this means is that when a number multiplies an expression inside parentheses, you can distribute the multiplication to each term of the expression individually. Then, you can follow the steps we have already practiced to isolate the variable and solve the equation.

## Example

Solve for $a .4(2 a+3)=28$
Answer
Apply the distributive property to expand $4(2 a+3)$ to $8 a+12$
$4(2 a+3)=28$
$8 a+12=28$
Subtract 12 from both sides to isolate the variable term.
$8 a+12=28$

| -12 | -12 |
| :--- | :--- |

$8 a=16$
Divide both terms by 8 to get a coefficient of 1 .
$\underline{8 a}=\underline{16}$
$8 \quad 8$
$a=2$

In the video that follows, we show another example of how to use the distributive property to solve a multistep linear equation.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/? $\mathrm{p}=71$

In the next example, there are parentheses on both sides of the equal sign. To clear them, you'll need to use the distributive property on both sides of the equation.

Solve for $t$. $2(4 t-5)=-3(2 t+1)$

## Answer

Apply the distributive property to expand $2(4 t-5)$ to $8 t-10$ and $-3(2 t+1)$ to $-6 t-3$. Be careful in this step-you are distributing a negative number, so keep track of the sign of each number after you multiply.
$2(4 t-5)=-3(2 t+1)$
$8 t-10=-6 t-3$
Add $6 t$ to both sides to begin combining like terms.

$$
\begin{aligned}
& 8 t-10=-6 t-3 \\
& +6 t \quad+6 t \\
& \hline 14 t-10=-3
\end{aligned}
$$

Add 10 to both sides of the equation to isolate $t$.

$$
\begin{aligned}
& 14 t-10=-3 \\
& +10+10 \\
& 14 t=7
\end{aligned}
$$

The last step is to divide both sides by 14 to completely isolate $t$.

$$
\frac{14 t}{14}=\frac{7}{14}
$$

We simplify the fraction $\frac{7}{14}$ into the final answer of $t=\frac{1}{2}$
Watch the following video for a demonstration of how to solve a multi-step equation with two sets of parentheses.


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## Clearing Fractions and Decimals from Equations

Sometimes, you will encounter a multi-step equation containing fractions. Before attempting to solve the equation, first use the multiplication property of equality to multiply both sides of the equation by a common denominator of all of the fractions in the equation. This will clear all the fractions out of the equation. See the example below.

## Example

Solve $\frac{1}{2} x-3=2-\frac{3}{4} x$ by clearing the fractions in the equation first.

## Answer

Multiply both sides of the equation by 4, the common denominator of the fractional coefficients.
$\frac{1}{2} x-3=2-\frac{3}{4} x$
$4\left(\frac{1}{2} x-3\right)=4\left(2-\frac{3}{4} x\right)$
Use the distributive property to expand the expressions on both sides. Multiply.

$$
\begin{aligned}
& 4\left(\frac{1}{2} x\right)-4(3)=4(2)-4\left(-\frac{3}{4} x\right) \\
& \frac{4}{2} x-12=8-\frac{12}{4} x \\
& 2 x-12=8-3 x
\end{aligned}
$$

Add $3 x$ to both sides to move the variable terms to only one side.

$$
\begin{aligned}
& 2 x-12=8-3 x \\
& +3 x \quad+3 x \\
& \hline
\end{aligned}
$$

$5 x-12=8$
Add 12 to both sides to move the constant terms to the other side.

$$
\begin{array}{r}
5 x-12=8 \\
+12+12
\end{array}
$$

$5 x=20$
Divide to isolate the variable.

$$
\underline{5 x}=\underline{20}
$$

$5 \quad 5$
$x=4$

It is acceptable to simply do the operations on the fractions without clearing them first, but the technique used here will apply to more complicated situations you'll encounter later, so it is worthwhile to practice it.

Watch the following video for a demonstration of how to solve a multi-step equation containing fractions by using the least common denominator to clear the fractions first.


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Regardless of which method you use to solve equations containing variables, you will get the same answer. You can choose the method you find the easiest. Remember to check your answer by substituting your solution into the original equation.

Sometimes, you will encounter a multi-step equation with decimals. To clear the decimals from the equation, use the multiplication property of equality to multiply both sides of the equation by a a factor of 10 that will help clear the decimals. See the example below.

## Example

Solve $3 y+10.5=6.5+2.5 y$ by clearing the decimals in the equation first.

## Answer

Since the smallest decimal place value represented in the equation is 0.10 , we want to multiply by 10 to clear the decimals from the equation.
$3 y+10.5=6.5+2.5 y$
$10(3 y+10.5)=10(6.5+2.5 y)$
Use the distributive property to expand the expressions on both sides.
$10(3 y)+10(10.5)=10(6.5)+10(2.5 y)$
Multiply.
$30 y+105=65+25 y$
Move the smaller variable term, $25 y$, by subtracting it from both sides.
$30 y+105=65+25 y$
$-25 y \quad-25 y$
$5 y+105=65$
Subtract 105 from both sides to isolate the term with the variable.
$5 y+105=65$
$5 y=-40$

Divide both sides by 5 to isolate the $y$.
$\underline{5 y}=\underline{-40}$
$5 \quad 5$
$y=-8$

Watch the following example to see how to clear decimals first to solve a multi-step linear equation containing decimals.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=71

Here are some steps to follow when you solve multi-step equations.

## SOLVING MULTI-STEP EQUATIONS

1. (Optional) Multiply to clear any fractions or decimals.
2. Simplify each side by clearing parentheses and combining like terms.
3. Add or subtract to isolate the variable term-you may have to move a term with the variable.
4. Multiply or divide to isolate the variable.
5. Check the solution.

Complex, multi-step equations often require multi-step solutions. Before you can begin to isolate a variable, you may need to simplify the equation first. This may mean using the distributive property to remove parentheses or multiplying both sides of an equation by a common denominator to get rid of fractions. Sometimes it requires both techniques. If your multi-step equation has an absolute value, you will need to solve two equations, sometimes isolating the absolute value expression first.

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## CLASSIFY SOLUTIONS TO LINEAR EQUATIONS

## Learning Outcomes

- Solve equations that have one solution, no solution, or an infinite number of solutions
- Recognize when a linear equation that contains absolute value does not have a solution

There are three cases that can come up as we are solving linear equations. We have already seen one, where an equation has one solution. Sometimes we come across equations that do not have any solutions and even some that have an infinite number of solutions.

## Equations with No Solutions

## Example

Solve for $x .12+2 x-8=7 x+5-5 x$
Answer
Combine like terms on both sides of the equation.

$$
\begin{gathered}
12+2 x-8=7 x+5-5 x \\
2 x+4=2 x+5
\end{gathered}
$$

Isolate the $x$ term by subtracting $2 x$ from both sides.

$$
\begin{aligned}
& 2 x+4=2 x+5 \\
&-2 x \quad-2 x \\
& \hline 4=5
\end{aligned}
$$

This false statement implies there are no solutions to this equation. Sometimes, we say the solution does not exist, or DNE for short.

In the example above, a solution was not obtained. Using the the properties of equality to isolate the variable resulted instead in the false statement $4=5$. Certainly, 4 is not equal to 5 .

Note that in the second line of the solution above, the statement $2 x+4=2 x+5$ was obtained after combining like terms on both sides. If we examine that statement carefully, we can see that it was false even before we attempted to solve it. It would not be possible for the quantity $2 x$ with 4 added to it to be equal to
the same quantity $2 x$ with 5 added to it. The two sides of the equation do not balance. Since there is no value of $x$ that will ever make this a true statement, we say that the equation has no solution.

Be careful that you do not confuse the solution $x=0$ with no solution. The solution $x=0$ means that the value 0 satisfies the equation, so there is a solution. To say that a statement has no solution means that there is no value of the variable, not even 0 , which would satisfy the equation (that is, make the original statement true).

## Think About It

Try solving these equations. How many steps do you need to take before you can tell whether the equation has no solution or one solution?
a) Solve $8 y=3(y+4)+y$

Use the textbox below to record how many steps you think it will take before you can tell whether there is no solution or one solution.

## Answer

Solve $8 y=3(y+4)+y$
First, distribute the 3 into the parentheses on the right-hand side.
$8 y=3 y+12+y$
Next, begin combining like terms.
$8 y=4 y+12$
Now move the variable terms to one side. Moving the $4 y$ will help avoid a negative sign.

$$
\begin{aligned}
& 8 y=4 y+12 \\
& -4 y-4 y
\end{aligned}
$$

$$
4 y=12
$$

Now, divide each side by $4 y$.
$\frac{4 y}{4}=\frac{12}{4}$
$y=3$
Because we were able to isolate $y$ on one side and a number on the other side, we have one solution to this equation.
b) Solve $2(3 x-5)-4 x=2 x+7$

Use the textbox below to record how many steps you think it will take before you can tell whether there is no solution or one solution.

Answer
Solve $2(3 x-5)-4 x=2 x+7$.
First, distribute the 2 into the parentheses on the left-hand side.
$6 x-10-4 x=2 x+7$

Now begin simplifying. You can combine the $x$ terms on the left-hand side.

$$
2 x-10=2 x+7
$$

Now, take a moment to ponder this equation. It says that $2 x-10$ is equal to $2 x+7$. Can some number times two minus 10 be equal to that same number times two plus seven?

Pretend $x=3$.
Is it true that $2(3)-10=-4$ is equal to $2(3)+7=13$. NO! We do not even really need to continue solving the equation, but we can just to be thorough.

Add 10 to both sides.

$$
\begin{array}{cc}
2 x-10= & 2 x+7 \\
+10 \quad+10
\end{array}
$$

$$
2 x=2 x+17
$$

Now subtract $2 x$ from both sides.

$$
\begin{gathered}
2 x=2 x+17 \\
-2 x-2 x \\
0=17
\end{gathered}
$$

We know that 0 and 17 are not equal, so there is no number that $x$ could be to make this equation true.
This false statement implies there are no solutions to this equation, or DNE (does not exist) for short.

## Equations with Many Solutions

You have seen that if an equation has no solution, you end up with a false statement instead of a value for $x$. It is possible to have an equation where any value for $x$ will provide a solution to the equation. In the example below, notice how combining the terms $5 x$ and $-4 x$ on the left leaves us with an equation with exactly the same terms on both sides of the equal sign.

## Example

Solve for $x .5 x+3-4 x=3+x$

## Answer

Combine like terms on both sides of the equation.

$$
\begin{array}{r}
5 x+3-4 x=3+x \\
x+3=3+x
\end{array}
$$

Isolate the $x$ term by subtracting $x$ from both sides.

$$
x+3=3+x
$$

$$
\begin{aligned}
& -x \quad-x \\
& \hline 3=3
\end{aligned}
$$

This true statement implies there are an infinite number of solutions to this equation, or we can also write the solution as "All Real Numbers"

When solving, the true statement " $3=3$ " was obtained. When solving an equation reveals a true statement like this, it means that the solution to the equation is all real numbers, that is, there are infinitely many solutions. Try substituting $x=0$ into the original equation-you will get a true statement! Try $x=-\frac{3}{4}$. It will also satisfy the equation. In fact any real value of x will make the original statement true.

Indeed, after combining like terms, the equation $x+3=3+x$ was obtained. It is certainly true that the quantity $x$ with 3 added to it is equal to 3 with $x$ added to it by the commutative property of addition.

## Example

Solve for $x .3(2 x-5)=6 x-15$

## Answer

Distribute the 3 through the parentheses on the left-hand side.
$3(2 x-5)=6 x-15$
$6 x-15=6 x-15$
Wait! This looks just like the previous example. You have the same expression on both sides of an equal sign. No matter what number you choose for $x$, you will have a true statement. We can finish the algebra:

$$
\begin{gathered}
6 x-15=6 x-15 \\
-6 x \quad-6 x \\
\hline-15=-15
\end{gathered}
$$

This true statement implies there are an infinite number of solutions to this equation.
Watch the following video for demonstrations of equations with no solutions and infinitely many solutions.


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The next video demonstrates equations with no or infinitely many solutions involving parentheses.

## Absolute Value Equations with No Solutions

As we are solving absolute value equations, it is important to be aware of special cases. An absolute value is defined as the distance of a number from 0 on a number line, so the absolute value of a number must be


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a positive. When an absolute value expression is given to be equal to a negative number, we say the equation has no solution (DNE, for short). Notice how this happens in the next two examples.

## Example

Solve for $x .7+|2 x-5|=4$
Answer
Notice absolute value is not alone. Subtract 7 from each side to isolate the absolute value.
$7+|2 x-5|=4$

| -7 | -7 |
| :--- | :--- |

$|2 x-5|=-3$
Result of absolute value is negative! The result of an absolute value must always be positive, so we say there is no solution to this equation, or DNE.

## Example

Solve for $x .-\frac{1}{2}|x+3|=6$

## Answer

Notice absolute value is not alone. Multiply both sides by the reciprocal of $-\frac{1}{2}$, which is -2 .

$$
-\frac{1}{2}|x+3|=6
$$

$$
(-2)-\frac{1}{2}|x+3|=(-2) 6
$$

$$
|x+3|=-12
$$

Again, we have a result where an absolute value is negative!
There is no solution to this equation, or DNE.
In this last video, see show more examples of absolute value equations that have no solutions.


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We have seen that solutions to equations can fall into three categories:

1. exactly one solution;
2. no solution (also called DNE for does not exist)); or
3. many solutions (also called infinitely many solutions, or we may say the solution is all real numbers).

Keep in mind that sometimes we do not need to do much algebra to see what the outcome will be.

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## SUMMARY: REVIEW

## Key Concepts

- The addition and multiplication properties of equality allow the same operation to be performed on both sides of an equation
- To solve an equation containing an absolute value, first isolate the absolute value then write two equations: $|u|=c$ and $|u|=-c$
- Use the distributive property to eliminate parentheses in equations before collecting and combining like terms.
- Use the multiplication property of equality to eliminate fractions from equations by multiplying both sides by the LCD.
- Eliminate decimals from an equation by multiplying both sides by a sufficient factor of 10 .
- Equations may have one unique solution, no solutions, or an infinite number of solutions.


## Glossary

coefficient a number multiplying a variable
equation a mathematical statement of equivalency between two expressions
expression groups of terms connected by addition or subtraction
term a single number, or variables and numbers connected by multiplication
variable a symbol that stands for an unknown quantity, often represented by letters such as $x, y$, or $z$.
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# INTRODUCTION TO POINTS AND LINES IN THE COORDINATE PLANE 

## Learning Outcomes

By the end of this section, you will be able to:

- Define the components of the Cartesian coordinate system.
- Plot points on the Cartesian coordinate plane.
- Use the distance formula to find the distance between two points in the plane.
- Use the midpoint formula to find the midpoint between two points.
- Plot linear equations in two variables on the coordinate plane.
- Use intercepts to plot lines.
- Use a graphing utility to graph a linear equation on a coordinate plane.


Tracie set out from Elmhurst, IL to go to Franklin Park. On the way, she made a few stops to do errands. Each stop is indicated by a red dot in the figure below. Laying a rectangular coordinate grid over the map, we can see that each stop aligns with an intersection of grid lines. In this section, we will learn how to use grid lines to describe locations and changes in locations.

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## PLOTTING POINTS ON THE COORDINATE PLANE

## LEARNING OUTCOMES

- Define the components of the Cartesian coordinate system.
- Plot points on the Cartesian coordinate plane.

An old story describes how seventeenth-century philosopher/mathematician René Descartes invented the system that has become the foundation of algebra while sick in bed. According to the story, Descartes was staring at a fly crawling on the ceiling when he realized that he could describe the fly's location in relation to the perpendicular lines formed by the adjacent walls of his room. He viewed the perpendicular lines as horizontal and vertical axes. Further, by dividing each axis into equal unit lengths, Descartes saw that it was possible to locate any object in a two-dimensional plane using just two numbers-the displacement from the horizontal axis and the displacement from the vertical axis.

While there is evidence that ideas similar to Descartes' grid system existed centuries earlier, it was Descartes who introduced the components that comprise the Cartesian coordinate system, a grid system having perpendicular axes. Descartes named the horizontal axis the $\boldsymbol{x}$-axis and the vertical axis the $\boldsymbol{y}$-axis.

The Cartesian coordinate system, also called the rectangular coordinate system, is based on a twodimensional plane consisting of the $x$-axis and the $y$-axis. Perpendicular to each other, the axes divide the plane into four sections. Each section is called a quadrant; the quadrants are numbered counterclockwise as shown in the figure below.

## Quadrant II

## Quadrant I

$y$-axis

$x$-axis and up the $y$-axis; decreasing, negative numbers to the left on the $x$-axis and down the $y$-axis. The axes extend to positive and negative infinity as shown by the arrowheads in the figure below.


Each point in the plane is identified by its $\boldsymbol{x}$-coordinate, or horizontal displacement from the origin, and its $\boldsymbol{y}$ coordinate, or vertical displacement from the origin. Together we write them as an ordered pair indicating the combined distance from the origin in the form $(x, y)$. An ordered pair is also known as a coordinate pair because it consists of $x$ and $y$-coordinates. For example, we can represent the point $(3,-1)$ in the plane by moving three units to the right of the origin in the horizontal direction and one unit down in the vertical direction.


An illustration of how to plot the point (3,-1).
When dividing the axes into equally spaced increments, note that the $x$-axis may be considered separately from the $y$-axis. In other words, while the $x$-axis may be divided and labeled according to consecutive integers, the $y$-axis may be divided and labeled by increments of 2 or 10 or 100. In fact, the axes may represent other units such as years against the balance in a savings account or quantity against cost. Consider the rectangular coordinate system primarily as a method for showing the relationship between two quantities.

## A GENERAL NOTE: CARTESIAN COORDINATE SYSTEM

A two-dimensional plane where the

- $x$-axis is the horizontal axis
- $y$-axis is the vertical axis

A point in the plane is defined as an ordered pair, $(x, y)$, such that $x$ is determined by its horizontal distance from the origin and $y$ is determined by its vertical distance from the origin.

## EXAMPLE: PLOTTING POINTS IN A RECTANGULAR COORDINATE SYSTEM

Plot the points $(-2,4),(3,3)$, and $(0,-3)$ in the coordinate plane.

## Answer

To plot the point $(-2,4)$, begin at the origin. The $x$-coordinate is -2 , so move two units to the left. The $y$ coordinate is 4 , so then move four units up in the positive $y$ direction.
To plot the point $(3,3)$, begin again at the origin. The $x$-coordinate is 3 , so move three units to the right. The $y$-coordinate is also 3 , so move three units up in the positive $y$ direction.
To plot the point $(0,-3)$, begin again at the origin. The $x$-coordinate is 0 . This tells us not to move in either direction along the $x$-axis. The $y$-coordinate is -3 , so move three units down in the negative $y$ direction.


## Analysis of the Solution

Note that when either coordinate is zero, the point must be on an axis. If the $x$-coordinate is zero, the point is on the $y$-axis. If the $y$-coordinate is zero, the point is on the $x$-axis.

## TRY IT

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## DISTANCE IN THE COORDINATE PLANE

## LEARNING OUTCOMES

- Use the distance formula to find the distance between two points in the plane.
- Use the midpoint formula to find the midpoint between two points.

Derived from the Pythagorean Theorem, the distance formula is used to find the distance between two points in the plane. The Pythagorean Theorem, $a^{2}+b^{2}=c^{2}$, is based on a right triangle where $a$ and $b$ are the lengths of the legs adjacent to the right angle, and $c$ is the length of the hypotenuse.


The relationship of sides $\left|x_{2}-x_{1}\right|$ and $\left|y_{2}-y_{1}\right|$ to side $d$ is the same as that of sides $a$ and $b$ to side $c$. We use the absolute value symbol to indicate that the length is a positive number because the absolute value of any number is positive. (For example, $|-3|=3$.) The symbols $\left|x_{2}-x_{1}\right|$ and $\left|y_{2}-y_{1}\right|$ indicate that the lengths of the sides of the triangle are positive. To find the length $c$, take the square root of both sides of the Pythagorean Theorem.

$$
c^{2}=a^{2}+b^{2} \rightarrow c=\sqrt{a^{2}+b^{2}}
$$

It follows that the distance formula is given as

$$
d^{2}=\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2} \rightarrow d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}
$$

We do not have to use the absolute value symbols in this definition because any number squared is positive.

## A GENERAL NOTE: THE DISTANCE FORMULA

Given endpoints $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$, the distance between two points is given by

$$
d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}
$$

In the following video, we present more worked examples of how to use the distance formula to find the distance between two points in the coordinate plane.

## Using the Midpoint Formula

When the endpoints of a line segment are known, we can find the point midway between them. This point is known as the midpoint and the formula is known as the midpoint formula. Given the endpoints of a line segment, $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$, the midpoint formula states how to find the coordinates of the midpoint $M$.

$$
M=\left(\frac{x_{1}+x_{2}}{2}, \frac{y_{1}+y_{2}}{2}\right)
$$

A graphical view of a midpoint is shown below. Notice that the line segments on either side of the midpoint are congruent.

## EXAMPLE: FINDING THE DISTANCE BETWEEN TWO POINTS

Find the distance between the points $(-3,-1)$ and $(2,3)$.

## Answer

Let us first look at the graph of the two points. Connect the points to form a right triangle.


Then, calculate the length of $d$ using the distance formula.

$$
\begin{aligned}
& d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}} \\
& d=\sqrt{(2-(-3))^{2}+(3-(-1))^{2}} \\
& =\sqrt{(5)^{2}+(4)^{2}} \\
& =\sqrt{25+16} \\
& =\sqrt{41}
\end{aligned}
$$

Find the distance between two points: $(1,4)$ and $(11,9)$.

## Answer

$\sqrt{125}=5 \sqrt{5}$


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## EXAMPLE: FINDING THE DISTANCE BETWEEN TWO LOCATIONS

Let's return to the situation introduced at the beginning of this section.
Tracie set out from Elmhurst, IL to go to Franklin Park. On the way, she made a few stops to do errands.
Each stop is indicated by a red dot. Find the total distance that Tracie traveled. Compare this with the distance between her starting and final positions.

## Answer

The first thing we should do is identify ordered pairs to describe each position. If we set the starting position at the origin, we can identify each of the other points by counting units east (right) and north (up) on the grid. For example, the first stop is 1 block east and 1 block north, so it is at $(1,1)$. The next stop is 5 blocks to the east so it is at $(5,1)$. After that, she traveled 3 blocks east and 2 blocks north to $(8,3)$. Lastly, she traveled 4 blocks north to ( 8,7 ). We can label these points on the grid.


Next, we can calculate the distance. Note that each grid unit represents 1,000 feet.

- From her starting location to her first stop at $(1,1)$, Tracie might have driven north 1,000 feet and then east 1,000 feet, or vice versa. Either way, she drove 2,000 feet to her first stop.
- Her second stop is at $(5,1)$. So from $(1,1)$ to $(5,1)$, Tracie drove east 4,000 feet.
- Her third stop is at $(8,3)$. There are a number of routes from $(5,1)$ to $(8,3)$. Whatever route Tracie decided to use, the distance is the same, as there are no angular streets between the two points. Let's say she drove east 3,000 feet and then north 2,000 feet for a total of 5,000 feet.
- Tracie's final stop is at $(8,7)$. This is a straight drive north from $(8,3)$ for a total of 4,000 feet.

Next, we will add the distances listed in the table.

| From/To | Number of Feet Driven |
| :--- | :--- |
| $(0,0)$ to $(1,1)$ | 2,000 |
| $(1,1)$ to $(5,1)$ | 4,000 |
| $(5,1)$ to $(8,3)$ | 5,000 |
| $(8,3)$ to $(8,7)$ | 4,000 |
| Total | 15,000 |

The total distance Tracie drove is 15,000 feet or 2.84 miles. This is not, however, the actual distance between her starting and ending positions. To find this distance, we can use the distance formula between the points $(0,0)$ and $(8,7)$.
$d=\sqrt{(8-0)^{2}+(7-0)^{2}}$
$=\sqrt{64+49}$
$=\sqrt{113}$
$=10.63$ units

At 1,000 feet per grid unit, the distance between Elmhurst, IL to Franklin Park is 10,630.14 feet, or 2.01 miles. The distance formula results in a shorter calculation because it is based on the hypotenuse of a right triangle, a straight diagonal from the origin to the point $(8,7)$. Perhaps you have heard the saying "as the crow flies," which means the shortest distance between two points because a crow can fly in a straight line even though a person on the ground has to travel a longer distance on existing roadways.

## EXAMPLE: FINDING THE MIDPOINT OF THE LINE SEGMENT

Find the midpoint of the line segment with the endpoints $(7,-2)$ and $(9,5)$.
Answer
Use the formula to find the midpoint of the line segment.

$$
\begin{aligned}
\left(\frac{x_{1}+x_{2}}{2}, \frac{y_{1}+y_{2}}{2}\right) & =\left(\frac{7+9}{2}, \frac{-2+5}{2}\right) \\
& =\left(8, \frac{3}{2}\right)
\end{aligned}
$$

## TRY IT

Find the midpoint of the line segment with endpoints $(-2,-1)$ and $(-8,6)$.

## Answer

$\left(-5, \frac{5}{2}\right)$
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## EXAMPLE: FINDING THE CENTER OF A CIRCLE

The diameter of a circle has endpoints $(-1,-4)$ and $(5,-4)$. Find the center of the circle.

## Answer

The center of a circle is the center or midpoint of its diameter. Thus, the midpoint formula will yield the center point.

$$
\begin{aligned}
& \left(\frac{x_{1}+x_{2}}{2}, \frac{y_{1}+y_{2}}{2}\right) \\
& \left(\frac{-1+5}{2}, \frac{-4-4}{2}\right)=\left(\frac{4}{2},-\frac{8}{2}\right)=(2,-4)
\end{aligned}
$$

## TRY IT

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## GRAPHING LINEAR EQUATIONS

## LEARNING OUTCOMES

- Plot linear equations in two variables on the coordinate plane.
- Use intercepts to plot lines.
- Use a graphing utility to graph a linear equation on a coordinate plane.

We can plot a set of points to represent an equation. When such an equation contains both an $x$ variable and a $y$ variable, it is called an equation in two variables. Its graph is called a graph in two variables. Any graph on a two-dimensional plane is a graph in two variables.

Suppose we want to graph the equation $y=2 x-1$. We can begin by substituting a value for $x$ into the equation and determining the resulting value of $y$. Each pair of $x$ and $y$-values is an ordered pair that can be plotted. The table below lists values of $x$ from -3 to 3 and the resulting values for $y$.

| $x$ | $y=2 x-1$ | $(x, y)$ |
| :--- | :--- | :--- |
| -3 | $y=2(-3)-1=-7$ | $(-3,-7)$ |
| -2 | $y=2(-2)-1=-5$ | $(-2,-5)$ |
| -1 | $y=2(-1)-1=-3$ | $(-1,-3)$ |
| 0 | $y=2(0)-1=-1$ | $(0,-1)$ |
| 1 | $y=2(1)-1=1$ | $(1,1)$ |
| 2 | $y=2(2)-1=3$ | $(2,3)$ |
| 3 | $y=2(3)-1=5$ | $(3,5)$ |

We can plot these points from the table. The points for this particular equation form a line, so we can connect them. This is not true for all equations.


Note that the $x$-values chosen are arbitrary regardless of the type of equation we are graphing. Of course, some situations may require particular values of $x$ to be plotted in order to see a particular result. Otherwise, it is logical to choose values that can be calculated easily, and it is always a good idea to choose values that are both negative and positive. There is no rule dictating how many points to plot, although we need at least two to graph a line. Keep in mind, however, that the more points we plot, the more accurately we can sketch the graph.

## HOW TO: GIVEN AN EQUATION, GRAPH BY PLOTTING POINTS

1. Make a table with one column labeled $x$, a second column labeled with the equation, and a third column listing the resulting ordered pairs.
2. Enter $x$-values down the first column using positive and negative values. Selecting the $x$-values in numerical order will make graphing easier.
3. Select $x$-values that will yield $y$-values with little effort, preferably ones that can be calculated mentally.
4. Plot the ordered pairs.
5. Connect the points if they form a line.

## Using Intercepts to Plot Lines in the Coordinate Plane

The intercepts of a graph are points where the graph crosses the axes. The $\boldsymbol{x}$-intercept is the point where the graph crosses the $x$-axis. At this point, the $y$-coordinate is zero. The $y$-intercept is the point where the graph crosses the $y$-axis. At this point, the $x$-coordinate is zero.

To determine the $x$-intercept, we set $y$ equal to zero and solve for $x$. Similarly, to determine the $y$-intercept,

## EXAMPLE: GRAPHING AN EQUATION IN TWO VARIABLES BY PLOTTING POINTS

Graph the equation $y=-x+2$ by plotting points.

## Answer

First, we construct a table similar to the one below. Choose $x$ values and calculate $y$.

| $x$ | $y=-x+2$ | $(x, y)$ |
| :--- | :--- | :--- |
| -5 | $y=-(-5)+2=7$ | $(-5,7)$ |
| -3 | $y=-(-3)+2=5$ | $(-3,5)$ |
| -1 | $y=-(-1)+2=3$ | $(-1,3)$ |
| 0 | $y=-(0)+2=2$ | $(0,2)$ |
| 1 | $y=-(1)+2=1$ | $(1,1)$ |
| 3 | $y=-(3)+2=-1$ | $(3,-1)$ |
| 5 | $y=-(5)+2=-3$ | $(5,-3)$ |

Now, plot the points. Connect them if they form a line.


## TRY IT

Construct a table and graph the equation by plotting points: $y=\frac{1}{2} x+2$.
Answer

| $x$ | $y=\frac{1}{2} x+2$ | $(x, y)$ |
| :--- | :--- | :--- |
| -2 | $y=\frac{1}{2}(-2)+2=1$ | $(-2,1)$ |
| -1 | $y=\frac{1}{2}(-1)+2=\frac{3}{2}$ | $\left(-1, \frac{3}{2}\right)$ |
| 0 | $y=\frac{1}{2}(0)+2=2$ | $(0,2)$ |
| 1 | $y=\frac{1}{2}(1)+2=\frac{5}{2}$ | $\left(1, \frac{5}{2}\right)$ |
| 2 | $y=\frac{1}{2}(2)+2=3$ | $(2,3)$ |



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we set $x$ equal to zero and solve for $y$. For example, lets find the intercepts of the equation $y=3 x-1$.
To find the $x$-intercept, set $y=0$.
$y=3 x-1$
$0=3 x-1$
$1=3 x$
$\frac{1}{3}=x$
$\left(\frac{1}{3}, 0\right) \quad x$-intercept
To find the $y$-intercept, set $x=0$.
$y=3 x-1$
$y=3(0)-1$
$y=-1$
$(0,-1) \quad y$-intercept

We can confirm that our results make sense by observing a graph of the equation. Notice that the graph crosses the axes where we predicted it would.


## HOW TO: GIVEN AN EQUATION, FIND THE INTERCEPTS

1. Find the $x$-intercept by setting $y=0$ and solving for $x$.
2. Find the $y$-intercept by setting $x=0$ and solving for $y$.

## Using a Graphing Utility to Plot Lines

https://www.geogebra.org/graphing https://www.desmos.com/calculator https://www.mathway.com/Graph
https://www.symbolab.com/graphing-calculator

## EXAMPLE: FINDING THE INTERCEPTS OF THE GIVEN EQUATION

Find the intercepts of the equation $y=-3 x-4$. Then sketch the graph using only the intercepts.
Answer
Set $y=0$ to find the $x$-intercept.
$y=-3 x-4$
$0=-3 x-4$
$4=-3 x$
$-\frac{4}{3}=x$
$\left(-\frac{4}{3}, 0\right) x$-intercept
Set $x=0$ to find the $y$-intercept.
$y=-3 x-4$
$y=-3(0)-4$
$y=-4$
$(0,-4) y$-intercept
Plot both points and draw a line passing through them.


## TRY IT

Find the intercepts of the equation and sketch the graph: $y=-\frac{3}{4} x+3$.
Answer
$x$-intercept is $(4,0)$; $y$-intercept is $(0,3)$.


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## TRY IT NOW

These graphing utilities have features that allow you to turn a constant (number) into a variable. Follow these steps to learn how:

1. Graph the line $y=-\frac{2}{3} x-\frac{4}{3}$.
2. On the next line enter $y=-a x-\frac{4}{3}$. You will see a button pop up that says "add slider: a", click on the button. You will see the next line populated with the variable a and the interval on which a can take values.
3. What part of a line does the variable a represent? The slope or the y-intercept?

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## SUMMARY: POINTS AND LINES IN THE COORDINATE PLANE

## Key Concepts

- We can locate or plot points in the Cartesian coordinate system using ordered pairs which are defined as displacement from the $x$-axis and displacement from the $y$-axis.
- An equation can be graphed in the plane by creating a table of values and plotting points.
- Using a graphing calculator or a computer program makes graphing equations faster and more accurate. Equations usually have to be entered in the form $y=$ $\qquad$ .
- Finding the $x$ - and $y$-intercepts can define the graph of a line. These are the points where the graph crosses the axes.
- The distance formula is derived from the Pythagorean Theorem and is used to find the length of a line segment.
- The midpoint formula provides a method of finding the coordinates of the midpoint by dividing the sum of the $x$-coordinates and the sum of the $y$-coordinates of the endpoints by 2 .


## Glossary

Cartesian coordinate system a grid system designed with perpendicular axes invented by René Descartes equation in two variables a mathematical statement, typically written in $x$ and $y$, in which two expressions are equal
distance formula a formula that can be used to find the length of a line segment if the endpoints are known
graph in two variables the graph of an equation in two variables, which is always shown in two variables in the two-dimensional plane
intercepts the points at which the graph of an equation crosses the $x$-axis and the $y$-axis
midpoint formula a formula to find the point that divides a line segment into two parts of equal length
ordered pair a pair of numbers indicating horizontal displacement and vertical displacement from the origin; also known as a coordinate pair, $(x, y)$
origin the point where the two axes cross in the center of the plane, described by the ordered pair $(0,0)$
quadrant one quarter of the coordinate plane, created when the axes divide the plane into four sections
$\boldsymbol{x}$-axis the common name of the horizontal axis on a coordinate plane; a number line increasing from left to right
$x$-coordinate the first coordinate of an ordered pair, representing the horizontal displacement and direction from the origin
$x$-intercept the point where a graph intersects the $x$-axis; an ordered pair with a $y$-coordinate of zero
$\boldsymbol{y}$-axis the common name of the vertical axis on a coordinate plane; a number line increasing from bottom to top
$\boldsymbol{y}$-coordinate the second coordinate of an ordered pair, representing the vertical displacement and direction from the origin
$\boldsymbol{y}$-intercept a point where a graph intercepts the $y$-axis; an ordered pair with an $x$-coordinate of zero

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## INTRODUCTION TO EQUATIONS OF LINES

## LEARNING OUTCOMES

By the end of this lesson, you will be able to:

- Use slope-intercept form to plot and write equations of lines.
- Use point-slope form to write the equation of a line.
- Write the equation of a line in standard form.
- Recognize vertical and horizontal lines from their graphs and equations.
- Determine whether two lines are parallel or perpendicular.
- Find the equations of parallel and perpendicular lines.
- Write the equations of lines that are parallel or perpendicular to a given line.

Now that we have learned how to plot points on a coordinate plane and graph linear equations, we can begin to analyze the equations of lines and evaluate the different characteristics of these lines. In this section, we will learn about the commonly used forms for writing linear equations and the properties of lines that can be determined from their equations.

For example, without creating a table of values, you will be able to match each equation below to its corresponding graph. You will also be able to explain the similarities and differences of each line, how they relate to each other, and why they behave that way.
(a) $y=3 x+2$
(b) $y-4=-\frac{1}{2}(x+2)$

(c) $x=5$
(d) $y=-2$
(e) $3 x=y+1$
(f) $2 y-3 x=6$

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## WRITING EQUATIONS OF LINES

## LEARNING OUTCOMES

- Use slope-intercept form to plot and write equations of lines.
- Use point-slope form to write the equation of a line.
- Write the equation of a line in standard form.
- Recognize vertical and horizontal lines from their graphs and equations.


## Slope-Intercept Form

Perhaps the most familiar form of a linear equation is slope-intercept form written as $y=m x+b$, where $m=$ slope and $b=y$-intercept. Let us begin with the slope.

## The Slope of a Line

The slope of a line refers to the ratio of the vertical change in $y$ over the horizontal change in $x$ between any two points on a line. It indicates the direction in which a line slants as well as its steepness. Slope is sometimes described as rise over run.

$$
m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
$$

If the slope is positive, the line slants upward to the right. If the slope is negative, the line slants downward to the right. As the slope increases, the line becomes steeper. Some examples are shown below. The lines
indicate the following slopes: $m=-3, m=2$, and $m=\frac{1}{3}$.


## A GENERAL NOTE: THE SLOPE OF A LINE

The slope of a line, $m$, represents the change in $y$ over the change in $x$. Given two points, $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$, the following formula determines the slope of a line containing these points:

$$
m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
$$

## EXAMPLE: FINDING THE SLOPE OF A LINE GIVEN TWO POINTS

Find the slope of a line that passes through the points $(2,-1)$ and $(-5,3)$.

## Answer

We substitute the $y$-values and the $x$-values into the formula.

$$
\begin{aligned}
m & =\frac{3-(-1)}{-5-2} \\
& =\frac{4}{-7} \\
& =-\frac{4}{7}
\end{aligned}
$$

The slope is $-\frac{4}{7}$.

## Analysis of the Solution

It does not matter which point is called $\left(x_{1}, y_{1}\right)$ or $\left(x_{2}, y_{2}\right)$. As long as we are consistent with the order of the $y$ terms and the order of the $x$ terms in the numerator and denominator, the calculation will yield the same result.

## TRY IT

Find the slope of the line that passes through the points $(-2,6)$ and $(1,4)$.

Answer
$x=-5$
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## EXAMPLE: IDENTIFYING THE SLOPE AND $Y$-INTERCEPT OF A LINE GIVEN AN EQUATION

Identify the slope and $y$-intercept given the equation $y=-\frac{3}{4} x-4$.
Answer
As the line is in $y=m x+b$ form, the given line has a slope of $m=-\frac{3}{4}$. The $y$-intercept is $b=-4$.

## Analysis of the Solution

The $y$-intercept is the point at which the line crosses the $y$-axis. On the $y$-axis, $x=0$. We can always identify the $y$-intercept when the line is in slope-intercept form, as it will always equal $b$. Or, just substitute $x=0$ and solve for $y$.

## RECALL OPERATIONS ON FRACTIONS

Fractions are commonly encountered when writing equations for lines using the point-slope formula. Being able to multiply or add and subtract fractions will enable you to avoid decimal approximations for numbers like $\frac{1}{3}$ or $\frac{5}{7}$.
Adding or subtracting fractions: $\frac{a}{b} \pm \frac{c}{d}=\frac{a d \pm b c}{b d}$
Multiplying fractions: $\frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}$
Don't forget to simplify fractions in your final answer!

## The Point-Slope Formula

Given the slope and one point on a line, we can find the equation of the line using point-slope form.

$$
y-y_{1}=m\left(x-x_{1}\right)
$$

This is an important formula, as it will be used in other areas of College Algebra and often in Calculus to find the equation of a tangent line. We need only one point and the slope of the line to use the formula. After substituting the slope and the coordinates of one point into the formula, we simplify it and write it in slopeintercept form.

## A GENERAL NOTE: THE POINT-SLOPE FORMULA

Given one point and the slope, using point-slope form will lead to the equation of a line:

$$
y-y_{1}=m\left(x-x_{1}\right)
$$

## Standard Form of a Line

## EXAMPLE: FINDING THE EQUATION OF A LINE GIVEN THE SLOPE AND ONE POINT

Write the equation of the line with slope $m=-3$ and passing through the point $(4,8)$. Write the final equation in slope-intercept form.

## Answer

Using point-slope form, substitute -3 for $m$ and the point $(4,8)$ for $\left(x_{1}, y_{1}\right)$.

$$
\begin{aligned}
& y-y_{1}=m\left(x-x_{1}\right) \\
& y-8=-3(x-4) \\
& y-8=-3 x+12 \\
& y=-3 x+20
\end{aligned}
$$

## Analysis of the Solution

Note that any point on the line can be used to find the equation. If done correctly, the same final equation will be obtained.

## TRY IT

Given $m=4$, find the equation of the line in slope-intercept form passing through the point $(2,5)$.
Answer
$y=4 x-3$
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Another way that we can represent the equation of a line is in standard form. Standard form is given as

$$
A x+B y=C
$$

where $A, B$, and $C$ are integers. The $x$ and $y$-terms are on one side of the equal sign and the constant term is on the other side.

## Vertical and Horizontal Lines

Most of the lines we have worked with so far have been slanted, or oblique. In other words, they were neither horizontal nor vertical lines. The equations of vertical and horizontal lines do not require any of the preceding formulas, although we can use the formulas to prove that the equations are correct. The equation of a vertical line is given as

$$
x=c
$$

where $c$ is a constant. The slope of a vertical line is undefined, and regardless of the $y$-value of any point on the line, the $x$-coordinate of the point will be $c$.

Suppose that we want to find the equation of a line containing the following points: $(-3,-5),(-3,1),(-3,3)$, and $(-3,5)$. First, we will find the slope.

$$
m=\frac{5-3}{-3-(-3)}=\frac{2}{0}
$$

## EXAMPLE: FINDING THE EQUATION OF A LINE PASSING THROUGH TWO GIVEN POINTS

Find the equation of the line passing through the points $(3,4)$ and $(0,-3)$. Write the final equation in slope-intercept form.
Answer
First, we calculate the slope using the slope formula and two points.

$$
\begin{aligned}
& m=\frac{-3-4}{0-3} \\
& =\frac{-7}{-3} \\
& =\frac{7}{3}
\end{aligned}
$$

Next, we use point-slope form with the slope of $\frac{7}{3}$ and either point. Let's pick the point $(3,4)$ for $\left(x_{1}, y_{1}\right)$.

$$
\begin{aligned}
& y-4=\frac{7}{3}(x-3) \\
& y-4=\frac{7}{3} x-7 \\
& y=\frac{7}{3} x-3
\end{aligned}
$$

In slope-intercept form, the equation is written as $y=\frac{7}{3} x-3$.

## Analysis of the Solution

To prove that either point can be used, let us use the second point $(0,-3)$ and see if we get the same equation.

$$
\begin{aligned}
& y-(-3)=\frac{7}{3}(x-0) \\
& y+3=\frac{7}{3} x \\
& y=\frac{7}{3} x-3
\end{aligned}
$$

We see that the same line will be obtained using either point. This makes sense because we used both points to calculate the slope.

Zero in the denominator means that the slope is undefined and, therefore, we cannot use point-slope form. However, we can plot the points. Notice that all of the $x$-coordinates are the same and we find a vertical line through $x=-3$.

The equation of a horizontal line is given as

$$
y=c
$$

where $c$ is a constant. The slope of a horizontal line is zero, and for any $x$-value of a point on the line, the $y$ coordinate will be $c$.

Suppose we want to find the equation of a line that contains the following set of points:
$(-2,-2),(0,-2),(3,-2)$, and $(5,-2)$. We can use point-slope form. First, we find the slope using any two points on the line.

## EXAMPLE: FINDING THE EQUATION OF A LINE AND WRITING IT IN STANDARD FORM

Find the equation of the line with $m=-6$ and passing through the point $\left(\frac{1}{4},-2\right)$. Write the equation in standard form.
Answer
We begin by using point-slope form.

$$
\begin{aligned}
& y-(-2)=-6\left(x-\frac{1}{4}\right) \\
& y+2=-6 x+\frac{3}{2}
\end{aligned}
$$

From here, we multiply through by 2 as no fractions are permitted in standard form. Then we move both variables to the left aside of the equal sign and move the constants to the right.
$2(y+2)=\left(-6 x+\frac{3}{2}\right) 2$
$2 y+4=-12 x+3$
$12 x+2 y=-1$
This equation is now written in standard form.

## TRY IT

Find the equation of the line in standard form with slope $m=-\frac{1}{3}$ which passes through the point $\left(1, \frac{1}{3}\right)$.
Answer
$x+3 y=2$
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$m=\frac{-2-(-2)}{0-(-2)}$
$=\frac{0}{2}$
$=0$
Use any point for $\left(x_{1}, y_{1}\right)$ in the formula, or use the $y$-intercept.

$$
\begin{aligned}
& y-(-2)=0(x-3) \\
& y+2=0 \\
& y=-2
\end{aligned}
$$

The graph is a horizontal line through $y=-2$. Notice that all of the $y$-coordinates are the same.


The line $x=-3$ is a vertical line. The line $y=-2$ is a horizontal line.

## TRY IT

Use an online graphing calculator to graph the following:

1. A horizontal line that passes through the point $(-5,2)$
2. A vertical line that passes through the point $(3,3)$

## EXAMPLE: FINDING THE EQUATION OF A LINE PASSING THROUGH THE GIVEN POINTS

Find the equation of the line passing through the given points: $(1,-3)$ and $(1,4)$.
Answer
The $x$-coordinate of both points is 1 . Therefore, we have a vertical line, $x=1$.

## TRY IT

Find the equation of the line passing through $(-5,2)$ and $(2,2)$.
Answer

Horizontal line: $y=2$
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## PARALLEL AND PERPENDICULAR LINES

## LEARNING OUTCOMES

- Determine whether two lines are parallel or perpendicular.
- Find the equations of parallel and perpendicular lines.
- Write the equations of lines that are parallel or perpendicular to a given line.

Parallel lines have the same slope and different $y$-intercepts. Lines that are parallel to each other will never intersect. For example, the figure below shows the graphs of various lines with the same slope, $m=2$.


Parallel lines have slopes that are the same.
All of the lines shown in the graph are parallel because they have the same slope and different $y$-intercepts.
Lines that are perpendicular intersect to form a $90^{\circ}$ angle. The slope of one line is the negative reciprocal of the other. We can show that two lines are perpendicular if the product of the two slopes is $-1: m_{1} \cdot m_{2}=-1$. For example, the figure below shows the graph of two perpendicular lines. One line has a slope of 3 ; the other line has a slope of $-\frac{1}{3}$.
$m_{1} \cdot m_{2}=-1$
$3 \cdot\left(-\frac{1}{3}\right)=-1$


Perpendicular lines have slopes that are negative reciprocals of each other.
If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

## Writing Equations of Parallel Lines

Suppose we are given the following equation:
$y=3 x+1$
We know that the slope of the line formed by the function is 3 . We also know that the $y$-intercept is $(0,1)$. Any other line with a slope of 3 will be parallel to $y=3 x+1$. So all of the following lines will be parallel to the given line.

$$
\begin{aligned}
& y=3 x+6 \\
& y=3 x+1 \\
& y=3 x+\frac{2}{3}
\end{aligned}
$$

Suppose then we want to write the equation of a line that is parallel to $y=3 x+6$ and passes through the point $(1,7)$. We already know that the slope is 3 . We just need to determine which value for $b$ will give the correct line. We can begin with point-slope form of a line and then rewrite it in slope-intercept form.

## EXAMPLE: GRAPHING TWO EQUATIONS, AND DETERMINING WHETHER THE LINES ARE PARALLEL, PERPENDICULAR, OR NEITHER

Graph the equations of the given lines and state whether they are parallel, perpendicular, or neither:
$3 y=-4 x+3$ and $3 x-4 y=8$.

## Answer

The first thing we want to do is rewrite the equations so that both equations are in slope-intercept form.
First equation:

$$
\begin{aligned}
& 3 y=-4 x+3 \\
& y=-\frac{4}{3} x+1
\end{aligned}
$$

Second equation:

$$
\begin{aligned}
& 3 x-4 y=8 \\
& -4 y=-3 x+8 \\
& y=\frac{3}{4} x-2
\end{aligned}
$$

See the graph of both lines in the graph below.


From the graph, we can see that the lines appear perpendicular, but we must compare the slopes.

$$
\begin{aligned}
& m_{1}=-\frac{4}{3} \\
& m_{2}=\frac{3}{4} \\
& m_{1} \cdot m_{2}=\left(-\frac{4}{3}\right)\left(\frac{3}{4}\right)=-1
\end{aligned}
$$

The slopes are negative reciprocals of each other confirming that the lines are perpendicular.

## TRY IT

Graph the two lines and determine whether they are parallel, perpendicular, or neither: $2 y-x=10$ and $2 y=x+4$.
Answer

Parallel lines. Write the equations in slope-intercept form.


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$$
\begin{aligned}
& y-y_{1}=m\left(x-x_{1}\right) \\
& y-7=3(x-1) \\
& y-7=3 x-3 \\
& y=3 x+4
\end{aligned}
$$

So $y=3 x+4$ is parallel to $y=3 x+1$ and passes through the point $(1,7)$.

## HOW TO: GIVEN THE EQUATION OF A LINE, WRITE THE EQUATION OF A LINE PARALLEL TO THE GIVEN LINE THAT PASSES THROUGH A GIVEN POINT

1. Find the slope of the line.
2. Substitute the given values into either point-slope form or slope-intercept form.
3. Simplify.

## Writing Equations of Perpendicular Lines

We can use a very similar process to write the equation for a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the following line:
$y=2 x+4$
The slope of the line is 2 and its negative reciprocal is $-\frac{1}{2}$. Any function with a slope of $-\frac{1}{2}$ will be perpendicular to $y=2 x+4$. So all of the following lines will be perpendicular to $y=2 x+4$.

## EXAMPLE: FINDING A LINE PARALLEL TO A GIVEN LINE

Find a line parallel to the graph of $y=3 x+6$ that passes through the point $(3,0)$.
Answer
The slope of the given line is 3 . If we choose to use slope-intercept form, we can substitute $m=3, x=3$, and $\mathrm{y}=0$ into slope-intercept form to find the $y$-intercept.
$y=3 x+b$
$0=3(3)+b$
$b=-9$
The line parallel to $y=3 x+6$ that passes through $(3,0)$ is $y=3 x-9$.

## Analysis of the Solution

We can confirm that the two lines are parallel by graphing them. The graph below shows that the two lines will never intersect.


## TRY IT

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$y=-\frac{1}{2} x+4$
$y=-\frac{1}{2} x+2$
$y=-\frac{1}{2} x-\frac{1}{2}$
As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose then we want to write the equation of a line that is perpendicular to $y=2 x+4$ and passes through the point $(4,0)$. We already know that the slope is $-\frac{1}{2}$. Now we can use the point to find the $y$-intercept by substituting the given values into slope-intercept form and solving for $b$.
$y=m x+b$
$0=-\frac{1}{2}(4)+b$
$0=-2+b$
$2=b$
$b=2$
The equation for the function with a slope of $-\frac{1}{2}$ and a $y$-intercept of 2 is $y=-\frac{1}{2} x+2$.
So $y=-\frac{1}{2} x+2$ is perpendicular to $y=2 x+4$ and passes through the point (4, 0). Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

## Q \& A

A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not $\mathbf{- 1}$. Doesn't this fact contradict the definition of perpendicular lines?
No. For two perpendicular linear functions, the product of their slopes is -1. As you will learn later, a vertical line is not a function so the definition is not contradicted.

## HOW TO: GIVEN THE EQUATION OF A LINE, WRITE THE EQUATION OF A LINE PERPENDICULAR TO THE GIVEN LINE THAT PASSES THROUGH A GIVEN POINT

1. Find the slope of the given line.
2. Determine the negative reciprocal of the slope.
3. Substitute the slope and point into either point-slope form or slope-intercept form.
4. Simplify.

## Writing the Equations of Lines Parallel or Perpendicular to a

## Given Line

As we have learned, determining whether two lines are parallel or perpendicular is a matter of finding the slopes. To write the equation of a line parallel or perpendicular to another line, we follow the same principles as we do for finding the equation of any line. After finding the slope, use point-slope form to write the equation of the new line.

## EXAMPLE: FINDING THE EQUATION OF A PERPENDICULAR LINE

Find the equation of a line perpendicular to $y=3 x+3$ that passes through the point $(3,0)$.

## Answer

The original line has slope $m=3$, so the slope of the perpendicular line will be its negative reciprocal or $-\frac{1}{3}$. Using this slope and the given point, we can find the equation for the line.

$$
\begin{aligned}
& =-\frac{1}{3} x+b \\
& 0=-\frac{1}{3}(3)+b \\
& 1=b \\
& b=1
\end{aligned}
$$

The line perpendicular to $y=3 x+3$ that passes through $(3,0)$ is $y=-\frac{1}{3} x+1$.

Analysis of the Solution

A graph of the two lines is shown below.


## TRY IT

Given the line $y=2 x-4$, write an equation for the line passing through $(0,0)$ that is

1. parallel to $y$
2. perpendicular to $y$

## Answer

1. $y=2 x$ is parallel
2. $y=-\frac{1}{2} x$ is perpendicular

## HOW TO: GIVEN TWO POINTS ON A LINE, WRITE THE EQUATION OF A PERPENDICULAR LINE THAT PASSES THROUGH A THIRD POINT

1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Substitute the slope and point into either point-slope form or slope-intercept form.
4. Simplify.

## EXAMPLE: FINDING THE EQUATION OF A PERPENDICULAR LINE

A line passes through the points $(-2,6)$ and $(4,5)$. Find the equation of a perpendicular line that passes through the point $(4,5)$.

## Answer

From the two points of the given line, we can calculate the slope of that line.

$$
m_{1}=\frac{5-6}{4-(-2)}=\frac{-1}{6}=-\frac{1}{6}
$$

Find the negative reciprocal of the slope.

$$
m_{2}=\frac{-1}{-\frac{1}{6}}=-1\left(-\frac{6}{1}\right)=6
$$

We can then solve for the $y$-intercept of the line passing through the point $(4,5)$.
$y=6 x+b$
$5=6(4)+b$
$5=24+b$
$-19=b$
$b=-19$
The equation for the line that is perpendicular to the line passing through the two given points and also passes through point $(4,5)$ is:
$y=6 x-19$

## TRY IT

A line passes through the points $(-2,-15)$ and $(2,-3)$. Find the equation of a perpendicular line that passes through the point $(6,4)$.
Answer
$y=-\frac{1}{3} x+6$

## EXAMPLE: WRITING THE EQUATION OF A LINE PARALLEL TO A GIVEN LINE

Write the equation of line parallel to a $5 x+3 y=1$ which passes through the point $(3,5)$.

## Answer

First, we will write the equation in slope-intercept form to find the slope.

$$
\begin{aligned}
& 5 x+3 y=1 \\
& 3 y=-5 x+1 \\
& y=-\frac{5}{3} x+\frac{1}{3}
\end{aligned}
$$

The slope is $m=-\frac{5}{3}$. The $y$-intercept is $\frac{1}{3}$, but that really does not enter into our problem, as the only thing we need for two lines to be parallel is the same slope.
The one exception is that if the $y$-intercepts are the same, then the two lines are the same line. The next step is to use this slope and the given point in point-slope form.

$$
\begin{aligned}
& y-5=-\frac{5}{3}(x-3) \\
& y-5=-\frac{5}{3} x+5 \\
& y=-\frac{5}{3} x+10
\end{aligned}
$$

The equation of the line is $y=-\frac{5}{3} x+10$.


## TRY IT

Find the equation of the line parallel to $5 x=7+y$ which passes through the point $(-1,-2)$.
Answer
$y=5 x+3$
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## EXAMPLE: FINDING THE EQUATION OF A PERPENDICULAR LINE

Find the equation of the line perpendicular to $5 x-3 y+4=0$ which goes through the point $(-4,1)$.
Answer
The first step is to write the equation in slope-intercept form.

$$
\begin{aligned}
& 5 x-3 y+4=0 \\
& -3 y=-5 x-4 \\
& y=\frac{5}{3} x+\frac{4}{3}
\end{aligned}
$$

We see that the slope is $m=\frac{5}{3}$. This means that the slope of the line perpendicular to the given line is the negative reciprocal or $-\frac{3}{5}$. Next, we use point-slope form with this new slope and the given point.

$$
\begin{aligned}
& y-1=-\frac{3}{5}(x-(-4)) \\
& y-1=-\frac{3}{5} x-\frac{12}{5} \\
& y=-\frac{3}{5} x-\frac{12}{5}+\frac{5}{5} \\
& y=-\frac{3}{5} x-\frac{7}{5}
\end{aligned}
$$

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## SUMMARY: EQUATIONS OF LINES

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## Key Concepts

- Given two points, we can find the slope of a line using the slope formula.
- We can identify the slope and $y$-intercept of an equation in slope-intercept form.
- We can find the equation of a line given the slope and a point.
- We can also find the equation of a line given two points. Find the slope and use point-slope form.
- The standard form of a line has no fractions.
- Horizontal lines have a slope of zero and are defined as $y=c$, where $c$ is a constant.
- Vertical lines have an undefined slope (zero in the denominator) and are defined as $x=c$, where $c$ is a constant.
- Parallel lines have the same slope and different $y$-intercepts.
- Perpendicular lines have slopes that are negative reciprocals of each other unless one is horizontal and the other is vertical.


## Glossary

slope the change in $y$-values over the change in $x$-values

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## PUTTING IT TOGETHER: THE RECTANGULAR COORDINATE SYSTEM AND EQUATIONS OF LINES

Now that you have become familiar with the rectangular coordinate system, you are ready to consider the problem posed at the beginning of the module:

Caroline wants to take a spring break trip that will cost approximately $\$ 2,500$.

- How many hours will she have to work to earn enough to pay for her vacation?
- If she can only work 4 hours per day, how many days per week will she have to work?
- How many weeks will it take?

The amount Caroline has at any given time depends on the number of hours she works. If you write an equation that models the amount of money in her account at a given time, you will be able to answer the questions posed above. Begin by recalling what you know:

- Caroline has a job that pays $\$ 15.00 / \mathrm{hr}$
- She is starting with an initial deposit in her savings account of $\$ 400$.

Next, choose a variable to represent the number of hours, such as $h$. Her savings account balance is equal to the amount Caroline earns from working plus the initial deposit.

You can calculate the total amount Caroline gets paid by multiplying the amount per hour by the number of hours, $15 h$. Now you can write your equation:

Savings account balance $=$ amount earned from working + initial deposit
Savings account balance $=15 h+400$
Now that you have an equation, you can determine how many hours Caroline needs to work to earn \$2500
Savings account balance $=15 h+400$
$2500=15 h+400$
$2100=15 h$
$140=h$
So Caroline needs to work 140 hours to have enough money for her trip. Okay, we are making progress now.

The next question asked you to determine how many days per week Caroline would have to work if she can work only 4 hours per day. How can you find the number of days if you know the total number of hours and the number of hours per day? Divide!
$\frac{140}{4}=35$
Caroline will have to work 35 days to reach her goal. Since there are 7 days in a week, she will need to work 5 weeks, assuming she works every day.

Now you have answered all the questions, but you can enhance your solution by considering the second way to look at this problem-creating and analyzing a graph. You were provided a graph at the beginning of the module, but now you can create the graph for yourself. Go back to the equation you wrote to represent the problem.

Savings account balance $=15 h+400$
You can set the savings account balance as $y$ and the number of hours as $x$.
$y=15 x+400$
Then you can create a table of points using the equation.

| $x$ | $y=15 x+400$ | $y$ | $(x, y)$ |
| :--- | :--- | :--- | :--- |
| 0 | $y=15(0)+400$ | 400 | $(0,400)$ |
| 10 | $y=15(10)+400$ | 550 | $(10,550)$ |
| 20 | $y=15(20)+400$ | 700 | $(20,700)$ |
| 30 | $y=15(30)+400$ | 850 | $(30,850)$ |

Plot the points in the table and then connect them. Congratulations, you created the graph you saw at the beginning of the module!


The slope, or slant, of the line is determined by the equation.
$y=m x+b$
$y=15 x+400$
So the slope is 15 and depends on the amount Caroline earns per hour. How do you think the graph would change if she were paid a different rate? If she were paid less, the line would be less steep and if she were paid more, the line would be steeper. Both would affect how long it would take her to reach her goal as indicated by the orange line.

The orange horizontal line is the line $y=2500$ because that is the amount Caroline needs for her trip. How can you tell the number of hours she needs to work just by looking at the graph? The point at which the two lines intersect gives the answer. So find that point on the graph and then trace down to the x -axis. You wind up at 140 hours which is exactly what you calculated earlier.

Another nice feature of the graph is that you can see how much Caroline has after any number of hours worked and how many hours she has left to go. For example, suppose Caroline has worked 40 hours. How much savings does she have? To figure it out, find 40 hours on the $x$-axis and then trace vertically up to the line. Read left to the $y$-axis to find that she has $\$ 1000$ dollars. Every point on the graph relates a number of hours worked to the amount in her savings account. So the point you just looked at, (40, 1000), relates 40 hours to $\$ 1000$. The point Caroline needs to get to, $(140,2500)$ relates 140 hours of work to her goal of $\$ 2,500$. Now there is only one thing left to do. Caroline needs to get to work so she can enjoy her spring break vacation!

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# MODULE 4: EQUATIONS AND INEQUALITIES 

## WHY IT MATTERS: EQUATIONS AND INEQUALITIES

## How do Equations and Inequalities Help in the Real World?

You are in charge of planning and purchasing materials for a new school to be built overseas in a small village. The plans for the foundation indicate that it must be rectangular with a length of exactly 100 feet greater than the width. However, the total area must be $12,000 \mathrm{sq}$. ft. How long and how wide should the foundation be?

First we have to model the problem using mathematics. Let $x$ represent the width (shorter side). Then the length must be $x+100$. Using the area formula for a rectangle, we find:

Area $=($ length $) \times($ width $)=(x+100)(x)=x^{2}+100 x$

U.S. Navy Seabees help build a classroom for a school in Thailand.

Now since the total area must be equal to 12,000 , we can write:
$x^{2}+100 x=12,000$
or equivalently,

$$
x^{2}+100 x-12,000=0
$$

This is an equation, which is a mathematical statement consisting of two expressions set equal to one another. In order to find out the proper length and width of the floor, we would have to solve the equation for the variable x using techniques that you will soon discover.

Suppose the walls of the school will be built from concrete blocks costing $\$ 1.20$ per block (which includes tax) plus a flat fee of $\$ 100$ for hauling the blocks to the site. Having only a limited budget, you must be very careful to order only what you can afford. Suppose the school requires at least 1800 concrete blocks (but having more blocks around for future projects would not be a bad idea). Your budgeted amount for concrete blocks is $\$ 2,400$. What is the maximum number of concrete blocks that you can purchase? Do you have enough money to buy at least the minimum number of concrete blocks required for the project?

To answer the question above, we must be familiar with inequalities. An inequality is a statement involving relations like "less than," "more than," "at least," or "at most." For example, if the number of blocks to be purchased is n , then there are two inequalities implied by the story.
$n \geq 1800$ and $(1.20) n+100 \leq 2400$
The first inequality expresses the fact that at least 1800 blocks are required. The second inequality states that the total cost of the blocks, together with the hauling fee, cannot exceed the budgeted amount of \$2,400.

In this module, you will learn how to analyze this situation as well as many others that involve building and solving equations and inequalities. We will revisit our construction site again at the end of the module.

## LEARNING OUTCOMES

## Review Topics for Success

- Solve rational equations
- Recognize formulas for perimeter, area, and volume.
- Recognize the formula that relates distance, rate, and time.
- Rearrange formulas to isolate specific variables.
- Represent inequalities using an inequality symbol.
- Represent inequalities on a number line.


## Linear Equations

- Solve equations in one variable algebraically.
- Solve rational equations.


## Models and Applications

- Write a linear equation to express the relationship between unknown quantities.
- Write a linear equation that models two different cell phone packages.
- Use a linear model to answer questions.
- Set up a linear equation involving distance, rate, and time.
- Find the dimensions of a rectangle given the area.
- Find the dimensions of a box given information about its side lengths.


## Quadratic Equations

- Factor a quadratic equation to solve it.
- Use the square root property to solve a quadratic equation.
- Use the Pythagorean Theorem and the square root property to find the unknown length of the side of a right triangle.
- Complete the square to solve a quadratic equation.
- Use the quadratic formula to solve a quadratic equation.
- Use the discriminant to determine the number and type of solutions to a quadratic equation.


## Equation-Solving Techniques

- Solve a radical equation, identify extraneous solution.
- Solve an equation with rational exponents.
- Solve polynomial equations.
- Solve absolute value equations.

Linear Inequalities and Absolute Value Inequalities

- Use interval notation to express inequalities.
- Use properties of inequalities.
- Solve compound inequalities.
- Solve absolute value inequalities.
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## REVIEW TOPICS FOR SUCCESS

## Learning Outcomes

By the end of this section, you will be able to:

- Solve rational equations
- Recognize formulas for perimeter, area, and volume
- Recognize the formula that relates distance, rate, and time
- Rearrange formulas to isolate specific variables
- Represent inequalities using an inequality symbol
- Represent inequalities on a number line

In this module, you'll learn how to use linear equations to model real-world situations and explore several specific types of equations. You'll also define a process for translating inequalities stated in words and symbols or as graphs on the number line into a special kind of notation called interval notation. Before beginning, it will be helpful to refresh techniques for eliminating denominators from rational equations in one variable, and see some formulas that commonly occur during modeling and learn how to manipulate them. A review of representing inequalities using words and symbols and as graphs on the number line is also included in this section.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

## RECALL FOR SUCCESS

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

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## SOLVE RATIONAL EQUATIONS

## LEARNING OUTCOMES

- Solve rational equations by clearing denominators
- Identify extraneous solutions in a rational equation

Equations that contain rational expressions are called rational equations. For example, $\frac{2 x+1}{4}=\frac{x}{3}$ is a rational equation. Rational equations can be useful for representing real-life situations and for finding answers to real problems. In particular, they are quite good for describing a variety of proportional relationships.

One of the most straightforward ways to solve a rational equation is to eliminate denominators with the common denominator and then use properties of equality to isolate the variable. This method is often used to solve linear equations that involve fractions as in the following example:

Solve $\frac{1}{2} x-3=2-\frac{3}{4} x$ by clearing the fractions in the equation first.
Multiply both sides of the equation by 4 , the common denominator of the fractional coefficients.

$$
\begin{aligned}
& \frac{1}{2} x-3=2-\frac{3}{4} x \\
& 4\left(\frac{1}{2} x-3\right)=4\left(2-\frac{3}{4} x\right) \\
& 4\left(\frac{1}{2} x\right)-4(3)=4(2)+4\left(-\frac{3}{4} x\right) \\
& 2 x-12=8-3 x \\
& \underline{+3 x} \quad \underline{+3 x} \\
& 5 x-12=8 \\
& \begin{array}{l}
+12 \\
5 x=20 \\
x=4
\end{array}
\end{aligned}
$$

We could have found a common denominator and worked with fractions, but that often leads to more mistakes. We can apply the same idea to solving rational equations. The difference between a linear equation and a rational equation is that rational equations can have polynomials in the numerator and denominator of the fractions. This means that clearing the denominator may sometimes mean multiplying the whole rational equation by a polynomial. In the next example, we will clear the denominators of a rational equation with a term that has a polynomial in the numerator.

## Example

Solve the equation $\frac{x+5}{8}=\frac{7}{4}$.

## Answer

Find the least common denominator of 4 and 8 . Remember, to find the LCD, identify the greatest number of times each factor appears in each factorization. Here, 2 appears 3 times, so $2 \cdot 2 \cdot 2$, or 8 , will be the LCD.

Multiply both sides of the equation by the common denominator, 8 , to keep the equation balanced and to eliminate the denominators.
$8 \cdot \frac{x+5}{8}=\frac{7}{4} \cdot 8$

$$
\begin{aligned}
& \frac{8(x+5)}{8}=\frac{7(8)}{4} \\
& \frac{8}{8} \cdot(x+5)=\frac{7(4 \cdot 2)}{4} \\
& \frac{8}{8} \cdot(x+5)=7 \cdot 2 \cdot \frac{4}{4} \\
& 1 \cdot(x+5)=14 \cdot 1
\end{aligned}
$$

Simplify and solve for $x$.

$$
\begin{aligned}
& x+5=14 \\
& x=9
\end{aligned}
$$

Check the solution by substituting 9 for $x$ in the original equation.

$$
\begin{aligned}
& \frac{x+5}{8}=\frac{7}{4} \\
& \frac{9+5}{8}=\frac{7}{4} \\
& \frac{14}{8}=\frac{7}{4} \\
& \frac{7}{4}=\frac{7}{4}
\end{aligned}
$$

Therefore, $x=9$.

In the next example, we show how to solve a rational equation with a binomial in the denominator of one term. We will use the common denominator to eliminate the denominators from both fractions. Note that the LCD is the product of both denominators because they do not share any common factors.

## Example

Solve the equation $\frac{8}{x+1}=\frac{4}{3}$.
Answer
Clear the denominators by multiplying each side by the common denominator. The common denominator is $3(x+1)$ since 3 and $x+1$ do not have any common factors.

$$
3(x+1)\left(\frac{8}{x+1}\right)=3(x+1)\left(\frac{4}{3}\right)
$$

Simplify common factors.

$$
3(x+1)\left(\frac{8}{x+1}\right)=\not p(x+1)\left(\frac{4}{\not x}\right)
$$

$$
24=4(x+1)
$$

$$
24=4 x+4
$$

Now this looks like a linear equation, and we can use the addition and multiplication properties of equality to solve it.
$24=4 x+4$
$\underline{-4} \quad \underline{-4}$
$20=4 x$
$x=5$
Check the solution in the original equation.

$$
\begin{aligned}
& \frac{8}{(x+1)}=\frac{4}{3} \\
& \frac{8}{(5+1)}=\frac{4}{3} \\
& \frac{8}{6}=\frac{4}{3} \\
& \frac{4}{3}=\frac{4}{3}
\end{aligned}
$$

Therefore, $x=5$.

## Example

Solve the equation $\frac{x}{3}+1=\frac{4}{3}$.
Answer
Both fractions in the equation have a denominator of 3 . Multiply every term on both sides of the equation (not just the fractions!) by 3 .
$3\left(\frac{x}{3}+1\right)=3\left(\frac{4}{3}\right)$
Apply the distributive property and multiply 3 by each term within the parentheses. Then simplify and solve for $x$.

$$
\begin{aligned}
& 3\left(\frac{x}{3}\right)+3(1)=3\left(\frac{4}{3}\right) \\
& \not 2\left(\frac{x}{\not \partial}\right)+3(1)=\not 2\left(\frac{4}{\not \partial}\right) \\
& x+3=4 \\
& \quad \underline{-3} \text {-3 }
\end{aligned}
$$

$$
x=1
$$

Therefore, $x=1$.
In the video that follows, we present two ways to solve rational equations with both integer and variable denominators.

## Excluded Values and Extraneous Solutions

As you have seen, some rational expressions have a variable in the denominator. When this is the case, there is an extra step in solving them. Since division by 0 is undefined, you must exclude values of the


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variable that would result in a denominator of 0 . These values are called excluded values. Let us look at an example.

## Example

Solve the equation $\frac{2 x-5}{x-5}=\frac{15}{x-5}$.
Answer
Determine any values for $x$ that would make the denominator 0 .
$\frac{2 x-5}{x-5}=\frac{15}{x-5}$
5 is an excluded value because it makes the denominator $x-5$ equal to 0 .
Since the denominator of each expression in the equation is the same, the numerators must be equal. Set the numerators equal to one another and solve for $x$.

$$
\begin{aligned}
& 2 x-5=15 \\
& 2 x=20 \\
& x=10
\end{aligned}
$$

Check the solution in the original equation.

$$
\begin{aligned}
& \frac{2 x-5}{x-5}=\frac{15}{x-5} \\
& \frac{2(10)-5}{10-5}=\frac{15}{10-5} \\
& \frac{20-5}{10-5}=\frac{15}{10-5} \\
& \frac{15}{5}=\frac{15}{5}
\end{aligned}
$$

Therefore, $x=10$.

In the following video, we present an example of solving a rational equation with variables in the denominator.


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You have seen that there is more than one way to solve rational equations. Because both of these techniques manipulate and rewrite terms, sometimes they can produce solutions that do not work in the original form of the equation. These types of answers are called extraneous solutions. That is why it is always important to check all solutions in the original equations-you may find that they yield untrue statements or produce undefined expressions.

## Example

Solve the equation $\frac{16}{m+4}=\frac{m^{2}}{m+4}$.

## Answer

Determine any values for $m$ that would make the denominator $0 .-4$ is an excluded value because it makes $m+4$ equal to 0 .

Since the denominator of each expression in the equation is the same, the numerators must be equal. Set the numerators equal to one another and solve for $m$.

$$
\begin{aligned}
16 & =m^{2} \\
0 & =m^{2}-16 \\
0 & =(m+4)(m-4) \\
0 & =m+4 \quad \text { or } \quad 0=m-4 \\
m & =-4 \quad \text { or } \quad m=4 \\
m & =4,-4
\end{aligned}
$$

Check the solutions in the original equation.
Since $m=-4$ leads to division by 0 , it is an extraneous solution.
Notice, however, that latex]m=4[/latex] is a solution that results in a true statement.
$\frac{16}{m+4}=\frac{m^{2}}{m+4}$
$\frac{16}{-4+4}=\frac{(-4)^{2}}{-4+4}$
$\frac{16}{0}=\frac{16}{0}$
-4 is excluded because it leads to division by 0 .

$$
\frac{16}{4+4}=\frac{(4)^{2}}{4+4}
$$

$$
\frac{16}{8}=\frac{16}{8}
$$

Therefore, $m=4$.

## Summary

You can solve rational equations by finding a common denominator. By rewriting the equation so that all terms have the common denominator, you can solve for the variable using just the numerators. Or, you can multiply both sides of the equation by the least common denominator of all fractions so that all terms become polynomials instead of rational expressions.

An important step in solving rational equations is to reject any extraneous solutions from the final answer. Extraneous solutions are solutions that do not satisfy the original form of the equation because they produce untrue statements or are excluded values that cause a denominator to equal 0 .

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## USING AND MANIPULATING FORMULAS

## LEARNING OUTCOMES

- Recognize formulas for perimeter, area, and volume
- Recognize the formula that relates distance, rate, and time
- Rearrange formulas to isolate specific variables

Many real-world applications (problems that can be modeled by a mathematical equation and solved for an unknown quantity) involve formulas that describe relationships between quantities. For example, wall-to-wall carpeting is sold by the square foot. To find out how much carpeting would cover the floor of a $12^{\prime} \times 15^{\prime}$ foot room, we can multiply the length and width of the room to discover how many square feet of carpeting is required. $12 \times 15=180$. So, we would need to purchase 180 square feet of carpeting. Examples of formulas that commonly occur in applications include:

- the perimeter of a rectangle of length $L$ and width $W$
- $P=2 L+2 W$
- the area of a rectangular region of length $L$ and width $W$

$$
\circ A=L W
$$

- the volume of a rectangular solid with length L , width W , and height H
- $V=L W H$
- the distance $d$ covered when traveling at a constant rate $r$ for some time $t$
- $d=r t$.

Formulas such as these may be used to solve problems by substituting known values and solving for an unknown value. You should know these formulas and be able to recognize when to apply them to a problem.

## Isolate Variables in Formulas

Sometimes, we need to isolate one of the variables in a formula in order to solve for the unknown. In the carpeting example above, we found that it would take 180 square feet of carpet to cover a $12^{\prime} \times 15^{\prime}$ room. We knew the dimensions of the room and solved for how many square feet of carpet we needed, But what if, instead, we had tiles that we could use to cover the floor and we knew the length of the room, but needed to know what width room could be fully tiled?

Let's say you have a banquet hall with one wall that can slide to create a room of variable width. The length of the room (the length of the movable wall) is 85 feet. If you have 4,590 square feet of tile, what must the width be to ensure the entire room is fully tiled?

Use the formula $A=L W$, but solve it for W , width. $\frac{A}{L}=W$. Then, substitute what you know:
$\frac{4590}{85}=W=54$. It looks like we need to slide the wall out to create a width of 54 feet in order to use all the tile. .

This technique, isolating a variable of choice in any formula, is especially helpful if you have to perform the same calculation repeatedly, or you are having a computer perform the calculation repeatedly, for different values of the unknown variable.

## EXAMPLE: ISOLATE A VARIABLE IN A FORMULA

Isolate the variable for width $w$ from the formula for the perimeter of a rectangle:
$P=2(l)+2(w)$.
Answer
First, isolate the term with $w$ by subtracting $2 /$ from both sides of the equation.

$$
\begin{gathered}
P=2 l+2 w \\
-2 l-2 l \\
\hline P-2 l=\quad 2 w
\end{gathered}
$$

Next, clear the coefficient of $w$ by dividing both sides of the equation by 2 .

$$
\begin{aligned}
\frac{P-2 l}{2} & =\frac{2 w}{2} \\
\frac{P-2 l}{2} & =w \\
w & =\frac{P-2 l}{2}
\end{aligned}
$$

You can rewrite the equation so the isolated variable is on the left side.
$w=\frac{P-2 l}{2}$
Hopefully, you can see the value in being able to isolate a variable of interest! Here is another formula for which the skill is particularly useful.

## EXAMPLE: ISOLATE A VARIABLE IN A FORMULA

Use the multiplication and division properties of equality to isolate the variable $b$ given $A=\frac{1}{2} b h$
Answer

$$
A=\frac{1}{2} b h
$$

(2) $A=(2) \frac{1}{2} b h$

$$
2 A=b h
$$

$$
\frac{2 A}{h}=\frac{b h}{h}
$$

$$
\frac{2 A}{h}=\frac{b \npreceq}{\not K}
$$

Write the equation with the desired variable on the left-hand side as a matter of convention:
$b=\frac{2 A}{h}$
Use the multiplication and division properties of equality to isolate the variable $h$ given $A=\frac{1}{2} b h$
Answer

$$
A=\frac{1}{2} b h
$$

(2) $A=(2) \frac{1}{2} b h$

$$
2 A=b h
$$

$$
\frac{2 A}{b}=\frac{b h}{b}
$$

$$
\frac{2 A}{b}=\frac{h \not b}{\not b}
$$

Write the equation with the desired variable on the left-hand side as a matter of convention:
$h=\frac{2 A}{b}$

## THINK ABOUT IT

Isolate the variable for height, $h$, from the formula for the surface area of a cylinder, $s=2 \pi r h+2 \pi r^{2}$. In this example, the variable $h$ is buried pretty deeply in the formula for surface area of a cylinder. Using the order of operations, it can be isolated. Before you look at the solution, use the box below to write down what you think is the best first step to take to isolate $h$.

Answer
Isolate the term containing the variable $h$ by subtracting $2 \pi r^{2}$ from both sides.

$$
\begin{aligned}
& S=2 \pi r h+2 \pi r^{2} \\
& -2 \pi r^{2} \quad-2 \pi r^{2} \\
& \hline S-2 \pi r^{2}=2 \pi r h
\end{aligned}
$$

Next, isolate the variable $h$ by dividing both sides of the equation by $2 \pi r$.

$$
\begin{aligned}
& \frac{S-2 \pi r^{2}}{2 \pi r}=\frac{2 \pi r h}{2 \pi r} \\
& \frac{S-2 \pi r^{2}}{2 \pi r}=h
\end{aligned}
$$

You can rewrite the equation so the isolated variable is on the left side.
$h=\frac{S-2 \pi r^{2}}{2 \pi r}$
The formula for converting from the Fahrenheit temperature scale to the Celsius scale is given by $C=(F-32) \cdot \frac{5}{9}$. This formula takes a temperature in Fahrenheit, $F$, and coverts it to an equivalent temperature in Celsius, $C$.

## EXAMPLE: USE A FORMULA TO CONVERT FAHRENHEIT TO CELSIUS

Given a temperature of $12^{\circ} \mathrm{C}$, find the equivalent in ${ }^{\circ} \mathrm{F}$.

## Answer

Substitute the given temperature in ${ }^{\circ} C$ into the conversion formula:
$12=(F-32) \cdot \frac{5}{9}$
Isolate the variable $F$ to obtain the equivalent temperature.

$$
\begin{aligned}
& 12=(F-32) \cdot \frac{5}{9} \\
& \left(\frac{9}{5}\right) 12=F-32 \\
& \left(\frac{108}{5}\right) 12=F-32 \\
& 21.6=F-32 \\
& +32 \quad+32
\end{aligned}
$$

$$
53.6={ }^{\circ} F
$$

But what if we have the temperature in Celsius already, and wish to convert it to Fahrenheit? We can solve the formula for $F$. This will give us a formula that takes a temperature on the Celsius scale and converts it to an equivalent Fahrenheit temperature.

## EXAMPLE: ISOLATE A VARIABLE TO CONVERT CELSIUS TO FAHRENHEIT

Solve the formula shown below for converting from the Fahrenheit scale to the Celsius scale for F . Then use the new formula to convert a temperature of $21^{\circ} \mathrm{C}$ to Fahrenheit.
$C=(F-32) \cdot \frac{5}{9}$

## Answer

To isolate the variable $F$, it would be best to clear the fraction involving $F$ first. Multiply both sides of the equation by $\frac{9}{5}$.

$$
\begin{aligned}
\left(\frac{9}{5}\right) C & =(F-32)\left(\frac{5}{9}\right)\left(\frac{9}{5}\right) \\
\frac{9}{5} C & =F-32
\end{aligned}
$$

Add 32 to both sides.

$$
\begin{aligned}
& \frac{9}{5} C+32=F-32+32 \\
& \frac{9}{5} C+32=F \\
& F=\frac{9}{5} C+32
\end{aligned}
$$

And we can convert $21^{\circ} \mathrm{C}$ by substituting it into the new formula:

$$
\begin{aligned}
& F=\frac{9}{5}(21)+32 \\
& 21^{\circ} C=69.8^{\circ} F
\end{aligned}
$$

## INEQUALITY SYMBOLS AND GRAPHS

## Learning Outcomes

- Represent inequalities using an inequality symbol
- Represent inequalities on a number line

An inequality is a mathematical statement that compares two expressions using a phrase such as greater than or less than. Special symbols are used in these statements. In algebra, inequalities are used to
describe sets of values, as opposed to single values, of a variable. Sometimes, several numbers will satisfy an inequality, but at other times infinitely many numbers may provide solutions. Rather than try to list a possibly infinitely large set of numbers, mathematicians have developed some efficient ways to describe such large lists.

## Inequality Symbols

One way to represent such a list of numbers, an inequality, is by using an inequality symbol:

- $x<9$ indicates the list of numbers that are less than 9 . Since this list is infinite, it would be impossilbe to list all numbers less than 9 .
- $-5 \leq t$ indicates all the numbers that are greater than or equal to -5 .

If you were to read the above statement from left to right, it would translate as -5 is less than or equal to $t$. The direction of the symbol is dependent upon the statement you wish to make. For example, the following statements are equivalent. Both represent the list of all numbers less than 9 . Note how the open end of the inequality symbol faces the larger value while the smaller, pointy end points to the smaller of the values:

- $x<9$
- $9>x$

Here's another way of looking at is:

- $x<5$ means all the real numbers that are less than 5 , whereas;
- $5<x$ means that 5 is less than x , or we could rewrite this with the x on the left: $x>5$. Note how the inequality is still pointing the same direction relative to $x$. This statement represents all the real numbers that are greater than 5 which is easier to interpret than 5 is less than $x$.

The box below shows the symbol, meaning, and an example for each inequality sign, as they would be translated reading from left to right.

| Symbol | Words | Example |
| :--- | :--- | :--- |
| $\neq$ | not equal to | $2 \neq 8,2$ is not equal to 8. |
| $>$ | greater than | $5>1,5$ is greater than 1 |
| $<$ | less than | $2<11,2$ is less than 11 |
| $\geq$ | greater than or equal to | $4 \geq 4,4$ is greater than or equal to 4 |
| $\leq$ | less than or equal to | $7 \leq 9,7$ is less than or equal to 9 |

The inequality $x>y$ can also be written as $y<x$. The sides of any inequality can be switched as long as the inequality symbol between them is also reversed.

## Graphing an Inequality

Another way to represent an inequality is by graphing it on a number line:


Below are three examples of inequalities and their graphs. Graphs are often helpful for visualizing information.
$x \leq-4$. This translates to all the real numbers on a number line that are less than or equal to 4 .

$x \geq-3$. This translates to all the real numbers on the number line that are greater than or equal to -3 .


Each of these graphs begins with a circle-either an open or closed (shaded) circle. This point is often called the end point of the solution. A closed, or shaded, circle is used to represent the inequalities greater than or equal to $(\geq)$ or less than or equal to $(\leq)$. The end point is part of the solution. An open circle is used for greater than $(>)$ or less than (<). The end point is not part of the solution. When the end point is not included in the solution, we often say we have strict inequality rather than inequality with equality.

The graph then extends endlessly in one direction. This is shown by a line with an arrow at the end. For example, notice that for the graph of $x \geq-3$ shown above, the end point is -3 , represented with a closed circle since the inequality is greater than or equal to -3 . The blue line is drawn to the right on the number line because the values in this area are greater than -3 . The arrow at the end indicates that the solutions continue infinitely.

## EXAMPLE

Graph the inequality $x \geq 4$
Answer
We can use a number line as shown. Because the values for $x$ include 4 , we place a solid dot on the number line at 4.
Then we draw a line that begins at $x=4$ and, as indicated by the arrowhead, continues to positive infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4 .


This video shows an example of how to draw the graph of an inequality.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=167

## EXAMPLE

Write an inequality describing all the real numbers on the number line that are strictly less than 2 . Then draw the corresponding graph.

## Answer

We need to start from the left and work right, so we start from negative infinity and end at 2 . We will not include either because infinity is not a number, and the inequality does not include 2.
Inequality: $x<2$
To draw the graph, place an open dot on the number line first, and then draw a line extending to the left. Draw an arrow at the leftmost point of the line to indicate that it continues for infinity.


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## SUMMARY: REVIEW

## Key Concepts

- Denominators may be eliminated from rational equations by multiplying each term in the equation by the lowest common denominator (LCD).
- Declare values of the variable that could cause any denominator to equal zero before reducing or solving, then exclude those values from the solution set for the equation.
- The formula for the perimeter of a rectangle of length L and width W is $P=2 L+2 W$
- The formula for the area of a rectangular region of legth L and width W is $A=L W$
- The formula for the volume of a rectangular solid of length L , width W , and height H is $V=L W H$
- The formula that relates distance, rate, and time is $d=r t$
- A formula may be rearranged by algebraically isolating any one of the variables as desired
- Inequalities may be expressed in words, by using an inequality symbol, or graphed on a number line.


## Glossary

area the size of the surface of a two-dimensional figure
inequality a mathematical statement that compares the size or value of two mathematical expressions
LCD lowest common denominator, the smallest possible expression into which each denominator will factor evenly.
perimeter a measure of the linear distance around the edges of a two-dimensional figure
rational equation any equation containing a variable expression in one or more denominators
volume the size of the space contained within the edges of a three-dimensional figure

## INTRODUCTION TO MODELS AND APPLICATIONS

## Learning Outcomes

By the end of this section, you will be able to:

- Write a linear equation to express the relationship between unknown quantities.
- Write a linear equation that models two different cell phone packages.
- Use a linear model to answer questions.
- Set up a linear equation involving distance, rate, and time.
- Find the dimensions of a rectangle given the area.
- Find the dimensions of a box given information about its side lengths.


College students taking an exam. Credit: Kevin Dooley

Josh is hoping to get an A in his college algebra class. He has scores of 75, 82, 95, 91, and 94 on his first five tests. Only the final exam remains, and the maximum of points that can be earned is 100 . Is it possible for Josh to end the course with an A? A linear equation will give Josh his answer.

Many real-world applications can be modeled by linear equations. For example, a cell phone package may include a monthly service fee plus a charge per minute of talk-time; it costs a widget manufacturer a certain amount to produce $x$ widgets per month plus monthly operating charges; a car rental company charges a daily fee plus an amount per mile driven. These are examples of applications we come across every day that are modeled by linear equations. In this section, we will set up and use linear equations to solve such problems.

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## WRITING A LINEAR EQUATION TO SOLVE AN APPLICATION

## LEARNING OUTCOMES

- Write a linear equation to express the relationship between unknown quantities.
- Write a linear equation to model a real-world situation.
- Write a linear equation in one variable to solve problems with two unknowns.


## RECALL EXPRESSIONS AND EQUATIONS

Recall that a mathematical expression consists of terms connected by addition or subtraction, each term of which consists of variables and numbers connected by multiplication or division. A number that multiplies a variable, such as the 2 in the term $2 x$ is called the coefficient of the variable.
An equation is a mathematical statement of the equivalency of two expressions, one on either side of an equal sign.
Expressions may be combined or simplified by rearranging or doing operations on the terms. Equations may be solved for the value or values of the variable that make the statement of equivalency true.

Consider a car rental agency that charges $\$ 150$ per week plus $\$ 0.10$ per mile driven to rent a compact car. We can use these quantities to write an equation that models the cost of renting the car for a week $C$ given a certain number of miles $x$ driven.

$$
C=150+0.10 x
$$

To set up a linear equation that models a real-world situation, we must first determine the known quantities and define the unknown quantity as a variable. Then, we interpret the words as mathematical expressions using mathematical symbols. Let us use the car rental example above. In this case, a known cost, such as $\$ 0.10$ per mile, is multiplied by an unknown quantity, the number of miles driven. Therefore, we can write $0.10 x$ to model the portion of the weekly cost generated by miles driven. This expression represents a variable cost because it changes according to the number of miles driven.

If a quantity is independent of a variable, we usually just add or subtract it according to the problem. In the car rental example, there is a flat fee of $\$ 150$ to rent the car, independent of the number of miles driven. In applications involving costs, amounts such as this flat fee that do not change are often called fixed costs.

When dealing with real-world applications, there are certain expressions that we can translate directly into math. The table below lists some common verbal expressions and their equivalent mathematical expressions.

| Verbal | Translation to Math <br> Operations |
| :--- | :--- |
| One number exceeds another by $a$ | $x, x+a$ |
| Twice a number | $2 x$ |
| One number is $a$ more than another number | $x, x+a$ |
| One number is $a$ less than twice another number | $x, 2 x-a$ |
| The product of a number and $a$, decreased by $b$ | $a x-b$ |
| The quotient of a number and the number plus $a$ is three times the <br> number | $\frac{x}{x+a}=3 x$ |
| The product of three times a number and the number decreased by $b$ is <br> $c$ | $3 x(x-b)=c$ |

In the car-rental example above, we identified the unknown quantity (number of miles driven), and assigned it the variable $x$. Then we identified known quantities, and translated the words in the situation into relationships between the known and unknown quantities to write an equation that models the situation. We could use the model to answer questions about the situation such as finding the total cost for a week if 500 miles were driven or how many miles could be driven in a week on a budge of $\$ 375$.

## HOW TO: GIVEN A REAL-WORLD SITUATION, WRITE A LINEAR EQUATION TO MODEL IT

1. Identify known and unknown quantities.
2. Assign a variable to represent the unknown quantity.
3. If there is more than one unknown quantity, find a way to write the second unknown in terms of the first.
4. Write an equation interpreting the words in the problem as mathematical operations.
5. Solve the equation, check to be sure your answer is reasonable, and give the answer using the language and units of the original situation.

The following video shows another example of using linear equations to model and compare two cell phone plans.

We can also use a linear equation in one variable to solve a problem with two unknowns by writing an expression for one unknown in terms of the other.

Watch the following video for another example of using a linear equation in one variable to solve an application.

## EXAMPLE

Find a linear equation to solve for the following unknown quantities: One number exceeds another number by 17 and their sum is 31 . Find the two numbers.

## Answer

Let $x$ equal the first number. Then, as the second number exceeds the first by 17, we can write the second number as $x+17$. The sum of the two numbers is 31 . We usually interpret the word is as an equal sign.
$x+(x+17)=31$
$2 x+17=31$ Simplify and solve.
$2 x \quad=14$
$x=7$

The second number would then be $x+17=7+17=24$
The two numbers are 7 and 24 .

## TRY IT

Find a linear equation to solve for the following unknown quantities: One number is three more than twice another number. If the sum of the two numbers is 36 , find the numbers.
Answer
11 and 25

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## RECALL THE FORMS OF EQUATIONS OF LINES

Slope-intercept form: $y=m x+b$, where $x$ and $y$ represent the coordinates of any point on the line, $m$ represents the slope of the line, and $b$ represents the initial value, or the y-intercept. You can solve for $b$ by substituting a known point for $x$ and $y$ and the slope of the line for $m$.
The point-slope form, $y-y_{1}=m\left(x-x_{1}\right)$ simplifies to slope-intercept form when solved for $y$. Substitute the coordinates of a point for $y_{1}$ and $x_{1}$ and the slope for $m$ then simplify.

## EXAMPLE: WRITE A LINEAR EQUATION IN THE FORM OF TO SOLVE A REAL-WORLD APPLICATION

There are two cell phone companies that offer different packages. Company A charges a monthly service fee of $\$ 34$ plus $\$ .05 / \mathrm{min}$ talk-time. Company B charges a monthly service fee of $\$ 40$ plus $\$ .04 / \mathrm{min}$ talktime.

1. Write a linear equation that models the packages offered by both companies.
2. If the average number of minutes used each month is 1,160 , which company offers the better plan?
3. If the average number of minutes used each month is 420 , which company offers the better plan?
4. How many minutes of talk-time would yield equal monthly statements from both companies?

## Answer

1. The model for Company $A$ can be written as $A=0.05 x+34$. This includes the variable cost of $0.05 x$ plus the monthly service charge of $\$ 34$. Company $B$ 's package charges a higher monthly fee of $\$ 40$, but a lower variable cost of $0.04 x$. Company $B$ 's model can be written as $B=0.04 x+40$.
2. If the average number of minutes used each month is 1,160 , we have the following:

$$
\begin{aligned}
\text { Company } A & =0.05(1,160)+34 \\
& =58+34 \\
& =92
\end{aligned}
$$

Company $B=0.04(1,160)+40$

$$
=46.4+40
$$

$$
=86.4
$$

So, Company $B$ offers the lower monthly cost of $\$ 86.40$ as compared with the $\$ 92$ monthly cost offered by Company $A$ when the average number of minutes used each month is 1,160 .
3 . If the average number of minutes used each month is 420 , we have the following:

$$
\begin{aligned}
\text { Company } A & =0.05(420)+34 \\
& =21+34 \\
& =55
\end{aligned}
$$

$$
\begin{aligned}
\text { Company } B & =0.04(420)+40 \\
& =16.8+40 \\
& =56.8
\end{aligned}
$$

If the average number of minutes used each month is 420 , then Company $A$ offers a lower monthly cost of $\$ 55$ compared to Company B's monthly cost of $\$ 56.80$.
4. To answer the question of how many talk-time minutes would yield the same bill from both companies, we should think about the problem in terms of $(x, y)$ coordinates: At what point are both the $x$-value and the $y$-value equal? We can find this point by setting the equations equal to each other and solving for $x$.
$0.05 x+34=0.04 x+40$
$0.01 x=6$
$x=600$

Check the $x$-value in each equation.

$$
\begin{aligned}
& 0.05(600)+34=64 \\
& 0.04(600)+40=64
\end{aligned}
$$

Therefore, a monthly average of 600 talk-time minutes renders the plans equal.

1.

Minutes


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## TRY IT

There are two cell phone companies that offer different packages. Company A charges a monthly service fee of $\$ 34$ plus $\$ 0.04 / \mathrm{min}$ talk-time. Company B charges a monthly service fee of $\$ 45$ plus $\$ 0.03 / \mathrm{min}$ talktime. Use $x$ for your variable.
a) Write an equation that models the monthly cost for company $A$.
b) Write an equation that models the monthly cost for company $B$.
C) If the average number of minutes used each month is 1,162 , how much is the monthly cost for each company?
Answer
a) $A=0.04 x+34$
b) $B=0.03 x+45$
c) Company A's cost would be $\$ 80.48$, and Company B's cost would be $\$ 79.86$

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## TRY IT

Find a linear equation to model this real-world application: It costs ABC electronics company $\$ 2.50$ per unit to produce a part used in a popular brand of desktop computers. The company has monthly operating expenses of $\$ 350$ for utilities and $\$ 3,300$ for salaries. What are the company's monthly expenses?

## Answer

$C=2.5 x+3,650$
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## EXAMPLE: WRITE A LINEAR EQUATION IN ONE VARIABLE TO MODEL AND SOLVE AN APPLICATION

A bag is filled with green and blue marbles. There are 77 marbles in the bag. If there are 17 more green marbles than blue marbles, find the number of green marbles and the number of blue marbles in the bag.
How many marbles of each color are in the bag?

## Answer

Let $G$ represent the number of green marbles in the bag and $B$ represent the number of blue marbles. We have that $G+B=77$. But we also have that there are 17 more green marbles than blue. That is, the number of green marbles is the same as the number of blue marbles plus 17 . We can translate that as $G=B+17$. Since we've found a way to express the variable $G$ in terms of $B$, we can write one equation in one variable.
$B+17+B=77$, which we can solve for $B$.
$2 B=60$, so $B=30$
There are 30 Blue marbles in the bag and $30+17=47$ is the number of green marbles.


## TRY IT

A cash register contains only five dollar and ten dollar bills. It contains twice as many five's as ten's and the total amount of money in the cash register is 560 dollars. How many ten's are in the cash register?

## Answer

There are 28 ten dollar bills in the cash register.
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## USING FORMULAS TO SOLVE PROBLEMS

## LEARNING OUTCOMES

- Set up a linear equation involving distance, rate, and time.
- Find the dimensions of a rectangle given the area.
- Find the dimensions of a box given information about its side lengths.

Many applications are solved using known formulas. The problem is stated, a formula is identified, the known quantities are substituted into the formula, the equation is solved for the unknown, and the problem's question is answered. Sometimes, these problems involve two equations representing two unknowns, which can be written using one equation in one variable by expressing one unknown in terms of the other. Examples of formulas include the perimeter of a rectangle, $P=2 L+2 W$; the area of a rectangular region, $A=L W$; and the volume of a rectangular solid, $V=L W H$.

## RECALL THE RELATIONSHIP BETWEEN DISTANCE, RATE AND TIME

The distance $d$ covered when traveling at a constant rate $r$ for some time $t$ is given by the formula $d=r t$.

## EXAMPLE: SOLVING AN APPLICATION USING A FORMULA

It takes Andrew 30 minutes to drive to work in the morning. He drives home using the same route, but it takes 10 minutes longer, and he averages $10 \mathrm{mi} / \mathrm{h}$ less than in the morning. How far does Andrew drive to work?
Answer
This is a distance problem, so we can use the formula $d=r t$, where distance equals rate multiplied by time. Note that when rate is given in mi/h, time must be expressed in hours. Consistent units of measurement are key to obtaining a correct solution.
First, we identify the known and unknown quantities. Andrew's morning drive to work takes 30 min , or $\frac{1}{2} \mathrm{~h}$ at rate $r$. His drive home takes 40 min , or $\frac{2}{3} \mathrm{~h}$, and his speed averages $10 \mathrm{mi} / \mathrm{h}$ less than the morning drive. Both trips cover distance $d$. A table, such as the one below, is often helpful for keeping track of information in these types of problems.

|  | $d$ | $r$ | $t$ |
| :--- | :--- | :--- | :--- |
| To Work | $d$ | $r$ | $\frac{1}{2}$ |
| To Home | $d$ | $r-10$ | $\frac{2}{3}$ |

Write two equations, one for each trip.

$$
\begin{array}{ll}
d=r\left(\frac{1}{2}\right) & \text { To work } \\
d=(r-10)\left(\frac{2}{3}\right) & \text { To home }
\end{array}
$$

As both equations equal the same distance, we set them equal to each other and solve for $r$.

$$
\begin{aligned}
& r\left(\frac{1}{2}\right)=(r-10)\left(\frac{2}{3}\right) \\
& \frac{1}{2} r=\frac{2}{3} r-\frac{20}{3} \\
& \frac{1}{2} r-\frac{2}{3} r=-\frac{20}{3} \\
& -\frac{1}{6} r=-\frac{20}{3} \\
& r=-\frac{20}{3}(-6) \\
& r=40
\end{aligned}
$$

We have solved for the rate of speed to work, 40 mph . Substituting 40 into the rate on the return trip yields $30 \mathrm{mi} / \mathrm{h}$. Now we can answer the question. Substitute the rate back into either equation and solve for $d$.

$$
\begin{aligned}
d & =40\left(\frac{1}{2}\right) \\
& =20
\end{aligned}
$$

The distance between home and work is 20 mi .

## Analysis of the Solution

Note that we could have cleared the fractions in the equation by multiplying both sides of the equation by the LCD to solve for $r$.
$r\left(\frac{1}{2}\right)=(r-10)\left(\frac{2}{3}\right)$
$6 \times r\left(\frac{1}{2}\right)=6 \times(r-10)\left(\frac{2}{3}\right)$
$3 r=4(r-10)$
$3 r=4 r-40$
$-r=-40$
$r=40$

## HOW WERE THE FRACTIONS HANDLED IN THE EXAMPLE ABOVE?

Recall that when solving multi-step equations, it is helpful to multiply by the LCD to clear the denominators from the equation.
But it is also permissible to use operations on fractions to combine like terms. The example above demonstrates both.

## Answer

- In the text of the solution when solving for $r$ the first time, the parentheses were eliminated using the distributive property, then operations on fractions were used to combine like terms.
$r\left(\frac{1}{2}\right)=(r-10)\left(\frac{2}{3}\right)$
$\frac{1}{2} r=\frac{2}{3} r-\frac{20}{3}$
$\frac{1}{2} r-\frac{2}{3} r=-\frac{20}{3}$
$-\frac{1}{6} r=-\frac{20}{3}$
$r=-\frac{20}{3}(-6)$
$r=40$
- Later, in the analysis of the solution, the LCD between the denominators 2 and 3 was multiplied on both sides of the equation to cancel out the denominators so that operations on fractions were not necessary. Do you see how the LCD wasn't actually multiplied through on both sides, but that the denominators cancelled out, resulting in a linear equation in one variable without denominators?
$r\left(\frac{1}{2}\right)=(r-10)\left(\frac{2}{3}\right)$
$6 \times r\left(\frac{1}{2}\right)=6 \times(r-10)\left(\frac{2}{3}\right)$
$3 r=4(r-10)$
$3 r=4 r-40$
$-r=-40$
$r=40$
Both methods are equally correct unless your instructor requires you to specifically demonstrate knowledge of one or the other.
Which do you prefer?


## How To: Solve Multi-Step Equations

1. (Optional) Multiply to clear any fractions or decimals.
2. Simplify each side by clearing parentheses and combining like terms.
3. Add or subtract to isolate the variable term-you may have to move a term with the variable.
4. Multiply or divide to isolate the variable.
5. Check the solution.

## TRY IT

On Saturday morning, it took Jennifer 3.6 hours to drive to her mother's house for the weekend. On Sunday evening, due to heavy traffic, it took Jennifer 4 hours to return home. Her speed was $5 \mathrm{mi} / \mathrm{h}$ slower on Sunday than on Saturday. What was her speed on Sunday?

## Answer

$45 \frac{\mathrm{mi}}{\mathrm{h}}$
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## EXAMPLE: SOLVING A PERIMETER PROBLEM

The perimeter of a rectangular outdoor patio is 54 ft . The length is 3 ft . greater than the width. What are the dimensions of the patio?

## Answer

The perimeter formula is standard: $P=2 L+2 W$. We have two unknown quantities, length and width. However, we can write the length in terms of the width as $L=W+3$. Substitute the perimeter value and the expression for length into the formula. It is often helpful to make a sketch and label the sides as shown below.


Now we can solve for the width and then calculate the length.

$$
\begin{aligned}
& P=2 L+2 W \\
& 54=2(W+3)+2 W \\
& 54=2 W+6+2 W \\
& 54=4 W+6 \\
& 48=4 W \\
& 12=W \\
& (12+3)=L \\
& 15=L
\end{aligned}
$$

The dimensions are $L=15 \mathrm{ft}$ and $W=12 \mathrm{ft}$.

In the example above, the length was written in terms of the width in order to substitute it for the variable $L$ to be able to write one equation in one variable.
This is a powerful technique in algebra and should be practiced until it becomes familiar, so it's worth taking a closer look.

## Answer

We had $P=2 L+2 W$, which is one equation in two variables. Given the value for $P$, we don't have enough information to solve for either $L$ or $W$.
But we do know that $L=W+3$. We can use this to substitute for the $L$ in the original equation. Be careful to retain the original statement exactly, substituting only the value for $L$. A handy way to handle the substitution is with a set of parentheses.
$P=2 L+2 W$
$54=2(L)+2 W \quad$ wrap the $L$ to see where to make the substitution
$54=2(W+3)+2 W \quad$ drop the expression for $L$ into the parentheses
$54=2 W+6+2 W$
$54=4 W+6$

Now we have one equation in one variable. We can use the properties of equality to isolate the variable $W$ on one side, with its numerical value on the other.

## TRY IT

Find the dimensions of a rectangle given that the perimeter is 110 cm . and the length is 1 cm . more than twice the width.

## Answer

$\mathrm{L}=37 \mathrm{~cm}, \mathrm{~W}=18 \mathrm{~cm}$
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## EXAMPLE: SOLVING AN AREA PROBLEM

The perimeter of a tablet of graph paper is $48 \mathrm{in}^{2}$. The length is 6 in . more than the width. Find the area of the graph paper.
Answer
The standard formula for area is $A=L W$; however, we will solve the problem using the perimeter formula. The reason we use the perimeter formula is because we know enough information about the perimeter that the formula will allow us to solve for one of the unknowns. As both perimeter and area use length and width as dimensions, they are often used together to solve a problem such as this one.
We know that the length is 6 in . more than the width, so we can write length as $L=W+6$. Substitute the value of the perimeter and the expression for length into the perimeter formula and find the length.

$$
\begin{aligned}
& P=2 L+2 W \\
& 48=2(W+6)+2 W \\
& 48=2 W+12+2 W \\
& 48=4 W+12 \\
& 36=4 W \\
& 9=W \\
& (9+6)=L \\
& 15=L
\end{aligned}
$$

Now, we find the area given the dimensions of $L=15 \mathrm{in}$. and $W=9 \mathrm{in}$.

$$
\begin{aligned}
A & =L W \\
A & =15(9) \\
& =135 \mathrm{in}^{2}
\end{aligned}
$$

The area is $135 \mathrm{in}^{2}$.

## TRY IT

A game room has a perimeter of 70 ft . The length is five more than twice the width. How many $\mathrm{ft}^{2}$ of new carpeting should be ordered?
Answer
$250 \mathrm{ft}^{2}$
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## EXAMPLE: SOLVING A VOLUME PROBLEM

Find the dimensions of a shipping box given that the length is twice the width, the height is 8 inches, and the volume is $1,600 \mathrm{in}{ }^{3}$.
Answer
The formula for the volume of a box is given as $V=L W H$, the product of length, width, and height. We are given that $L=2 W$, and $H=8$. The volume is 1,600 cubic inches.
$V=L W H$
$1,600=(2 W) W(8)$
$1,600=16 W^{2}$
$100=W^{2}$
$10=W$
The dimensions are $L=20 \mathrm{in}$., $W=10$ in., and $H=8 \mathrm{in}$.

## Analysis of the Solution

Note that the square root of $W^{2}$ would result in a positive and a negative value. However, because we are describing width, we can use only the positive result.

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## SUMMARY: MODELS AND APPLICATIONS

## Key Concepts

- A linear equation can be used to solve for an unknown in a number problem.
- Applications can be written as mathematical problems by identifying known quantities and assigning a variable to unknown quantities.
- There are many known formulas that can be used to solve applications. Distance problems, for example, are solved using the $d=r t$ formula.
- Many geometry problems are solved using the perimeter formula $P=2 L+2 W$, the area formula $A=L W$, or the volume formula $V=L W H$.


## Glossary

area in square units, the area formula used in this section is used to find the area of any two-dimensional rectangular region: $A=L W$
perimeter in linear units, the perimeter formula is used to find the linear measurement, or outside length and width, around a two-dimensional regular object; for a rectangle: $P=2 L+2 W$
volume in cubic units, the volume measurement includes length, width, and depth: $V=L W H$

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## INTRODUCTION TO QUADRATIC EQUATIONS

## Learning Outcomes

By the end of this section, you will be able to:

- Factor a quadratic equation to solve it.
- Use the square root property to solve a quadratic equation.
- Use the Pythagorean Theorem and the square root property to find the unknown length of the side of a right triangle.
- Complete the square to solve a quadratic equation.
- Use the quadratic formula to solve a quadratic equation.
- Use the discriminant to determine the number and type of solutions to a quadratic equation.


The left computer monitor in the image above is a 23.6 -inch model and the one on the right is a 27 -inch model. Proportionally, the monitors appear very similar. If there is a limited amount of space and we desire the largest monitor possible, how do we decide which one to choose? In this section, we will learn how to solve problems such as this using four different methods.

## FACTORING AND THE SQUARE ROOT PROPERTY

## LEARNING OUTCOMES

- Factor a quadratic equation to solve it.
- Use the square root property to solve a quadratic equation.
- Use the Pythagorean Theorem and the square root property to find the unknown length of the side of a right triangle.


## RECALL: IDENTIFYING POLYNOMIALS

Recall that a polynomial is an expression containing variable terms joined together by addition or subtraction. We can identify the degree of the polynomial by finding the term having the highest power (largest exponent on its variable). That term will be the leading term of the polynomial and its degree is the degree of the polynomial.

An equation containing a second-degree polynomial is called a quadratic equation. For example, equations such as $2 x^{2}+3 x-1=0$ and $x^{2}-4=0$ are quadratic equations. They are used in countless ways in the fields of engineering, architecture, finance, biological science, and, of course, mathematics.

Often the easiest method of solving a quadratic equation is factoring. Factoring means finding expressions that can be multiplied together to give the expression on one side of the equation.

If a quadratic equation can be factored, it is written as a product of linear terms. Solving by factoring depends on the zero-product property which states that if $a \cdot b=0$, then $a=0$ or $b=0$, where $a$ and $b$ are real numbers or algebraic expressions. In other words, if the product of two numbers or two expressions equals zero, then one of the numbers or one of the expressions must equal zero because zero multiplied by anything equals zero.

Multiplying the factors expands the equation to a string of terms separated by plus or minus signs. So, in that sense, the operation of multiplication undoes the operation of factoring. For example, expand the factored expression $(x-2)(x+3)$ by multiplying the two factors together.

$$
\begin{aligned}
(x-2)(x+3) & =x^{2}+3 x-2 x-6 \\
& =x^{2}+x-6
\end{aligned}
$$

The product is a quadratic expression. Set equal to zero, $x^{2}+x-6=0$ is a quadratic equation. If we were to factor the equation, we would get back the factors we multiplied.

The process of factoring a quadratic equation depends on the leading coefficient, whether it is 1 or another integer. We will look at both situations; but first, we want to confirm that the equation is written in standard form, $a x^{2}+b x+c=0$, where $a, b$, and $c$ are real numbers and $a \neq 0$. The equation $x^{2}+x-6=0$ is in standard form.

## RECALL THE GREATEST COMMON FACTOR

The greatest common factor (GCF) is the largest number or variable expression that can divided evenly into two or more numbers or expressions.
To find the GCF of an expression containing variable terms, first find the GCF of the coefficients, then find the GCF of the variables. The GCF of the variables will be the smallest degree of each variable that appears in each term.
For example, $4 x$ is the GCF of $16 x$ and $20 x^{2}$ because it is the largest expression that divides evenly into both $16 x$ and $20 x^{2}$.
When factoring, remember to look first for a greatest common factor. If you can find one to factor out, it will usually make your factoring task less challenging.

We can use the zero-product property to solve quadratic equations in which we first have to factor out the greatest common factor (GCF), and for equations that have special factoring formulas as well, such as the difference of squares, both of which we will see later in this section.

## A GENERAL NOTE: THE ZERO-PRODUCT PROPERTY AND QUADRATIC EQUATIONS

## The zero-product property states

$$
\text { If } a \cdot b=0, \text { then } a=0 \text { or } b=0,
$$

where $a$ and $b$ are real numbers or algebraic expressions.
A quadratic equation is an equation containing a second-degree polynomial; for example

$$
a x^{2}+b x+c=0
$$

where $a, b$, and $c$ are real numbers, and $a \neq 0$. It is in standard form.

## Solving Quadratics with a Leading Coefficient of 1

In the quadratic equation $x^{2}+x-6=0$, the leading coefficient, or the coefficient of $x^{2}$, is 1 . We have one method of factoring quadratic equations in this form.

## HOW TO: GIVEN A QUADRATIC EQUATION WITH THE LEADING COEFFICIENT OF 1, FACTOR IT

1. Find two numbers whose product equals $c$ and whose sum equals $b$.
2. Use those numbers to write two factors of the form $(x+k)$ or $(x-k)$, where $k$ is one of the numbers found in step 1 . Use the numbers exactly as they are. In other words, if the two numbers are 1 and -2 , the factors are $(x+1)(x-2)$.
3. Solve using the zero-product property by setting each factor equal to zero and solving for the variable.

## Using the Square Root Property

When there is no linear term in the equation, another method of solving a quadratic equation is by using the square root property, in which we isolate the $x^{2}$ term and take the square root of the number on the other side of the equal sign. Keep in mind that sometimes we may have to manipulate the equation to isolate the

## EXAMPLE: FACTORING AND SOLVING A QUADRATIC WITH LEADING COEFFICIENT OF 1

Factor and solve the equation: $x^{2}+x-6=0$.

## Answer

To factor $x^{2}+x-6=0$, we look for two numbers whose product equals -6 and whose sum equals 1 . Begin by looking at the possible factors of -6 .
$1 \cdot(-6)$
$(-6) \cdot 1$
$2 \cdot(-3)$
$3 \cdot(-2)$
The last pair, $3 \cdot(-2)$ sums to 1 , so these are the numbers. Note that only one pair of numbers will work. Then, write the factors.

$$
(x-2)(x+3)=0
$$

To solve this equation, we use the zero-product property. Set each factor equal to zero and solve.

$$
\begin{aligned}
& (x-2)(x+3)=0 \\
& (x-2)=0 \\
& x=2 \\
& (x+3)=0 \\
& x=-3
\end{aligned}
$$

The two solutions are $x=2$ and $x=-3$. We can see how the solutions relate to the graph below. The solutions are the $x$-intercepts of the graph of $x^{2}+x-6$.


## TRY IT

Factor and solve the quadratic equation: $x^{2}-5 x-6=0$.

## Answer

$(x-6)(x+1)=0 ; x=6, x=-1$
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$x^{2}$ term so that the square root property can be used.

## A GENERAL NOTE: THE SQUARE ROOT PROPERTY

With the $x^{2}$ term isolated, the square root propty states that:

$$
\text { if } x^{2}=k \text {, then } x= \pm \sqrt{k}
$$

where $k$ is a nonzero real number.

## RECALL: PRINCIPAL SQUARE ROOT

Recall that the princicpal square root of a number such as $\sqrt{9}$ is the non-negative root, 3 .
And note that there is a difference between $\sqrt{9}$ and $x^{2}=9$. In the case of $x^{2}=9$, we seek all numbers whose square is 9 , that is $x= \pm 3$.

## HOW TO: GIVEN A QUADRATIC EQUATION WITH AN $x^{2}$ TERM BUT NO $x$ TERM, USE THE SQUARE ROOT PROPERTY TO SOLVE IT

1. Isolate the $x^{2}$ term on one side of the equal sign.
2. Take the square root of both sides of the equation, putting a $\pm$ sign before the expression on the side opposite the squared term.
3. Simplify the numbers on the side with the $\pm$ sign.

## EXAMPLE: SOLVING A SIMPLE QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY

Solve the quadratic using the square root property: $x^{2}=8$.

## Answer

Take the square root of both sides, and then simplify the radical. Remember to use $\mathrm{a} \pm$ sign before the radical symbol.

$$
\begin{aligned}
x^{2} & =8 \\
x & = \pm \sqrt{8} \\
& = \pm 2 \sqrt{2}
\end{aligned}
$$

The solutions are $x=2 \sqrt{2}, x=-2 \sqrt{2}$.

## EXAMPLE: SOLVING A QUADRATIC EQUATION USING THE SQUARE ROOT PROPERTY

Solve the quadratic equation: $4 x^{2}+1=7$

## Answer

First, isolate the $x^{2}$ term. Then take the square root of both sides.
$4 x^{2}+1=7$
$4 x^{2}=6$
$x^{2}=\frac{6}{4}$
$x= \pm \frac{\sqrt{6}}{2}$
The solutions are $x=\frac{\sqrt{6}}{2}, x=-\frac{\sqrt{6}}{2}$.

## TRY IT

Solve the quadratic equation using the square root property: $3(x-4)^{2}=15$.
Answer
$x=4 \pm \sqrt{5}$

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One of the most famous formulas in mathematics is the Pythagorean Theorem. It is based on a right triangle and states the relationship among the lengths of the sides as $a^{2}+b^{2}=c^{2}$, where $a$ and $b$ refer to the legs of a right triangle adjacent to the $90^{\circ}$ angle, and $c$ refers to the hypotenuse. It has immeasurable uses in architecture, engineering, the sciences, geometry, trigonometry, and algebra, and in everyday applications.

We use the Pythagorean Theorem to solve for the length of one side of a triangle when we have the lengths of the other two. Because each of the terms is squared in the theorem, when we are solving for a side of a triangle, we have a quadratic equation. We can use the methods for solving quadratic equations that we learned in this section to solve for the missing side.

The Pythagorean Theorem is given as
$a^{2}+b^{2}=c^{2}$
where $a$ and $b$ refer to the legs of a right triangle adjacent to the $90^{\circ}$ angle, and $c$ refers to the hypotenuse.


## EXAMPLE: FINDING THE LENGTH OF THE MISSING SIDE OF A RIGHT TRIANGLE

Find the length of the missing side of the right triangle.


## Answer

As we have measurements for side $b$ and the hypotenuse, the missing side is $a$.

$$
\begin{aligned}
& a^{2}+b^{2}=c^{2} \\
& a^{2}+(4)^{2}=(12)^{2} \\
& a^{2}+16=144 \\
& a^{2}=128 \\
& a=\sqrt{128} \\
& a=8 \sqrt{2}
\end{aligned}
$$

Use the Pythagorean Theorem to solve the right triangle problem: Leg a measures 4 units, leg $b$ measures 3 units. Find the length of the hypotenuse.
Answer
5 units
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## COMPLETING THE SQUARE AND THE QUADRATIC FORMULA

## LEARNING OUTCOMES

- Complete the square to solve a quadratic equation.
- Use the quadratic formula to solve a quadratic equation.
- Use the discriminant to determine the number and type of solutions to a quadratic equation.

Not all quadratic equations can be factored or can be solved in their original form using the square root property. In these cases, we may use a method for solving a quadratic equation known as completing the square. Using this method, we add or subtract terms to both sides of the equation until we have a perfect square trinomial on one side of the equal sign. We then apply the square root property. To complete the square, the leading coefficient, $a$, must equal 1 . If it does not, then divide the entire equation by $a$. Then, we can use the following procedures to solve a quadratic equation by completing the square.

We will use the example $x^{2}+4 x+1=0$ to illustrate each step.

1. Given a quadratic equation that cannot be factored and with $a=1$, first add or subtract the constant term to the right sign of the equal sign.

$$
x^{2}+4 x=-1
$$

2. Multiply the $b$ term by $\frac{1}{2}$ and square it.

$$
\begin{aligned}
& \frac{1}{2}(4)=2 \\
& 2^{2}=4
\end{aligned}
$$

3. Add $\left(\frac{1}{2} b\right)^{2}$ to both sides of the equal sign and simplify the right side. We have

$$
x^{2}+4 x+4=-1+4
$$

$$
x^{2}+4 x+4=3
$$

4. The left side of the equation can now be factored as a perfect square.

$$
\begin{aligned}
& x^{2}+4 x+4=3 \\
& (x+2)^{2}=3
\end{aligned}
$$

5. Use the square root property and solve.

$$
\begin{aligned}
& \sqrt{(x+2)^{2}}= \pm \sqrt{3} \\
& x+2= \pm \sqrt{3} \\
& x=-2 \pm \sqrt{3}
\end{aligned}
$$

6. The solutions are $x=-2+\sqrt{3}, x=-2-\sqrt{3}$.

## PROPERTIES OF EQUALITY AND TAKING THE SQUARE ROOT OF BOTH SIDES

Remember that we are permitted, by the properties of equality, to add, subtract, multiply, or divide the same amount to both sides of an equation. Doing so won't change the value of the equation but it will enable us to isolate the variable on one side (that is, to solve the equation for the variable).
The square root property gives us another operation we can do to both sides of an equation, taking the square root. We just have to remember when taking the square root (or any even root, as we'll see later), to consider both the positive and negative possibilities of the constant.
The square root property
If $\sqrt{x}=k$
Then $x= \pm k$

## Using the Quadratic Formula

The fourth method of solving a quadratic equation is by using the quadratic formula, a formula that will solve all quadratic equations. Although the quadratic formula works on any quadratic equation in standard form, it is easy to make errors in substituting the values into the formula. Pay close attention when substituting, and use parentheses when inserting a negative number.

We can derive the quadratic formula by completing the square. We will assume that the leading coefficient is positive; if it is negative, we can multiply the equation by -1 and obtain a positive a. Given $a x^{2}+b x+c=0, a \neq 0$, we will complete the square as follows:

1. First, move the constant term to the right side of the equal sign:

$$
a x^{2}+b x=-c
$$

2. As we want the leading coefficient to equal 1 , divide through by a:

$$
x^{2}+\frac{b}{a} x=-\frac{c}{a}
$$

3. Then, find $\frac{1}{2}$ of the middle term, and add $\left(\frac{1}{2} \frac{b}{a}\right)^{2}=\frac{b^{2}}{4 a^{2}}$ to both sides of the equal sign:

$$
x^{2}+\frac{b}{a} x+\frac{b^{2}}{4 a^{2}}=\frac{b^{2}}{4 a^{2}}-\frac{c}{a}
$$

4. Next, write the left side as a perfect square. Find the common denominator of the right side and write it as a single fraction:

$$
\left(x+\frac{b}{2 a}\right)^{2}=\frac{b^{2}-4 a c}{4 a^{2}}
$$

5. Now, use the square root property, which gives

## EXAMPLE: SOLVING A QUADRATIC BY COMPLETING THE SQUARE

Solve the quadratic equation by completing the square: $x^{2}-3 x-5=0$.
Answer
First, move the constant term to the right side of the equal sign by adding 5 to both sides of the equation.

$$
x^{2}-3 x=5
$$

Then, take $\frac{1}{2}$ of the $b$ term and square it.

$$
\begin{aligned}
& \frac{1}{2}(-3)=-\frac{3}{2} \\
& \left(-\frac{3}{2}\right)^{2}=\frac{9}{4}
\end{aligned}
$$

Add the result to both sides of the equal sign.

$$
\begin{aligned}
& x^{2}-3 x+\left(-\frac{3}{2}\right)^{2}=5+\left(-\frac{3}{2}\right)^{2} \\
& x^{2}-3 x+\frac{9}{4}=5+\frac{9}{4}
\end{aligned}
$$

Factor the left side as a perfect square and simplify the right side.

$$
\left(x-\frac{3}{2}\right)^{2}=\frac{29}{4}
$$

Use the square root property and solve.
$\sqrt{\left(x-\frac{3}{2}\right)^{2}}= \pm \sqrt{\frac{29}{4}}$
$\left(x-\frac{3}{2}\right)= \pm \frac{\sqrt{29}}{2}$
$x=\frac{3}{2} \pm \frac{\sqrt{29}}{2}$
The solutions are $x=\frac{3}{2}+\frac{\sqrt{29}}{2}, x=\frac{3}{2}-\frac{\sqrt{29}}{2}$.

## TRY IT

Solve by completing the square: $x^{2}-6 x=13$.

## Answer

$x=3 \pm \sqrt{22}$
Note that when solving a quadratic by completing the square, a negative value will sometimes arise under the square root symbol. Later, we'll see that this value can be represented by a complex number (as shown in the video help for the problem below). We may also treat this type of solution as unreal, stating that no real solutions exist for this equation, by writing DNE. We will study complex numbers more thoroughly in a later module.

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$$
x+\frac{b}{2 a}= \pm \sqrt{\frac{b^{2}-4 a c}{4 a^{2}}}
$$

$x+\frac{b}{2 a}=\frac{ \pm \sqrt{b^{2}-4 a c}}{2 a}$
6. Finally, add $-\frac{b}{2 a}$ to both sides of the equation and combine the terms on the right side. Thus,

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

## A GENERAL NOTE: THE QUADRATIC FORMULA

Written in standard form, $a x^{2}+b x+c=0$, any quadratic equation can be solved using the quadratic formula:

$$
x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}
$$

where $a, b$, and $c$ are real numbers and $a \neq 0$.

## HOW TO: GIVEN A QUADRATIC EQUATION, SOLVE IT USING THE QUADRATIC FORMULA

1. Make sure the equation is in standard form: $a x^{2}+b x+c=0$.
2. Make note of the values of the coefficients and constant term, $a, b$, and $c$.
3. Carefully substitute the values noted in step 2 into the equation. To avoid needless errors, use parentheses around each number input into the formula.
4. Calculate and solve.

## EXAMPLE : SOLVE A QUADRATIC EQUATION USING THE QUADRATIC FORMULA

Solve the quadratic equation: $x^{2}+5 x+1=0$.
Answer
Identify the coefficients: $a=1, b=5, c=1$. Then use the quadratic formula.
$x=\frac{-(5) \pm \sqrt{(5)^{2}-4(1)(1)}}{2(1)}$
$=\frac{-5 \pm \sqrt{25-4}}{2}$
$=\frac{-5 \pm \sqrt{21}}{2}$

## RECALL ADDING AND SUBTRACTING FRACTIONS

The form we have used to recall adding and subtracting fractions can help us write the solutions to quadratic equations.
$\frac{a}{b} \pm \frac{c}{d}=\frac{a d \pm b c}{b d}$
Ex. The solutions
$x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$
can also be written as two separate fractions
$x=\frac{-b}{2 a} \pm \frac{\sqrt{b^{2}-4 a c}}{2 a}$.

## EXAMPLE: SOLVING A QUADRATIC EQUATION WITH THE QUADRATIC FORMULA

Use the quadratic formula to solve $x^{2}+x+2=0$.

## Answer

First, we identify the coefficients: $a=1, b=1$, and $c=2$.
Substitute these values into the quadratic formula.

$$
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-(1) \pm \sqrt{(1)^{2}-(4) \cdot(1) \cdot(2)}}{2 \cdot 1} \\
& =\frac{-1 \pm \sqrt{1-8}}{2} \\
& =\frac{-1 \pm \sqrt{-7}}{2}
\end{aligned}
$$

The solutions to the equation are unreal because the square root of a negative number does not exist in the real numbers. If asked to find all real solutions, we would indicate that they do not exist by writing DNE.
We may also express unreal solutions as complex numbers by writing
$x=\frac{-1 \pm i \sqrt{7}}{2}$
or
$x=-\frac{1}{2}+\frac{\sqrt{7}}{2} i$ and $x=-\frac{1}{2}-\frac{\sqrt{7}}{2} i$, where $\sqrt{-1}$ is expressed as $i$.

## TRY IT

Solve the quadratic equation using the quadratic formula: $9 x^{2}+3 x-2=0$.

## Answer

$x=-\frac{2}{3}, x=\frac{1}{3}$
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## The Discriminant

The quadratic formula not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions when we consider the discriminant, or the expression under the radical, $b^{2}-4 a c$. The discriminant tells us whether the solutions are real numbers or complex numbers as well as how many solutions of each type to expect. The table below relates the value of the discriminant to the solutions of a quadratic equation.

## Value of Discriminant

$b^{2}-4 a c=0$
$b^{2}-4 a c>0$, perfect square
$b^{2}-4 a c>0$, not a perfect square
$b^{2}-4 a c<0$

## Results

One rational solution (double solution)
Two rational solutions
Two irrational solutions
Two complex solutions

## RECALL RATIONAL AND IRRATIONAL NUMBERS ARE REAL NUMBERS

- Rational numbers: A rational number is a number that can be expressed as a ratio of integers (a fraction with an integer numerator and a positive, non-zero integer denominator). The square root of a perfect square, such as $\sqrt{25}$ is rational because it can be expressed as $\frac{5}{1}=5$. Solutions containing only rational numbers are themselves rational solutions.
- Irrational numbers: All the real numbers that are not rational are called irrational numbers. These numbers cannot be expressed as a fraction of integers. As quadratic solutions, irrational numbers arise in the form of a square root of a numbers that is not a perfect square. For example, $\sqrt{6}$ cannot be expressed as a ratio of two integers. $\sqrt{6}$ is an irrational number. Solutions containing an irrational number are themselves irrational solutions.
- Both rational and irrational numbers are real numbers. Quadratic solutions that are either rational or irrational are real solutions.
- Quadratic solutions that are unreal (they have a negative under the square root symbol, the radical symbol) are called complex solutions.


## A GENERAL NOTE: THE DISCRIMINANT

For $a x^{2}+b x+c=0$, where $a, b$, and $c$ are real numbers, the discriminant is the expression under the radical in the quadratic formula: $b^{2}-4 a c$. It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

## EXAMPLE: USING THE DISCRIMINANT TO FIND THE NATURE OF THE SOLUTIONS TO A QUADRATIC EQUATION

Use the discriminant to find the nature of the solutions to the following quadratic equations:

1. $x^{2}+4 x+4=0$
2. $8 x^{2}+14 x+3=0$
3. $3 x^{2}-5 x-2=0$
4. $3 x^{2}-10 x+15=0$

## Answer

Calculate the discriminant $b^{2}-4 a c$ for each equation and state the expected type of solutions.

1. $x^{2}+4 x+4=0: b^{2}-4 a c=(4)^{2}-4(1)(4)=0$. There will be one rational double solution.
2. $8 x^{2}+14 x+3=0: b^{2}-4 a c=(14)^{2}-4(8)(3)=100$. As 100 is a perfect square, there will be two rational solutions.
3. $3 x^{2}-5 x-2=0: b^{2}-4 a c=(-5)^{2}-4(3)(-2)=49$. As 49 is a perfect square, there will be two rational solutions.
4. $3 x^{2}-10 x+15=0: b^{2}-4 a c=(-10)^{2}-4(3)(15)=-80$. There will be two complex solutions.

## TRY IT

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## SUMMARY: QUADRATIC EQUATIONS

## Key Concepts

- Many quadratic equations can be solved by factoring when the equation has a leading coefficient of 1 or if the equation is a difference of squares. The zero-factor property is then used to find solutions.
- Many quadratic equations with a leading coefficient other than 1 can be solved by factoring using the grouping method.
- Another method for solving quadratics is the square root property. The variable is squared. We isolate the squared term and take the square root of both sides of the equation. The solution will yield a positive and negative solution.
- Completing the square is a method of solving quadratic equations when the equation cannot be factored.
- A highly dependable method for solving quadratic equations is the quadratic formula based on the coefficients and the constant term in the equation.
- The discriminant is used to indicate the nature of the solutions that the quadratic equation will yield: real or complex, rational or irrational, and how many of each.
- The Pythagorean Theorem, among the most famous theorems in history, is used to solve right-triangle problems and has applications in numerous fields. Solving for the length of one side of a right triangle requires solving a quadratic equation.


## Glossary

completing the square a process for solving quadratic equations in which terms are added to or subtracted from both sides of the equation in order to make one side a perfect square
discriminant the expression under the radical in the quadratic formula that indicates the nature of the solutions, real or complex, rational or irrational, single or double roots.

Pythagorean Theorem a theorem that states the relationship among the lengths of the sides of a right triangle, used to solve right triangle problems
quadratic equation an equation containing a second-degree polynomial; can be solved using multiple methods
quadratic formula a formula that will solve all quadratic equations
square root property one of the methods used to solve a quadratic equation in which the $x^{2}$ term is isolated so that the square root of both sides of the equation can be taken to solve for $x$
zero-product property the property that formally states that multiplication by zero is zero so that each factor of a quadratic equation can be set equal to zero to solve equations

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# INTRODUCTION TO EQUATION-SOLVING TECHNIQUES 

## Learning Outcomes

By the end of this section, you will be able to:

- Solve a radical equation, identify extraneous solution.
- Solve an equation with rational exponents.
- Solve polynomial equations.
- Solve absolute value equations.

We have solved linear equations, rational equations, and quadratic equations using several methods. However, there are many other types of equations, and we will investigate a few more types in this section. We will look at equations involving rational exponents, polynomial equations, radical equations, absolute value equations, equations in quadratic form, and some rational equations that can be transformed into quadratics. Solving any equation, however, employs the same basic algebraic rules. We will learn some new techniques as they apply to certain equations, but the algebra never changes.

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## EQUATIONS WITH RADICALS AND RATIONAL EXPONENTS

## LEARNING OUTCOMES

- Solve a radical equation, identify extraneous solution.
- Solve an equation with rational exponents.

Radical equations are equations that contain variables in the radicand (the expression under a radical symbol), such as

$$
\begin{aligned}
& \sqrt{3 x+18}=x \\
& \sqrt{x+3}=x-3 \\
& \sqrt{x+5}-\sqrt{x-3}=2
\end{aligned}
$$

Radical equations may have one or more radical terms and are solved by eliminating each radical, one at a time. We have to be careful when solving radical equations as it is not unusual to find extraneous solutions, roots that are not, in fact, solutions to the equation. These solutions are not due to a mistake in the solving method, but result from the process of raising both sides of an equation to a power. Checking each answer in the original equation will confirm the true solutions.

## A GENERAL NOTE: RADICAL EQUATIONS

An equation containing terms with a variable in the radicand is called a radical equation.

## HOW TO: GIVEN A RADICAL EQUATION, SOLVE IT

1. Isolate the radical expression on one side of the equal sign. Put all remaining terms on the other side.
2. If the radical is a square root, then square both sides of the equation. If it is a cube root, then raise both sides of the equation to the third power. In other words, for an $n$th root radical, raise both sides to the $n$th power. Doing so eliminates the radical symbol.
3 . Solve the resulting equation.
3. If a radical term still remains, repeat steps 1-2.
4. Check solutions by substituting them into the original equation.

## RECALL MULTIPLYING POLYNOMIAL EXPRESSIONS

When squaring (or raising to any power) both sides of an equation as in step (2) above, don't forget to apply the properties of exponents carefully and distribute all the terms appropriately.
$(x+3)^{2} \neq x^{2}+9$
$(x+3)^{2}=(x+3)(x+3)=x^{2}+6 x+9$
The special form for perfect square trinomials comes in handy when solving radical equations.
$(a+b)^{2}=a^{2}+2 a b+b^{2}$
$(a-b)^{2}=a^{2}-2 a b+b^{2}$
This enables us to square binomials containing radicals by following the form.

$$
\begin{aligned}
(x-\sqrt{3 x-7})^{2} & =x^{2}-2 \sqrt{3 x-7}+(\sqrt{3 x-7})^{2} \\
& =x^{2}-2 \sqrt{3 x-7}+3 x-7
\end{aligned}
$$

Rational exponents are exponents that are fractions, where the numerator is a power and the denominator is a root. For example, $16^{\frac{1}{2}}$ is another way of writing $\sqrt{16}$ and $8^{\frac{2}{3}}$ is another way of writing $(\sqrt[3]{8})^{2}$.

We can solve equations in which a variable is raised to a rational exponent by raising both sides of the equation to the reciprocal of the exponent. The reason we raise the equation to the reciprocal of the exponent is because we want to eliminate the exponent on the variable term, and a number multiplied by its reciprocal equals 1 . For example, $\frac{2}{3}\left(\frac{3}{2}\right)=1$.

## EXAMPLE: SOLVING AN EQUATION WITH ONE RADICAL

Solve $\sqrt{15-2 x}=x$.
Answer
The radical is already isolated on the left side of the equal sign, so proceed to square both sides.
$\sqrt{15-2 x}=x$
$(\sqrt{15-2 x})^{2}=(x)^{2}$
$15-2 x=x^{2}$
We see that the remaining equation is a quadratic. Set it equal to zero and solve.

$$
\begin{aligned}
& 0=x^{2}+2 x-15 \\
& 0=(x+5)(x-3) \\
& x=-5 \\
& x=3
\end{aligned}
$$

The proposed solutions are $x=-5$ and $x=3$. Let us check each solution back in the original equation.
First, check $x=-5$.
$\sqrt{15-2 x}=x$
$\sqrt{15-2(-5)}=-5$
$\sqrt{25}=-5$
$5 \neq-5$
This is an extraneous solution. While no mistake was made solving the equation, we found a solution that does not satisfy the original equation.
Check $x=3$.
$\sqrt{15-2 x}=x$
$\sqrt{15-2(3)}=3$
$\sqrt{9}=3$
$3=3$
The solution is $x=3$.

## TRY IT

Solve the radical equation: $\sqrt{x+3}=3 x-1$
Answer
$x=1$; extraneous solution $x=-\frac{2}{9}$
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## EXAMPLE: SOLVING A RADICAL EQUATION CONTAINING TWO RADICALS

Solve $\sqrt{2 x+3}+\sqrt{x-2}=4$.
Answer
As this equation contains two radicals, we isolate one radical, eliminate it, and then isolate the second radical.

$$
\begin{array}{ll}
\sqrt{2 x+3}+\sqrt{x-2}=4 & \\
\sqrt{2 x+3}=4-\sqrt{x-2} & \text { Subtract } \sqrt{x-2} \text { from both sides. } \\
(\sqrt{2 x+3})^{2}=(4-\sqrt{x-2})^{2} & \text { Square both sides. }
\end{array}
$$

Use the perfect square formula to expand the right side: $(a-b)^{2}=a^{2}-2 a b+b^{2}$.

$$
\begin{array}{ll}
2 x+3=(4)^{2}-2(4) \sqrt{x-2}+(\sqrt{x-2})^{2} & \\
2 x+3=16-8 \sqrt{x-2}+(x-2) & \\
2 x+3=14+x-8 \sqrt{x-2} & \text { Combine like terms. } \\
x-11=-8 \sqrt{x-2} & \text { Isolate the second radical. } \\
(x-11)^{2}=(-8 \sqrt{x-2})^{2} & \text { Square both sides. } \\
x^{2}-22 x+121=64(x-2) &
\end{array}
$$

Now that both radicals have been eliminated, set the quadratic equal to zero and solve.

$$
\begin{aligned}
& x^{2}-22 x+121=64 x-128 \\
& x^{2}-86 x+249=0 \\
& (x-3)(x-83)=0 \quad \text { Factor and solve. } \\
& x=3 \\
& x=83
\end{aligned}
$$

The proposed solutions are $x=3$ and $x=83$. Check each solution in the original equation.

$$
\begin{aligned}
& \sqrt{2 x+3}+\sqrt{x-2}=4 \\
& \sqrt{2 x+3}=4-\sqrt{x-2} \\
& \sqrt{2(3)+3}=4-\sqrt{(3)-2} \\
& \sqrt{9}=4-\sqrt{1} \\
& 3=3
\end{aligned}
$$

One solution is $x=3$.
Check $x=83$.

$$
\begin{aligned}
& \sqrt{2 x+3}+\sqrt{x-2}=4 \\
& \sqrt{2 x+3}=4-\sqrt{x-2} \\
& \sqrt{2(83)+3}=4-\sqrt{(83-2)} \\
& \sqrt{169}=4-\sqrt{81}
\end{aligned}
$$

The only solution is $x=3$. We see that $x=83$ is an extraneous solution.

## TRY IT

Solve the equation with two radicals: $\sqrt{3 x+7}+\sqrt{x+2}=1$.
Answer
$x=-2$; extraneous solution $x=-1$
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## RECALL REWRITING EXPRESSIONS CONTAINING EXPONENTS

Recall the properties used to simplify expressions containing exponents. They work the same whether the exponent is an integer or a fraction.
It is helpful to remind yourself of these properties frequently throughout the course. They will by handy from now on in all the mathematics you'll do.

Product Rule: $\quad a^{m} \cdot a^{n}=a^{m+n}$
Quotient Rule: $\quad \frac{a^{m}}{a^{n}}=a^{m-n}$
Power Rule: $\quad\left(a^{m}\right)^{n}=a^{m \cdot n}$
Zero Exponent: $\quad a^{0}=1$
Negative Exponent: $a^{-n}=\frac{1}{a^{n}}$ and $a^{n}=\frac{1}{a^{-n}}$
Power of a Product: $(a b)^{n}=a^{n} b^{n}$
Power of a Quotient: $\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}$

## A GENERAL NOTE: RATIONAL EXPONENTS

A rational exponent indicates a power in the numerator and a root in the denominator. There are multiple ways of writing an expression, a variable, or a number with a rational exponent:

$$
a^{\frac{m}{n}}=\left(a^{\frac{1}{n}}\right)^{m}=\left(a^{m}\right)^{\frac{1}{n}}=\sqrt[n]{a^{m}}=(\sqrt[n]{a})^{m}
$$

## EXAMPLE: EVALUATING A NUMBER RAISED TO A RATIONAL EXPONENT

Evaluate $8^{\frac{2}{3}}$.

## Answer

Whether we take the root first or the power first depends on the number. It is easy to find the cube root of 8 , so rewrite $8^{\frac{2}{3}}$ as $\left(8^{\frac{1}{3}}\right)^{2}$.

$$
\begin{aligned}
\left(8^{\frac{1}{3}}\right)^{2} & =(2)^{2} \\
& =4
\end{aligned}
$$

## TRY IT

Evaluate $64^{-\frac{1}{3}}$.
Answer
$\frac{1}{4}$
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## EXAMPLE: SOLVING AN EQUATION INVOLVING A VARIABLE RAISED TO A RATIONAL EXPONENT

Solve the equation in which a variable is raised to a rational exponent: $x^{\frac{5}{4}}=32$.

## Answer

The way to remove the exponent on $x$ is by raising both sides of the equation to a power that is the reciprocal of $\frac{5}{4}$, which is $\frac{4}{5}$.
$x^{\frac{5}{4}}=32$
$\left(x^{\frac{5}{4}}\right)^{\frac{4}{5}}=(32)^{\frac{4}{5}}$
$x=(2)^{4} \quad$ The fifth root of 32 is 2.
$x=16$

## TRY IT

Solve the equation $x^{\frac{3}{2}}=125$.

## Answer

25

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## RECALL FACTORING WHEN THE GCF IS A VARIABLE

Remember, when factoring a GCF (greatest common factor) from a polynomial expression, factor out the smallest power of the variable present in each term. This works whether the exponent on the variable is an integer or a fraction.

## EXAMPLE: SOLVING AN EQUATION INVOLVING RATIONAL EXPONENTS AND FACTORING

Solve $3 x^{\frac{3}{4}}=x^{\frac{1}{2}}$.

## Answer

This equation involves rational exponents as well as factoring rational exponents. Let us take this one step at a time. First, put the variable terms on one side of the equal sign and set the equation equal to zero.

$$
\begin{aligned}
& 3 x^{\frac{3}{4}}-\left(x^{\frac{1}{2}}\right)=x^{\frac{1}{2}}-\left(x^{\frac{1}{2}}\right) \\
& 3 x^{\frac{3}{4}}-x^{\frac{1}{2}}=0
\end{aligned}
$$

Now, it looks like we should factor the left side, but what do we factor out? We can always factor the term with the lowest exponent. Rewrite $x^{\frac{1}{2}}$ as $x^{\frac{2}{4}}$. Then, factor out $x^{\frac{2}{4}}$ from both terms on the left.

$$
\begin{aligned}
& 3 x^{\frac{3}{4}}-x^{\frac{2}{4}}=0 \\
& x^{\frac{2}{4}}\left(3 x^{\frac{1}{4}}-1\right)=0
\end{aligned}
$$

Where did $x^{\frac{1}{4}}$ come from? Remember, when we multiply two numbers with the same base, we add the exponents. Therefore, if we multiply $x^{\frac{2}{4}}$ back in using the distributive property, we get the expression we had before the factoring, which is what should happen. We need an exponent such that when added to $\frac{2}{4}$ equals $\frac{3}{4}$. Thus, the exponent on $x$ in the parentheses is $\frac{1}{4}$.
Let us continue. Now we have two factors and can use the zero factor theorem.

$$
\begin{array}{ll}
x^{\frac{2}{4}}\left(3 x^{\frac{1}{4}}-1\right)=0 & \\
x^{\frac{2}{4}}=0 & \\
x=0 & \\
3 x^{\frac{1}{4}}-1=0 & \\
3 x^{\frac{1}{4}}=1 & \text { Divide both sides by } 3 . \\
x^{\frac{1}{4}}=\frac{1}{3} & \text { Raise both sides to the reciprocal of } \frac{1}{4} . \\
\left(x^{\frac{1}{4}}\right)^{4}=\left(\frac{1}{3}\right)^{4} & \\
x=\frac{1}{81} &
\end{array}
$$

The two solutions are $x=0, x=\frac{1}{81}$.

## TRY IT

Solve: $(x+5)^{\frac{3}{2}}=8$.

## Answer

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## SOLVING OTHER TYPES OF EQUATIONS

## LEARNING OUTCOMES

- Solve polynomial equations.
- Solve absolute value equations.

We have used factoring to solve quadratic equations, but it is a technique that we can use with many types of polynomial equations which are equations that contain a string of terms including numerical coefficients and variables. When we are faced with an equation containing polynomials of degree higher than 2 , we can often solve them by factoring.

## RECALL THE ZERO-PRODUCT PROPERTY

Solving by factoring depends on the zero-product property which states that if $a \cdot b=0$, then $a=0$ or $b=0$, where $a$ and $b$ are real numbers or algebraic expressions.
In other words if the product of two expressions equals zero, then at least one of the expressions must equal zero.
We may apply the zero-product property to polynomial equations in the same way we did to quadratic equations.

## A GENERAL NOTE: POLYNOMIAL EQUATIONS

A polynomial of degree $n$ is an expression of the type

$$
a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{2} x^{2}+a_{1} x+a_{0}
$$

where $n$ is a positive integer and $a_{n}, \ldots, a_{0}$ are real numbers and $a_{n} \neq 0$.
Setting the polynomial equal to zero gives a polynomial equation. The total number of solutions (real and complex) to a polynomial equation is equal to the highest exponent $n$.

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## Solving an Absolute Value Equation

An absolute value equation is an equation in which the variable of interest is contained within absolute value bars. Absolute value is defined as a distance. That is, the bars are used to designate that the number inside the absolute value represents its distance from zero on the number line. For example, to solve an equation such as $|2 x-6|=8$, notice that the absolute value will be equal to 8 if the quantity inside the absolute value bars is 8 or -8 . This leads to two different equations we can solve independently.

$$
\begin{aligned}
& 2 x-6=8 \text { or } 2 x-6=-8 \\
& 2 x=14 \quad 2 x=-2 \\
& x=7 \quad x=-1
\end{aligned}
$$

Checking both solutions by evaluating them in the original equation shows that both 7 and -1 both make the statement true.

In applications, we may need to identify numbers or points on a line that are a specified distance from a given reference point. We can use an absolute value to accomplish such a task.

## A GENERAL NOTE: ABSOLUTE VALUE EQUATIONS

The absolute value of $x$ is written as $|x|$. It has the following properties:
If $x \geq 0$, then $|x|=x$.
If $x<0$, then $|x|=-x$.
For real numbers $A$ and $B$, an equation of the form $|A|=B$, with $B \geq 0$, will have solutions when $A=B$ or $A=-B$. If $B<0$, the equation $|A|=B$ has no solution.
An absolute value equation in the form $|a x+b|=c$ has the following properties:
If $c<0,|a x+b|=c$ has no solution.
If $c=0,|a x+b|=c$ has one solution.
If $c>0,|a x+b|=c$ has two solutions.

## HOW TO: GIVEN AN ABSOLUTE VALUE EQUATION, SOLVE IT

1. Isolate the absolute value expression on one side of the equal sign.
2. If $c>0$, write and solve two equations: $a x+b=c$ and $a x+b=-c$.

## Other Types of Equations

There are many other types of equations in addition to the ones we have discussed so far. We will see more of them throughout the text. Here, we will discuss equations that are in quadratic form and rational equations that result in a quadratic.

## EXAMPLE: SOLVING ABSOLUTE VALUE EQUATIONS

Solve the following absolute value equations:
a. $|6 x+4|=8$
b. $|3 x+4|=-9$
c. $|3 x-5|-4=6$
d. $|-5 x+10|=0$

## Answer

a. $|6 x+4|=8$

Write two equations and solve each:

$$
\begin{array}{ll}
6 x+4=8 & \text { or } \\
6 x+4=-8 \\
6 x=4 & 6 x=-12 \\
x=\frac{2}{3} & x=-2
\end{array}
$$

The two solutions are $x=\frac{2}{3}, x=-2$.
b. $|3 x+4|=-9$

There is no solution as an absolute value cannot be negative.
c. $|3 x-5|-4=6$

Isolate the absolute value expression and then write two equations.

$$
\begin{aligned}
& |3 x-5|-4=6 \\
& |3 x-5|=10
\end{aligned}
$$

$3 x-5=10$
$3 x-5=-10$
$3 x=15$
$3 x=-5$
$x=5$
$x=-\frac{5}{3}$

There are two solutions: $x=5, x=-\frac{5}{3}$.
d. $|-5 x+10|=0$

The equation is set equal to zero, so we have to write only one equation.
$-5 x+10=0$
$-5 x=-10$
$x=2$
There is one solution: $x=2$.

## RECALL PROPERTIES OF EQUALITY

When solving absolute value equations, the absolute value expression must be isolated on one side of the equation before setting up the two cases to remove the absolute value bars.
Use the properties of equality to isolate the absolute value expression, but avoid multiplying into or dividing from any expression inside the bars.

Ex. use the properties of equality and the techniques of solving linear equations to isolate the absolute value expression in the equation $5-2|3 x-4|=-7$, then solve the equation.

## Answer

$5-2|3 x-4|=-7$
$-2|3 x-4|=-12 \quad$ subtract the 5 to the right-hand side $|3 x-4|=6 \quad$ divide by -2

Solution
$x=-\frac{2}{3}, x=\frac{10}{3}$

## TRY IT

Solve the absolute value equation: $|1-4 x|+8=13$.

## Answer

$x=-1, x=\frac{3}{2}$
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## Solving Equations in Quadratic Form

Equations in quadratic form are equations with three terms. The first term has a power other than 2 . The middle term has an exponent that is one-half the exponent of the leading term. The third term is a constant. We can solve equations in this form as if they were quadratic. A few examples of these equations include $x^{4}-5 x^{2}+4=0, x^{6}+7 x^{3}-8=0$, and $x^{\frac{2}{3}}+4 x^{\frac{1}{3}}+2=0$. In each one, doubling the exponent of the middle term equals the exponent on the leading term. We can solve these equations by substituting a variable for the middle term.

## A GENERAL NOTE: QUADRATIC FORM

If the exponent on the middle term is one-half of the exponent on the leading term, we have an equation in quadratic form which we can solve as if it were a quadratic. We substitute a variable for the middle term to solve equations in quadratic form.

## HOW TO: GIVEN AN EQUATION QUADRATIC IN FORM, SOLVE IT

1. Identify the exponent on the leading term and determine whether it is double the exponent on the middle term.
2. If it is, substitute a variable, such as $u$, for the variable portion of the middle term.
3. Rewrite the equation so that it takes on the standard form of a quadratic.
4. Solve using one of the usual methods for solving a quadratic.
5. Replace the substitution variable with the original term.

6 . Solve the remaining equation.

## Solving Rational Equations Resulting in a Quadratic

Earlier, we solved rational equations. Sometimes, solving a rational equation results in a quadratic. When

## EXAMPLE: SOLVING A FOURTH-DEGREE EQUATION IN QUADRATIC FORM

Solve this fourth-degree equation: $3 x^{4}-2 x^{2}-1=0$.
Answer
This equation fits the main criteria: that the power on the leading term is double the power on the middle term. Next, we will make a substitution for the variable term in the middle. Let $u=x^{2}$. Rewrite the equation in $u$.

$$
3 u^{2}-2 u-1=0
$$

Now solve the quadratic.
$3 u^{2}-2 u-1=0$
$(3 u+1)(u-1)=0$
Solve for $u$.

$$
\begin{aligned}
& 3 u+1=0 \\
& 3 u=-1 \\
& u=-\frac{1}{3} \\
& x^{2}=-\frac{1}{3} \\
& x= \pm i \sqrt{\frac{1}{3}}
\end{aligned}
$$

Replace $u$ with its original term.

$$
\begin{aligned}
& u-1=0 \\
& u=1 \\
& x^{2}=1 \\
& x= \pm 1
\end{aligned}
$$

The solutions are $x= \pm i \sqrt{\frac{1}{3}}$ and $x= \pm 1$.

## TRY IT

Solve using substitution: $x^{4}-8 x^{2}-9=0$.
Answer
$x=-3,3,-i, i$
this happens, we continue the solution by simplifying the quadratic equation by one of the methods we have seen. It may turn out that there is no solution.

## EXAMPLE: SOLVING AN EQUATION IN QUADRATIC FORM CONTAINING A BINOMIAL

Solve the equation in quadratic form: $(x+2)^{2}+11(x+2)-12=0$.

## Answer

This equation contains a binomial in place of the single variable. The tendency is to expand what is presented. However, recognizing that it fits the criteria for being in quadratic form makes all the difference in the solving process. First, make a substitution letting $u=x+2$. Then rewrite the equation in $u$.

$$
\begin{aligned}
& u^{2}+11 u-12=0 \\
& (u+12)(u-1)=0
\end{aligned}
$$

Solve using the zero-factor property and then replace $u$ with the original expression.
$u+12=0$
$u=-12$
$x+2=-12$
$x=-14$
The second factor results in

$$
\begin{aligned}
& u-1=0 \\
& u=1 \\
& x+2=1 \\
& x=-1
\end{aligned}
$$

We have two solutions: $x=-14, x=-1$.

## TRY IT

Solve: $(x-5)^{2}-4(x-5)-21=0$.
Answer
$x=2, x=12$
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## RECALL DIVISION BY ZERO

Dividing zero by a number results in zero.
$\frac{0}{a}=0$.
But we cannot divide by zero. A zero in a denominator results in an undefined expression.
$\frac{b}{0}=$ undefined

When solving rational equations in which a variable is contained in a denominator, always declare the value of the variable which would result in an undefined equation so that you won't forget to exclude it from the solution set.

## EXAMPLE: SOLVING A RATIONAL EQUATION LEADING TO A QUADRATIC

Solve the following rational equation: $\frac{-4 x}{x-1}+\frac{4}{x+1}=\frac{-8}{x^{2}-1}$.

## Answer

We want all denominators in factored form to find the LCD. Two of the denominators cannot be factored further. However, $x^{2}-1=(x+1)(x-1)$. Then, the LCD is $(x+1)(x-1)$. Next, we multiply the whole equation by the LCD.

$$
\begin{aligned}
& (x+1)(x-1)\left[\frac{-4 x}{x-1}+\frac{4}{x+1}\right]=\left[\frac{-8}{(x+1)(x-1)}\right](x+1)(x-1) \\
& -4 x(x+1)+4(x-1)=-8 \\
& -4 x^{2}-4 x+4 x-4=-8 \\
& -4 x^{2}+4=0 \\
& -4\left(x^{2}-1\right)=0 \\
& -4(x+1)(x-1)=0 \\
& x=-1 \\
& x=1
\end{aligned}
$$

In this case, either solution produces a zero in the denominator in the original equation. Thus, there is no solution.

## TRY IT

Solve $\frac{3 x+2}{x-2}+\frac{1}{x}=\frac{-2}{x^{2}-2 x}$.

## Answer

$x=-1, x=0$ is not a solution.
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## SUMMARY: EQUATION - SOLVING TECHNIQUES

## Key Concepts

- Rational exponents can be rewritten several ways depending on what is most convenient for the problem. To solve a radical equation, both sides of the equation are raised to a power that will render the exponent on the variable equal to 1.
- Factoring extends to higher-order polynomials when it involves factoring out the GCF or factoring by grouping.
- We can solve radical equations by isolating the radical and raising both sides of the equation to a power that matches the index.
- To solve absolute value equations, we need to write two equations, one for the positive value and one for the negative value.
- Equations in quadratic form are easy to spot, as the exponent on the first term is double the exponent on the second term and the third term is a constant. We may also see a binomial in place of the single variable. We use substitution to solve.
- Solving a rational equation may also lead to a quadratic equation or an equation in quadratic form.


## Glossary

absolute value equation an equation in which the variable appears in absolute value bars, typically with two solutions, one accounting for the positive expression and one for the negative expression
equations in quadratic form equations with a power other than 2 but with a middle term with an exponent that is one-half the exponent of the leading term
extraneous solutions any solutions obtained that are not valid in the original equation
polynomial equation an equation containing a string of terms including numerical coefficients and variables raised to whole-number exponents
radical equation an equation containing at least one radical term where the variable is part of the radicand

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# INTRODUCTION TO LINEAR INEQUALITIES AND ABSOLUTE VALUE INEQUALITIES 

## Learning Outcomes

By the end of this section, you will be able to:

- Use interval notation to express inequalities.
- Use properties of inequalities.
- Solve compound inequalities.
- Solve absolute value inequalities.


It is not easy to make the honor roll at most top universities. Suppose students were required to carry a course load of at least 12 credit hours and maintain a grade point average of 3.5 or above. How could these honor roll requirements be expressed mathematically? In this section, we will explore various ways to express different sets of numbers, inequalities, and absolute value inequalities.

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## WRITING AND MANIPULATING INEQUALITIES

## LEARNING OUTCOMES

- Use interval notation to express inequalities.
- Use properties of inequalities.

Indicating the solution to an inequality such as $x \geq 4$ can be achieved in several ways.
We can use a number line as shown below. The blue ray begins at $x=4$ and, as indicated by the arrowhead, continues to infinity, which illustrates that the solution set includes all real numbers greater than or equal to 4 .


We can use set-builder notation: $\{x \mid x \geq 4\}$, which translates to "all real numbers $x$ such that $x$ is greater than or equal to 4 ." Notice that braces are used to indicate a set.

The third method is interval notation, where solution sets are indicated with parentheses or brackets. The solutions to $x \geq 4$ are represented as $[4, \infty)$. This is perhaps the most useful method as it applies to concepts studied later in this course and to other higher-level math courses.

The main concept to remember is that parentheses represent solutions greater or less than the number, and brackets represent solutions that are greater than or equal to or less than or equal to the number. Use parentheses to represent infinity or negative infinity, since positive and negative infinity are not numbers in the usual sense of the word and, therefore, cannot be "equaled." A few examples of an interval, or a set of numbers in which a solution falls, are $[-2,6)$, or all numbers between -2 and 6 , including -2 , but not including $6 ;(-1,0)$, all real numbers between, but not including -1 and 0 ; and $(-\infty, 1]$, all real numbers less than and including 1 . The table below outlines the possibilities.

| Inequality | Words | Interval Notation |
| :--- | :--- | :--- |
| $a<x<b$ | all real numbers between $a$ and $b$, not including a and $b$ | $(a, b)$ |
| $x>a$ | All real numbers greater than $a$, but not including $a$ | $(a, \infty)$ |
| $x<b$ | All real numbers less than $b$, but not including $b$ | $(-\infty, b)$ |
| $x \geq a$ | All real numbers greater than $a$, including $a$ | $[a, \infty)$ |
| $x \leq b$ | All real numbers less than $b$, including $b$ | $(-\infty, b]$ |
| $a \leq x<b$ | All real numbers between $a$ and $b$, including $a$ | $[a, b)$ |
| $a<x \leq b$ | All real numbers between $a$ and $b$, including $b$ | $(a, b]$ |
| $a \leq x \leq b$ | All real numbers between $a$ and $b$, including $a$ and $b$ | $[a, b]$ |
| $x<a$ or $x>b$ | All real numbers less than $a$ or greater than $b$ | $(-\infty, a) \cup(b, \infty)$ |
| All real numbers | All real numbers | $(-\infty, \infty)$ |

## Example: Using Interval Notation to Express an inequality

Use interval notation to indicate all real numbers greater than or equal to -2 .

## Answer

Use a bracket on the left of -2 and parentheses after infinity: $[-2, \infty)$. The bracket indicates that -2 is included in the set with all real numbers greater than -2 to infinity.

## EXAMPLE: USING INTERVAL NOTATION TO EXPRESS AN INEQUALITY

Describe the inequality $x \geq 4$ using interval notation

## Answer

The solutions to $x \geq 4$ are represented as $[4, \infty)$.
Note the use of a bracket on the left because 4 is included in the solution set.

## TRY IT

Use interval notation to indicate all real numbers between and including -3 and 5 .
Answer
$[-3,5]$
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## EXAMPLE: USING INTERVAL NOTATION TO EXPRESS A COMPOUND INEQUALITY

Write the interval expressing all real numbers less than or equal to -1 or greater than or equal to 1 .
Answer
We have to write two intervals for this example. The first interval must indicate all real numbers less than or equal to 1 . So, this interval begins at $-\infty$ and ends at -1 , which is written as $(-\infty,-1]$.
The second interval must show all real numbers greater than or equal to 1 , which is written as $[1, \infty)$. However, we want to combine these two sets. We accomplish this by inserting the union symbol, $\cup$, between the two intervals.

$$
(-\infty,-1] \cup[1, \infty)
$$

## TRY IT

Express all real numbers less than -2 or greater than or equal to 3 in interval notation.
Answer
$(-\infty,-2) \cup[3, \infty)$

## TRY IT

We are going to look at a line with endpoints along the x -axis.

1. First we will adjust the left endpoint to $(-15,0)$, and the right endpoint to $(5,0)$

2. Write an inequality that represents the line you created.
3. If we were to slide the left endpoint to $(2,0)$, what do you think will happen to the line?
4. Now what if we were to slide the right endpoint to $(11,0)$, what do you think will happen to the line? Sketch on a piece of paper what you think this new inequality graph will look like.

## Answer

With endpoints ( $-15,0$ ) and ( 5,0 ), the values for x on the line are between -15 and 5 , so we can write $-15<x<5$. We made it a strict inequality because the dots on the endpoints of the lines are open.
Moving the left endpoint towards the right endpoint shortens the line. Then moving the right endpoint away from the left endpoint lengthens the line again.


## THINK ABOUT IT

In the previous examples you were given an inequality or a description of one with words and asked to draw the corresponding graph and write the interval. In this example you are given an interval and asked to write the inequality and draw the graph.
Given $(-\infty, 10)$, write the associated inequality and draw the graph.
In the box below, write down whether you think it will be easier to draw the graph first or write the inequality first.

## Answer

We will draw the graph first.
The interval reads "all real numbers less than 10," so we will start by placing an open dot on 10 and drawing a line to the left with an arrow indicating the solution continues to negative infinity.


To write the inequality, we will use $<$ since the parentheses indicate that 10 is not included. $x<10$
In the following video, you will see examples of how to write inequalities in the three ways presented here: as an inequality, in interval notation, and with a graph.


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Using the Properties of Inequalities

## RECALL SOLVING MULTI-STEP EQUATIONS

When solving inequalities, all the properties of equality and real numbers apply. We are permitted to add, subtract, multiply, or divide the same quantity to both sides of the inequality.

Likewise, we may apply the distributive, commutative, and associative properties as desired to help isolate the variable.
We may also distribute the LCD on both sides of an inequality to eliminate denominators.
The only difference is that if we multiply or divide both sides by a negative quantity, we must reverse the direction of the inequality symbol.

When we work with inequalities, we can usually treat them similarly to but not exactly as we treat equations. We can use the addition property and the multiplication property to help us solve them. The one exception is when we multiply or divide by a negative number, we must reverse the inequality symbol.

## A GENERAL NOTE: PROPERTIES OF INEQUALITIES

Addition Property If $a<b$, then $a+c<b+c$.
Multiplication Property If $a<b$ and $c>0$, then $a c<b c$.
If $a<b$ and $c<0$, then $a c>b c$.
These properties also apply to $a \leq b, a>b$, and $a \geq b$.

## EXAMPLE: DEMONSTRATING THE ADDITION PROPERTY

Illustrate the addition property for inequalities by solving each of the following:

1. $x-15<4$
2. $6 \geq x-1$
3. $x+7>9$

## Answer

The addition property for inequalities states that if an inequality exists, adding or subtracting the same number on both sides does not change the inequality.
1.
$x-15<4$
$x-15+15<4+15$ Add 15 to both sides.
$x<19$
2.
$6 \geq x-1$
$6+1 \geq x-1+1$ Add 1 to both sides.
$7 \geq x$
3.
$x+7>9$
$x+7-7>9-7$ Subtract 7 from both sides.
$x>2$

Watch the following two videos for a demonstration of using the addition and multiplication properties to solve inequalities.

## TRY IT

Solve $3 x-2<1$.
Answer
$x<1$
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## EXAMPLE: DEMONSTRATING THE MULTIPLICATION PROPERTY

Illustrate the multiplication property for inequalities by solving each of the following:

1. $3 x<6$
2. $-2 x-1 \geq 5$
3. $5-x>10$

Answer
1.
$3 x<6$
$\frac{1}{3}(3 x)<(6) \frac{1}{3}$
$x<2$
2.
$-2 x-1 \geq 5$
$-2 x \geq 6$
$\left(-\frac{1}{2}\right)(-2 x) \geq(6)\left(-\frac{1}{2}\right) \quad$ Multiply by $-\frac{1}{2}$.
$x \leq-3$
Reverse the inequality.
3.
$5-x>10$
$-x>5$
$(-1)(-x)>(5)(-1)$ Multiply by -1 .
$x<-5 \quad$ Reverse the inequality.

## TRY IT

Solve $4 x+7 \geq 2 x-3$.
Answer
$x \geq-5$

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## Solving Inequalities in One Variable Algebraically

As the examples have shown, we can perform the same operations on both sides of an inequality, just as we do with equations; we combine like terms and perform operations. To solve, we isolate the variable.

## EXAMPLE: SOLVING AN INEQUALITY ALGEBRAICALLY

Solve the inequality: $13-7 x \geq 10 x-4$.
Answer
Solving this inequality is similar to solving an equation up until the last step.
$13-7 x \geq 10 x-4$
$13-17 x \geq-4 \quad$ Move variable terms to one side of the inequality.
$-17 x \geq-17 \quad$ Isolate the variable term.
$x \leq 1 \quad$ Dividing both sides by -17 reverses the inequality.
The solution set is given by the interval $(-\infty, 1]$, or all real numbers less than and including 1 .

## TRY IT

Solve the inequality and write the answer using interval notation: $-x+4<\frac{1}{2} x+1$.
Answer
$(2, \infty)$
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## EXAMPLE: SOLVING AN INEQUALITY WITH FRACTIONS

Solve the following inequality and write the answer in interval notation: $-\frac{3}{4} x \geq-\frac{5}{8}+\frac{2}{3} x$.
Answer
We begin solving in the same way we do when solving an equation.
$-\frac{3}{4} x \geq-\frac{5}{8}+\frac{2}{3} x$
$-\frac{3}{4} x-\frac{2}{3} x \geq-\frac{5}{8} \quad$ Put variable terms on one side.
$-\frac{9}{12} x-\frac{8}{12} x \geq-\frac{5}{8} \quad$ Write fractions with common denominator.
$-\frac{17}{12} x \geq-\frac{5}{8}$
$x \leq-\frac{5}{8}\left(-\frac{12}{17}\right) \quad$ Multiplying by a negative number reverses the inequality.
$x \leq \frac{15}{34}$
The solution set is the interval $\left(-\infty, \frac{15}{34}\right]$.

## TRY IT

Solve the inequality and write the answer in interval notation: $-\frac{5}{6} x \leq \frac{3}{4}+\frac{8}{3} x$.
Answer

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## COMPOUND AND ABSOLUTE VALUE INEQUALITIES

## LEARNING OUTCOMES

- Solve compound inequalities.
- Solve absolute value inequalities.

A compound inequality includes two inequalities in one statement. A statement such as $4<x \leq 6$ means $4<x$ and $x \leq 6$. There are two ways to solve compound inequalities: separating them into two separate inequalities or leaving the compound inequality intact and performing operations on all three parts at the same time. We will illustrate both methods.

## Solving Absolute Value Inequalities

As we know, the absolute value of a quantity is a positive number or zero. From the origin, a point located at $(-x, 0)$ has an absolute value of $x$ as it is $x$ units away. Consider absolute value as the distance from one point to another point. Regardless of direction, positive or negative, the distance between the two points is represented as a positive number or zero.

An absolute value inequality is an equation of the form

$$
|A|<B,|A| \leq B,|A|>B, \text { or }|A| \geq B,
$$

where $A$, and sometimes $B$, represents an algebraic expression dependent on a variable $x$. Solving the inequality means finding the set of all $x$-values that satisfy the problem. Usually this set will be an interval or the union of two intervals and will include a range of values.

There are two basic approaches to solving absolute value inequalities: graphical and algebraic. The advantage of the graphical approach is we can read the solution by interpreting the graphs of two equations.

## EXAMPLE: SOLVING A COMPOUND INEQUALITY

Solve the compound inequality: $3 \leq 2 x+2<6$.

## Answer

The first method is to write two separate inequalities: $3 \leq 2 x+2$ and $2 x+2<6$. We solve them independently.
$3 \leq 2 x+2$ and $2 x+2<6$
$1 \leq 2 x$
$2 x<4$
$\frac{1}{2} \leq x$
$x<2$

Then, we can rewrite the solution as a compound inequality, the same way the problem began.

$$
\frac{1}{2} \leq x<2
$$

In interval notation, the solution is written as $\left[\frac{1}{2}, 2\right)$.
The second method is to leave the compound inequality intact and perform solving procedures on the three parts at the same time.
$3 \leq 2 x+2<6$
$1 \leq 2 x<4 \quad$ Isolate the variable term and subtract 2 from all three parts.
$\frac{1}{2} \leq x<2 \quad$ Divide through all three parts by 2.
We get the same solution: $\left[\frac{1}{2}, 2\right)$.

## TRY IT

Solve the compound inequality $4<2 x-8 \leq 10$.
Answer
$6<x \leq 9$ or $(6,9]$
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The advantage of the algebraic approach is that solutions are exact, as precise solutions are sometimes difficult to read from a graph.

Suppose we want to know all possible returns on an investment if we could earn some amount of money within $\$ 200$ of $\$ 600$. We can solve algebraically for the set of $x$-values such that the distance between $x$ and 600 is less than 200. We represent the distance between $x$ and 600 as $|x-600|$, and therefore, $|x-600| \leq 200$ or
$-200 \leq x-600 \leq 200$
$-200+600 \leq x-600+600 \leq 200+600$
$400 \leq x \leq 800$
This means our returns would be between $\$ 400$ and $\$ 800$.
To solve absolute value inequalities, just as with absolute value equations, we write two inequalities and then solve them independently.

## EXAMPLE: SOLVING A COMPOUND INEQUALITY WITH THE VARIABLE IN ALL THREE PARTS

Solve the compound inequality with variables in all three parts: $3+x>7 x-2>5 x-10$.

## Answer

Let's try the first method. Write two inequalities:
$3+x>7 x-2$ and $7 x-2>5 x-10$
$3>6 x-2 \quad 2 x-2>-10$
$5>6 x \quad 2 x>-8$
$\frac{5}{6}>x \quad x>-4$
$x<\frac{5}{6}$
$-4<x$
The solution set is $-4<x<\frac{5}{6}$ or in interval notation $\left(-4, \frac{5}{6}\right)$. Notice that when we write the solution in interval notation, the smaller number comes first. We read intervals from left to right as they appear on a number line.


## TRY IT

Solve the compound inequality: $3 y<4-5 y<5+3 y$.
Answer
$\left(-\frac{1}{8}, \frac{1}{2}\right)$
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## A GENERAL NOTE: ABSOLUTE VALUE INEQUALITIES

For an algebraic expression $X$ and $k>0$, an absolute value inequality is an inequality of the form: $|X|<k$ which is equivalent to $-k<X<k$ or $|X|>k$ which is equivalent to $X<-k$ or $X>k$ These statements also apply to $|X| \leq k$ and $|X| \geq k$.

## EXAMPLE: DETERMINING A NUMBER WITHIN A PRESCRIBED DISTANCE

Describe all values $x$ within a distance of 4 from the number 5 .

## Answer

We want the distance between $x$ and 5 to be less than or equal to 4 . We can draw a number line to represent the condition to be satisfied.


The distance from $x$ to 5 can be represented using an absolute value symbol, $|x-5|$. Write the values of $x$ that satisfy the condition as an absolute value inequality.

$$
|x-5| \leq 4
$$

We need to write two inequalities as there are always two solutions to an absolute value equation.

$$
x-5 \leq 4 \text { and } x-5 \geq-4
$$

$$
x \leq 9 \quad x \geq 1
$$

If the solution set is $x \leq 9$ and $x \geq 1$, then the solution set is an interval including all real numbers between and including 1 and 9 .
So $|x-5| \leq 4$ is equivalent to $[1,9]$ in interval notation.

## TRY IT

Describe all $x$-values within a distance of 3 from the number 2 .
Answer
$|x-2| \leq 3$

## EXAMPLE: SOLVING AN ABSOLUTE VALUE INEQUALITY

Solve $|x-1| \leq 3$.
Answer
$|x-1| \leq 3$
$-3 \leq x-1 \leq 3$
$-2 \leq x \leq 4$
$[-2,4]$

## EXAMPLE: USING A GRAPHICAL APPROACH TO SOLVE ABSOLUTE VALUE INEQUALITIES

Given the equation $y=-\frac{1}{2}|4 x-5|+3$, determine the $x$-values for which the $y$-values are negative.

## Answer

We are trying to determine where $y<0$ which is when $-\frac{1}{2}|4 x-5|+3<0$. We begin by isolating the absolute value.
$-\frac{1}{2}|4 x-5|<-3$ Multiply both sides by -2 , and reverse the inequality.

$$
|4 x-5|>6
$$

Next, we solve $|4 x-5|=6$.

$$
\begin{array}{lll}
4 x-5=6 & & 4 x-5=-6 \\
4 x=11 & \text { or } & 4 x=-1 \\
x=\frac{11}{4} & & x=-\frac{1}{4}
\end{array}
$$

Now, we can examine the graph to observe where the $y$-values are negative. We observe where the branches are below the $x$-axis. Notice that it is not important exactly what the graph looks like, as long as we know that it crosses the horizontal axis at $x=-\frac{1}{4}$ and $x=\frac{11}{4}$ and that the graph opens downward.


Solve $-2|k-4| \leq-6$.
Answer
$k \leq 1$ or $k \geq 7$; in interval notation, this would be $(-\infty, 1] \cup[7, \infty)$.


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## TRY IT

Sometimes a picture is worth a thousand words. You can turn a single variable inequality into a two variable inequality and make a graph. The x-intercepts of the graph will correspond with the solution to the inequality you can find by hand.
Let's use the last example to try it. We will change the variable to x to make it easier to enter in an online graphing calculator.
To turn $-2|x-4| \leq-6$ into a two variable equation, move everything to one side, and place the variable y on the other side like this:
$-2|x-4| \leq-6$
$-2|x-4|+6 \leq y$
Now enter this inequality in an online graphing calculator and hover over the x-intercepts.
If you need instruction on how to enter inequalities in an online graphing calculator, watch this tutorial for the process within Desmos. Other calculators will behave slightly differently.
Are the $x$-values of the intercepts the same values as the solution we found above?
Now you try turning this single variable inequality into a two variable inequality:
$5|9-2 x| \geq 10$
Graph your inequality with an online graphing calculator, and write the solution interval.


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## SUMMARY: COMPOUND AND ABSOLUTE VALUE INEQUALITIES

## Key Concepts

- Interval notation is a method to give the solution set of an inequality. Highly applicable in calculus, it is a system of parentheses and brackets that indicate what numbers are included in a set and whether the endpoints are included as well.
- Solving inequalities is similar to solving equations. The same algebraic rules apply, except for one: multiplying or dividing by a negative number reverses the inequality.
- Compound inequalities often have three parts and can be rewritten as two independent inequalities. Solutions are given by boundary values which are indicated as a beginning boundary or an ending boundary in the solutions to the two inequalities.
- Absolute value inequalities will produce two solution sets due to the nature of absolute value. We solve by writing two equations: one equal to a positive value and one equal to a negative value.
- Absolute value inequality solutions can be verified by graphing. We can check the algebraic solutions by graphing as we cannot depend on a visual for a precise solution.


## Glossary

compound inequality a problem or a statement that includes two inequalities
interval an interval describes a set of numbers where a solution falls
interval notation a mathematical statement that describes a solution set and uses parentheses or brackets to indicate where an interval begins and ends
linear inequality similar to a linear equation except that the solutions will include an interval of numbers

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## PUTTING IT TOGETHER: EQUATIONS AND INEQUALITIES

At the beginning of this module, we explored a few problems associated with the construction of a school. First we had to set up a mathematical model that would accurately reflect the following requirements for the foundation of the building.

- The shape must be rectangular.
- The length must be 100 ft . longer than the width.
- There must be a total of $12,000 \mathrm{sq}$. ft . in the foundation.


Letting $x$ stand for the width of the rectangle, the length must then be $x+100$. The area of any rectangle is equal to the product of its length and width, and since the area must be $12,000 \mathrm{sq}$. ft., this leads to the equation,
$(x+100)(x)=12,000$
$x^{2}+100 x=12,000$
$x^{2}+100 x-12,000=0$
The equation is quadratic, that is, degree 2. In this case, the trinomial can be factored.
$(x-300)(x+400)=0$

This implies either $x=300$, or $x=-400$. However, since negative width does not make much sense (have you ever tried to order a negative 8 foot fence post?), we only use the positive answer, $x=300$. This means the width is 300 ft ., and the length is $300+100=400 \mathrm{ft}$. A quick check verifies that the dimensions yield the correct square footage:
$300 \times 400=12,000$
Ok, now that we have our foundation planned, let's move on to the walls. Recall, this project requires at least 1800 concrete blocks. The budget for this material is $\$ 2,400$, and each block costs $\$ 1.20$. Furthermore, there is a fee of $\$ 100$ for hauling the blocks to the construction site, regardless of how many blocks are ordered.

Let $n$ stand for the number of blocks that would be purchased. Then two inequalities can be set up based on the constraints in the problem.
$n \geq 1800$
$(1.20) n+100 \leq 2400$
What is the maximum number of blocks that you can purchase? Let's solve the second inequality to find out.
$(1.20) n \leq 2400-100=2300$
$n \leq \frac{2300}{1.20} \approx 1916.7$
There is no use for a fraction of a block, so we might as well say that $n \leq 1916$. A maximum of 1916 whole blocks can be purchased. Notice that $1916 \geq 1800$, so the first inequality is also satisfied, meaning that our budget allows for more than enough blocks to complete this phase of the project.

With the help of equations and inequalities, as well as specialized techniques for solving them, the plans for the new school are coming together nicely!

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## MODULE 5: FUNCTION BASICS

## WHY IT MATTERS: FUNCTION BASICS

When Galileo was about 26 years old and working at the University of Pisa, he began considering the rates at which objects fall. Aristotle before him had predicted that the rate at which an object falls is dependent on its mass. In other words, a heavy object should fall faster than a lighter one. Galileo devised an experiment to test this idea. Perhaps you have heard the famous story of Galileo dropping two balls of different masses from the top of the leaning Tower of Pisa.

There are two aspects of this story that are truly important. One is that controlled scientific experiments were simply not performed during his time. And quite frankly, the evidence is unclear as to whether he actually performed the experiment or just thought about the experiment based on
 observations on smaller scales, such as ramps. The story was recounted by one of Galileo's students years after his death, but was never described by Galileo himself.

The other aspect is that Galileo's conclusion, whether determined through experiment or thought, was that in the absence of air resistance all objects fall at the same rate regardless of mass. That rate is the acceleration due to gravity, which on Earth is $9.8 \mathrm{~m} / \mathrm{s}^{2}$. This conclusion not only surprised Galileo and provided the first opposition to Aristotle's theory, but forever changed our understanding of falling motion.

Knowing that all objects fall at the same rate, you can predict both the velocity of an object and the distance it has fallen after some amount of time.

It turns out that the Tower of Pisa, at just over 55 meters, is a little too short for substantial experiments with falling bodies. But suppose Galileo were around today. He could actually perform his experiment by hanging out of a window in the tallest building in the world, the Burj Khalifa in Dubai, which stands 828 meters tall!

The table below lists the velocity and distance traveled by an object dropped from some height at 1 second time intervals.


Chart shows the correlation between time (seconds 1-8), velocity and distance.
In this module, you'll find out how to answer these questions. It turns out that both velocity and distance can be described by functions. Read on to find out what a function is, how a function can be described, and what you can do to evaluate a function.

## LEARNING OUTCOMES

## Review Topics for Success

- Define a function using tables.
- Define a function from a set of ordered pairs.
- Define the domain and range of a function given as a table or a set of ordered pairs.
- Write functions using algebraic notation.
- Use the vertical line test to determine whether a graph represents a function.
- Given a function described by an equation, find function values (outputs) for numerical inputs.
- Given a function described by an equation, find function values (outputs) for variable inputs.


## Characteristics of Functions and Their Graphs

- Determine whether a relation represents a function.
- Find function values.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions in the library of functions.


## Domain and Range of Functions

- Find the domain of a function defined by an equation.
- Write the domain and range using standard notations.
- Find Domain and Range from a Graph.
- Give domain and range of Toolkit Functions.
- Graph piecewise-defined functions.


## Rates of Change and Behavior of Graphs

- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

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## REVIEW TOPICS FOR SUCCESS

## Learning Outcomes

By the end of this section, you will be able to:

- Define a function using tables
- Define a function from a set of ordered pairs
- Define the domain and range of a function given as a table or a set of ordered pairs
- Write functions using algebraic notation
- Use the vertical line test to determine whether a graph represents a function
- Given a function described by an equation, find function values (outputs) for numerical inputs
- Given a function described by an equation, find function values (outputs) for variable inputs

In this module you will investigate the general characteristics of mathematical functions. Functions describe special relationships called relations between sets of items or values. This ability to describe well-defined relations is useful for making mathematical models to predict behavior or outcomes in certain situations.

Before diving into a detailed investigation of functions, though, it will be good to build up an intuitive understanding of what they are. In this reveiw section we'll define what a function is and see the notation we use to write them. We'll also get an introduction to how functions are evaluated and how they appear on a graph.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

## RECALL FOR SUCCESS

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

## DEFINE A FUNCTION

## LEARNING OUTCOMES

- Define a function using tables
- Define a function from a set of ordered pairs
- Define the domain and range of a function given as a table or a set of ordered pairs

There are many kinds of relations. A relation is simply a correspondence between sets of values or information. Think about members of your family and their ages. The pairing of each member of your family and their age is a relation. Each family member can be paired with an age in the set of ages of your family members. Another example of a relation is the pairing of a state with its U.S. senators. Each state can be matched with two individuals who have each been elected to serve as a senator. In turn, each senator can be matched with one specific state that he or she represents. Both of these are real-life examples of relations.

The first value of a relation is an input value and the second value is the output value. A function is a specific type of relation in which each input value corresponds to one and only one output value. An input is called the independent value, and an output is called the dependent value, as it depends on the value of the input.

Notice in the first table below, where the input is "name" and the output is "age," each input matches with exactly one output. This is an example of a function.

| Family Member's Name (Input) | Family Member's Age (Output) |
| :---: | :---: |
| Nellie | 13 |
| Marcos | 11 |
| Esther | 46 |
| Samuel | 47 |
| Nina | 47 |
| Paul | 47 |
| Katrina | 21 |
| Andrew | 16 |
| Maria | 13 |
| Ana | 81 |

Compare this with the next table where the input is "age" and the output is "name." Some of the inputs result in more than one output. This is an example of a correspondence that is not a function.

| Family Member's Age <br> (Input) | Family Member's Name <br> (Output) |
| :---: | :---: |
| 11 | Marcos |
| 13 | Nellie, Maria |
| 16 | Andrew |
| 21 | Katrina |
| 46 | Esther |
| 47 | Samuel, Nina, Paul |
| 81 | Ana |

Now let us look at some other examples to determine whether the given relations are functions, and under what circumstances. Remember that a relation is a function if there is only one output for each input.

Relations can be written as ordered pairs of numbers $(x, y)$ or as numbers in a table of values the columns of which each contain inputs or outputs. By examining the inputs ( $x$-coordinates) and outputs ( $y$ coordinates), you can determine whether or not the relation is a function. Remember, in a function, each input corresponds to only one output. That is, each $x$ value corresponds to exactly one $y$ value.

There is a name for the set of input values and another name for the set of output values for a function. The set of input values is called the domain of the function. The set of output values is called the range of the function.

Given a set of ordered pairs, find the domain by listing all of the input values, which are the $x$-coordinates. To find the range, list all of the output values, which are the $y$-coordinates.

Consider the following set of ordered pairs:
$\{(-2,0),(0,6),(2,12),(4,18)\}$
You have the following:
Domain: $\{-2,0,2,4\}$
Range: $\{0,6,12,18\}$
Now try it yourself.
Watch the following video for another example of how to identify whether a table of values represents a function and how to determine the domain and range.

The following video shows how to determine whether a relation given in non-tabular form is a function and how to find the domain and range.

## Summary: Determining Whether a Relation is a Function

1. Identify the input values - this is your domain.
2. Identify the output values - this is your range.
3. If each value in the domain leads to only one value in the range, classify the relationship as a function. If any value in the domain leads to two or more values in the range, do not classify the relationship as a function.

## EXAMPLE

Fill in the table.


## EXAMPLE

List the domain and range for the following table of values where $x$ is the input and $y$ is the output.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | :---: |
| -3 | 4 |
| -2 | 4 |
| -1 | 4 |
| 2 | 4 |
| 3 | 4 |

Answer
The domain describes all the inputs, and we can use set notation with brackets $\}$ to make the list.
Domain : $\{-3,-2,-1,2,3\}$
The range describes all the outputs.
Range : \{4\}
We only listed 4 once because it is not necessary to list it every time it appears in the range.


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## EXAMPLE

Define the domain and range for the following set of ordered pairs, and determine whether the relation given is a function.
$\{(-3,-6),(-2,-1),(1,0),(1,5),(2,0)\}$

## Answer

We list all of the input values as the domain. The input values are represented first in the ordered pair as a matter of convention.
Domain: $\{-3,-2,1,2\}$
Note how we did not enter repeated values more than once; it is not necessary.
The range is the list of outputs for the relation; they are entered second in the ordered pair.
Range: $\{-6,-1,0,5\}$
Organizing the ordered pairs in a table can help you tell whether this relation is a function. By definition, the inputs in a function have only one output.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | :---: |
| -3 | -6 |
| -2 | -1 |
| 1 | 0 |
| 1 | 5 |
| 2 | 0 |

The relation is not a function because the input 1 has two outputs: 0 and 5 .


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## EXAMPLE

Find the domain and range of the relation and determine whether it is a function.
$\{(-3,4),(-2,4),(-1,4),(2,4),(3,4)\}$
Answer
Domain: $\{-3,-2,-1,2,3\}$
Range: \{4\}
To help you determine whether this is a function, you could reorganize the information by creating a table.

| $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| :---: | :---: |
| -3 | 4 |
| -2 | 4 |
| -1 | 4 |
| 2 | 4 |
| 3 | 4 |

Each input has only one output, and the fact that it is the same output (4) does not matter.
This relation is a function.

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## FUNCTION NOTATION AND GRAPHS OF FUNCTIONS

## LEARNING OUTCOMES

- Write functions using algebraic notation
- Use the vertical line test to determine whether a graph represents a function

Some people think of functions as "mathematical machines." Imagine you have a machine that changes a number according to a specific rule such as "multiply by 3 and add 2 " or "divide by 5 , add 25 , and multiply by -1 ." If you put a number into the machine, a new number will pop out the other end having been changed according to the rule. The number that goes in is called the input, and the number that is produced is called the output.

You can also call the machine " $f$ " for function. If you put $x$ into the box, $f(x)$, comes out. Mathematically speaking, $x$ is the input, or the "independent variable," and $f(x)$ is the output, or the "dependent variable,"
since it depends on the value of $x$.
$f(x)=4 x+1$ is written in function notation and is read " $f$ of $x$ equals $4 x$ plus 1 ." It represents the following situation: A function named $f$ acts upon an input, $x$, and produces $f(x)$ which is equal to $4 x+1$. This is the same as the equation $y=4 x+1$.

Function notation gives you more flexibility because you do not have to use $y$ for every equation. Instead, you could use $f(x)$ or $g(x)$ or $c(x)$. This can be a helpful way to distinguish equations of functions when you are dealing with more than one at a time.

## Using Function Notation

Once we determine that a relationship is a function, we need to display and define the functional relationship so that we can understand it, use it, and possibly even program it into a computer. There are various ways of representing functions. A standard function notation is one representation that facilitates working with functions.

Now you try it.

## EXAMPLE

Represent height as a function of age using function notation.
Answer
To represent "height as a function of age," we start by identifying the variables: $h$ for height and $a$ for age.
$h$ is $f$ of $a \quad$ We name the function $f$; height is a function of age.
$h=f(a) \quad$ We use parentheses to indicate the function input.
$f(a) \quad$ We name the function $f$; the expression is read as " $f$ of $a$."
Note: We can use any letter to name the function; the notation $h=f(a)$ shows us that $h$ depends on, or is a function of, $a$. The value $a$ must be put into the function $f$ to get a result (height). The parentheses indicate that age is the input for the function; they do not indicate multiplication.

Now try another.

## EXAMPLE

1. Write the formula for perimeter of a square, $P=4 s$, as a function.
2. Write the formula for area of a square, $A=l^{2}$, as a function.

## Answer

1. Name the function $P$. $P$ is a function of the length of the sides, s. Perimeter, as a function of side length, is equal to 4 times side length. $P(s)=4 s$
2. Name the function A. Area, as a function of the length of the sides, is equal to the length squared. $A(l)=l^{2}$.

This would make it easy to graph both functions on the same graph without confusion about the variables.
We can also use an algebraic expression as the input to a function. For example $f(a+b)$ means "first add $a$ and $b$, and the result is the input for the function $f$." The operations must be performed in this order to obtain the correct result.

Note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels

## A GENERAL NOTE: FUNCTION NOTATION

The notation $y=f(x)$ defines a function named $f$. This is read as " $y$ is a function of $x$." The letter $x$ represents the input value, or independent variable. The letter $y$ or $f(x)$, represents the output value, or dependent variable.

## EXAMPLE

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

## Answer

The number of days in a month is a function of the name of the month, so if we name the function $f$, we write days $=f$ (month) or $d=f(m)$. The name of the month is the input to a "rule" that associates a specific number (the output) with each input.


For example, $f($ March $)=31$, because March has 31 days. The notation $d=f(m)$ reminds us that the number of days, $d$ (the output), is dependent on the name of the month, $m$ (the input).
of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

## EXAMPLE

A function $N=f(y)$ gives the number of police officers, $N$, in a town in year $y$. What does $f(2005)=300$ represent?

## Answer

When we read $f(2005)=300$, we see that the input year is 2005 . The value for the output, the number of police officers, $N$, is 300 . Remember, $N=f(y)$. The statement $f(2005)=300$ tells us that in the year 2005 there were 300 police officers in the town.

In the following videos we show two more examples of how to express a relationship using function notation.



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## Graphs of Functions

When both the independent quantity (input) and the dependent quantity (output) are real numbers, a function can be represented by a graph in the coordinate plane. The independent value is plotted on the $x$ axis and the dependent value is plotted on the $y$-axis. The fact that each input value has exactly one output value means graphs of functions have certain characteristics. For each input on the graph, there will be exactly one output. For a function defined as $y=f(x)$, or y is a function of $x$, we would write ordered pairs $(x, f(x))$ using function notation instead of $(x, y)$ as you may have seen previously.


We can identify whether the graph of a relation represents a function because for each $x$-coordinate there will be exactly one $y$-coordinate.


When a vertical line is placed across the plot of this relation, it does not intersect the graph more than once for any values of $x$.

If, on the other hand, a graph shows two or more intersections with a vertical line, then an input ( $x$ coordinate) can have more than one output ( $y$-coordinate), and $y$ is not a function of $x$. Examining the graph of a relation to determine if a vertical line would intersect with more than one point is a quick way to determine if the relation shown by the graph is a function. This method is often called the "vertical line test."

You try it.
The vertical line method can also be applied to a set of ordered pairs plotted on a coordinate plane to determine if the relation is a function.

Now consider the set of ordered pairs $\{(3,-1),(5,-2),(3,-3),(-3,5)\}$. One of the inputs, 3 , can produce two different outputs, -1 and -3 . You know what that means-this set of ordered pairs is not a function. A plot confirms this.


Notice that a vertical line passes through two plotted points. One $x$-coordinate has multiple $y$-coordinates. This relation is not a function.

In the following video, we show another example of determining whether a graph represents a function using the vertical line test.

## EXAMPLE

Use the vertical line test to determine whether the relation plotted on this graph is a function.


## Answer

This relationship cannot be a function, because some of the $x$-coordinates have two corresponding $y$ coordinates.


## EXAMPLE

Consider the ordered pairs $\{(-1,3),(-2,5),(-3,3),(-5,-3)\}$ plotted on the graph below. Use the vertical line test to determine whether the set of ordered pairs represents a function.


## Answer

Drawing vertical lines through each point results in each line only touching one point. This means that none of the $x$-coordinates have two corresponding $y$-coordinates, so this is a function.



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## EVALUATE FUNCTIONS

## LEARNING OUTCOMES

- Given a function described by an equation, find function values (outputs) for numerical inputs
- Given a function described by an equation, find function values (outputs) for variable inputs

Throughout this course, you have been working with algebraic equations. Many of these equations are functions. For example, $y=4 x+1$ is an equation that represents a function. When you input values for $x$, you can determine a single output for $y$. In this case, if you substitute $x=10$ into the equation you will find that y must be 41 ; there is no other value of y that would make the equation true.

Rather than using the variable $y$, the equations of functions can be written using function notation. Function notation is very useful when you are working with more than one function at a time and substituting more than one value in for $x$.

Equations written using function notation can also be evaluated. With function notation, you might see the following:

Given $f(x)=4 x+1$, find $f(2)$.
You read this problem like this: "given $f$ of $x$ equals $4 x$ plus one, find $f$ of 2 ." While the notation and wording is different, the process of evaluating a function is the same as evaluating an equation. In both cases, you substitute 2 for $x$, multiply it by 4 and add 1 , simplifying to get 9 . In both a function and an equation, an input of 2 results in an output of 9 .
$f(x)=4 x+1$
$f(2)=4(2)+1=8+1=9$
You can simply apply what you already know about evaluating expressions to evaluate a function. It is important to note that the parentheses that are part of function notation do not mean multiply. The notation $f(x)$ does not mean $f$ multiplied by $x$. Instead, the notation means " $f$ of $x$ " or "the function of $x$."To evaluate the function, take the value given for $x$, and substitute that value in for $x$ in the expression. Let us look at a couple of examples.

Functions can be evaluated for negative values of $x$, too. Keep in mind the rules for integer operations.
You may also be asked to evaluate a function for more than one value as shown in the example that follows.

## Variable Inputs

## EXAMPLE

Given $f(x)=3 x-4$, find $f(5)$.
Answer
Substitute 5 in for $x$ in the function.
$f(5)=3(5)-4$
Simplify the expression on the right side of the equation.
$f(5)=15-4$
$f(5)=11$

## EXAMPLE

Given $p(x)=2 x^{2}+5$, find $p(-3)$.
Answer
Substitute -3 in for $x$ in the function.
$p(-3)=2(-3)^{2}+5$
Simplify the expression on the right side of the equation.
$p(-3)=2(9)+5$
$p(-3)=18+5$
$p(-3)=23$

## EXAMPLE

Given $f(x)=|4 x-3|$, find $f(0), f(2)$, and $f(-1)$.

## Answer

Treat each of these like three separate problems. In each case, you substitute the value in for $x$ and simplify.
Start with $x=0$.
$f(0)=|4(0)-3|=|-3|=3$
$f(0)=3$
Evaluate for $x=2$.
$f(2)=|4(2)-3|=|5|=5$
$f(2)=5$
Evaluate for $x=-1$.
$f(-1)=|4(-1)-3|=|-7|=7$
$f(-1)=7$
So far, you have evaluated functions for inputs that have been constants. Functions can also be evaluated for inputs that are variables or expressions. The process is the same, but the simplified answer will contain a variable. The following examples show how to evaluate a function for a variable input.

## Example

Given $f(x)=3 x^{2}+2 x+1$, find $f(b)$.

## Answer

This problem is asking you to evaluate the function for $b$. This means substitute $b$ in the equation for $x$.
$f(b)=3 b^{2}+2 b+1$
(That is all-you are done.)

In the following example, you evaluate a function for an expression. So here you will substitute the entire expression in for $x$ and simplify.

## Example

Given $f(x)=4 x+1$, find $f(h+1)$.

## Answer

This time, you substitute $(h+1)$ into the equation for $x$.
$f(h+1)=4(h+1)+1$
Use the distributive property on the right side, and then combine like terms to simplify.
$f(h+1)=4 h+4+1=4 h+5$
Given $f(x)=4 x+1, f(h+1)=4 h+5$.

In the following video, we show more examples of evaluating functions for both integer and variable inputs.


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## Summary

Function notation takes the form such as $f(x)=18 x-10$ and is read " $f$ of $x$ equals 18 times $x$ minus 10 ." Function notation can use letters other than $f$, such as $c(x), g(x)$, or $h(x)$. As you go further in your study of functions, this notation will provide you more flexibility, allowing you to examine and compare different functions more easily. Just as an algebraic equation written in $x$ and $y$ can be evaluated for different values of the input $x$, an equation written in function notation can also be evaluated for different values of $x$. To evaluate a function, substitute in values for $x$ and simplify to find the related output.

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\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
- A function is a special relation in which each input corresponds to exactly one output. An input is called the independent value, and an output is called the dependent value.
- The set of all input values makes up the domain, and the set of all output variables makes up the range of the function.
- A function may be described using notation stating that the output \(y\) is dependent upon the input \(x\) by writing \(y=f(x)\).
- We may identify whether a relation represents a function from a table of values, a set of ordered pairs, or a graph.
- Functions may be evaluated for numerical or variable inputs

\section*{Glossary}
domain the set of all input values into a function
function a specific type of relation in which each input value corresponds to one and only one output value.
function notation a notation used for representing output as a function of input, \(y=f(x)\), that is \(y\) is a function of \(x\)
range the set of all output values of a function
relation a correspondence between sets of values or information
vertical line test a method for determining graphically whether a relation represents a function

\section*{INTRODUCTION TO CHARACTERISTICS OF FUNCTIONS AND THEIR GRAPHS}

\section*{Learning Outcomes}

By the end of this lesson, you will be able to:
- Determine whether a relation represents a function.
- Find the value of a function.
- Determine whether a function is one-to-one.
- Use the vertical line test to identify functions.
- Graph the functions listed in the library of functions.

A jetliner changes altitude as its distance from the starting point of a flight increases. The weight of a growing child increases with time. In each case, one quantity depends on another. There is a relationship between the two quantities that we can describe, analyze, and use to make predictions. In this section, we will analyze such relationships.

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\section*{CHARACTERISTICS OF FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Identify functions given in tabular form.
- Write and evaluate functional relationships using standard notation.
- Determine whether a function is one-to-one.

\section*{ABOUT THE NOTATION}

Recall that ordered pairs of the form \((x, y)\) contain elements from two different sets of values, \(x\)-values and \(y\)-values.
We use curly braces to denote a set of elements. The set of elements named \(a, b\), and \(c\) can be written as \(\{a, b, c\}\).

A relation is a set of ordered pairs. The set of the first components of each ordered pair is called the domain of the relation and the set of the second components of each ordered pair is called the range of the relation. Consider the following set of ordered pairs. The first numbers in each pair are the first five natural numbers. The second number in each pair is twice the first.
\(\{(1,2),(2,4),(3,6),(4,8),(5,10)\}\)
The domain is \(\{1,2,3,4,5\}\). The range is \(\{2,4,6,8,10\}\).
Note the values in the domain are also known as an input values, or values of the independent variable, and are often labeled with the lowercase letter \(x\). Values in the range are also known as an output values, or values of the dependent variable, and are often labeled with the lowercase letter \(y\).

A function \(f\) is a relation that assigns a single value in the range to each value in the domain. In other words, no \(x\)-values are used more than once. For our example that relates the first five natural numbers to numbers double their values, this relation is a function because each element in the domain, \(\{1,2,3,4,5\}\), is paired with exactly one element in the range, \(\{2,4,6,8,10\}\).

Now let's consider the set of ordered pairs that relates the terms "even" and "odd" to the first five natural numbers. It would appear as
\(\{(\) odd, 1\(),(\) even, 2\(),(\) odd, 3\(),(\) even, 4\(),(\) odd, 5\()\}\)
Notice that each element in the domain, \{even,odd\} is not paired with exactly one element in the range, \(\{1,2,3,4,5\}\). For example, the term "odd" corresponds to three values from the domain, \(\{1,3,5\}\) and the term "even" corresponds to two values from the range, \(\{2,4\}\). This violates the definition of a function, so this relation is not a function.

This image compares relations that are functions and not functions.

(a) This relationship is a function because each input is associated with a single output. Note that input \(q\) and \(r\) both give output \(n\). (b) This relationship is also a function. In this case, each input is associated with a single output. (c) This relationship is not a function because input \(q\) is associated with two different outputs.

\section*{A GENERAL NOTE: FUNCTIONS}

A function is a relation in which each possible input value leads to exactly one output value. We say "the output is a function of the input."
The input values make up the domain, and the output values make up the range.

\section*{HOW TO: GIVEN A RELATIONSHIP BETWEEN TWO QUANTITIES, DETERMINE WHETHER THE RELATIONSHIP IS A FUNCTION.}
1. Identify the input values.
2. Identify the output values.
3. If each input value leads to only one output value, the relationship is a function. If any input value leads to two or more outputs, the relationship is not a function.

\section*{Using Function Notation}

Once we determine that a relationship is a function, we need to display and define the functional relationships so that we can understand and use them, and sometimes also so that we can program them into computers. There are various ways of representing functions. A standard function notation is one representation that makes it easier to work with functions.

To represent "height is a function of age," we start by identifying the descriptive variables \(h\) for height and \(a\) for age. The letters \(f, g\), and \(h\) are often used to represent functions just as we use \(x, y\), and \(z\) to represent numbers and \(A, B\), and \(C\) to represent sets.
\(h\) is \(f\) of \(a \quad\) We name the function \(f\); height is a function of age.
\(h=f(a) \quad\) We use parentheses to indicate the function input.
\(f(a) \quad\) We name the function \(f\); the expression is read as " \(f\) of \(a^{\prime \prime}\).

\section*{EXAMPLE: DETERMINING IF MENU PRICE LISTS ARE FUNCTIONS}

The coffee shop menu consists of items and their prices.
1. Is price a function of the item?
2. Is the item a function of the price?


\section*{Answer}
1. Let's begin by considering the input as the items on the menu. The output values are then the prices.


Each item on the menu has only one price, so the price is a function of the item.
2. Two items on the menu have the same price. If we consider the prices to be the input values and the items to be the output, then the same input value could have more than one output associated with it.


Therefore, the item is a not a function of price.

Remember, we can use any letter to name the function; we can use the notation \(h(a)\) to show that \(h\) depends on \(a\). The input value \(a\) must be put into the function \(h\) to get an output value. The parentheses indicate that age is input into the function; they do not indicate multiplication.

We can also give an algebraic expression as the input to a function. For example \(f(a+b)\) means "first add \(a\) and \(b\), and the result is the input for the function \(f\)." We must perform the operations in this order to obtain the correct result.

\section*{EXAMPLE: DETERMINING IF CLASS GRADE RULES ARE FUNCTIONS}

In a particular math class, the overall percent grade corresponds to a grade point average. Is grade point average a function of the percent grade? Is the percent grade a function of the grade point average? The table below shows a possible rule for assigning grade points.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|}
\hline Percent Grade & \(0-56\) & \(57-61\) & \(62-66\) & \(67-71\) & \(72-77\) & \(78-86\) & \(87-91\) & \(92-100\) \\
\hline Grade Point Average & 0.0 & 1.0 & 1.5 & 2.0 & 2.5 & 3.0 & 3.5 & 4.0 \\
\hline
\end{tabular}

Answer
For any percent grade earned, there is an associated grade point average, so the grade point average is a function of the percent grade. In other words, if we input the percent grade, the output is a specific grade point average.
In the grading system given, there is a range of percent grades that correspond to the same grade point average. For example, students who receive a grade point average of 3.0 could have a variety of percent grades ranging from 78 all the way to 86 . Thus, percent grade is not a function of grade point average.

\section*{TRY IT}

The table below lists the five greatest baseball players of all time in order of rank.
\begin{tabular}{|l|l|}
\hline Player & Rank \\
\hline Babe Ruth & 1 \\
\hline Willie Mays & 2 \\
\hline Ty Cobb & 3 \\
\hline Walter Johnson & 4 \\
\hline Hank Aaron & 5 \\
\hline
\end{tabular}
1. Is the rank a function of the player name?
2. Is the player name a function of the rank?

\section*{Answer}
1. yes
2. yes. (Note: If two players had been tied for, say, 4th place, then the name would not have been a function of rank.)

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\section*{RECALL FUNCTION NOTATION}

Some of the information in this section may look familiar from the Review for Success: Function Notation and Graphs of Functions. Learning to work with functions is naturally challenging, so it's worth another look at it. Don't be discouraged if the notation feels awkward or difficult to understand. Learning mathematics is very similar to learning a new language; it's normal that it takes lots of repetition before it starts to feel natural.

\section*{A GENERAL NOTE: FUNCTION NOTATION}

The notation \(y=f(x)\) defines a function named \(f\). This is read as " \(y\) is a function of \(x\)." The letter \(x\) represents the input value, or independent variable. The letter \(y\), or \(f(x)\), represents the output value, or dependent variable.

\section*{EXAMPLE: USING FUNCTION NOTATION FOR DAYS IN A MONTH}

Use function notation to represent a function whose input is the name of a month and output is the number of days in that month.

\section*{Answer}

The number of days in a month is a function of the name of the month, so if we name the function \(f\), we write days \(=f\) (month) or \(d=f(m)\). The name of the month is the input to a "rule" that associates a specific number (the output) with each input.


For example, \(f(\) April \()=30\), because March has 31 days. The notation \(d=f(m)\) reminds us that the number of days, \(d\) (the output), is dependent on the name of the month, \(m\) (the input).

\section*{Analysis of the Solution}

We must restrict the function to non-leap years. Otherwise, February would have 2 outputs and this would not be a function. Also note that the inputs to a function do not have to be numbers; function inputs can be names of people, labels of geometric objects, or any other element that determines some kind of output. However, most of the functions we will work with in this book will have numbers as inputs and outputs.

\section*{EXAMPLE: INTERPRETING FUNCTION NOTATION}

A function \(N=f(y)\) gives the number of police officers, \(N\), in a town in year \(y\). What does \(f(2005)=300\) represent?

\section*{Answer}

When we read \(f(2005)=300\), we see that the input year is 2005 . The value for the output, the number of police officers, \(N\), is 300 . Remember, \(N=f(y)\). The statement \(f(2005)=300\) tells us that in the year 2005 there were 300 police officers in the town.

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\section*{Q \& A}

Instead of a notation such as \(y=f(x)\), could we use the same symbol for the output as for the function, such as \(y=y(x)\), meaning " \(\boldsymbol{y}\) is a function of \(\boldsymbol{x}\) ?"

\section*{Representing Functions Using Tables}

A common method of representing functions is in the form of a table. The table rows or columns display the corresponding input and output values. In some cases these values represent all we know about the relationship; other times the table provides a few select examples from a more complete relationship.

The table below lists the input number of each month (January = 1, February =2, and so on) and the output value of the number of days in that month. This information represents all we know about the months and days for a given year (that is not a leap year). Note that, in this table, we define a days-in-a-month function \(f\), where \(D=f(m)\) identifies months by an integer rather than by name.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|l|l|l|l|}
\hline Month number, \(m\) (input) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\hline Days in month, \(D\) (output) & 31 & 28 & 31 & 30 & 31 & 30 & 31 & 31 & 30 & 31 & 30 & 31
\end{tabular}

The table below defines a function \(Q=g(n)\). Remember, this notation tells us that \(g\) is the name of the function that takes the input \(n\) and gives the output \(Q\).
\begin{tabular}{l|l|l|l|l|l}
\hline\(n\) & 1 & 2 & 3 & 4 & 5 \\
\hline\(Q\) & 8 & 6 & 7 & 6 & 8 \\
\hline
\end{tabular}

The table below displays the age of children in years and their corresponding heights. This table displays just some of the data available for the heights and ages of children. We can see right away that this table does not represent a function because the same input value, 5 years, has two different output values, 40 in . and 42 in .
\begin{tabular}{|l|l|l|l|l|l|l|l|}
\hline Age in years, \(a\) (input) & 5 & 5 & 6 & 7 & 8 & 9 & 10 \\
\hline Height in inches, \(h\) (output) & 40 & 42 & 44 & 47 & 50 & 52 & 54 \\
\hline
\end{tabular}

\section*{HOW TO: GIVEN A TABLE OF INPUT AND OUTPUT VALUES, DETERMINE WHETHER THE TABLE REPRESENTS A FUNCTION.}
1. Identify the input and output values.
2. Check to see if each input value is paired with only one output value. If so, the table represents a function.

\section*{READING FUNCTIONS FROM A TABLE}

Note that the tables above indicate which values are the inputs and which are the outputs to make it easier to clearly identify the relationship. Sometimes, though, the relationship is not labeled. For instance, if a table consists of measures of distance in one column and measures of time in another, you might fairly assume that time is the input and that a distance traveled would be dependent upon the time spent traveling. Either way, when reading a function from a table, it's a good idea to try to discern which values are intended to be or make the most sense as the inputs and which make most sense as outcome values, dependent upon the inputs.

\section*{EXAMPLE: IDENTIFYING TABLES THAT REPRESENT FUNCTIONS}

Which table, \(\mathrm{A}, \mathrm{B}\), or C , represents a function (if any)?

\section*{Table A}
\begin{tabular}{|l|l|}
\hline & Table A \\
\hline Input & Output \\
\hline 2 & 1 \\
\hline 5 & 3 \\
\hline 8 & 6 \\
\hline & Table B \\
\hline Input & Output \\
\hline-3 & 5 \\
\hline 0 & 1 \\
\hline 4 & 5 \\
\hline & Table C \\
\hline Input & Output \\
\hline 1 & 0 \\
\hline 5 & 2 \\
\hline 5 & 4 \\
\hline
\end{tabular}

Answer
a) and b) define functions. In both, each input value corresponds to exactly one output value. c) does not define a function because the input value of 5 corresponds to two different output values.
When a table represents a function, corresponding input and output values can also be specified using function notation.
The function represented by a) can be represented by writing
\(f(2)=1, f(5)=3\), and \(f(8)=6\)
Similarly, the statements \(g(-3)=5, g(0)=1\), and \(g(4)=5\) represent the function in b).
c) cannot be expressed in a similar way because it does not represent a function.

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\section*{RECALL EVALUATING FUNCTIONS}

Since a function \(y=f(x)\) is an equation, we can evaluate it for a numerical or algebraic input.
Ex. if \(f(x)=5 x-11\), we can find \(f(2)\) or \(f(a+b)\).
\(f(2)=5(2)-11\)
\(=10-11\)
\(=-1\)
\[
\begin{aligned}
\text { or } & \\
f(a+b) & =5(a+b)-11 \\
& =5 a+5 b-11
\end{aligned}
\]

Note that we can also solve for what value of the input makes the output a certain value. That is, given our function above, \(f(x)=5 x-11\), what value of \(x\) makes \(f(x)=1\) ?

\section*{Answer}

The idea is to replace \(f(x)\) the value of 1 , then solve for the input \(x\) that makes that a true statement.
\[
\begin{aligned}
f(x) & =5 x-11 \\
1 & =5 x-11 \\
12 & =5 x \\
\frac{12}{5} & =x
\end{aligned}
\]

When we know an input value and want to determine the corresponding output value for a function, we evaluate the function. Evaluating will always produce one result because each input value of a function corresponds to exactly one output value.

When we know an output value and want to determine the input values that would produce that output value, we set the output equal to the function's formula and solve for the input. Solving can produce more than one solution because different input values can produce the same output value.

\section*{Determine whether a function is one-to-one}



Some functions have a given output value that corresponds to two or more input values. For example, in the following stock chart the stock price was \(\$ 1000\) on five different dates, meaning that there were five different input values that all resulted in the same output value of \(\$ 1000\).

However, some functions have only one input value for each output value, as well as having only one output for each input. We call these functions one-to-one functions. As an example, consider a school that uses only letter grades and decimal equivalents, as listed in.
\begin{tabular}{|l|l|}
\hline Letter grade & Grade point average \\
\hline A & 4.0 \\
\hline B & 3.0 \\
\hline C & 2.0 \\
\hline D & 1.0 \\
\hline
\end{tabular}

This grading system represents a one-to-one function, because each letter input yields one particular grade point average output and each grade point average corresponds to one input letter.

To visualize this concept, let's look again at the two simple functions sketched in (a) and (b) below.


The function in part (a) shows a relationship that is not a one-to-one function because inputs \(q\) and \(r\) both give output \(n\). The function in part (b) shows a relationship that is a one-to-one function because each input is associated with a single output.

\section*{A GENERAL NOTE: ONE-TO-ONE FUNCTION}

A one-to-one function is a function in which each output value corresponds to exactly one input value.

\section*{EXAMPLE: DETERMINING WHETHER A RELATIONSHIP IS A ONE-TO-ONE FUNCTION}

Is the area of a circle a function of its radius? If yes, is the function one-to-one?

\section*{Answer}

A circle of radius \(r\) has a unique area measure given by \(A=\pi r^{2}\), so for any input, \(r\), there is only one output, \(A\). The area is a function of radius \(r\).
If the function is one-to-one, the output value, the area, must correspond to a unique input value, the radius. Any area measure \(A\) is given by the formula \(A=\pi r^{2}\). Because areas and radii are positive numbers, there is exactly one solution: \(r=\sqrt{\frac{A}{\pi}}\). So the area of a circle is a one-to-one function of the circle's radius.

\section*{RECALL MANIPULATING FORMULAS TO ISOLATE A CERTAIN VARIABLE}

In the example above, a formula was solved for one of the variables. We often need to do this in order to rewrite the formula in a useful way.

\section*{Answer}

In the example above, the formula for the area of a circle, \(A=\pi r^{2}\), was solved for A using the properties of equality.
\[
A=\pi r^{2}
\]
\(\frac{A}{\pi}=r^{2} \quad\) divide both sides by \(\pi\)
\(r=\sqrt{\frac{A}{\pi}} \quad\) take the square root of both sides
Only the positive root was required since radius cannot be negative.

\section*{TRY IT}
1. Is a balance a function of the bank account number?
2. Is a bank account number a function of the balance?
3. Is a balance a one-to-one function of the bank account number?

\section*{Answer}
1. yes, because each bank account (input) has a single balance (output) at any given time.
2. no, because several bank accounts (inputs) may have the same balance (output).
3. no, because the more than one bank account (input) can have the same balance (output).

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\section*{EVALUATING AND SOLVING FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Evaluate and solve functions in algebraic form.
- Evaluate functions given tabular or graphical data.

When we have a function in formula form, it is usually a simple matter to evaluate the function. For example, the function \(f(x)=5-3 x^{2}\) can be evaluated by squaring the input value, multiplying by 3 , and then subtracting the product from 5 .

\section*{HOW TO: EVALUATE A FUNCTION GIVEN ITS FORMULA.}
1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

Watch the video below for more examples of evaluating a function for specific values of the input.
In addition to evaluating functions for a particular input, we can also solve functions for the input that creates a particular output.

\section*{EXAMPLE: EVALUATING FUNCTIONS}

Given the function \(h(p)=p^{2}+2 p\), evaluate \(h(4)\).
Answer
To evaluate \(h(4)\), we substitute the value 4 for the input variable \(p\) in the given function.
\[
\begin{aligned}
h(p) & =p^{2}+2 p \\
h(4) & =(4)^{2}+2(4) \\
& =16+8 \\
& =24
\end{aligned}
\]

Therefore, for an input of 4 , we have an output of 24 or \(h(4)=24\).


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\section*{EVALUATING FUNCTIONS}

When evaluating functions, it's handy to wrap the input variable in parentheses before making the substitution.
Ex. Given \(f(x)=x^{2}-8\), find \(f(-3)\)
\[
\begin{aligned}
f(x) & =(x)^{2}-8 \\
& =(-3)^{2}-8 \\
& =9-8 \\
& =1
\end{aligned}
\]

The value of the function \(f(x)=x^{2}-8\), at the input \(x=-3\), is 1 .
The next video shows another example of how to solve a function.

\section*{Evaluating Functions Expressed in Formulas}

Some functions are defined by mathematical rules or procedures expressed in equation form. If it is possible to express the function output with a formula involving the input quantity, then we can define a function in algebraic form. For example, the equation \(2 n+6 p=12\) expresses a functional relationship between \(n\) and \(p\). We can rewrite it to decide if \(p\) is a function of \(n\).

\section*{EXAMPLE: EVALUATING FUNCTIONS AT SPECIFIC VALUES}

For the function, \(f(x)=x^{2}+3 x-4\), evaluate each of the following.
1. \(f(2)\)
2. \(f(a)\)
3. \(f(a+h)\)
4. \(\frac{f(a+h)-f(a)}{h}\)

\section*{Answer}

Replace the \(x\) in the function with each specified value.
1. Because the input value is a number, 2 , we can use algebra to simplify.
\[
\begin{aligned}
f(2) & =2^{2}+3(2)-4 \\
& =4+6-4 \\
& =6
\end{aligned}
\]
2. In this case, the input value is a letter so we cannot simplify the answer any further.
\[
f(a)=a^{2}+3 a-4
\]
3. With an input value of \(a+h\), we must use the distributive property.
\[
\begin{aligned}
f(a+h) & =(a+h)^{2}+3(a+h)-4 \\
& =a^{2}+2 a h+h^{2}+3 a+3 h-4
\end{aligned}
\]
4. In this case, we apply the input values to the function more than once, and then perform algebraic operations on the result. We already found that
\[
f(a+h)=a^{2}+2 a h+h^{2}+3 a+3 h-4
\]
and we know that
\[
f(a)=a^{2}+3 a-4
\]

Now we combine the results and simplify.
\[
\begin{aligned}
\frac{f(a+h)-f(a)}{h} & =\frac{\left(a^{2}+2 a h+h^{2}+3 a+3 h-4\right)-\left(a^{2}+3 a-4\right)}{h} \\
& =\frac{2 a h+h^{2}+3 h}{h} \\
& =\frac{h(2 a+h+3)}{h} \\
& =2 a+h+3
\end{aligned}
\]

Factor out \(h\).
Simplify.

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\section*{TRY IT}

Given the function \(g(m)=\sqrt{m-4}\), evaluate \(g(5)\).
Answer
\(g(5)=\sqrt{5-4}=1\)
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\section*{EXAMPLE: SOLVING FUNCTIONS}

Given the function \(h(p)=p^{2}+2 p\), solve for \(h(p)=3\).

\section*{Answer}
\(h(p)=3\)
\(p^{2}+2 p=3 \quad\) Substitute the original function \(h(p)=p^{2}+2 p\).
\(p^{2}+2 p-3=0 \quad\) Subtract 3 from each side.
\((p+3)(p-1)=0 \quad\) Factor.
If \((p+3)(p-1)=0\), either \((p+3)=0\) or \((p-1)=0\) (or both of them equal 0 ). We will set each factor equal to 0 and solve for \(p\) in each case.
\(p+3=0, \quad p=-3\)
\(p-1=0, \quad p=1\)
This gives us two solutions. The output \(h(p)=3\) when the input is either \(p=1\) or \(p=-3\).

\begin{tabular}{|l|r|r|r|r|r|}
\hline \(\boldsymbol{p}\) & -3 & -2 & 0 & 1 & 4 \\
\hline \(\boldsymbol{h}(\boldsymbol{p})\) & 3 & 0 & 0 & 3 & 24 \\
\hline
\end{tabular}

We can also verify by graphing as in Figure 5. The graph verifies that \(h(1)=h(-3)=3\) and \(h(4)=24\).

\section*{HOW TO: SOLVE A FUNCTION.}
1. Replace the output in the formula with the value provided.
2. Solve for the input variable that makes the statement true.

Watch this video to see another example of how to express an equation as a function.
Sometimes a relationship between variables cannot be expressed as a function. See the example below for more information.

\section*{Evaluating a Function Given in Tabular Form}

As we saw above, we can represent functions in tables. Conversely, we can use information in tables to


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\section*{TRY IT}

Given the function \(g(m)=\sqrt{m-4}\), solve \(g(m)=2\).

\section*{Answer}
\(m=8\)
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\section*{FUNCTIONS, EQUATIONS, AND FORMULAS}

We've seen that an equation such as \(a x+b y=c\) can be written in a different form by solving the equation for one of the variables. If we solve this linear equation for \(y\) it can be written in the slopeintercept form of a line, \(y=m x+b\).
Certain formulas can be written in function form by solving for one of the variables. For instance, can you see how to solve the formula for a rectangle having a perimeter of 21 feet, \(21=2 l+2 w\), for length?

\section*{Answer}
\[
\begin{aligned}
21 & =2 l+2 w \\
21-2 w & =2 l \\
\frac{21-2 w}{2} & =l
\end{aligned}
\]

We can now declare a function, \(l=f(x)\) that returns an output length for a rectangle having a perimeter of 21 feet based on different width inputs.

\section*{HOW TO: GIVEN A FUNCTION IN EQUATION FORM, WRITE ITS ALGEBRAIC FORMULA.}
1. Solve the equation to isolate the output variable on one side of the equal sign, with the other side as an expression that involves only the input variable.
2. Use all the usual algebraic methods for solving equations, such as adding or subtracting the same quantity to or from both sides, or multiplying or dividing both sides of the equation by the same

\section*{EXAMPLE: FINDING AN EQUATION OF A FUNCTION}

Express the relationship \(2 n+6 p=12\) as a function \(p=f(n)\), if possible.
Answer
To express the relationship in this form, we need to be able to write the relationship where \(p\) is a function of \(n\), which means writing it as \(p=\) expression involving \(n\).
\(2 n+6 p=12\)
\(6 p=12-2 n \quad\) Subtract \(2 n\) from both sides.
\(p=\frac{12-2 n}{6} \quad\) Divide both sides by 6 and simplify.
\(p=\frac{12}{6}-\frac{2 n}{6}\)
\(p=2-\frac{1}{3} n\)
Therefore, \(p\) as a function of \(n\) is written as
\(p=f(n)=2-\frac{1}{3} n\)

\section*{Analysis of the Solution}

It is important to note that not every relationship expressed by an equation can also be expressed as a function with a formula.


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write functions, and we can evaluate functions using the tables. For example, how well do our pets recall the fond memories we share with them? There is an urban legend that a goldfish has a memory of 3 seconds, but this is just a myth. Goldfish can remember up to 3 months, while the beta fish has a memory of up to 5 months. And while a puppy's memory span is no longer than 30 seconds, the adult dog can remember for 5 minutes. This is meager compared to a cat, whose memory span lasts for 16 hours.

The function that relates the type of pet to the duration of its memory span is more easily visualized with the use of a table. See the table below.

\section*{EXAMPLE: EXPRESSING THE EQUATION OF A CIRCLE AS A FUNCTION}

Does the equation \(x^{2}+y^{2}=1\) represent a function with \(x\) as input and \(y\) as output? If so, express the relationship as a function \(y=f(x)\).
Answer
First we subtract \(x^{2}\) from both sides.
\(y^{2}=1-x^{2}\)
We now try to solve for \(y\) in this equation.
\(y= \pm \sqrt{1-x^{2}}\)
\(=\sqrt{1-x^{2}}\) and \(-\sqrt{1-x^{2}}\)
We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function \(y=f(x)\).

\section*{TRY IT}

If \(x-8 y^{3}=0\), express \(y\) as a function of \(x\).

\section*{Answer}
\(y=f(x)=\frac{\sqrt[3]{x}}{2}\)
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\section*{Q \& A}

Are there relationships expressed by an equation that do represent a function but which still cannot be represented by an algebraic formula?
Yes, this can happen. For example, given the equation \(x=y+2^{y}\), if we want to express \(y\) as a function of \(x\), there is no simple algebraic formula involving only \(x\) that equals \(y\). However, each \(x\) does determine a unique value for \(y\), and there are mathematical procedures by which \(y\) can be found to any desired accuracy. In this case, we say that the equation gives an implicit (implied) rule for \(y\) as a function of \(x\), even though the formula cannot be written explicitly.
\begin{tabular}{|l|l|}
\hline Pet & Memory span in hours \\
\hline Puppy & 0.008 \\
\hline Adult dog & 0.083 \\
\hline Cat & 16 \\
\hline Goldfish & 2160 \\
\hline Beta fish & 3600 \\
\hline
\end{tabular}

At times, evaluating a function in table form may be more useful than using equations. Here let us call the function \(P\).

The domain of the function is the type of pet and the range is a real number representing the number of hours the pet's memory span lasts. We can evaluate the function \(P\) at the input value of "goldfish." We
would write \(P\) (goldfish) \(=2160\). Notice that, to evaluate the function in table form, we identify the input value and the corresponding output value from the pertinent row of the table. The tabular form for function \(P\) seems ideally suited to this function, more so than writing it in paragraph or function form.

\section*{HOW TO: GIVEN A FUNCTION REPRESENTED BY A TABLE, IDENTIFY SPECIFIC OUTPUT AND INPUT VALUES.}
1. Find the given input in the row (or column) of input values.
2. Identify the corresponding output value paired with that input value.

3 . Find the given output values in the row (or column) of output values, noting every time that output value appears.
4. Identify the input value(s) corresponding to the given output value.

\section*{EXAMPLE: EVALUATING AND SOLVING A TABULAR FUNCTION}

Using the table below,
1. Evaluate \(g(3)\).
2. Solve \(g(n)=6\).
\begin{tabular}{|l|l|l|l|l|l|}
\hline\(n\) & 1 & 2 & 3 & 4 & 5 \\
\hline\(g(n)\) & 8 & 6 & 7 & 6 & 8 \\
\hline
\end{tabular}

Answer
- Evaluating \(g(3)\) means determining the output value of the function \(g\) for the input value of \(n=3\). The table output value corresponding to \(n=3\) is 7 , so \(g(3)=7\).
- Solving \(g(n)=6\) means identifying the input values, \(n\), that produce an output value of 6 . The table below shows two solutions: \(n=2\) and \(n=4\).
\begin{tabular}{|l|l|l|l|l|l|}
\hline\(n\) & 1 & 2 & 3 & 4 & 5 \\
\hline\(g(n)\) & 8 & 6 & 7 & 6 & 8 \\
\hline
\end{tabular}

When we input 2 into the function \(g\), our output is 6 . When we input 4 into the function \(g\), our output is also 6.

\section*{TRY IT}

Using the table from the previous example, evaluate \(g(1)\).

\section*{Answer}
\(g(1)=8\)
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\section*{Finding Function Values from a Graph}

We can view a function as a set of inputs and their corresponding outputs. That is, we can see a function as a set of ordered pairs, \((x, y)\).
Remember that, in function notation, \(y=f(x)\), so the ordered pairs containing inputs and outputs can be written in the form of (input, output) or \((x, f(x))\).

Evaluating a function using a graph also requires finding the corresponding output value for a given input value, only in this case, we find the output value by looking at the graph. Solving a function equation using a graph requires finding all instances of the given output value on the graph and observing the corresponding input value(s).

\section*{EXAMPLE: READING FUNCTION VALUES FROM A GRAPH}

Given the graph below,
1. Evaluate \(f(2)\).
2. Solve \(f(x)=4\).


\section*{Answer}
1. To evaluate \(f(2)\), locate the point on the curve where \(x=2\), then read the \(y\)-coordinate of that point. The point has coordinates \((2,1)\), so \(f(2)=1\).

2. To solve \(f(x)=4\), we find the output value 4 on the vertical axis. Moving horizontally along the line \(y=4\), we locate two points of the curve with output value \(4:(-1,4)\) and \((3,4)\). These points represent the two solutions to \(f(x)=4: x=-1\) or \(x=3\). This means \(f(-1)=4\) and \(f(3)=4\), or when the input is -1 or 3 , the output is 4 . See the graph below.


\section*{TRY IT}

Using the graph, solve \(f(x)=1\).


\section*{Answer}
\(x=0\) or \(x=2\)
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\section*{TRY IT}

You can use an online graphing calculator to graph functions, find function values, and evaluate functions. Watch this short tutorial to learn how to within Desmos. Other online graphing tools will be slightly different.


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Now try the following with an online graphing calculator:
1. Graph the function \(f(x)=-\frac{1}{2} x^{2}+x+4\) using function notation.
2. Evaluate the function at \(x=1\)
3. Make a table of values that references the function. Include at least the interval \([-5,5]\) for \(x\)-values.
4. Solve the function for \(f(0)\)

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\section*{IDENTIFY FUNCTIONS USING GRAPHS}

\section*{LEARNING OUTCOMES}
- Verify a function using the vertical line test
- Verify a one-to-one function with the horizontal line test
- Identify the graphs of the toolkit functions

As we have seen in examples above, we can represent a function using a graph. Graphs display many input-output pairs in a small space. The visual information they provide often makes relationships easier to understand. We typically construct graphs with the input values along the horizontal axis and the output values along the vertical axis.

The most common graphs name the input value \(x\) and the output value \(y\), and we say \(y\) is a function of \(x\), or \(y=f(x)\) when the function is named \(f\). The graph of the function is the set of all points \((x, y)\) in the plane that satisfies the equation \(y=f(x)\). If the function is defined for only a few input values, then the graph of the function is only a few points, where the \(x\)-coordinate of each point is an input value and the \(y\)-coordinate of each point is the corresponding output value. For example, the black dots on the graph in the graph below tell us that \(f(0)=2\) and \(f(6)=1\). However, the set of all points \((x, y)\) satisfying \(y=f(x)\) is a curve. The curve shown includes \((0,2)\) and \((6,1)\) because the curve passes through those points.


The vertical line test can be used to determine whether a graph represents a function. A vertical line includes all points with a particular \(x\) value. The \(y\) value of a point where a vertical line intersects a graph represents an output for that input \(x\) value. If we can draw any vertical line that intersects a graph more than once, then the graph does not define a function because that \(x\) value has more than one output. A function has only one output value for each input value.

Function


Not a Function


Not a Function


\section*{HOW TO: GIVEN A GRAPH, USE THE VERTICAL LINE TEST TO DETERMINE IF THE GRAPH REPRESENTS A FUNCTION.}
1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
2. If there is any such line, the graph does not represent a function.
3. If no vertical line can intersect the curve more than once, the graph does represent a function.

\section*{EXAMPLE: APPLYING THE VERTICAL LINE TEST}

Which of the graphs represent(s) a function \(y=f(x)\) ?


\section*{Answer}

If any vertical line intersects a graph more than once, the relation represented by the graph is not a function. Notice that any vertical line would pass through only one point of the two graphs shown in parts (a) and (b) of the graph above. From this we can conclude that these two graphs represent functions. The third graph does not represent a function because, at most \(x\)-values, a vertical line would intersect the graph at more than one point.


\section*{The Horizontal Line Test}

Once we have determined that a graph defines a function, an easy way to determine if it is a one-to-one function is to use the horizontal line test. Draw horizontal lines through the graph. A horizontal line includes all points with a particular \(y\) value. The \(x\) value of a point where a vertical line intersects a function


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\section*{TRY IT}

Does the graph below represent a function?


\section*{Answer}

Yes.
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\section*{TRY IT}

In the graph below, use the vertical line test to determine which relations are functions and which are not.


\section*{RELATIONS, FUNCTIONS, AND ONE-TO-ONE FUNCTIONS}

A relation is a set of ordered pairs, inputs and their corresponding outputs, \((x, y)\).
A function is a particular relation in which each input corresponds to exactly one output, \((x, f(x))\). A one-to-one function is a particular function in which each output corresponds to exactly one input.
represents the input for that output \(y\) value. If we can draw any horizontal line that intersects a graph more than once, then the graph does not represent a function because that \(y\) value has more than one input.

\section*{HOW TO: GIVEN A GRAPH OF A FUNCTION, USE THE HORIZONTAL LINE TEST TO DETERMINE IF THE GRAPH REPRESENTS A ONE-TO-ONE FUNCTION.}
1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
2. If there is any such line, the function is not one-to-one.
3. If no horizontal line can intersect the curve more than once, the function is one-to-one.

\section*{Identifying Basic Toolkit Functions}

In this text we explore functions-the shapes of their graphs, their unique characteristics, their algebraic formulas, and how to solve problems with them. When learning to read, we start with the alphabet. When learning to do arithmetic, we start with numbers. When working with functions, it is similarly helpful to have a base set of building-block elements. We call these our "toolkit functions," which form a set of basic named functions for which we know the graph, formula, and special properties. Some of these functions are programmed to individual buttons on many calculators. For these definitions we will use \(x\) as the input variable and \(y=f(x)\) as the output variable.

We will see these toolkit functions, combinations of toolkit functions, their graphs, and their transformations frequently throughout this book. It will be very helpful if we can recognize these toolkit functions and their features quickly by name, formula, graph, and basic table properties. The graphs and sample table values are included with each function shown below.

\section*{EXAMPLE: APPLYING THE HORIZONTAL LINE TEST}

Consider the functions (a), and (b)shown in the graphs below.


Are either of the functions one-to-one?

\section*{Answer}

The function in (a) is not one-to-one. The horizontal line shown below intersects the graph of the function at two points (and we can even find horizontal lines that intersect it at three points.)


The function in (b) is one-to-one. Any horizontal line will intersect a diagonal line at most once.
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\section*{TOOLKIT FUNCTIONS}

You should memorize the shapes of these graphs along with their names in words and in equation form. Be able to plot a few points of each to the left and right of the \(y\)-axis. You'll be using these function families throughout the rest of the course.

\section*{Toolkit Functions}
\begin{tabular}{l|l|l} 
Name & Function & Graph
\end{tabular}

\begin{tabular}{|c|c|c|c|c|}
\hline Name & Function & \multicolumn{3}{|l|}{Graph} \\
\hline Cubic & \(f(x)=x^{3}\) &  & \begin{tabular}{|c|}
\hline-1 \\
\hline-0.5 \\
\hline 0 \\
\hline 0.5 \\
\hline 1 \\
\hline
\end{tabular} & \begin{tabular}{|l|}
\hline \multicolumn{1}{|c|}{\(\boldsymbol{f}(\boldsymbol{x})\)} \\
\hline-1 \\
\hline-0.125 \\
\hline 0 \\
\hline 0.125 \\
\hline
\end{tabular} \\
\hline Reciprocal/ Rational & \(f(x)=\frac{1}{x}\) &  & \begin{tabular}{|c|}
\hline-2 \\
\hline-1 \\
\hline-0.5 \\
\hline 0.5 \\
\hline 1 \\
\hline
\end{tabular} & \begin{tabular}{|c|}
\hline\(f(x)\) \\
\hline-0.5 \\
\hline-1 \\
\hline-2 \\
\hline 2 \\
\hline 1 \\
\hline 0.5 \\
\hline
\end{tabular} \\
\hline Reciprocal / Rational squared & \(f(x)=\frac{1}{x^{2}}\) &  & \begin{tabular}{|l|}
\hline \(\boldsymbol{x}\) \\
\hline-2 \\
\hline-1 \\
\hline-0.5 \\
\hline 0.5 \\
\hline 1 \\
\hline 2 \\
\hline
\end{tabular} & \begin{tabular}{l}
\(f(\boldsymbol{x})\) \\
\hline 0.25 \\
\hline 1 \\
\hline 4 \\
\hline 4 \\
\hline 1 \\
\hline 0.25 \\
\hline
\end{tabular} \\
\hline Square root & \(f(x)=\sqrt{x}\) &  & \begin{tabular}{l} 
\\
\\
\hline \\
\hline 0 \\
\hline 1 \\
\hline 4
\end{tabular} & \[
\begin{gathered}
\frac{f(x)}{0} \\
\hline 1
\end{gathered}
\] \\
\hline
\end{tabular}

Name
Function

Cube root
\(f(x)=\sqrt[3]{x}\)

Graph


\section*{TRY IT}

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\section*{TRY IT}

In this exercise, you will reference the nine toolkit functions presented above.
1. Graph each toolkit function using function notation on an online graphing calculator.
2. Make a table of values that references the function and includes at least the interval \([-5,5]\).
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\section*{SUMMARY: CHARACTERISTICS OF FUNCTIONS AND THEIR GRAPHS}

\section*{Key Equations}
\begin{tabular}{|l|l|}
\hline Constant function & \(f(x)=c\), where \(c\) is a constant \\
\hline Identity function & \(f(x)=x\) \\
\hline Absolute value function & \(f(x)=|x|\) \\
\hline Quadratic function & \(f(x)=x^{2}\) \\
\hline Cubic function & \(f(x)=x^{3}\) \\
\hline Reciprocal function & \(f(x)=\frac{1}{x}\) \\
\hline Reciprocal squared function & \(f(x)=\frac{1}{x^{2}}\) \\
\hline Square root function & \(f(x)=\sqrt{x}\) \\
\hline Cube root function & \(f(x)=\sqrt[3]{x}\) \\
\hline
\end{tabular}

\section*{Key Concepts}
- A relation is a set of ordered pairs. A function is a specific type of relation in which each domain value, or input, leads to exactly one range value, or output.
- Function notation is a shorthand method for relating the input to the output in the form \(y=f(x)\).
- In table form, a function can be represented by rows or columns that relate to input and output values.
- To evaluate a function we determine an output value for a corresponding input value. Algebraic forms of a function can be evaluated by replacing the input variable with a given value.
- To solve for a specific function value, we determine the input values that yield the specific output value.
- An algebraic form of a function can be written from an equation.
- Input and output values of a function can be identified from a table.
- Relating input values to output values on a graph is another way to evaluate a function.
- A function is one-to-one if each output value corresponds to only one input value.
- A graph represents a function if any vertical line drawn on the graph intersects the graph at no more than one point.
- A graph represents a one-to-one function if any horizontal line drawn on the graph intersects the graph at no more than one point.

\section*{Glossary}
dependent variable an output variable
domain the set of all possible input values for a relation
function a relation in which each input value yields a unique output value
horizontal line test a method of testing whether a function is one-to-one by determining whether any horizontal line intersects the graph more than once
independent variable an input variable
input each object or value in a domain that relates to another object or value by a relationship known as a function
one-to-one function a function for which each value of the output is associated with a unique input value output each object or value in the range that is produced when an input value is entered into a function range the set of output values that result from the input values in a relation
relation a set of ordered pairs
vertical line test a method of testing whether a graph represents a function by determining whether a vertical line intersects the graph no more than once

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\section*{INTRODUCTION TO DOMAIN AND RANGE OF FUNCTIONS}

\section*{Learning Outcomes}

By the end of this lesson, you will be able to:
- Find the domain of a function defined by an equation.
- Write the domain and range using standard notations.
- Find domain and range from a graph.
- Give domain and range of Toolkit Functions.
- Graph piecewise-defined functions.

If you're in the mood for a scary movie, you may want to check out one of the five most popular horror movies of all time-I am Legend, Hannibal, The Ring, The Grudge, and The Conjuring. Figure 1 shows the amount, in dollars, each of those movies grossed when they were released as well as the ticket sales for horror movies in general by year. Notice that we can use the data to create a function of the amount each movie earned or the total ticket sales for all horror movies by year. In creating various functions using the data, we can identify different independent and dependent variables, and we can analyze the data and the functions to determine the domain and range. In this section we will investigate methods for determining the domain and range of functions such as these.

Top-Five Grossing Horror Movies for years 2000-2013


Market Share of Horror Movies, by Year


Figure 1. Based on data compiled by www.the-numbers.com.

\section*{STANDARD NOTATION FOR DEFINING SETS}

\section*{LEARNING OUTCOMES}
- Write sets using set-builder, inequality, and interval notation
- Describe sets on the real number line using set builder, interval, and inequality notation

There are several ways to define sets of numbers or mathematical objects. The reason we are introducing this here is because we often need to define the sets of numbers that make up the inputs and outputs of a function.

How we write sets that make up the domain and range of functions often depends on how the relation or function are defined or presented to us. For example, we can use lists to describe the domain of functions that are given as sets of ordered pairs. If we are given an equation or graph, we might use inequalities or intervals to describe domain and range.

In this section, we will introduce the standard notation used to define sets, and give you a chance to practice writing sets in three ways, inequality notation, set-builder notation, and interval notation.

Consider the set \(\{x \mid 10 \leq x<30\}\), which describes the behavior of \(x\) in set-builder notation. The braces \(\}\) are read as "the set of," and the vertical bar | is read as "such that," so we would read \(\{x \mid 10 \leq x<30\}\) as "the set of \(x\)-values such that 10 is less than or equal to \(x\), and \(x\) is less than 30 ."

The table below compares inequality notation, set-builder notation, and interval notation.
\begin{tabular}{|c|c|c|c|}
\hline & Inequality Notation & Set-bullder Notation & Interval Notation \\
\hline  & \(5<h \leq 10\) & \(\{h \mid 5<h \leq 10\}\) & \((5,10]\) \\
\hline  & \(5 \leq h<10\) & \(\{h \mid 5 \leq h<10\}\) & \([5,10)\) \\
\hline  & \(5<h<10\) & \(\{h \mid 5<h<10\}\) & \((5,10)\) \\
\hline  & \(h<10\) & \(\{h \mid h<10\}\) & \((-\infty, 10)\) \\
\hline  & \(h>10\) & \(\{h \mid h>10\}\) & \((10, \infty)\) \\
\hline  & All real numbers & R & \((-\infty, \infty)\) \\
\hline
\end{tabular}

To combine two intervals using inequality notation or set-builder notation, we use the word "or." As we saw in earlier examples, we use the union symbol, \(\cup\), to combine two unconnected intervals. For example, the union of the sets \(\{2,3,5\}\) and \(\{4,6\}\) is the set \(\{2,3,4,5,6\}\). It is the set of all elements that belong to one or the other (or both) of the original two sets. For sets with a finite number of elements like these, the elements do not have to be listed in ascending order of numerical value. If the original two sets have some elements in common, those elements should be listed only once in the union set. For sets of real numbers on intervals, another example of a union is
\(\{x||x| \geq 3\}=(-\infty,-3] \cup[3, \infty)\)
This video describes how to use interval notation to describe a set.
This video describes how to use Set-Builder notation to describe a set.
The table below gives a summary of interval notation.


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\section*{A GENERAL NOTE: SET-BUILDER NOTATION AND INTERVAL NOTATION}

Set-builder notation is a method of specifying a set of elements that satisfy a certain condition. It takes the form \(\{x \mid\) statement about \(x\}\) which is read as, "the set of all \(x\) such that the statement about \(x\) is true." For example,
\(\{x \mid 4<x \leq 12\}\)
Interval notation is a way of describing sets that include all real numbers between a lower limit that may or may not be included and an upper limit that may or may not be included. The endpoint values are listed between brackets or parentheses. A square bracket indicates inclusion in the set, and a parenthesis indicates exclusion from the set. For example,
\((4,12]\)

\section*{HOW TO: GIVEN A LINE GRAPH, DESCRIBE THE SET OF VALUES USING INTERVAL NOTATION.}
1. Identify the intervals to be included in the set by determining where the heavy line overlays the real line.
2. At the left end of each interval, use [ with each end value to be included in the set (solid dot) or ( for each excluded end value (open dot).
3. At the right end of each interval, use ] with each end value to be included in the set (filled dot) or ) for each excluded end value (open dot).
4. Use the union symbol \(\cup\) to combine all intervals into one set.

\section*{EXAMPLE: DESCRIBING SETS ON THE REAL-NUMBER LINE}

Describe the intervals of values shown below using inequality notation, set-builder notation, and interval notation.


Answer
To describe the values, \(x\), included in the intervals shown, we would say, " \(x\) is a real number greater than or equal to 1 and less than or equal to 3 , or a real number greater than 5 ."

\section*{Inequality}
\(1 \leq x \leq 3\) or \(x>5\)
Set-builder notation
\[
\{x \mid 1 \leq x \leq 3 \text { or } x>5\}
\]

\section*{Interval notation}
\[
[1,3] \cup(5, \infty)
\]

Remember that, when writing or reading interval notation, using a square bracket means the boundary is included in the set. Using a parenthesis means the boundary is not included in the set.

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\section*{TRY IT}

Given the graph below, specify the graphed set in
1. words
2. set-builder notation
3. interval notation


\section*{Answer}

Words: values that are less than or equal to -2 , or values that are greater than or equal to -1 and less than 3.

Set-builder notation: \(\{x \mid x \leq-2\) or \(-1 \leq x<3\}\);
Interval notation: \((-\infty,-2] \cup[-1,3)\)
\begin{tabular}{|c|c|c|c|}
\hline Inequality & Interval Notation & Graph on Number Line & Description \\
\hline \(x>a\) & \((a, \infty)\) &  & \(x\) is greater than a \\
\hline \(x<a\) & \((-\infty, a)\) &  & \(x\) is less than \(a\) \\
\hline \(x \geq a\) & \([a, \infty)\) &  & \(x\) is greater than or equal to a \\
\hline \(x \leq a\) & \((-\infty, a]\) &  & \(x\) is less than or equal to a \\
\hline \(a<x<b\) & \((a, b)\) &  & \(x\) is strictly between \(a\) and \(b\) \\
\hline \(a \leq x<b\) & \([a, b)\) &  & \(x\) is between \(a\) and \(b\), to include a \\
\hline \(a<x \leq b\) & \((a, b]\) &  & \(x\) is between \(a\) and \(b\), to include \(b\) \\
\hline \(a \leq x \leq b\) & \([a, b]\) &  & \(x\) is between \(a\) and \(b\), to include \(a\) and \(b\) \\
\hline
\end{tabular}

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\section*{WRITE DOMAIN AND RANGE GIVEN AN EQUATION}

\section*{LEARNING OUTCOMES}
- Find the domain of a function defined by an equation.
- Write domain and range using standard notations.

In Functions and Function Notation, we were introduced to the concepts of domain and range. In this section we will practice determining domains and ranges for specific functions. Keep in mind that, in determining domains and ranges, we need to consider what is physically possible or meaningful in realworld examples, such as tickets sales and year in the horror movie example above. We also need to consider what is mathematically permitted. For example, we cannot include any input value that leads us to take an even root of a negative number if the domain and range consist of real numbers. Or in a function expressed as a formula, we cannot include any input value in the domain that would lead us to divide by 0.


We can visualize the domain as a "holding area" that contains "raw materials" for a "function machine" and the range as another "holding area" for the machine's products.

We can write the domain and range in interval notation, which uses values within brackets to describe a set of numbers. In interval notation, we use a square bracket [ when the set includes the endpoint and a parenthesis ( to indicate that the endpoint is either not included or the interval is unbounded. For example, if a person has \(\$ 100\) to spend, he or she would need to express the interval that is more than 0 and less than or equal to 100 and write ( 0,100 ]. We will discuss interval notation in greater detail later.

Let's turn our attention to finding the domain of a function whose equation is provided. Oftentimes, finding the domain of such functions involves remembering three different forms. First, if the function has no denominator or an even root, consider whether the domain could be all real numbers. Second, if there is a denominator in the function's equation, exclude values in the domain that force the denominator to be zero. Third, if there is an even root, consider excluding values that would make the radicand negative.

Before we begin, let us review the conventions of interval notation:
- The smallest term from the interval is written first.
- The largest term in the interval is written second, following a comma.
- Parentheses, ( or ), are used to signify that an endpoint is not included, called exclusive.
- Brackets, [ or ], are used to indicate that an endpoint is included, called inclusive.

Watch the following video to see more examples of how to find the domain of a rational function (one with a fraction).

The next video gives more examples of how to define the domain of a function that contains an even root.

\section*{EXAMPLE: FINDING THE DOMAIN OF A FUNCTION AS A SET OF ORDERED PAIRS}

Find the domain of the following function: \(\{(2,10),(3,10),(4,20),(5,30),(6,40)\}\).

\section*{Answer}

First identify the input values. The input value is the first coordinate in an ordered pair. There are no restrictions, as the ordered pairs are simply listed. The domain is the set of the first coordinates of the ordered pairs.
\(\{2,3,4,5,6\}\)

\section*{TRY IT}

Find the domain of the function:
\(\{(-5,4),(0,0),(5,-4),(10,-8),(15,-12)\}\)
Answer
\(\{-5,0,5,10,15\}\)
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\section*{HOW TO: GIVEN A FUNCTION WRITTEN IN EQUATION FORM, FIND THE DOMAIN.}
1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.

3 . Write the domain in interval form, if possible.

\section*{EXAMPLE: FINDING THE DOMAIN OF A FUNCTION}

Find the domain of the function \(f(x)=x^{2}-1\).
Answer
The input value, shown by the variable \(x\) in the equation, is squared and then the result is lowered by one.
Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.
In interval form the domain of \(f\) is \((-\infty, \infty)\).

\section*{TRY IT}

Find the domain of the function: \(f(x)=5-x+x^{3}\).
Answer
\((-\infty, \infty)\)
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\section*{HOW TO: GIVEN A FUNCTION WRITTEN IN AN EQUATION FORM THAT INCLUDES A FRACTION, FIND THE DOMAIN.}
1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for \(x\). These are the values that cannot be inputs in the function.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

\section*{EXAMPLE: FINDING THE DOMAIN OF A FUNCTION INVOLVING A DENOMINATOR (RATIONAL FUNCTION)}

Find the domain of the function \(f(x)=\frac{x+1}{2-x}\).

\section*{Answer}

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for \(x\).
\(2-x=0\)
\(-x=-2\)
\(x=2\)
Now, we will exclude 2 from the domain. The answers are all real numbers where \(x<2\) or \(x>2\). We can use a symbol known as the union, \(\cup\), to combine the two sets. In interval notation, we write the solution:
\((-\infty, 2) \cup(2, \infty)\).


In interval form, the domain of \(f\) is \((-\infty, 2) \cup(2, \infty)\).


\section*{TRY IT}

Find the domain of the function: \(f(x)=\frac{1+4 x}{2 x-1}\).
Answer
\(\left(-\infty, \frac{1}{2}\right) \cup\left(\frac{1}{2}, \infty\right)\)
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\section*{HOW TO: GIVEN A FUNCTION WRITTEN IN EQUATION FORM INCLUDING AN EVEN ROOT, FIND THE DOMAIN.}
1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for \(x\).
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

\section*{DIVISION BY ZERO AND EVEN ROOTS OF NEGATIVE NUMBERS}

While zero divided by any number equals zero, division by zero results in an undefined ratio.
\(\frac{0}{a}=0 \quad\) but \(\quad \frac{b}{0}=u n d\)
An even root of a negative number does not exist in the real numbers.
\(\sqrt{-1}=i\)
Since the domain of any function defined in the real plane is the set of all real input into the function, we must exclude any values of the input variable that create undefined expressions or even roots of a negative.

\section*{EXAMPLE: FINDING THE DOMAIN OF A FUNCTION WITH AN EVEN ROOT}

Find the domain of the function \(f(x)=\sqrt{7-x}\).
Answer
When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.
Set the radicand greater than or equal to zero and solve for \(x\).
\(7-x \geq 0\)
\(-x \geq-7\)
\(x \leq 7\)
Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7 , or \((-\infty, 7]\).


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\section*{TRY IT}

Find the domain of the function \(f(x)=\sqrt{5+2 x}\).
Answer
\(\left[-\frac{5}{2}, \infty\right)\)
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\section*{Q \& A}

\section*{Can there be functions in which the domain and range do not intersect at all?}

Yes. For example, the function \(f(x)=-\frac{1}{\sqrt{x}}\) has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

\section*{TRY IT}

When you are defining the domain of a function, it can help to graph it, especially when you have a rational or a function with an even root.
First, determine the domain restrictions for the following functions, then graph each one to check whether your domain agrees with the graph.
1. \(f(x)=\sqrt{2 x-4}+5\)
2. \(g(x)=\frac{2 x+4}{x-1}\)

\section*{HOW TO: GIVEN THE FORMULA FOR A FUNCTION, DETERMINE THE DOMAIN AND RANGE.}
1. Exclude from the domain any input values that result in division by zero.
2. Exclude from the domain any input values that have nonreal (or undefined) number outputs.
3. Use the valid input values to determine the range of the output values.
4. Look at the function graph and table values to confirm the actual function behavior.

\section*{EXAMPLE: FINDING THE DOMAIN AND RANGE USING TOOLKIT FUNCTIONS}

Find the domain and range of \(f(x)=2 x^{3}-x\).
Answer
There are no restrictions on the domain, as any real number may be cubed and then subtracted from the result.
The domain is \((-\infty, \infty)\) and the range is also \((-\infty, \infty)\).

\section*{EXAMPLE: FINDING THE DOMAIN AND RANGE}

Find the domain and range of \(f(x)=\frac{2}{x+1}\).
Answer
We cannot evaluate the function at -1 because division by zero is undefined. The domain is \((-\infty,-1) \cup(-1, \infty)\). Because the function is never zero, we exclude 0 from the range. The range is \((-\infty, 0) \cup(0, \infty)\).

\section*{EXAMPLE: FINDING THE DOMAIN AND RANGE}

Find the domain and range of \(f(x)=2 \sqrt{x+4}\).
Answer
We cannot take the square root of a negative number, so the value inside the radical must be nonnegative.
\(x+4 \geq 0\) when \(x \geq-4\)
The domain of \(f(x)\) is \([-4, \infty)\).
We then find the range. We know that \(f(-4)=0\), and the function value increases as \(x\) increases without any upper limit. We conclude that the range of \(f\) is \([0, \infty)\).

\section*{Analysis of the Solution}

The graph below represents the function \(f\).


\section*{TRY IT}

Find the domain and range of \(f(x)=-\sqrt{2-x}\).
Answer
Domain: \((-\infty, 2]\) Range: \((-\infty, 0]\)

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\section*{DETERMINE DOMAIN AND RANGE FROM A GRAPH}

\section*{LEARNING OUTCOMES}
- Find domain and range from a graph, and an equation.
- Give the domain and range of the toolkit functions.

Another way to identify the domain and range of functions is by using graphs. Because the domain refers to the set of possible input values, the domain of a graph consists of all the input values shown on the \(x\)-axis. The range is the set of possible output values, which are shown on the \(y\)-axis. Keep in mind that if the graph continues beyond the portion of the graph we can see, the domain and range may be greater than the visible values.


We can observe that the graph extends horizontally from -5 to the right without bound, so the domain is \([-5, \infty)\). The vertical extent of the graph is all range values 5 and below, so the range is \((-\infty, 5]\). Note that the domain and range are always written from smaller to larger values, or from left to right for domain, and from the bottom of the graph to the top of the graph for range.

\section*{Domain and Range of Toolkit Functions}

\section*{EXAMPLE: FINDING DOMAIN AND RANGE FROM A GRAPH}

Find the domain and range of the function \(f\).


Answer
We can observe that the horizontal extent of the graph is -3 to 1 , so the domain of \(f\) is \((-3,1]\).


The vertical extent of the graph is 0 to -4 , so the range is \([-4,0]\).


\section*{EXAMPLE: FINDING DOMAIN AND RANGE FROM A GRAPH OF OIL PRODUCTION}

Find the domain and range of the function \(f\).
Alaska Crude Oil Production

(credit: modification of work by the U.S. Energy Information Administration)

\section*{Answer}

The input quantity along the horizontal axis is "years," which we represent with the variable \(t\) for time. The output quantity is "thousands of barrels of oil per day," which we represent with the variable \(b\) for barrels.
The graph may continue to the left and right beyond what is viewed, but based on the portion of the graph that is visible, we can determine the domain as \(1973 \leq t \leq 2008\) and the range as approximately \(180 \leq b \leq 2010\).
In interval notation, the domain is [1973, 2008], and the range is about [180, 2010]. For the domain and the range, we approximate the smallest and largest values since they do not fall exactly on the grid lines.

\section*{TRY IT}

Given the graph, identify the domain and range using interval notation.


\section*{Answer}

Domain \(=[1950,2002]\) Range \(=[47,000,000,89,000,000]\)
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\section*{Q \& A}

Can a function's domain and range be the same?
Yes. For example, the domain and range of the cube root function are both the set of all real numbers.
We will now return to our set of toolkit functions to determine the domain and range of each.


Domain: \((-\infty, \infty)\)
Range: \([c, c]\)
For the constant function \(f(x)=c\), the domain consists of all real numbers; there are no restrictions on the input. The only output value is the constant \(c\), so the range is the set \(\{c\}\) that contains this single element. In interval notation, this is written as \([c, c]\), the interval that both begins and ends with \(c\).


Domain: \((-\infty, \infty)\)
Range: \((-\infty, \infty)\)
For the identity function \(f(x)=x\), there is no restriction on \(x\). Both the domain and range are the set of all real numbers.


Domain: \((-\infty, \infty)\) Range: \([0, \infty)\)

For the absolute value function \(f(x)=|x|\), there is no restriction on \(x\). However, because absolute value is defined as a distance from 0 , the output can only be greater than or equal to 0 .


Domain: \((-\infty, \infty)\)
Range: \([0, \infty)\)
For the quadratic function \(f(x)=x^{2}\), the domain is all real numbers since the horizontal extent of the graph is the whole real number line. Because the graph does not include any negative values for the range, the range is only nonnegative real numbers.


Domain: \((-\infty, \infty)\)
Range: \((-\infty, \infty)\)
For the cubic function \(f(x)=x^{3}\), the domain is all real numbers because the horizontal extent of the graph is the whole real number line. The same applies to the vertical extent of the graph, so the domain and range include all real numbers.

\[
\begin{gathered}
\text { Domain: }(-\infty, 0) \cup(0, \infty) \\
\text { Range: }(-\infty, 0) \cup(0, \infty)
\end{gathered}
\]

For the reciprocal function \(f(x)=\frac{1}{x}\), we cannot divide by 0 , so we must exclude 0 from the domain. Further, 1 divided by any value can never be 0 , so the range also will not include 0 . In set-builder notation, we could also write \(\{x \mid x \neq 0\}\), the set of all real numbers that are not zero.


Domain: \((-\infty, 0) \cup(0, \infty)\)
Range: \((0, \infty)\)
For the reciprocal squared function \(f(x)=\frac{1}{x^{2}}\), we cannot divide by 0 , so we must exclude 0 from the domain. There is also no \(x\) that can give an output of 0 , so 0 is excluded from the range as well. Note that the output of this function is always positive due to the square in the denominator, so the range includes only positive numbers.


\section*{Domain: \([0, \infty)\) \\ Range: [0, \(\infty\) )}

For the square root function \(f(x)=\sqrt{x}\), we cannot take the square root of a negative real number, so the domain must be 0 or greater. The range also excludes negative numbers because the square root of a positive number \(x\) is defined to be positive, even though the square of the negative number \(-\sqrt{x}\) also gives us \(x\).


Domain: \((-\infty, \infty)\)
Range: \((-\infty, \infty)\)
For the cube root function \(f(x)=\sqrt[3]{x}\), the domain and range include all real numbers. Note that there is no problem taking a cube root, or any odd-integer root, of a negative number, and the resulting output is negative (it is an odd function).

\section*{TIP FOR SUCCESS}

When evaluating functions to plot points, remember to wrap the variable in parentheses to be sure you handle negative input appropriately.
Ex. For the function \(f(x)=x^{2}\), evaluate \(f(-3)\).
Answer
\[
\begin{aligned}
f(-3) & =(-3)^{2} \\
& =9
\end{aligned}
\]

\section*{TRY IT}

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\section*{PIECEWISE-DEFINED FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Write piecewise defined functions.
- Graph piecewise-defined functions.

Sometimes, we come across a function that requires more than one formula in order to obtain the given output. For example, in the toolkit functions, we introduced the absolute value function \(f(x)=|x|\). With a domain of all real numbers and a range of values greater than or equal to 0 , absolute value can be defined as the magnitude, or modulus, of a real number value regardless of sign. It is the distance from 0 on the number line. All of these definitions require the output to be greater than or equal to 0 .

If we input 0 , or a positive value, the output is the same as the input.
\(f(x)=x\) if \(x \geq 0\)
If we input a negative value, the output is the opposite of the input.
\(f(x)=-x\) if \(x<0\)
Because this requires two different processes or pieces, the absolute value function is an example of a piecewise function. A piecewise function is a function in which more than one formula is used to define the output over different pieces of the domain.

We use piecewise functions to describe situations in which a rule or relationship changes as the input value crosses certain "boundaries." For example, we often encounter situations in business for which the cost per piece of a certain item is discounted once the number ordered exceeds a certain value. Tax brackets are another real-world example of piecewise functions. For example, consider a simple tax system in which
incomes up to \(\$ 10,000\) are taxed at 10 , and any additional income is taxed at \(20 \%\). The tax on a total income, \(S\), would be \(0.1 S\) if \(S \leq \$ 10,000\) and \(1000+0.2(S-\$ 10,000)\), if \(S>\$ 10,000\).

\section*{A GENERAL NOTE: PIECEWISE FUNCTIONS}

A piecewise function is a function in which more than one formula is used to define the output. Each formula has its own domain, and the domain of the function is the union of all these smaller domains. We notate this idea like this:
\(f(x)=\left\{\begin{array}{l}\text { formula } 1 \text { if } \mathrm{x} \text { is in domain } 1 \\ \text { formula } 2 \text { if } \mathrm{x} \text { is in domain } 2 \\ \text { formula } 3 \text { if } \mathrm{x} \text { is in domain } 3\end{array}\right.\)
In piecewise notation, the absolute value function is
\(|x|=\left\{\begin{array}{r}x \text { if } x \geq 0 \\ -x \text { if } x<0\end{array}\right.\)

\section*{HOW TO: GIVEN A PIECEWISE FUNCTION, WRITE THE FORMULA AND IDENTIFY THE DOMAIN FOR EACH INTERVAL.}
1. Identify the intervals for which different rules apply.
2. Determine formulas that describe how to calculate an output from an input in each interval.
3. Use braces and if-statements to write the function.

\section*{TIP FOR SUCCESS: PIECEWISE FUNCTIONS}

The curly brace that appears in a piecewise formula doesn't denote a set, since there is no matching brace on the other side of the statement. Instead, it indicates that the right hand side of the function formula is made of different pieces, each depending on the input.
You can read it as a list of if-then statements for the function value.
Piecewise functions are challenging to understand. Give yourself time to work with them repeatedly on paper, evaluating and graphing different functions. It's natural to need to work many problems before you feel comfortable with them.

\section*{EXAMPLE: WRITING A PIECEWISE FUNCTION}

A museum charges \(\$ 5\) per person for a guided tour with a group of 1 to 9 people or a fixed \(\$ 50\) fee for a group of 10 or more people. Write a function relating the number of people, \(n\), to the cost, \(C\).

\section*{Answer}

Two different formulas will be needed. For \(n\)-values under 10, \(C=5 n\). For values of \(n\) that are 10 or greater, \(C=50\).
\(C(n)=\left\{\begin{array}{l}5 n \text { if } 0<n<10 \\ 50 \text { if } n \geq 10\end{array}\right.\)

\section*{Analysis of the Solution}

The graph is a diagonal line from \(n=0\) to \(n=10\) and a constant after that. In this example, the two formulas agree at the meeting point where \(n=10\), but not all piecewise functions have this property.


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\section*{EXAMPLE: WORKING WITH A PIECEWISE FUNCTION}

A cell phone company uses the function below to determine the cost, \(C\), in dollars for \(g\) gigabytes of data transfer.
\(C(g)=\left\{\begin{aligned} 25 & \text { if } 0<g<2 \\ 25+10(g-2) & \text { if } g \geq 2\end{aligned}\right.\)
Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

\section*{Answer}

To find the cost of using 1.5 gigabytes of data, \(C(1.5)\), we first look to see which part of the domain our input falls in. Because 1.5 is less than 2, we use the first formula.
\(C(1.5)=\$ 25\)
To find the cost of using 4 gigabytes of data, \(C(4)\), we see that our input of 4 is greater than 2 , so we use the second formula.
\(C(4)=25+10(4-2)=\$ 45\)

\section*{Analysis of the Solution}

We can see where the function changes from a constant to a shifted and stretched identity at \(g=2\). We plot the graphs for the different formulas on a common set of axes, making sure each formula is applied on its proper domain.


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\section*{HOW TO: GIVEN A PIECEWISE FUNCTION, SKETCH A GRAPH.}
1. Indicate on the \(x\)-axis the boundaries defined by the intervals on each piece of the domain.
2. For each piece of the domain, graph on that interval using the corresponding equation pertaining to that piece. Do not graph two functions over one interval because it would violate the criteria of a function.


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\section*{EXAMPLE: GRAPHING A PIECEWISE FUNCTION}

Sketch a graph of the function.
\(f(x)=\left\{\begin{array}{llr}x^{2} & \text { if } x \leq 1 \\ 3 & \text { if } 1 & <x \leq 2 \\ x & \text { if } x & >2\end{array}\right.\)

\section*{Answer}

Each of the component functions is from our library of toolkit functions, so we know their shapes. We can imagine graphing each function and then limiting the graph to the indicated domain. At the endpoints of the domain, we draw open circles to indicate where the endpoint is not included because of a less-than or greater-than inequality; we draw a closed circle where the endpoint is included because of a less-than-or-equal-to or greater-than-or-equal-to inequality.
Below are the three components of the piecewise function graphed on separate coordinate systems.
(a) \(f(x)=x^{2}\) if \(x \leq 1\); (b) \(f(x)=3\) if \(1<x \leq 2\); (c) \(f(x)=x\) if \(x>2\)

(a)

(b)

(c)

Now that we have sketched each piece individually, we combine them in the same coordinate plane.


\section*{Analysis of the Solution}

Note that the graph does pass the vertical line test even at \(x=1\) and \(x=2\) because the points \((1,3)\) and \((2,2)\) are not part of the graph of the function, though \((1,1)\) and \((2,3)\) are.

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\section*{TRY IT}

You can use an online graphing calculator to graph piecewise defined functions. Watch this tutorial video to learn how within Desmos. Other online graphing tools will be slightly different.


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Graph the following piecewise function with an online graphing calculator.
\(f(x)=\left\{\begin{array}{rll}x^{3} & \text { if } x & <-1 \\ -2 & \text { if }-1 & <x<4 \\ \sqrt{x} & \text { if } x & >4\end{array}\right.\)

\section*{TRY IT}

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\section*{Q\&A}

Can more than one formula from a piecewise function be applied to a value in the domain?
No. Each value corresponds to one equation in a piecewise formula.
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\section*{SUMMARY: DOMAIN AND RANGE OF FUNCTIONS}

\section*{Key Concepts}
- The domain of a function includes all real input values that would not cause us to attempt an undefined mathematical operation, such as dividing by zero or taking the square root of a negative number.
- The domain of a function can be determined by listing the input values of a set of ordered pairs.
- The domain of a function can also be determined by identifying the input values of a function written as an equation.
- Interval values represented on a number line can be described using inequality notation, set-builder notation, and interval notation.
- For many functions, the domain and range can be determined from a graph.
- An understanding of toolkit functions can be used to find the domain and range of related functions.
- A piecewise function is described by more than one formula.
- A piecewise function can be graphed using each algebraic formula on its assigned subdomain.

\section*{Glossary}
interval notation a method of describing a set that includes all numbers between a lower limit and an upper limit; the lower and upper values are listed between brackets or parentheses, a square bracket indicating inclusion in the set, and a parenthesis indicating exclusion
piecewise function a function in which more than one formula is used to define the output
set-builder notation a method of describing a set by a rule that all of its members obey; it takes the form \(\{x \mid\) statement about \(x\}\)

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\section*{INTRODUCTION TO RATES OF CHANGE AND BEHAVIORS OF GRAPHS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Find the average rate of change of a function.
- Use a graph to determine where a function is increasing, decreasing, or constant.
- Use a graph to locate local maxima and local minima.
- Use a graph to locate the absolute maximum and absolute minimum.

Gasoline costs have experienced some wild fluctuations over the last several decades. The table below (Note: http://www.eia.gov/totalenergy/data/annual/showtext.cfm?t=ptb0524. Accessed 3/5/2014.) lists the average cost, in dollars, of a gallon of gasoline for the years 2005-2012. The cost of gasoline can be considered as a function of year.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|}
\hline\(y\) & 2005 & 2006 & 2007 & 2008 & 2009 & 2010 & 2011 & 2012 \\
\hline\(C(y)\) & 2.31 & 2.62 & 2.84 & 3.30 & 2.41 & 2.84 & 3.58 & 3.68 \\
\hline
\end{tabular}

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \(\$ 2.31\) to \(\$ 3.68\), an increase of \(\$ 1.37\). While this is interesting, it might be more useful to look at how much the price changed per year. In this section, we will investigate changes such as these.

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\section*{RATES OF CHANGE}

\section*{LEARNING OUTCOMES}
- Find the average rate of change of a function.

Gasoline costs have experienced some wild fluctuations over the last several decades. The table below (Note: http://www.eia.gov/totalenergy/data/annual/showtext.cfm?t=ptb0524. Accessed 3/5/2014.) lists the
average cost, in dollars, of a gallon of gasoline for the years 2005-2012. The cost of gasoline can be considered as a function of year.
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\hline\(y\) & 2005 & 2006 & 2007 & 2008 & 2009 & 2010 & 2011 & 2012 \\
\hline\(C(y)\) & 2.31 & 2.62 & 2.84 & 3.30 & 2.41 & 2.84 & 3.58 & 3.68 \\
\hline
\end{tabular}

If we were interested only in how the gasoline prices changed between 2005 and 2012, we could compute that the cost per gallon had increased from \(\$ 2.31\) to \(\$ 3.68\), an increase of \(\$ 1.37\). While this is interesting, it might be more useful to look at how much the price changed per year. In this section, we will investigate changes such as these.

The price change per year is a rate of change because it describes how an output quantity changes relative to the change in the input quantity. We can see that the price of gasoline in the table above did not change by the same amount each year, so the rate of change was not constant. If we use only the beginning and ending data, we would be finding the average rate of change over the specified period of time. To find the average rate of change, we divide the change in the output value by the change in the input value.
\[
\begin{aligned}
\text { Average rate of change } & =\frac{\text { Change in output }}{\text { Change in input }} \\
& =\frac{\Delta y}{\Delta x} \\
& =\frac{y_{2}-y_{1}}{x_{2}-x_{1}} \\
& =\frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{x_{2}-x_{1}}
\end{aligned}
\]

\section*{RECALL THE SLOPE OF A LINE}

The formula for the average rate of change of any function looks a lot like the formula for finding the slope of a line. That's because the average rate of change between two points on the graph of a function is the slope of the straight line between those points.
Since \(y=f(x)\), we write \((x, y)\) as \((x, f(x))\) and \(\frac{\text { rise }}{\text { run }} \Rightarrow \frac{y_{2}-y_{1}}{x_{2}-x_{1}} \Rightarrow \frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{x_{2}-x_{1}}\).
The Greek letter \(\Delta\) (delta) signifies the change in a quantity; we read the ratio as "delta- \(y\) over delta- \(x\) " or "the change in \(y\) divided by the change in \(x\)." Occasionally we write \(\Delta f\) instead of \(\Delta y\), which still represents the change in the function's output value resulting from a change to its input value.

In our example, the gasoline price increased by \(\$ 1.37\) from 2005 to 2012 . Over 7 years, the average rate of change was
\(\frac{\Delta y}{\Delta x}=\frac{1.37}{7 \text { years }} \approx 0.196\) dollars per year
On average, the price of gas increased by about 19.6¢ each year.
Other examples of rates of change include:
- A population of rats increasing by 40 rats per week
- A car traveling 68 miles per hour (distance traveled changes by 68 miles each hour as time passes)
- A car driving 27 miles per gallon of gasoline (distance traveled changes by 27 miles for each gallon)
- The current through an electrical circuit increasing by 0.125 amperes for every volt of increased voltage
- The amount of money in a college account decreasing by \(\$ 4,000\) per quarter

The following video provides another example of how to find the average rate of change between two points

\section*{A GENERAL NOTE: RATE OF CHANGE}

A rate of change describes how an output quantity changes relative to the change in the input quantity. The units on a rate of change are "output units per input units."
The average rate of change between two input values is the total change of the function values (output values) divided by the change in the input values.
\(\frac{\Delta y}{\Delta x}=\frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{x_{2}-x_{1}}\)

\section*{HOW TO: GIVEN THE VALUE OF A FUNCTION AT DIFFERENT POINTS, CALCULATE THE AVERAGE RATE OF CHANGE OF A FUNCTION FOR THE INTERVAL BETWEEN TWO VALUES \(x_{1}\) AND \(x_{2}\).}
1. Calculate the difference \(y_{2}-y_{1}=\Delta y\).
2. Calculate the difference \(x_{2}-x_{1}=\Delta x\).
3. Find the ratio \(\frac{\Delta y}{\Delta x}\).

\section*{EXAMPLE: COMPUTING AN AVERAGE RATE OF CHANGE}

Using the data in the table below, find the average rate of change of the price of gasoline between 2007 and 2009.
\begin{tabular}{|l|l|l|l|l|l|l|l|}
\hline\(y\) & 2005 & 2006 & 2007 & 2008 & 2009 & 2010 & 2011 \\
\hline\(C(y)\) & 2.31 & 2.62 & 2.84 & 3.30 & 2.41 & 2.84 & 3.58 \\
\hline & 3.68 \\
\hline
\end{tabular}

Answer
In 2007, the price of gasoline was \(\$ 2.84\). In 2009, the cost was \(\$ 2.41\). The average rate of change is
\[
\frac{\Delta y}{\Delta x}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}
\]
\[
\begin{aligned}
& =\frac{\$ 2.41-\$ 2.84}{2009-2007} \\
& =\frac{-\$ 0.43}{2 \text { years }} \\
& =-\$ 0.22 \text { per year }
\end{aligned}
\]

\section*{Analysis of the Solution}

Note that a decrease is expressed by a negative change or "negative increase." A rate of change is negative when the output decreases as the input increases or when the output increases as the input decreases.
from a table of values.
The following video provides another example of finding the average rate of change of a function given a formula and an interval.


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\section*{TRY IT}

Using the data in the table below, find the average rate of change between 2005 and 2010.
\begin{tabular}{|l|l|l|l|l|l|l|l|}
\hline\(y\) & 2005 & 2006 & 2007 & 2008 & 2009 & 2010 & 2011 \\
\hline\(C(y)\) & 2.31 & 2.62 & 2.84 & 3.30 & 2.41 & 2.84 & 3.58 \\
\hline & 3.68 \\
\hline
\end{tabular}

Answer
\(\frac{\$ 2.84-\$ 2.31}{5 \text { years }}=\frac{\$ 0.53}{5 \text { years }}=\$ 0.106\) per year.
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\section*{EXAMPLE: COMPUTING AVERAGE RATE OF CHANGE FROM A TABLE}

After picking up a friend who lives 10 miles away, Anna records her distance from home over time. The values are shown in the table below. Find her average speed over the first 6 hours.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|}
\hline \(\boldsymbol{t}\) (hours) & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline \(\boldsymbol{D}(\boldsymbol{t})\) (miles) & 10 & 55 & 90 & 153 & 214 & 240 & 282 & 300 \\
\hline Answer
\end{tabular}

Here, the average speed is the average rate of change. She traveled 282 miles in 6 hours, for an average speed of
\[
\frac{292-10}{6-0}=\frac{282}{6}
\]
\[
=47
\]

The average speed is 47 miles per hour.

\section*{Analysis of the Solution}

Because the speed is not constant, the average speed depends on the interval chosen. For the interval [2,3], the average speed is 63 miles per hour.

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\section*{EXAMPLE: COMPUTING AVERAGE RATE OF CHANGE FROM A GRAPH}

Given the function \(g(t)\), find the average rate of change on the interval \([-1,2]\).


\section*{Answer}


At \(t=-1\), the graph shows \(g(-1)=4\). At \(t=2\), the graph shows \(g(2)=1\).
The horizontal change \(\Delta t=3\) is shown by the red arrow, and the vertical change \(\Delta g(t)=-3\) is shown by the turquoise arrow. The output changes by -3 while the input changes by 3 , giving an average rate of change of
\(\frac{1-4}{2-(-1)}=\frac{-3}{3}=-1\)

\section*{Analysis of the Solution}

Note that the order we choose is very important. If, for example, we use \(\frac{y_{2}-y_{1}}{x_{1}-x_{2}}\), we will not get the correct answer. Decide which point will be 1 and which point will be 2, and keep the coordinates fixed as \(\left(x_{1}, y_{1}\right)\) and \(\left(x_{2}, y_{2}\right)\).

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\section*{EXAMPLE: COMPUTING AVERAGE RATE OF CHANGE FOR A FUNCTION EXPRESSED AS A FORMULA}

Compute the average rate of change of \(f(x)=x^{2}-\frac{1}{x}\) on the interval \([2,4]\).

\section*{Answer}

We can start by computing the function values at each endpoint of the interval.
\[
\begin{aligned}
f(2) & =2^{2}-\frac{1}{2} & f(4) & =4^{2}-\frac{1}{4} \\
& =4-\frac{1}{2} & & =16-\frac{1}{4} \\
& =\frac{7}{2} & & =\frac{63}{4}
\end{aligned}
\]

Now we compute the average rate of change.
Average rate of change \(=\frac{f(4)-f(2)}{4-2}\)
\[
\begin{aligned}
& =\frac{\frac{63}{4}-\frac{7}{2}}{4-2} \\
& =\frac{\frac{49}{4}}{2} \\
& =\frac{49}{8}
\end{aligned}
\]


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\section*{TRY IT}

Find the average rate of change of \(f(x)=x-2 \sqrt{x}\) on the interval \([1,9]\).
Answer
\(\frac{1}{2}\)

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\section*{EXAMPLE: FINDING THE AVERAGE RATE OF CHANGE OF A FORCE}

The electrostatic force \(F\), measured in newtons, between two charged particles can be related to the distance between the particles \(d\), in centimeters, by the formula \(F(d)=\frac{2}{d^{2}}\). Find the average rate of change of force if the distance between the particles is increased from 2 cm to 6 cm .
Answer
We are computing the average rate of change of \(F(d)=\frac{2}{d^{2}}\) on the interval \([2,6]\).
Average rate of change \(=\frac{F(6)-F(2)}{6-2}\)
\[
\begin{array}{lr}
=\frac{\frac{2}{6^{2}}-\frac{2}{2^{2}}}{6-2} & \text { Simplify } \\
=\frac{\frac{2}{36}-\frac{2}{4}}{4} & \\
=\frac{-\frac{16}{36}}{4} & \text { Combine numerator terms } \\
=-\frac{1}{9} & \text { Simplify }
\end{array}
\]

The average rate of change is \(-\frac{1}{9}\) newton per centimeter.

\section*{EXAMPLE: FINDING AN AVERAGE RATE OF CHANGE AS AN EXPRESSION}

Find the average rate of change of \(g(t)=t^{2}+3 t+1\) on the interval \([0, a]\). The answer will be an expression involving \(a\).

\section*{Answer}

We use the average rate of change formula.
\[
\begin{array}{rlr}
\text { Average rate of change } & =\frac{g(a)-g(0)}{a-0} & \text { Evaluate } \\
& =\frac{\left(a^{2}+3 a+1\right)-\left(0^{2}+3(0)+1\right)}{a-0} & \\
& =\frac{a^{2}+3 a+1-1}{a} & \text { Simplify } \\
& =\frac{a(a+3)}{a} & \text { Divide by the common factor } a \\
& =a+3 &
\end{array}
\]

This result tells us the average rate of change in terms of \(a\) between \(t=0\) and any other point \(t=a\). For example, on the interval \([0,5]\), the average rate of change would be \(5+3=8\).

\section*{TRY IT}

Find the average rate of change of \(f(x)=x^{2}+2 x-8\) on the interval \([5, a]\).

\section*{Answer}
\(a+7\)
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## BEHAVIORS OF FUNCTIONS

## LEARNING OUTCOMES

- Determine where a function is increasing, decreasing, or constant
- Find local extrema of a function from a graph
- Describe behavior of the toolkit functions

As part of exploring how functions change, we can identify intervals over which the function is changing in specific ways. We say that a function is increasing on an interval if the function values increase as the input values increase within that interval. Similarly, a function is decreasing on an interval if the function values decrease as the input values increase over that interval. The average rate of change of an increasing function is positive, and the average rate of change of a decreasing function is negative. The graph below shows examples of increasing and decreasing intervals on a function.


The function $f(x)=x^{3}-12 x$ is increasing on $(-\infty,-2) \cup(2, \infty)$ and is decreasing on $(-2,2)$.
This video further explains how to find where a function is increasing or decreasing.
While some functions are increasing (or decreasing) over their entire domain, many others are not. A value of the output where a function changes from increasing to decreasing (as we go from left to right, that is, as the input variable increases) is called a local maximum. If a function has more than one, we say it has local maxima. Similarly, a value of the output where a function changes from decreasing to increasing as the input variable increases is called a local minimum. The plural form is "local minima." Together, local maxima and minima are called local extrema, or local extreme values, of the function. (The singular form is "extremum.") Often, the term local is replaced by the term relative. In this text, we will use the term local.

A function is neither increasing nor decreasing on an interval where it is constant. A function is also neither increasing nor decreasing at extrema. Note that we have to speak of local extrema, because any given local extremum as defined here is not necessarily the highest maximum or lowest minimum in the function's entire


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domain.
For the function below, the local maximum is 16 , and it occurs at $x=-2$. The local minimum is -16 and it occurs at $x=2$.


To locate the local maxima and minima from a graph, we need to observe the graph to determine where the graph attains its highest and lowest points, respectively, within an open interval. Like the summit of a roller coaster, the graph of a function is higher at a local maximum than at nearby points on both sides. The graph will also be lower at a local minimum than at neighboring points. The graph below illustrates these ideas for a local maximum.


Definition of a local maximum.
These observations lead us to a formal definition of local extrema.

## A GENERAL NOTE: LOCAL MINIMA AND LOCAL MAXIMA

A function $f$ is an increasing function on an open interval if $f(b)>f(a)$ for any two input values $a$ and $b$ in the given interval where $b>a$.
A function $f$ is a decreasing function on an open interval if $f(b)<f(a)$ for any two input values $a$ and $b$ in the given interval where $b>a$.
A function $f$ has a local maximum at $x=b$ if there exists an interval ( $a, c$ ) with $a<b<c$ such that, for any $x$ in the interval $(a, c), f(x) \leq f(b)$. Likewise, $f$ has a local minimum at $x=b$ if there exists an interval ( $a, c$ ) with $a<b<c$ such that, for any $x$ in the interval $(a, c), f(x) \geq f(b)$.

## Analyzing the Toolkit Functions for Increasing or Decreasing Intervals

We will now return to our toolkit functions and discuss their graphical behavior in the table below.

## EXAMPLE: FINDING INCREASING AND DECREASING INTERVALS ON A GRAPH

Given the function $p(t)$ in the graph below, identify the intervals on which the function appears to be increasing.


## Answer

We see that the function is not constant on any interval. The function is increasing where it slants upward as we move to the right and decreasing where it slants downward as we move to the right. The function appears to be increasing from $t=1$ to $t=3$ and from $t=4$ on.
In interval notation, we would say the function appears to be increasing on the interval $(1,3)$ and the interval $(4, \infty)$.

## Analysis of the Solution

Notice in this example that we used open intervals (intervals that do not include the endpoints), because the function is neither increasing nor decreasing at $t=1, t=3$, and $t=4$. These points are the local extrema (two minima and a maximum).

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## TIP FOR SUCCESS: INCREASING/DECREASING BEHAVIOR

The behavior of the function values of the graph of a function is read over the $x$-axis, from left to right. That is,

- a function is said to be increasing if its function values increase as x increases;
- a function is said to be decreasing if its function values decrease as x increases.


## EXAMPLE: FINDING LOCAL EXTREMA FROM A GRAPH

Graph the function $f(x)=\frac{2}{x}+\frac{x}{3}$. Then use the graph to estimate the local extrema of the function and to determine the intervals on which the function is increasing.

## Answer

Using technology, we find that the graph of the function looks like that below. It appears there is a low point, or local minimum, between $x=2$ and $x=3$, and a mirror-image high point, or local maximum, somewhere between $x=-3$ and $x=-2$.


## Analysis of the Solution

Most graphing calculators and graphing utilities can estimate the location of maxima and minima. The graph below provides screen images from two different technologies, showing the estimate for the local maximum and minimum.

(a)

(b)

Based on these estimates, the function is increasing on the interval $(-\infty,-2.449)$ and $(2.449, \infty)$. Notice that, while we expect the extrema to be symmetric, the two different technologies agree only up to four decimals due to the differing approximation algorithms used by each. (The exact locations of the extrema are at $\pm \sqrt{6}$, but determining this requires calculus.)

## TIP FOR SUCCESS: READING EXTREMA

Recall that points on the graph of a function are ordered pairs in the form of (input, output) $=(x, f(x))$.
If a function's graph has a local minimum or maximum at some point $(x, f(x))$, we say
"the extrema occurs at $x$, and that the minimum or maximum is $f(x)$."

## TRY IT

Graph the function $f(x)=x^{3}-6 x^{2}-15 x+20$ to estimate the local extrema of the function. Use these to determine the intervals on which the function is increasing and decreasing.

## Answer

The local maximum appears to occur at $(-1,28)$, and the local minimum occurs at $(5,-80)$. The function is increasing on $(-\infty,-1) \cup(5, \infty)$ and decreasing on $(-1,5)$.


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## EXAMPLE: FINDING LOCAL MAXIMA AND MINIMA FROM A GRAPH

For the function $f$ whose graph is shown below, find all local maxima and minima.


Answer
Observe the graph of $f$. The graph attains a local maximum at $x=1$ because it is the highest point in an open interval around $x=1$. The local maximum is the $y$-coordinate at $x=1$, which is 2 .
The graph attains a local minimum at $x=-1$ because it is the lowest point in an open interval around $x=-1$. The local minimum is the $y$-coordinate at $x=-1$, which is -2 .

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## TIP FOR SUCCESS: TOOLKIT FUNCTIONS

The toolkit functions continue to appear throughout the course. Have you memorized them yet? Would you be able to sketch a quick graph of each from its equation?

| Function | Increasing/Decreasing |
| :--- | :--- |
| Constant Function $f(x)=c$ | Neither increasing nor decreasing |
| Quadratic Function $f(x)=x^{2}$ | Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ |
| Square Root $f(x)=\sqrt{x}$ | Minimum at $x=0$ |
| Cubic Function $f(x)=x^{3}$ | Increasing on $(0, \infty)$ |
| Reciprocal $\operatorname{Squared} f(x)=\frac{1}{x^{2}}$ | Increasing on $(-\infty, 0)$ Decreasing on $(0, \infty)$ |
|  |  |


| Function | Increasing/Decreasing |
| :--- | :--- |
| Absolute Value $f(x)=\|x\|$ | Increasing on $(0, \infty)$ Decreasing on $(-\infty, 0)$ |
|  | Minimum at $x=0$ |

## Use a graph to locate the absolute maximum and absolute minimum of a function

There is a difference between locating the highest and lowest points on a graph in a region around an open interval (locally) and locating the highest and lowest points on the graph for the entire domain. The $y$ coordinates (output) at the highest and lowest points are called the absolute maximum and absolute minimum, respectively.

To locate absolute maxima and minima from a graph, we need to observe the graph to determine where the graph attains it highest and lowest points on the domain of the function.


Not every function has an absolute maximum or minimum value. The toolkit function $f(x)=x^{3}$ is one such function.

## A GENERAL NOTE: ABSOLUTE MAXIMA AND MINIMA

The absolute maximum of $f$ at $x=c$ is $f(c)$ where $f(c) \geq f(x)$ for all $x$ in the domain of $f$. The absolute minimum of $f$ at $x=d$ is $f(d)$ where $f(d) \leq f(x)$ for all $x$ in the domain of $f$.

## EXAMPLE: FINDING ABSOLUTE MAXIMA AND MINIMA FROM A GRAPH

For the function $f$ shown below, find all absolute maxima and minima.


## Answer

Observe the graph of $f$. The graph attains an absolute maximum in two locations, $x=-2$ and $x=2$, because at these locations, the graph attains its highest point on the domain of the function. The absolute maximum is the $y$-coordinate at $x=-2$ and $x=2$, which is 16 .
The graph attains an absolute minimum at $x=3$, because it is the lowest point on the domain of the function's graph. The absolute minimum is the $y$-coordinate at $x=3$, which is -10 .

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# SUMMARY: RATES OF CHANGE AND BEHAVIORS OF GRAPHS 

## Key Equations

$$
\frac{\Delta y}{\Delta x}=\frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{x_{2}-x_{1}}
$$

Key Concepts

- A rate of change relates a change in an output quantity to a change in an input quantity. The average rate of change is determined using only the beginning and ending data.
- Identifying points that mark the interval on a graph can be used to find the average rate of change.
- Comparing pairs of input and output values in a table can also be used to find the average rate of change.
- An average rate of change can also be computed by determining the function values at the endpoints of an interval described by a formula.
- The average rate of change can sometimes be determined as an expression.
- A function is increasing where its rate of change is positive and decreasing where its rate of change is negative.
- A local maximum is where a function changes from increasing to decreasing and has an output value larger (more positive or less negative) than output values at neighboring input values.
- A local minimum is where the function changes from decreasing to increasing (as the input increases) and has an output value smaller (more negative or less positive) than output values at neighboring input values.
- Minima and maxima are also called extrema.
- We can find local extrema from a graph.
- The highest and lowest points on a graph indicate the maxima and minima.


## Glossary

absolute maximum the greatest value of a function over an interval
absolute minimum the lowest value of a function over an interval
average rate of change the difference in the output values of a function found for two values of the input divided by the difference between the inputs
decreasing function a function is decreasing in some open interval if $f(b)<f(a)$ for any two input values $a$ and $b$ in the given interval where $b>a$
increasing function a function is increasing in some open interval if $f(b)>f(a)$ for any two input values $a$ and $b$ in the given interval where $b>a$
local extrema collectively, all of a function's local maxima and minima
local maximum a value of the input where a function changes from increasing to decreasing as the input value increases.
local minimum a value of the input where a function changes from decreasing to increasing as the input value increases.
rate of change the change of an output quantity relative to the change of the input quantity

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## PUTTING IT TOGETHER: FUNCTION BASICS

At the beginning of the module, you were considering Galileo's famous experiment in which he dropped, or thought about dropping, two balls of different masses from the top of the Tower of Pisa. And you looked at data describing a falling object. Now that you learned about functions, let's take another look at the data.

| Time (s) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Velocity <br> $(\mathrm{m} / \mathrm{s})$ | 0 | 9.8 | 19.6 | 29.4 | 39.2 | 49.0 | 58.8 | 68.6 | 78.4 |
| Distance <br> $(\mathrm{m})$ | 0 | 4.9 | 19.6 | 44.1 | 78.4 | 122.5 | 176.4 | 240.1 | 313.6 |

Chart shows the correlation between time (seconds 1-8), velocity and distance.

Consider the set of ordered pairs relating time to velocity.
$\{(0,0),(1,9.8),(2,19.6),(3,29.4),(4,39.2),(5,49.0),(6,58.8),(7,68.6)$, and $(8,78.4)\}$

Is the relation a function? Indeed it is. Each input value corresponds to only one output value.


Now consider the set of ordered pairs relating time to distance.
$\{(0,0),(1,4.9),(2,19.6),(3,44.1),(4,78.4),(5,122.5),(6,176.4),(7,240.1)$, and $(8,313.6)\}$

As before, this relation is also a function.

| Inputs |  |  |
| :---: | :---: | :---: |
| 0 |  | Outputs |
| 1 |  |  |
| 1 |  | -0 |
| 2 |  |  |
| 3 |  | -19.9 |
| 4 |  | -44.1 |
| 5 |  | -122.5 |
| 6 |  | -176.4 |
| 7 |  | -240.1 |
| 8 |  | -313.6 |

Now that you know that both velocity and distance can be described by functions, you can evaluate them from the table.

- Suppose velocity as a function of time is represented as $V(t)$. Evaluate and explain $V(4)$. To evaluate, find the value of the function at 4 seconds. At 4 seconds, velocity is $39.2 \mathrm{~m} / \mathrm{s}$. Therefore, $V(4)=39.2$.
- Similarly, solve for $t$ when $V(t)=49.0$. Find $49.0 \mathrm{~m} / \mathrm{s}$ on the table and read the related time, 5 s . That means that the velocity is $49.0 \mathrm{~m} / \mathrm{s}$ after 5 seconds so $V(5)=49.0$.
- Suppose now that the distance is represented as $D(t)$. Evaluate and explain $D(6)$. To evaluate, find the value of the function at 6 seconds. It is 176.4 m , which means that the falling object travels 176.4 m in 6 seconds.
- We can also solve $D(t)=240.1$. Find 240.1 m in the table and read the related time, which is 7 seconds. So $D(7)=240.1$.

You can also use the table to try to determine the function formulas. Begin by reading across the velocity data to look for a pattern. Each value is 9.8 times the number of seconds. So the velocity as a function of time can be represented by the formula $V(t)=a t$, where a is acceleration (in this case due to gravity) and $t$ is time.

Finding the function formula for distance is a little trickier. The distance as a function of time can be represented by the formula, where $a$ is acceleration and $t$ is time.

Now that you know the function formulas, you can evaluate them for any value of time. Suppose it takes an object dropped from a higher floor of the Burj Khalifa 10 seconds to fall to the ground. How fast would the object be traveling when it hit, and how far would it have fallen in that time?

We can evaluate the functions for velocity and distance at 10 seconds to answer these questions.
Substitute 10 seconds for $t$ in each function. Remember that acceleration due to gravity is $9.8 \mathrm{~m} / \mathrm{s}^{2}$.

| $V(t)=a t$ | $D(10)=\frac{1}{2} a t^{2}$ |
| :--- | :--- |
| $V(10)=a(10)$ | $D(10)=\frac{1}{2} a(10)^{2}$ |
| $V(10)=(9.8)(10)$ | $D(10)=\frac{1}{2}(9.8)(10)^{2}$ |
| $V(10)=98$ | $D(10)=490$ |

Add these values to the table to the table and what do you see? Velocity and distance really start to increase the longer the object falls. Let's evaluate the functions at 13 seconds and add those results to the table as well.

| Time (s) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 10 | 13 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Velocity <br> $(\mathrm{m} / \mathrm{s})$ | 0 | 9.8 | 19.6 | 29.4 | 39.2 | 49.0 | 58.8 | 68.6 | 78.4 | 98 | 127.4 |
| Distance <br> $(\mathrm{m})$ | 0 | 4.9 | 19.6 | 44.1 | 78.4 | 122.5 | 176.4 | 240.1 | 313.6 | 490 | 828.1 |

Chart shows the correlation between time (seconds 1-8, 10 and 13), velocity and distance.

That means that it would take about 13 seconds for an object to fall the tremendous height of the tallest building in the world, and it would be traveling about $127 \mathrm{~m} / \mathrm{s}$. That is almost 300 miles per hour! You certainly don't want to be under that when it falls. Thankfully, due to your knowledge of functions, you would know to stay far away.

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# MODULE 6: ALGEBRAIC OPERATIONS ON FUNCTIONS 

## WHY IT MATTERS: ALGEBRAIC OPERATIONS ON FUNCTIONS

## Why Study Algebraic Operations on Functions?

If you could "see" the sound coming from your speakers or earphones, it might resemble a series of ups and downs corresponding to the vibrations caused by the sound as it travels through the air. In fact, with the help of a tool called an oscilloscope, it is possible to visualize sounds as waves (technically, sound travels through air as a series of compression
 waves, however the oscilloscope converts sound to transverse waves for better visualization).

The wave itself may be regarded as the graph of a function. In other words, the sound wave may be modeled by the rule
$y=f(x)$
It is not essential to understand exactly how this function is put together - you might learn more about the details of sound waves after studying a little trigonometry - however we can use what we learn in this module to transform the wave in certain ways.

For example, if we made the sound louder (increasing the amplitude), then the oscilloscope would display the same basic wave graph but with higher peaks and deeper valleys. The new function is a vertical stretch of the original.

Conversely, if we made the sound softer (decreasing the amplitude), then the new wave would have lower peaks and shallower valleys, which is a vertical compression.

Figure 1 illustrates how the graph of a function in black might stretch vertically into the blue curve, or compress vertically into the red curve. What is the effect on the original function? It turns out that vertical compression and stretching can be accomplished by multiplying or dividing the output of the function by a constant (a):

Some sounds, such as musical tones, also have pitch, or a measure of how high or how low the tone sounds. If run your finger along the keys of a piano from left to right, the tones increase steadily in pitch. Pitch is measured by the frequency of a wave, which is the number of peaks within a given time period. Frequency is easy to see on the oscilloscope: the more wave peaks you see on the screen, the higher the frequency, and the higher the pitch of the sound.


Figure 2

To change the frequency of the wave we must stretch or compress the graph horizontally. Figure 2 illustrates the effect of horizontal compression (blue curve) and stretching (red curve) of the original curve (in black). If the black curve represents a musical tone, then the blue curve will sound higher, while the red curve will sound lower in pitch. Horizontal stretching and compression is handled by multiplying (or dividing) a constant to the input $x$ :
$y=f(b \cdot x)$

In this module, you will learn all about these kinds of transformations along with other ways of changing and combining functions.

## LEARNING OUTCOMES

## Review Topics for Success

- Find the domain of a square root and rational function
- Find the domain of a function from its algebraic form.
- Define the domain of linear, quadratic, radical, and rational functions from graphs


## Compositions of Functions

- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.


## Transformations of Functions

- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the $x$-axis and the $y$-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.
- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

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## REVIEW TOPICS FOR SUCCESS

## Learning Outcomes

By the end of this section, you will be able to:

- Find the domain of a square root and rational function
- Find the domain of a function from its algebraic form.
- Define the domain of linear, quadratic, radical, and rational functions from graphs

You've seen that algebraic expressions such as polynomials and rational expressions can be manipulated arithmetically. That is, we can perform the operations of arithmetic on them by adding, subtracting, multiplying, and dividing expressions. When we do these arithmetic operations on algebraic expressions, we call them algebraic operations.

In this module, you'll learn how to manipulate functions by performing algebraic operations to combine them arithmetically. You'll also see a new way to combine functions called composition of function, in which a new function is created by using the output of one function as the input for another function.

Functions can be manipulated in graphical form, too. We can shift them up or down, left or right on the coordinate plane, stretch or shrink them, or flip them over vertically or horizontally. And we can perform two or more of these transformations at once. We'll see how these transformations of the graphs of functions appear in the plane and how they can be detected in the equation of a function.

Finally, we'll look at inverse functions. You are already familiar with pairs of inverse operations. We think of subtraction for instance as reversing the procedure of addition and division as the "opposite" operation of multiplication. Cubing the cube root of a number undoes both operations and returns to us just the number inside the radical. The cube root function is thus said to be the inverse of the cubing function. We'll examine inverse operations algebraically and graphically.

Each time we combine functions, we create a new function that must be carefully defined with regard to what values may be used as input to the function. Before we learn how to combine functions, it will be necessary to formally discuss an idea you've seen already regarding rational and radical expressions: domain restrictions.

When we transform functions, the domain or range can become altered. So we'll also take a look in this review section at reading the domain and range of a function from a graph.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

## RECALL FOR SUCCESS

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

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\section*{DOMAIN RESTRICTIONS}

\section*{LEARNING OUTCOMES}
- Find the domain of a square root and rational function
- Find the domain of a function from its algebraic form.

Functions are a correspondence between two sets, called the domain and the range. When defining a function, you usually state what kind of numbers the domain \((x)\) and range \((f(x))\) values can be. But even if you say they are real numbers, that does not mean that all real numbers can be used for \(x\). It also does not mean that all real numbers can be function values, \(f(x)\). There may be restrictions on the domain and range. The restrictions partly depend on the type of function.

In this topic, all functions will be restricted to real number values. That is, only real numbers can be used in the domain, and only real numbers can be in the range.

There are two main reasons why domains are restricted.
- You cannot divide by 0 .
- You cannot take the square (or other even) root of a negative number, as the result will not be a real number.

These two issues would be of concern when the function is either of the following:
- A rational function where the denominator could potentially become 0 for some value or values of \(x\), \(f(x)=\frac{x+1}{2-x}\) is a rational function.
- A radical function with an even index (such as a square root), where the radicand (quantity under the radical) could potentially be negative for some value or values of \(x . f(x)=\sqrt{7-x}\) is a radical function.

The following table gives examples of domain restrictions for several different rational functions. Note that a rational function has the variable present in the denominator.
Function Restrictions to the Domain
\(f(x)=\frac{1}{x} \quad\) If \(x=0\), you would be dividing by 0 , so \(x \neq 0\).
\(f(x)=\frac{2+x}{x-3} \quad\) If \(x=3\), you would be dividing by 0 , so \(x \neq 3\).
\(f(x)=\frac{2(x-1)}{x-1}\) Although you can simplify this function to \(f(x)=2\), when \(x=1\), the original function would include division by 0 , so \(x \neq 1\).

Both \(x=1\) and \(x=-1\) would make the denominator 0 . Again, this function can be
\(f(x)=\frac{x+1}{x^{2}-1} \quad\) simplified to \(f(x)=\frac{1}{x-1}\), but when \(x=1\) or \(x=-1\), the original function would include division by 0 , so \(x \neq 1\) and \(x \neq-1\).
\(f(x)=\frac{2(x-1)}{x^{2}+1}\)
This is an example with no domain restrictions, even though there is a variable in the denominator. Since \(x^{2} \geq 0, x^{2}+1\) can never be 0 . The least it can be is 1 , so there is no danger of division by 0 .

Square roots of negative numbers could happen whenever the function has a variable under a radical with an even root. Look at the following examples and note that "square root of a negative variable" doesn't necessarily mean that the value under the radical sign is negative. For example, if \(x=-4\), then \(-x=-(-4)=4\), a positive number.

\section*{Function}

\section*{Restrictions to the Domain}
\(f(x)=\sqrt{x} \quad\) If \(x<0\), you would be taking the square root of a negative number, so \(x \geq 0\).
\(f(x)=\sqrt{x+10} \quad\) If \(x<-10\), you would be taking the square root of a negative number, so \(x \geq-10\).
\(\begin{array}{ll}f(x)=\sqrt{-x} & \begin{array}{l}\text { When is }-x \text { negative? Only when } \mathrm{x} \text { is positive. (For example, if } x=1, \text { then }-x=-1 . \\ \text { But if } x=-3, \text { then }-x=3 \text {. This means } x \leq 0 .\end{array}\end{array}\)
\(f(x)=\sqrt{x^{2}-1}\)
\(x^{2}-1\) must be positive, \(x^{2}-1>0\). So \(x^{2}>1\). This happens only when x is greater than 1 or less than -1 : \(x \leq-1\) or \(x \geq 1\).

There are no domain restrictions even though there is a variable under the radical.
\(f(x)=\sqrt{x^{2}+10}\) Since \(x^{2} \geq 0, x^{2}+10\) can never be negative. The least it can be is \(\sqrt{10}\), so there is no danger of taking the square root of a negative number.

So, how exactly do you define the domain of a function?

\section*{HOW TO: GIVEN A FUNCTION WRITTEN IN EQUATION FORM, FIND THE DOMAIN}
1. Identify the input values.
2. Identify any restrictions on the input and exclude those values from the domain.
3. Write the domain in interval form, if possible.

The domain and range of a function are not necessarily the same. For example, the function \(f(x)=-\frac{1}{\sqrt{x}}\) has the set of all positive real numbers as its domain but the set of all negative real numbers as its range. As a more extreme example, a function's inputs and outputs can be completely different categories (for example, names of weekdays as inputs and numbers as outputs, as on an attendance chart), in such cases the domain and range have no elements in common.

\section*{EXAMPLE}

Find the domain of the function \(f(x)=x^{2}-1\).
Answer
The input value, shown by the variable \(x\) in the equation, is squared and then the result is lowered by one.
Any real number may be squared and then be lowered by one, so there are no restrictions on the domain of this function. The domain is the set of real numbers.
In interval form, the domain of \(f\) is \((-\infty, \infty)\).

\section*{HOW TO: GIVEN A FUNCTION WRITTEN IN EQUATION FORM THAT INCLUDES A FRACTION, FIND THE DOMAIN}
1. Identify the input values.
2. Identify any restrictions on the input. If there is a denominator in the function's formula, set the denominator equal to zero and solve for \(x\). If the function's formula contains an even root, set the radicand greater than or equal to 0 and then solve.
3. Write the domain in interval form, making sure to exclude any restricted values from the domain.

\section*{EXAMPLE}

Find the domain of the function \(f(x)=\frac{x+1}{2-x}\).

\section*{Answer}

When there is a denominator, we want to include only values of the input that do not force the denominator to be zero. So, we will set the denominator equal to 0 and solve for \(x\).
\(2-x=0\)
\(-x=-2\)
\(x=2\)
Now, we will exclude 2 from the domain. The answers are all real numbers where \(x<2\) or \(x>2\). We can use a symbol known as the union, \(\cup\), to combine the two sets. In interval notation, we write the solution: \((-\infty, 2) \cup(2, \infty)\).


In interval form, the domain of \(f\) is \((-\infty, 2) \cup(2, \infty)\).


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\section*{HOW TO: GIVEN A FUNCTION WRITTEN IN EQUATION FORM INCLUDING AN EVEN ROOT, FIND THE DOMAIN}
1. Identify the input values.
2. Since there is an even root, exclude any real numbers that result in a negative number in the radicand. Set the radicand greater than or equal to zero and solve for \(x\).
3. The solution(s) are the domain of the function. If possible, write the answer in interval form.

\section*{EXAMPLE}

Find the domain of the function \(f(x)=\sqrt{7-x}\).
Answer
When there is an even root in the formula, we exclude any real numbers that result in a negative number in the radicand.
Set the radicand greater than or equal to zero and solve for \(x\).
\(7-x \geq 0\)
\(-x \geq-7\)
\(x \leq 7\)
Now, we will exclude any number greater than 7 from the domain. The answers are all real numbers less than or equal to 7 , or \((-\infty, 7]\).


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FIND DOMAIN AND RANGE FROM A GRAPH

\section*{LEARNING OUTCOME}
- Define the domain of linear, quadratic, radical, and rational functions from graphs

Finding domain and range of different functions is often a matter of asking yourself, what values can this function not have? Pictures make it easier to visualize what the domain and range are, so we will show how to define the domain and range of functions given their graphs.

What are the domain and range of the real-valued function \(f(x)=x+3\) ?
This is a linear function. Remember that linear functions are lines that continue forever in each direction.


Any real number can be substituted for \(x\) and get a meaningful output. For any real number, you can always find an \(x\) value that gives you that number for the output. Unless a linear function is a constant, such as \(f(x)=2\), there is no restriction on the range.
The domain and range are all real numbers.
For the examples that follow, try to figure out the domain and range of the graphs before you look at the answer.

\section*{Example}

What are the domain and range of the real-valued function \(f(x)=-3 x^{2}+6 x+1\) ?


\section*{Answer}

This is a quadratic function. There are no rational (divide by zero) or radical (negative number under a root) expressions, so there is nothing to restrict from the domain. Any real number can be used for \(x\) to get a meaningful output.

Because the coefficient of \(x^{2}\) is negative, it will open downward. With quadratic functions, remember that there is either a maximum (greatest) value or a minimum (least) value. In this case, there is a maximum value.

The vertex, or high point, is at \((1,4)\). From the graph, you can see that \(f(x) \leq 4\).
The domain is all real numbers, and the range is all real numbers \(f(x)\) such that \(f(x) \leq 4\).

\section*{Example}

What is the domain and range of the real-valued function \(f(x)=-2+\sqrt{x+5}\) ?


\section*{Answer}

This is a radical function. The domain of a radical function is any \(x\) value for which the radicand (the value under the radical sign) is not negative. That means \(x+5 \geq 0\), so \(x \geq-5\).

Since the square root must always be positive or \(0, \sqrt{x+5} \geq 0\). That means \(-2+\sqrt{x+5} \geq-2\).
The domain is all real numbers \(x\) where \(x \geq-5\), and the range is all real numbers \(f(x)\) such that \(f(x) \geq-2\).

\section*{Example}

What is the domain of the real-valued function \(f(x)=\frac{3 x}{x+2}\) ?


\section*{Answer}

This is a rational function. The domain of a rational function is restricted where the denominator is 0 . In this case, \(x+2\) is the denominator, and this is 0 only when \(x=-2\).

The domain is all real numbers except -2

In the following video we show how to define the domain and range of functions from their graphs.


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\section*{Summary}

Although a function may be given as "real valued," it may be that the function has restrictions to its domain and range. There may be some real numbers that cannot be part of the domain or part of the range. This is particularly true with rational and radical functions which can have restrictions to their domain, range, or both. Other functions, such as quadratic functions and polynomial functions of even degree, can also have restrictions to their range.

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\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
- Division by 0 is undefined.
- Values of the input variable that would make the denominator of a rational expression equal to zero must be stated and excluded from the domain of a function containing the expression.
- To restrict the domain of a rational function, set the denominators each equal to zero. Solve for the variable in each denominator and exclude those solution sets.
- Taking an even root (e.g., a square root) of a negative number yields an unreal result.
- Values of the input variable that would place a negative amount under an even root must be stated and excluded from the domain of a function containing the expression.
- To restrict the domain of a function containing one radical, set the radicand greater than or equal to zero. Solve for the variable. The resulting solution set is the domain of the function.
- The domain of a function is read from the x-axis (the horizontal, or independent, axis) of the graph of the function.
- The range of a function is read from the \(y\)-axis (the vertical, or dependent axis) of the graph of the function.

\section*{Glossary}
domain the set of all possible input into a function
radical function a function containing a radical
radicand the value underneath the radical sign
range the set of all possible output from a function
rational function a function containing a rational expression (a fraction)

\section*{INTRODUCTION TO COMPOSITIONS OF FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Combine functions using algebraic operations.
- Create a new function by composition of functions.
- Evaluate composite functions.
- Find the domain of a composite function.
- Decompose a composite function into its component functions.

Suppose we want to calculate how much it costs to heat a house on a particular day of the year. The cost to heat a house will depend on the average daily temperature, and in turn, the average daily temperature depends on the particular day of the year. Notice how we have just defined two relationships: The cost depends on the temperature, and the temperature depends on the day.

\section*{Cost for the temperature}


\section*{Temperature on day 5}

Figure 1
Using descriptive variables, we can notate these two functions. The function \(C(T)\) gives the cost \(C\) of heating a house for a given average daily temperature in \(T\) degrees Celsius. The function \(T(d)\) gives the average daily temperature on day \(d\) of the year. For any given day, Cost \(=C(T(d))\) means that the cost depends on the temperature, which in turns depends on the day of the year. Thus, we can evaluate the cost function at the temperature \(T(d)\). For example, we could evaluate \(T(5)\) to determine the average daily temperature on the 5th day of the year. Then, we could evaluate the cost function at that temperature. We would write \(C(T(5))\).

By combining these two relationships into one function, we have performed function composition, which is the focus of this section.
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\section*{COMPOSITIONS OF FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Combine functions using algebraic operations.
- Create a new function by composition of functions.

Function composition, as we saw in the introduction and as we'll explore in detail later in this section, is a way to combine existing functions. Another way to combine functions is to carry out the usual algebraic operations on functions, such as addition, subtraction, multiplication and division. We do this by performing the operations with the function outputs, defining the result as the output of our new function.

Suppose we need to add two columns of numbers that represent a husband and wife's separate annual incomes over a period of years, with the result being their total household income. We want to do this for every year, adding only that year's incomes and then collecting all the data in a new column. If \(w(y)\) is the wife's income and \(h(y)\) is the husband's income in year \(y\), and we want \(T\) to represent the total income, then we can define a new function.
\[
T(y)=h(y)+w(y)
\]

If this holds true for every year, then we can focus on the relation between the functions without reference to a year and write
\[
T=h+w
\]

Just as for this sum of two functions, we can define difference, product, and ratio functions for any pair of functions that have the same kinds of inputs (not necessarily numbers) and also the same kinds of outputs (which do have to be numbers so that the usual operations of algebra can apply to them, and which also must have the same units or no units when we add and subtract). In this way, we can think of adding, subtracting, multiplying, and dividing functions.

For two functions \(f(x)\) and \(g(x)\) with real number outputs, we define new functions \(f+g, f-g, f g\), and \(\frac{f}{g}\) by the relations
\[
\begin{aligned}
c(f+g)(x) & =f(x)+g(x) \\
(f-g)(x) & =f(x)-g(x) \\
(f g)(x) & =f(x) g(x) \\
\left(\frac{f}{g}\right)(x) & =\frac{f(x)}{g(x)}
\end{aligned}
\]

\section*{EXAMPLE: PERFORMING ALGEBRAIC OPERATIONS ON FUNCTIONS}

Find and simplify the functions \((g-f)(x)\) and \(\left(\frac{g}{f}\right)(x)\), given \(f(x)=x-1\) and \(g(x)=x^{2}-1\). Give the domain of your result. Are they the same function?

\section*{Answer}

Begin by writing the general form, and then substitute the given functions.
\[
\begin{aligned}
(g-f)(x) & =g(x)-f(x) \\
(g-f)(x) & =x^{2}-1-(x-1) \\
& =x^{2}-x \\
& =x(x-1)
\end{aligned}
\]
\[
\begin{aligned}
\left(\frac{g}{f}\right)(x) & =\frac{g(x)}{f(x)} \\
\left(\frac{g}{f}\right)(x) & =\frac{x^{2}-1}{x-1} \\
& =\frac{(x+1)(x-1)}{x-1} \text { where } x \neq 1 \\
& =x+1
\end{aligned}
\]

The domain \((g-f)(x)\) is all real numbers and the domain of \(\left(\frac{g}{f}\right)(x)\) is \(x \neq 1\). The functions are not the same.
Note: For \(\left(\frac{g}{f}\right)(x)\), the condition \(x \neq 1\) is necessary because when \(x=1, f(x)=1\) is equal to 0 , which makes the quotient function undefined.

\section*{RECALL OPERATIONS ON ALGEBRAIC EXPRESSIONS}

See Module 2: Polynomial and Rational Expressions for a refresher on how to do operations on algebraic expressions.
- Use order of operations when multiple operations are present
- When subtracting one polynomial expression from another, be sure to subtract each term of the second from the first
- Use the distributive property when multiplying two polynomial expressions

\section*{TRY IT NOW}

Find and simplify the functions \((f g)(x)\) and \((f-g)(x)\).
\[
f(x)=x-1 \text { and } g(x)=x^{2}-1
\]

Are they the same function?

\section*{Answer}
\((f g)(x)=f(x) g(x)=(x-1)\left(x^{2}-1\right)=x^{3}-x^{2}-x+1\)
\((f-g)(x)=f(x)-g(x)=(x-1)-\left(x^{2}-1\right)=x-x^{2}\)
No, the functions are not the same.
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\section*{TRY IT}

Let's explore what happens to the graphs of functions that are combined using algebraic operations. Use an online graphing calculator to graph the following functions:
1. \(f(x)=x^{2}+3 x-4\)
2. \(g(x)=\frac{1}{x-1}\)

Now, enter \(h(x)=f(x)+g(x)\) into the next line.
Evaluate \(h(1)\), why do you think you get this result?

\section*{Create a New Function Using a Composition}

Performing algebraic operations on functions combines them into a new function, but we can also create functions by composing functions. When we wanted to compute a heating cost from a day of the year in the introductory example, we created a new function that takes a day as input and yields a cost as output. The process of combining functions so that the output of one function becomes the input of another is known as a composition of functions. The resulting function is known as a composite function. We represent this combination by the following notation:
\((f \circ g)(x)=f(g(x))\)
We read the left-hand side as " \(f\) composed with \(g\) at \(x\), " and the right-hand side as " \(f\) of \(g\) of \(x\)." The two sides of the equation have the same mathematical meaning and are equal. The open circle symbol o is called the composition operator. We use this operator mainly when we wish to emphasize the relationship between the functions themselves without referring to any particular input value. Composition is a binary operation that takes two functions and forms a new function, much as addition or multiplication takes two numbers and gives a new number. However, it is important not to confuse function composition with multiplication because, as we learned above, in most cases \(f(g(x)) \neq f(x) g(x)\).

It is also important to understand the order of operations in evaluating a composite function. We follow the usual convention with parentheses by starting with the innermost parentheses first, and then working to the
outside. In the equation above, the function \(g\) takes the input \(x\) first and yields an output \(g(x)\). Then the function \(f\) takes \(g(x)\) as an input and yields an output \(f(g(x))\).


\section*{Figure 2}

In general \(f \circ g\) and \(g \circ f\) are different functions. In other words in many cases \(f(g(x)) \neq g(f(x))\) for all \(x\).
For example if \(f(x)=x^{2}\) and \(g(x)=x+2\), then
\[
\begin{aligned}
f(g(x)) & =f(x+2) \\
& =(x+2)^{2} \\
& =x^{2}+4 x+4
\end{aligned}
\]
but
\[
\begin{aligned}
g(f(x)) & =g\left(x^{2}\right) \\
& =x^{2}+2
\end{aligned}
\]

These expressions are not equal for all values of \(x\), so the two functions are not equal. It is irrelevant that the expressions happen to be equal for the single input value \(x=-\frac{1}{2}\).

Note that the range of the inside function (the first function to be evaluated) needs to be within the domain of the outside function. Less formally, the composition has to make sense in terms of inputs and outputs. Additionally, in applied settings, function composition usually only makes sense in one specific order.

\section*{A GENERAL NOTE: COMPOSITION OF FUNCTIONS}

When the output of one function is used as the input of another, we call the entire operation a composition of functions. For any input \(x\) and functions \(f\) and \(g\), this action defines a composite function, which we write as \(f \circ g\) such that
\((f \circ g)(x)=f(g(x))\)
The domain of the composite function \(f \circ g\) is all \(x\) such that \(x\) is in the domain of \(g\) and \(g(x)\) is in the domain of \(f\).
It is important to realize that the product of functions \(f g\) is not the same as the function composition \(f(g(x))\), because, in general, \(f(x) g(x) \neq f(g(x))\).

\section*{EXAMPLE: DEFINE A COMPOSITION OF FUNCTIONS}

Using the functions provided, find \(f(g(x))\) and \(g(f(x))\).
\(f(x)=2 x+1\)
\(g(x)=3-x\)
Answer
Let's begin by substituting \(g(x)\) into \(f(x)\).
\[
\begin{aligned}
f(g(x)) & =2(3-x)+1 \\
& =6-2 x+1 \\
& =7-2 x
\end{aligned}
\]

Now we can substitute \(f(x)\) into \(g(x)\).
\(g(f(x))=3-(2 x+1)\)
\(=3-2 x-1\)
\(=-2 x+2\)
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\section*{TRY IT NOW}

Now we will explore the graph of a composition of functions. Use an online graphing calculator to graph the following functions:
1. \(f(x)=x^{2}+3 x-4\)
2. \(g(x)=\frac{1}{x-1}\)

Now define a new function:
\[
h(x)=f(f(x))
\]

Now zoom in (use the plus sign in the upper right-hand corner of the graph) on the point ( \(0.23607,-3.236\) ), notice that both graphs pass through this point. Continue this pattern for a few more iterations: define a new function that contains another composition of \(f(x)\) with itself. For example: \(p(x)=f(f(f(x)))\), and so on. Continue to use the zoom as you iterate.
The point ( \(0.23607,-3.236\) ) is called a fixed point of the function \(f(x)=x^{2}+3 x-4\). Fixed points are used in mathematical applications such as the page rank algorithm that Google uses to generate internet search results. Read more about fixed points here, and page rank here.

\section*{EXAMPLE: INTERPRETING COMPOSITE FUNCTIONS}

The function \(c(s)\) gives the number of calories burned completing \(s\) sit-ups, and \(s(t)\) gives the number of sit-ups a person can complete in \(t\) minutes. Interpret \(c(s(3))\).

\section*{Answer}

The inside expression in the composition is \(s(3)\). Because the input to the s-function is time, \(t=3\) represents 3 minutes, and \(s(3)\) is the number of sit-ups completed in 3 minutes.
Using \(s(3)\) as the input to the function \(c(s)\) gives us the number of calories burned during the number of sit-ups that can be completed in 3 minutes, or simply the number of calories burned in 3 minutes (by doing sit-ups).

\section*{EXAMPLE: INVESTIGATING THE ORDER OF FUNCTION COMPOSITION}

Suppose \(f(x)\) gives miles that can be driven in \(x\) hours and \(g(y)\) gives the gallons of gas used in driving \(y\) miles. Which of these expressions is meaningful: \(f(g(y))\) or \(g(f(x))\) ?

\section*{Answer}

The function \(y=f(x)\) is a function whose output is the number of miles driven corresponding to the number of hours driven.
number of miles \(=f\) (number of hours)
The function \(g(y)\) is a function whose output is the number of gallons used corresponding to the number of miles driven. This means:
number of gallons \(=g\) (number of miles)
The expression \(g(y)\) takes miles as the input and a number of gallons as the output. The function \(f(x)\) requires a number of hours as the input. Trying to input a number of gallons does not make sense. The expression \(f(g(y))\) is meaningless.
The expression \(f(x)\) takes hours as input and a number of miles driven as the output. The function \(g(y)\) requires a number of miles as the input. Using \(f(x)\) (miles driven) as an input value for \(g(y)\), where gallons of gas depends on miles driven, does make sense. The expression \(g(f(x))\) makes sense, and will yield the number of gallons of gas used, \(g\), driving a certain number of miles, \(f(x)\), in \(x\) hours.

\section*{TRY IT}

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\section*{Q \& A}

Are there any situations where \(f(g(y))\) and \(g(f(x))\) would both be meaningful or useful expressions?
Yes. For many pure mathematical functions, both compositions make sense, even though they usually produce different new functions. In real-world problems, functions whose inputs and outputs have the same units also may give compositions that are meaningful in either order.

\section*{TRY IT}

The gravitational force on a planet a distance \(r\) from the sun is given by the function \(G(r)\). The acceleration of a planet subjected to any force \(F\) is given by the function \(a(F)\). Form a meaningful composition of these two functions, and explain what it means.

\section*{Answer}

A gravitational force is still a force, so \(a(G(r))\) makes sense as the acceleration of a planet at a distance r from the Sun (due to gravity), but \(G(a(F))\) does not make sense.

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\section*{EVALUATE A COMPOSITION OF FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Evaluate a composition of functions using a table.
- Evaluate a composition of functions using an equation.

Once we compose a new function from two existing functions, we need to be able to evaluate it for any input in its domain. We will do this with specific numerical inputs for functions expressed as tables, graphs, and formulas and with variables as inputs to functions expressed as formulas. In each case we evaluate the inner function using the starting input and then use the inner function's output as the input for the outer function.

\section*{Evaluating Composite Functions Using Tables}

When working with functions given as tables, we read input and output values from the table entries and always work from the inside to the outside. We evaluate the inside function first and then use the output of the inside function as the input to the outside function.

\section*{Evaluating Composite Functions Using Graphs}

When we are given individual functions as graphs, the procedure for evaluating composite functions is similar to the process we use for evaluating tables. We read the input and output values, but this time, from the \(x\) - and \(y\) - axes of the graphs.

\section*{Evaluating Composite Functions Using Formulas}

\section*{EXAMPLE: USING A TABLE TO EVALUATE A COMPOSITE FUNCTION}

Using the table below, evaluate \(f(g(3))\) and \(g(f(3))\).
\begin{tabular}{|l|l|l|}
\hline\(x\) & \(f(x)\) & \(g(x)\) \\
\hline 1 & 6 & 3 \\
\hline 2 & 8 & 5 \\
\hline 3 & 3 & 2 \\
\hline 4 & 1 & 7 \\
\hline
\end{tabular}

Answer
To evaluate \(f(g(3))\), we start from the inside with the input value 3 . We then evaluate the inside expression \(g(3)\) using the table that defines the function \(g: g(3)=2\). We can then use that result as the input to the function \(f\), so \(g(3)\) is replaced by 2 and we get \(f(2)\). Then, using the table that defines the function \(f\), we find that \(f(2)=8\).
\(>(3)=2\)
\(f(g(3))=f(2)=8\)
To evaluate \(g(f(3))\), we first evaluate the inside expression \(f(3)\) using the first table: \(f(3)=3\). Then, using the table for \(g\), we can evaluate
\(g(f(3))=g(3)=2\)
The table below shows the composite functions \(f \circ g\) and \(g \circ f\) as tables.
\begin{tabular}{|l|l|l|l|l}
\(x\) & \(g(x)\) & \(\begin{array}{l}f \\
(g(x))\end{array}\) & \(f(x)\) & \(\begin{array}{l}g \\
(f(x))\end{array}\) \\
\hline 3 & 2 & 8 & 3 & 2
\end{tabular}

\section*{TRY IT}

Using the table below, evaluate \(f(g(1))\) and \(g(f(4))\).
\begin{tabular}{|l|l|l|}
\hline\(x\) & \(f(x)\) & \(g(x)\) \\
\hline 1 & 6 & 3 \\
\hline 2 & 8 & 5 \\
\hline 3 & 3 & 2 \\
\hline 4 & 1 & 7 \\
\hline
\end{tabular}

Answer
\(f(g(1))=f(3)=3\) and \(g(f(4))=g(1)=3\)
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When evaluating a composite function where we have either created or been given formulas, the rule of working from the inside out remains the same. The input value to the outer function will be the output of the

\section*{HOW TO: GIVEN A COMPOSITE FUNCTION AND GRAPHS OF ITS INDIVIDUAL FUNCTIONS, EVALUATE IT USING THE INFORMATION PROVIDED BY THE GRAPHS.}
1. Locate the given input to the inner function on the \(x\) - axis of its graph.
2. Read off the output of the inner function from the \(y\)-axis of its graph.
3. Locate the inner function output on the \(x\) - axis of the graph of the outer function.
4. Read the output of the outer function from the \(y\) - axis of its graph. This is the output of the composite function.
inner function, which may be a numerical value, a variable name, or a more complicated expression.
While we can compose the functions for each individual input value, it is sometimes helpful to find a single formula that will calculate the result of a composition \(f(g(x))\). To do this, we will extend our idea of function evaluation. Recall that, when we evaluate a function like \(f(t)=t^{2}-t\), we substitute the value inside the parentheses into the formula wherever we see the input variable.

\section*{EXAMPLE: USING A GRAPH TO EVALUATE A COMPOSITE FUNCTION}

Using the graphs below, evaluate \(f(g(1))\).

(a)

(b)

Answer


To evaluate \(f(g(1))\), we start with the inside evaluation.
We evaluate \(g(1)\) using the graph of \(g(x)\), finding the input of 1 on the \(x\) - axis and finding the output value of the graph at that input. Here, \(g(1)=3\). We use this value as the input to the function \(f\).
\(f(g(1))=f(3)\)
We can then evaluate the composite function by looking to the graph of \(f(x)\), finding the input of 3 on the \(x\) - axis and reading the output value of the graph at this input. Here, \(f(3)=6\), so \(f(g(1))=6\).

\section*{Analysis of the Solution}

The figure shows how we can mark the graphs with arrows to trace the path from the input value to the output value.



\section*{TRY IT}

Using the graphs below, evaluate \(g(f(2))\).


\(g(1)=3\)
\(f(3)=6\)
Answer
\(g(f(2))=g(5)=3\)
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\section*{HOW TO: GIVEN A FORMULA FOR A COMPOSITE FUNCTION, EVALUATE THE FUNCTION.}
1. Evaluate the inside function using the input value or variable provided.
2. Use the resulting output as the input to the outside function.

\section*{EXAMPLE: EVALUATING A COMPOSITION OF FUNCTIONS EXPRESSED AS FORMULAS WITH A NUMERICAL INPUT}

Given \(f(t)=t^{2}-t\) and \(h(x)=3 x+2\), evaluate \(f(h(1))\).

\section*{Answer}

Because the inside expression is \(h(1)\), we start by evaluating \(h(x)\) at 1.
\(h(1)=3(1)+2\)
\(h(1)=5\)
Then \(f(h(1))=f(5)\), so we evaluate \(f(t)\) at an input of 5 .
\(f(h(1))=f(5)\)
\(f(h(1))=5^{2}-5\)
\(f(h(1))=20\)

\section*{Analysis of the Solution}

It makes no difference what the input variables \(t\) and \(x\) were called in this problem because we evaluated for specific numerical values.

\section*{TRY IT}

Given \(f(t)=t^{2}-t\) and \(h(x)=3 x+2\), evaluate
a. \(h(f(2))\)
b. \(h(f(-2))\)

Answer
a. 8; b. 20

You can check your work with an online graphing calculator. Enter the functions above into an online graphing calculator as they are defined. In the next line enter \(h(f(2))\). You should see \(=8\) in the bottom right corner.

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\section*{DOMAIN OF A COMPOSITION}

\section*{LEARNING OUTCOMES}
- Find the domain of a composite function.
- Decompose a composite function.

As we discussed previously, the domain of a composite function such as \(f \circ g\) is dependent on the domain of \(g\) and the domain of \(f\). It is important to know when we can apply a composite function and when we cannot, that is, to know the domain of a function such as \(f \circ g\). Let us assume we know the domains of the functions \(f\) and \(g\) separately. If we write the composite function for an input \(x\) as \(f(g(x))\), we can see right away that \(x\) must be a member of the domain of \(g\) in order for the expression to be meaningful, because otherwise we cannot complete the inner function evaluation. However, we also see that \(g(x)\) must be a member of the domain of \(f\), otherwise the second function evaluation in \(f(g(x))\) cannot be completed, and the expression is still undefined. Thus the domain of \(f \circ g\) consists of only those inputs in the domain of \(g\) that produce outputs from \(g\) belonging to the domain of \(f\). Note that the domain of \(f\) composed with \(g\) is the set of all \(x\) such that \(x\) is in the domain of \(g\) and \(g(x)\) is in the domain of \(f\).

\section*{A GENERAL NOTE: DOMAIN OF A COMPOSITE FUNCTION}

The domain of a composite function \(f(g(x))\) is the set of those inputs \(x\) in the domain of \(g\) for which \(g(x)\) is in the domain of \(f\).

\section*{HOW TO: GIVEN A FUNCTION COMPOSITION \(f(g(x))\), DETERMINE ITS DOMAIN.}
1. Find the domain of \(g\).
2. Find the domain of \(f\).
3. Find those inputs, \(x\), in the domain of \(g\) for which \(g(x)\) is in the domain of \(f\). That is, exclude those inputs, \(x\), from the domain of \(g\) for which \(g(x)\) is not in the domain of \(f\). The resulting set is the domain of \(f \circ g\).

\section*{Decompose a Composite Function}

In some cases, it is necessary to decompose a complicated function. In other words, we can write it as a composition of two simpler functions. There is almost always more than one way to decompose a composite function, so we may choose the decomposition that appears to be most obvious.

\section*{EXAMPLE: FINDING THE DOMAIN OF A COMPOSITE FUNCTION}

Find the domain of
\((f \circ g)(x)\) where \(f(x)=\frac{5}{x-1}\) and \(g(x)=\frac{4}{3 x-2}\)

\section*{Answer}

The domain of \(g(x)\) consists of all real numbers except \(x=\frac{2}{3}\), since that input value would cause us to divide by 0 . Likewise, the domain of \(f\) consists of all real numbers except 1 . So we need to exclude from the domain of \(g(x)\) that value of \(x\) for which \(g(x)=1\).
\(\frac{4}{3 x-2}=1 \quad\) Set \(g(x)\) equal to 1
\(4=3 x-2 \quad\) Multiply by \(3 x-2\)
\(6=3 x \quad\) Add 2 to both sides
\(x=2 \quad\) Divide by 3
So the domain of \(f \circ g\) is the set of all real numbers except \(\frac{2}{3}\) and 2 . This means that
\(x \neq \frac{2}{3}\) or \(x \neq 2\)
We can write this in interval notation as
\(\left(-\infty, \frac{2}{3}\right) \cup\left(\frac{2}{3}, 2\right) \cup(2, \infty)\)

\section*{EXAMPLE: FINDING THE DOMAIN OF A COMPOSITE FUNCTION INVOLVING RADICALS}

Find the domain of
\((f \circ g)(x)\) where \(f(x)=\sqrt{x+2}\) and \(g(x)=\sqrt{3-x}\)
Answer
Because we cannot take the square root of a negative number, the domain of \(g\) is \((-\infty, 3]\). Now we check the domain of the composite function
\((f \circ g)(x)=\sqrt{3-\sqrt{x+2}}\)
For \((f \circ g)(x)\), we need \(3-\sqrt{x+2} \geq 0\), again because we cannot take the square root of a negative number. Since the output of a square root is always non-negative, when we add 2 , we see that \(3-\sqrt{x+2}>0\), so the domain of \((f \circ g)(x)=(-\infty, 3]\).

\section*{Analysis of the Solution}

This example shows that knowledge of the range of functions (specifically the inner function) can also be helpful in finding the domain of a composite function. It also shows that the domain of \(f \circ g\) can contain values that are not in the domain of \(f\), though they must be in the domain of \(g\).
You cannot rely on an algorithm to find the domain of a composite function. Rather, you will need to first ask yourself "what is the domain of the inner function", and determine whether this set will comply with the domain restrictions of the outer function. In this case, the set \((-\infty, 3]\) ensures a non-negative output for the inner function, which will in turn ensure a positive input for the composite function.

\section*{TRY IT}

Find the domain of
\((f \circ g)(x)\) where \(f(x)=\frac{1}{x-2}\) and \(g(x)=\sqrt{x+4}\)

\section*{Answer}
\([-4,0) \cup(0, \infty)\)
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\section*{TRY IT}

We can use graphs to visualize the domain that results from a composition of two functions.
Graph the two functions below with an online graphing calculator.
1. \(f(x)=\sqrt{3-x}\)
2. \(g(t)=\sqrt{x+4}\)

Next, create a new function, \(h(x)=g(f(x))\). Based on the graph, what is the domain of this function? Explain why \(g(f(x))\) and \(f(x)\) have the same domain.
Now define another composition, \(p(x)=f(g(x)\). What is the domain of this function? Explain why you can evaluate \(g(10)\), but not \(p(10)\).

\section*{EXAMPLE: DECOMPOSING A FUNCTION}

Write \(f(x)=\sqrt{5-x^{2}}\) as the composition of two functions.
Answer
We are looking for two functions, \(g\) and \(h\), so \(f(x)=g(h(x))\). To do this, we look for a function inside a function in the formula for \(f(x)\). As one possibility, we might notice that the expression \(5-x^{2}\) is the inside of the square root. We could then decompose the function as
\(h(x)=5-x^{2}\) and \(g(x)=\sqrt{x}\)
We can check our answer by recomposing the functions.
\(g(h(x))=g\left(5-x^{2}\right)=\sqrt{5-x^{2}}\)

\section*{Analysis of the Solution}

For every composition there are infinitely many possible function pairs that will work. In this case, another function pair where \(g(h(x))=\sqrt{5-x^{2}}\) is \(h(x)=x^{2}\) and \(g(x)=\sqrt{5-x}\)

\section*{TRY IT}

Write \(f(x)=\frac{4}{3-\sqrt{4+x^{2}}}\) as the composition of two functions.

\section*{Answer}

Possible answer:
\(g(x)=\sqrt{4+x^{2}}\)
\(h(x)=\frac{4}{3-x}\)
\(f=h \circ g\)
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\section*{SUMMARY: COMPOSITIONS OF FUNCTIONS}

\section*{Key Equation}

\section*{Composite function}
\[
(f \circ g)(x)=f(g(x))
\]

\section*{Key Concepts}
- We can perform algebraic operations on functions.
- When functions are combined, the output of the first (inner) function becomes the input of the second (outer) function.
- The function produced by combining two functions is a composite function.
- The order of function composition must be considered when interpreting the meaning of composite functions.
- A composite function can be evaluated by evaluating the inner function using the given input value and then evaluating the outer function taking as its input the output of the inner function.
- A composite function can be evaluated from a table.
- A composite function can be evaluated from a graph.
- A composite function can be evaluated from a formula.
- The domain of a composite function consists of those inputs in the domain of the inner function that correspond to outputs of the inner function that are in the domain of the outer function.
- Just as functions can be combined to form a composite function, composite functions can be decomposed into simpler functions.
- Functions can often be decomposed in more than one way.

\section*{Glossary}
composite function the new function formed by function composition, when the output of one function is used as the input of another

\section*{INTRODUCTION TO TRANSFORMATIONS OF FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Graph functions using vertical and horizontal shifts.
- Graph functions using reflections about the \(x\)-axis and the \(y\)-axis.
- Determine whether a function is even, odd, or neither from its graph.
- Graph functions using compressions and stretches.
- Combine transformations.


Figure 1. (credit: "Misko"/Flickr)

We all know that a flat mirror enables us to see an accurate image of ourselves and whatever is behind us. When we tilt the mirror, the images we see may shift horizontally or vertically. But what happens when we bend a flexible mirror? Like a carnival funhouse mirror, it presents us with a distorted image of ourselves, stretched or compressed horizontally or vertically. In a similar way, we can distort or transform mathematical functions to better adapt them to describing objects or processes in the real world. In this section, we will take a look at several kinds of transformations.

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\section*{SHIFTS}

\section*{LEARNING OUTCOMES}
- Graph functions using vertical and horizontal shifts.

One kind of transformation involves shifting the entire graph of a function up, down, right, or left. The simplest shift is a vertical shift, moving the graph up or down, because this transformation involves adding a positive or negative constant to the function. In other words, we add the same constant to the output value of the function regardless of the input. For a function \(g(x)=f(x)+k\), the function \(f(x)\) is shifted vertically \(k\) units.


Vertical shift by \(k=1\) of the cube root function \(f(x)=\sqrt[3]{x}\).
To help you visualize the concept of a vertical shift, consider that \(y=f(x)\). Therefore, \(f(x)+k\) is equivalent to \(y+k\). Every unit of \(y\) is replaced by \(y+k\), so the \(y\)-value increases or decreases depending on the value of \(k\). The result is a shift upward or downward.

\section*{A GENERAL NOTE: VERTICAL SHIFT}

Given a function \(f(x)\), a new function \(g(x)=f(x)+k\), where \(k\) is a constant, is a vertical shift of the function \(f(x)\). All the output values change by \(k\) units. If \(k\) is positive, the graph will shift up. If \(k\) is negative, the graph will shift down.

\section*{Identifying Horizontal Shifts}

We just saw that the vertical shift is a change to the output, or outside, of the function. We will now look at how changes to input, on the inside of the function, change its graph and meaning. A shift to the input results in a movement of the graph of the function left or right in what is known as a horizontal shift.

\section*{EXAMPLE: ADDING A CONSTANT TO A FUNCTION}

To regulate temperature in a green building, airflow vents near the roof open and close throughout the day. Figure 2 shows the area of open vents \(V\) (in square feet) throughout the day in hours after midnight, \(t\). During the summer, the facilities manager decides to try to better regulate temperature by increasing the amount of open vents by 20 square feet throughout the day and night. Sketch a graph of this new function.


\section*{Answer}

We can sketch a graph of this new function by adding 20 to each of the output values of the original function. This will have the effect of shifting the graph vertically up.


Notice that for each input value, the output value has increased by 20 , so if we call the new function \(S(t)\), we could write
\(S(t)=V(t)+20\)
This notation tells us that, for any value of \(t, S(t)\) can be found by evaluating the function \(V\) at the same input and then adding 20 to the result. This defines \(S\) as a transformation of the function \(V\), in this case a vertical shift up 20 units. Notice that, with a vertical shift, the input values stay the same and only the output values change.
\begin{tabular}{|l|l|l|l|l|l|l|}
\hline\(t\) & 0 & 8 & 10 & 17 & 19 & 24 \\
\hline\(V(t)\) & 0 & 0 & 220 & 220 & 0 & 0 \\
\hline\(S(t)\) & 20 & 20 & 240 & 240 & 20 & 20 \\
\hline
\end{tabular}

\section*{HOW TO: GIVEN A TABULAR FUNCTION, CREATE A NEW ROW TO REPRESENT A VERTICAL SHIFT.}
1. Identify the output row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each output cell. Add a positive value for up or a negative value for down.

\section*{EXAMPLE: SHIFIING A TABULAR FUNCTION VERTICALLY}

A function \(f(x)\) is given below. Create a table for the function \(g(x)=f(x)-3\).
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(f(x)\) & 1 & 3 & 7 & 11 \\
\hline
\end{tabular}

Answer
The formula \(g(x)=f(x)-3\) tells us that we can find the output values of \(g\) by subtracting 3 from the output values of \(f\). For example:
\(\begin{cases}f(2)=1 & \text { Given } \\ g(x)=f(x)-3 & \text { Given transformation } \\ g(2)=f(2)-3 & \\ =1-3 & \\ =-2 & \end{cases}\)
Subtracting 3 from each \(f(x)\) value, we can complete a table of values for \(g(x)\).
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(f(x)\) & 1 & 3 & 7 & 11 \\
\hline\(g(x)\) & -2 & 0 & 4 & 8 \\
\hline
\end{tabular}

\section*{Analysis of the Solution}

As with the earlier vertical shift, notice the input values stay the same and only the output values change.
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\section*{TRY IT}

The function \(h(t)=-4.9 t^{2}+30 t\) gives the height \(h\) of a ball (in meters) thrown upward from the ground after \(t\) seconds. Suppose the ball was instead thrown from the top of a 10-m building. Relate this new height function \(b(t)\) to \(h(t)\), and then find a formula for \(b(t)\).
Answer
\(b(t)=h(t)+10=-4.9 t^{2}+30 t+10\)
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Horizontal shift of the function \(f(x)=\sqrt[3]{x}\). Note that \(h=+1\) shifts the graph to the left, that is, towards negative values of \(x\).
For example, if \(f(x)=x^{2}\), then \(g(x)=(x-2)^{2}\) is a new function. Each input is reduced by 2 prior to squaring the function. The result is that the graph is shifted 2 units to the right, because we would need to increase the prior input by 2 units to yield the same output value as given in \(f\).

\section*{A GENERAL NOTE: HORIZONTAL SHIFT}

Given a function \(f\), a new function \(g(x)=f(x-h)\), where \(h\) is a constant, is a horizontal shift of the function \(f\). If \(h\) is positive, the graph will shift right. If \(h\) is negative, the graph will shift left.

\section*{EXAMPLE: ADDING A CONSTANT TO AN INPUT}

Returning to our building airflow example from Example 2, suppose that in autumn the facilities manager decides that the original venting plan starts too late, and wants to begin the entire venting program 2 hours earlier. Sketch a graph of the new function.

\section*{Answer}

We can set \(V(t)\) to be the original program and \(F(t)\) to be the revised program.
\(c V(t)=\) the original venting plan
\(F(t)=\) starting 2 hrs sooner
In the new graph, at each time, the airflow is the same as the original function \(V\) was 2 hours later. For example, in the original function \(V\), the airflow starts to change at 8 a.m., whereas for the function \(F\), the airflow starts to change at 6 a.m. The comparable function values are \(V(8)=F(6)\). Notice also that the vents first opened to \(220 \mathrm{ft}^{2}\) at 10 a .m. under the original plan, while under the new plan the vents reach \(220 \mathrm{ft}^{2}\) at 8 a.m., so \(V(10)=F(8)\).


In both cases, we see that, because \(F(t)\) starts 2 hours sooner, \(h=-2\). That means that the same output values are reached when \(F(t)=V(t-(-2))=V(t+2)\).

\section*{Analysis of the Solution}

Note that \(V(t+2)\) has the effect of shifting the graph to the left.
Horizontal changes or "inside changes" affect the domain of a function (the input) instead of the range and often seem counterintuitive. The new function \(F(t)\) uses the same outputs as \(V(t)\), but matches those outputs to inputs 2 hours earlier than those of \(V(t)\). Said another way, we must add 2 hours to the input of \(V\) to find the corresponding output for \(F: F(t)=V(t+2)\).

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\section*{HOW TO: GIVEN A TABULAR FUNCTION, CREATE A NEW ROW TO REPRESENT A HORIZONTAL SHIFT.}
1. Identify the input row or column.
2. Determine the magnitude of the shift.
3. Add the shift to the value in each input cell.

\section*{EXAMPLE: SHIFTING A TABULAR FUNCTION HORIZONTALLY}

A function \(f(x)\) is given below. Create a table for the function \(g(x)=f(x-3)\).
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(f(x)\) & 1 & 3 & 7 & 11 \\
\hline
\end{tabular}

\section*{Answer}

The formula \(g(x)=f(x-3)\) tells us that the output values of \(g\) are the same as the output value of \(f\) when the input value is 3 less than the original value. For example, we know that \(f(2)=1\). To get the same output from the function \(g\), we will need an input value that is 3 larger. We input a value that is 3 larger for \(g(x)\) because the function takes 3 away before evaluating the function \(f\).
\(\left\{\begin{array}{l}g(5)=f(5-3) \\ =f(2) \\ =1\end{array}\right.\)
We continue with the other values to create this table.
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 5 & 7 & 9 & 11 \\
\hline\(x-3\) & 2 & 4 & 6 & 8 \\
\hline\(f(x)\) & 1 & 3 & 7 & 11 \\
\hline\(g(x)\) & 1 & 3 & 7 & 11 \\
\hline
\end{tabular}

The result is that the function \(f(x)\) has been shifted to the right by 3 . Notice the output values for \(g(x)\) remain the same as the output values for \(f(x)\), but the corresponding input values, \(x\), have shifted to the right by 3 . Specifically, 2 shifted to 5,4 shifted to 7,6 shifted to 9 , and 8 shifted to 11 .

\section*{Analysis of the Solution}

The graph below represents both of the functions. We can see the horizontal shift in each point.


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\section*{EXAMPLE: IDENTIFYING A HORIZONTAL SHIFT OF A TOOLKIT FUNCTION}

This graph represents a transformation of the toolkit function \(f(x)=x^{2}\). Relate this new function \(g(x)\) to \(f(x)\), and then find a formula for \(g(x)\).


Answer
Notice that the graph is identical in shape to the \(f(x)=x^{2}\) function, but the \(x\)-values are shifted to the right 2 units. The vertex used to be at \((0,0)\), but now the vertex is at \((2,0)\). The graph is the basic quadratic function shifted 2 units to the right, so
\(g(x)=f(x-2)\)
Notice how we must input the value \(x=2\) to get the output value \(y=0\); the \(x\)-values must be 2 units larger because of the shift to the right by 2 units. We can then use the definition of the \(f(x)\) function to write a formula for \(g(x)\) by evaluating \(f(x-2)\).
\(\left\{\begin{array}{l}f(x)=x^{2} \\ g(x)=f(x-2) \\ g(x)=f(x-2)=(x-2)^{2}\end{array}\right.\)

\section*{Analysis of the Solution}

To determine whether the shift is +2 or -2 , consider a single reference point on the graph. For a quadratic, looking at the vertex point is convenient. In the original function, \(f(0)=0\). In our shifted function, \(g(2)=0\). To obtain the output value of 0 from the function \(f\), we need to decide whether a plus or a minus sign will work to satisfy \(g(2)=f(x-2)=f(0)=0\). For this to work, we will need to subtract 2 units from our input values.

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\section*{EXAMPLE: INTERPRETING HORIZONTAL VERSUS VERTICAL SHIFTS}

The function \(G(m)\) gives the number of gallons of gas required to drive \(m\) miles. Interpret \(G(m)+10\) and \(G(m+10)\).

\section*{Answer}
\(G(m)+10\) can be interpreted as adding 10 to the output, gallons. This is the gas required to drive \(m\) miles, plus another 10 gallons of gas. The graph would indicate a vertical shift.
\(G(m+10)\) can be interpreted as adding 10 to the input, miles. So this is the number of gallons of gas required to drive 10 miles more than \(m\) miles. The graph would indicate a horizontal shift.

\section*{TRY IT}

Given the function \(f(x)=\sqrt{x}\), graph the original function \(f(x)\) and the transformation \(g(x)=f(x+2)\) on the same axes. Is this a horizontal or a vertical shift? Which way is the graph shifted and by how many units?
Answer
A horizontal shift results when a constant is added to or subtracted from the input. A vertical shift results when a constant is added to or subtracted from the output.

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\section*{TRY IT}

Online graphing calculators can graph transformations using function notation. Use an online graphing calculator to graph the toolkit function \(f(x)=x^{2}\)
Now, enter \(f(x+5)\), and \(f(x)+5\) in the next two lines.
Now have the online graphing calculator make a table of values for the original function. Include integer values on the interval \([-5,5]\). Replace the column labeled \(y_{1}\) with \(f\left(x_{1}\right)\).
Now replace \(f\left(x_{1}\right)\) with \(f\left(x_{1}+3\right)\), and \(f\left(x_{1}\right)+3\).
What are the corresponding functions associated with the transformations you have graphed?
Answer
You have graphed the following transformations:
\(f(x+3)=(x+3)^{2}\)
\(f(x)+3=(x)^{2}+3\)

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\section*{REFLECTIONS}

\section*{LEARNING OUTCOMES}
- Graph functions using reflections about the \(x\)-axis and the \(y\)-axis.
- Determine whether a function is even, odd, or neither from its graph.

Another transformation that can be applied to a function is a reflection over the \(x\) - or \(y\)-axis. A vertical reflection reflects a graph vertically across the \(x\)-axis, while a horizontal reflection reflects a graph horizontally across the \(y\)-axis. The reflections are shown in Figure 9.


Vertical and horizontal reflections of a function.
Notice that the vertical reflection produces a new graph that is a mirror image of the base or original graph about the \(x\)-axis. The horizontal reflection produces a new graph that is a mirror image of the base or original graph about the \(y\)-axis.

\section*{A GENERAL NOTE: REFLECTIONS}

Given a function \(f(x)\), a new function \(g(x)=-f(x)\) is a vertical reflection of the function \(f(x)\), sometimes called a reflection about (or over, or through) the \(x\)-axis.
Given a function \(f(x)\), a new function \(g(x)=f(-x)\) is a horizontal reflection of the function \(f(x)\), sometimes called a reflection about the \(y\)-axis.

\section*{HOW TO: GIVEN A FUNCTION, REFLECT THE GRAPH BOTH VERTICALLY AND HORIZONTALLY.}
1. Multiply all outputs by -1 for a vertical reflection. The new graph is a reflection of the original graph about the \(x\)-axis.
2. Multiply all inputs by -1 for a horizontal reflection. The new graph is a reflection of the original graph about the \(y\)-axis.

\section*{Determine Whether a Functions is Even, Odd, or Neither}

Some functions exhibit symmetry so that reflections result in the original graph. For example, horizontally reflecting the toolkit functions \(f(x)=x^{2}\) or \(f(x)=|x|\) will result in the original graph. We say that these types of graphs are symmetric about the \(y\)-axis. Functions whose graphs are symmetric about the \(y\)-axis are called even functions.

If the graphs of \(f(x)=x^{3}\) or \(f(x)=\frac{1}{x}\) were reflected over both axes, the result would be the original graph.

\section*{EXAMPLE: REFLECTING A GRAPH HORIZONTALLY AND VERTICALLY}

Reflect the graph of \(s(t)=\sqrt{t}\) (a) vertically and (b) horizontally.
Answer
a. Reflecting the graph vertically means that each output value will be reflected over the horizontal \(t\)-axis as shown below.


Vertical reflection of the square root function
Because each output value is the opposite of the original output value, we can write
\(V(t)=-s(t)\) or \(V(t)=-\sqrt{t}\)
Notice that this is an outside change, or vertical shift, that affects the output \(s(t)\) values, so the negative sign belongs outside of the function.
b. Reflecting horizontally means that each input value will be reflected over the vertical axis as shown below.


Horizontal reflection of the square root function
Because each input value is the opposite of the original input value, we can write
\(H(t)=s(-t)\) or \(H(t)=\sqrt{-t}\)

Notice that this is an inside change or horizontal change that affects the input values, so the negative sign is on the inside of the function.
Note that these transformations can affect the domain and range of the functions. While the original square root function has domain \([0, \infty)\) and range \([0, \infty)\), the vertical reflection gives the \(V(t)\) function the range \((-\infty, 0]\) and the horizontal reflection gives the \(H(t)\) function the domain \((-\infty, 0]\).

\section*{TRY IT}

Use an online graphing calculator to reflect the graph of \(f(x)=|x-1|\) (a) vertically and (b) horizontally.
Answer
a)

b)


\section*{EXAMPLE: REFLECTING A TABULAR FUNCTION HORIZONTALLY AND VERTICALLY}

A function \(f(x)\) is given. Create a table for the functions below.
1. \(g(x)=-f(x)\)
2. \(h(x)=f(-x)\)
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(f(x)\) & 1 & 3 & 7 & 11 \\
\hline
\end{tabular}

\section*{Answer}
1. For \(g(x)\), the negative sign outside the function indicates a vertical reflection, so the \(x\)-values stay the same and each output value will be the opposite of the original output value.
\begin{tabular}{l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(g(x)\) & -1 & -3 & -7 & -11
\end{tabular}
2. For \(h(x)\), the negative sign inside the function indicates a horizontal reflection, so each input value will be the opposite of the original input value and the \(h(x)\) values stay the same as the \(f(x)\) values.
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & -2 & -4 & -6 & -8 \\
\hline\(h(x)\) & 1 & 3 & 7 & 11 \\
\hline
\end{tabular}

\section*{TRY IT}
\(x\)
\(f(x)\)
-2
5
0
10
2
4
15
20

Using the function \(f(x)\) given in the table above, create a table for the functions below.
a. \(g(x)=-f(x)\)
b. \(h(x)=f(-x)\)

Answer
1. \(g(x)=-f(x)\)
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & -2 & 0 & 2 & 4 \\
\hline\(g(x)\) & -5 & -10 & -15 & -20 \\
\hline
\end{tabular}
2. \(h(x)=f(-x)\)
\begin{tabular}{l|l|l|l|l|}
\hline\(x\) & -2 & 0 & 2 & 4 \\
\hline\(h(x)\) & 15 & 10 & 5 & unknown
\end{tabular}

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(a) The cubic toolkit function (b) Horizontal reflection of the cubic toolkit function (c) Horizontal and vertical reflections reproduce the original cubic function.

We say that these graphs are symmetric about the origin. A function with a graph that is symmetric about the origin is called an odd function.

Note: A function can be neither even nor odd if it does not exhibit either symmetry. For example, \(f(x)=2^{x}\) is neither even nor odd. Also, the only function that is both even and odd is the constant function \(f(x)=0\).


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\section*{A GENERAL NOTE: EVEN AND ODD FUNCTIONS}

A function is called an even function if for every input \(x\)
\(f(x)=f(-x)\)
The graph of an even function is symmetric about the \(y\) - axis.
A function is called an odd function if for every input \(x\)
\(f(x)=-f(-x)\)

\section*{HOW TO: GIVEN THE FORMULA FOR A FUNCTION, DETERMINE IF THE FUNCTION IS EVEN, ODD, OR NEITHER.}
1. Determine whether the function satisfies \(f(x)=f(-x)\). If it does, it is even.
2. Determine whether the function satisfies \(f(x)=-f(-x)\). If it does, it is odd.
3. If the function does not satisfy either rule, it is neither even nor odd.

\section*{RECALL EVALUATING FUNCTIONS}

When evaluating functions for specific input, we can wrap the variable in parentheses first, then drop in the value we want to use to evaluate it. This works whether the value is a constant or a variable.
Be extra careful when evaluating a function for a negative input. The negative sign goes in the parentheses.
Ex. Evaluate \(f(x)=x^{2}\) for -1
\[
\begin{aligned}
f(-1) & =(-1)^{2} \\
& =1
\end{aligned}
\]

\section*{EXAMPLE: DETERMINING WHETHER A FUNCTION IS EVEN, ODD, OR NEITHER}

Is the function \(f(x)=x^{3}+2 x\) even, odd, or neither?

\section*{Answer}

Without looking at a graph, we can determine whether the function is even or odd by finding formulas for the reflections and determining if they return us to the original function. Let's begin with the rule for even functions.
\(f(-x)=(-x)^{3}+2(-x)=-x^{3}-2 x\)
This does not return us to the original function, so this function is not even. We can now test the rule for odd functions.
\(-f(-x)=-\left(-x^{3}-2 x\right)=x^{3}+2 x\)
Because \(-f(-x)=f(x)\), this is an odd function.

\section*{Analysis of the Solution}

Consider the graph of \(f\). Notice that the graph is symmetric about the origin. For every point \((x, y)\) on the graph, the corresponding point \((-x,-y)\) is also on the graph. For example, \((1,3)\) is on the graph of \(f\), and the corresponding point \((-1,-3)\) is also on the graph.


\section*{TRY IT}

Is the function \(f(s)=s^{4}+3 s^{2}+7\) even, odd, or neither?

\section*{Answer}

Even
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\section*{COMPRESSIONS AND STRETCHES}

\section*{LEARNING OUTCOMES}
- Graph Functions Using Compressions and Stretches.

Adding a constant to the inputs or outputs of a function changed the position of a graph with respect to the axes, but it did not affect the shape of a graph. We now explore the effects of multiplying the inputs or outputs by some quantity.

We can transform the inside (input values) of a function or we can transform the outside (output values) of a function. Each change has a specific effect that can be seen graphically.

\section*{Vertical Stretches and Compressions}

When we multiply a function by a positive constant, we get a function whose graph is stretched or compressed vertically in relation to the graph of the original function. If the constant is greater than 1 , we get a vertical stretch; if the constant is between 0 and 1, we get a vertical compression. The graph below shows a function multiplied by constant factors 2 and 0.5 and the resulting vertical stretch and compression.


Vertical stretch and compression

\section*{A GENERAL NOTE: VERTICAL STRETCHES AND COMPRESSIONS}

Given a function \(f(x)\), a new function \(g(x)=a f(x)\), where \(a\) is a constant, is a vertical stretch or vertical compression of the function \(f(x)\).
- If \(a>1\), then the graph will be stretched.
- If \(0<a<1\), then the graph will be compressed.
- If \(a<0\), then there will be combination of a vertical stretch or compression with a vertical reflection.

\section*{HOW TO: GIVEN A FUNCTION, GRAPH ITS VERTICAL STRETCH.}
1. Identify the value of \(a\).
2. Multiply all range values by \(a\).
3. If \(a>1\), the graph is stretched by a factor of \(a\).

If \(0<a<1\), the graph is compressed by a factor of \(a\).
If \(a<0\), the graph is either stretched or compressed and also reflected about the \(x\)-axis.

\section*{EXAMPLE: GRAPHING A VERTICAL STRETCH}

A function \(P(t)\) models the number of fruit flies in a population over time, and is graphed below. A scientist is comparing this population to another population, \(Q\), whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.


\section*{Answer}

Because the population is always twice as large, the new population's output values are always twice the original function's output values.
If we choose four reference points, \((0,1),(3,3),(6,2)\) and \((7,0)\) we will multiply all of the outputs by 2.
The following shows where the new points for the new graph will be located.
\(\left\{\begin{array}{l}(0,1) \rightarrow(0,2) \\ (3,3) \rightarrow(3,6) \\ (6,2) \rightarrow(6,4) \\ (7,0) \rightarrow(7,0)\end{array}\right.\)


Figure 16
Symbolically, the relationship is written as
\(Q(t)=2 P(t)\)
This means that for any input \(t\), the value of the function \(Q\) is twice the value of the function \(P\). Notice that the effect on the graph is a vertical stretching of the graph, where every point doubles its distance from the horizontal axis. The input values, \(t\), stay the same while the output values are twice as large as before.

\section*{TRY IT}

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\section*{HOW TO: GIVEN A TABULAR FUNCTION AND ASSUMING THAT THE TRANSFORMATION IS A VERTICAL STRETCH OR COMPRESSION, CREATE A TABLE FOR A VERTICAL COMPRESSION.}
1. Determine the value of \(a\).
2. Multiply all of the output values by \(a\).

\section*{EXAMPLE: FINDING A VERTICAL COMPRESSION OF A TABULAR FUNCTION}

A function \(f\) is given in the table below. Create a table for the function \(g(x)=\frac{1}{2} f(x)\).
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(f(x)\) & 1 & 3 & 7 & 11 \\
\hline
\end{tabular}

Answer
The formula \(g(x)=\frac{1}{2} f(x)\) tells us that the output values of \(g\) are half of the output values of \(f\) with the same inputs. For example, we know that \(f(4)=3\). Then
\(g(4)=\frac{1}{2} \cdot f(4)=\frac{1}{2} \cdot(3)=\frac{3}{2}\)
We do the same for the other values to produce this table.
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(g(x)\) & \(\frac{1}{2}\) & \(\frac{3}{2}\) & \(\frac{7}{2}\) & \(\frac{11}{2}\) \\
\hline
\end{tabular}

\section*{Analysis of the Solution}

The result is that the function \(g(x)\) has been compressed vertically by \(\frac{1}{2}\). Each output value is divided in half, so the graph is half the original height.

\section*{TRY IT}

A function \(f\) is given below. Create a table for the function \(g(x)=\frac{3}{4} f(x)\).
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(f(x)\) & 12 & 16 & 20 & 0 \\
\hline
\end{tabular}

Answer
\begin{tabular}{l|l|l|l|l}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(g(x)\) & 9 & 12 & 15 & 0
\end{tabular}

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\section*{EXAMPLE: RECOGNIZING A VERTICAL STRETCH}


The graph is a transformation of the toolkit function \(f(x)=x^{3}\). Relate this new function \(g(x)\) to \(f(x)\), and then find a formula for \(g(x)\).
Answer
When trying to determine a vertical stretch or shift, it is helpful to look for a point on the graph that is relatively clear. In this graph, it appears that \(g(2)=2\). With the basic cubic function at the same input, \(f(2)=2^{3}=8\). Based on that, it appears that the outputs of \(g\) are \(\frac{1}{4}\) the outputs of the function \(f\) because \(g(2)=\frac{1}{4} f(2)\). From this we can fairly safely conclude that \(g(x)=\frac{1}{4} f(x)\).
We can write a formula for \(g\) by using the definition of the function \(f\).
\(g(x)=\frac{1}{4} f(x)=\frac{1}{4} x^{3}\)

\section*{TRY IT}

Write the formula for the function that we get when we stretch the identity toolkit function by a factor of 3, and then shift it down by 2 units.
Check your work with an online graphing calculator.
Answer
\(g(x)=3 x-2\)


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Now we consider changes to the inside of a function. When we multiply a function's input by a positive constant, we get a function whose graph is stretched or compressed horizontally in relation to the graph of the original function. If the constant is between 0 and 1 , we get a horizontal stretch; if the constant is greater than 1, we get a horizontal compression of the function.

Given a function \(y=f(x)\), the form \(y=f(b x)\) results in a horizontal stretch or compression. Consider the function \(y=x^{2}\). The graph of \(y=(0.5 x)^{2}\) is a horizontal stretch of the graph of the function \(y=x^{2}\) by a factor of 2 . The graph of \(y=(2 x)^{2}\) is a horizontal compression of the graph of the function \(y=x^{2}\) by a factor of 2 .

\section*{A GENERAL NOTE: HORIZONTAL STRETCHES AND COMPRESSIONS}

Given a function \(f(x)\), a new function \(g(x)=f(b x)\), where \(b\) is a constant, is a horizontal stretch or horizontal compression of the function \(f(x)\).
- If \(b>1\), then the graph will be compressed by \(\frac{1}{b}\).
- If \(0<b<1\), then the graph will be stretched by \(\frac{1}{b}\).
- If \(b<0\), then there will be combination of a horizontal stretch or compression with a horizontal reflection.

\section*{HOW TO: GIVEN A DESCRIPTION OF A FUNCTION, SKETCH A HORIZONTAL COMPRESSION OR STRETCH.}
1. Write a formula to represent the function.
2. Set \(g(x)=f(b x)\) where \(b>1\) for a compression or \(0<b<1\) for a stretch.

\section*{EXAMPLE: GRAPHING A HORIZONTAL COMPRESSION}

Suppose a scientist is comparing a population of fruit flies to a population that progresses through its lifespan twice as fast as the original population. In other words, this new population, \(R\), will progress in 1 hour the same amount as the original population does in 2 hours, and in 2 hours, it will progress as much as the original population does in 4 hours. Sketch a graph of this population.

\section*{Answer}

Symbolically, we could write
\(R(1)=P(2)\),
\(R(2)=P(4)\), and in general,
\(R(t)=P(2 t)\).
See below for a graphical comparison of the original population and the compressed population.

(a)

(b)
(a) Original population graph (b) Compressed population graph

\section*{Analysis of the Solution}

Note that the effect on the graph is a horizontal compression where all input values are half of their original distance from the vertical axis.

\section*{EXAMPLE: FINDING A HORIZONTAL STRETCH FOR A TABULAR FUNCTION}

A function \(f(x)\) is given below. Create a table for the function \(g(x)=f\left(\frac{1}{2} x\right)\).
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(f(x)\) & 1 & 3 & 7 & 11 \\
\hline
\end{tabular}

Answer
The formula \(g(x)=f\left(\frac{1}{2} x\right)\) tells us that the output values for \(g\) are the same as the output values for the function \(f\) at an input half the size. Notice that we do not have enough information to determine \(g(2)\) because \(g(2)=f\left(\frac{1}{2} \cdot 2\right)=f(1)\), and we do not have a value for \(f(1)\) in our table. Our input values to \(g\) will need to be twice as large to get inputs for \(f\) that we can evaluate. For example, we can determine \(g(4)\).
\(g(4)=f\left(\frac{1}{2} \cdot 4\right)=f(2)=1\)
We do the same for the other values to produce the table below.


This figure shows the graphs of both of these sets of points.

\section*{Analysis of the Solution}

Because each input value has been doubled, the result is that the function \(g(x)\) has been stretched horizontally by a factor of 2 .

\section*{EXAMPLE: RECOGNIZING A HORIZONTAL COMPRESSION ON A GRAPH}

Relate the function \(g(x)\) to \(f(x)\).


Answer
The graph of \(g(x)\) looks like the graph of \(f(x)\) horizontally compressed. Because \(f(x)\) ends at \((6,4)\) and \(g(x)\) ends at \((2,4)\), we can see that the \(x\) - values have been compressed by \(\frac{1}{3}\), because \(6\left(\frac{1}{3}\right)=2\). We might also notice that \(g(2)=f(6)\) and \(g(1)=f(3)\). Either way, we can describe this relationship as \(g(x)=f(3 x)\). This is a horizontal compression by \(\frac{1}{3}\).

\section*{Analysis of the Solution}

Notice that the coefficient needed for a horizontal stretch or compression is the reciprocal of the stretch or compression. So to stretch the graph horizontally by a scale factor of 4 , we need a coefficient of \(\frac{1}{4}\) in our function: \(f\left(\frac{1}{4} x\right)\). This means that the input values must be four times larger to produce the same result, requiring the input to be larger, causing the horizontal stretching.

\section*{TRY IT}

Write a formula for the toolkit square root function horizontally stretched by a factor of 3.
Use an online graphing calculator to check your work.
Answer
\(g(x)=\sqrt{\frac{1}{3} x}\)


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\section*{SEQUENCES OF TRANSFORMATIONS}

\section*{LEARNING OUTCOMES}
- Combine vertical and horizontal shifts.
- Follow a pattern when combining shifts and stretches.

Now that we have two transformations, we can combine them together. Vertical shifts are outside changes that affect the output ( \(y-\) ) axis values and shift the function up or down. Horizontal shifts are inside changes that affect the input ( \(x-\) ) axis values and shift the function left or right. Combining the two types of shifts will cause the graph of a function to shift up or down and right or left.

\section*{HOW TO: GIVEN A FUNCTION AND BOTH A VERTICAL AND A HORIZONTAL SHIFT, SKETCH THE GRAPH.}
1. Identify the vertical and horizontal shifts from the formula.
2. The vertical shift results from a constant added to the output. Move the graph up for a positive constant and down for a negative constant.
3. The horizontal shift results from a constant added to the input. Move the graph left for a positive constant and right for a negative constant.
4. Apply the shifts to the graph in either order.

\section*{Combine Shifts and Stretches}

When combining transformations, it is very important to consider the order of the transformations. For example, vertically shifting by 3 and then vertically stretching by 2 does not create the same graph as vertically stretching by 2 and then vertically shifting by 3, because when we shift first, both the original function and the shift get stretched, while only the original function gets stretched when we stretch first.

When we see an expression such as \(2 f(x)+3\), which transformation should we start with? The answer here follows nicely from the order of operations. Given the output value of \(f(x)\), we first multiply by 2 , causing the vertical stretch, and then add 3, causing the vertical shift. In other words, multiplication before addition.

\section*{EXAMPLE: GRAPHING COMBINED VERTICAL AND HORIZONTAL SHIFTS}

Given \(f(x)=|x|\), sketch a graph of \(h(x)=f(x+1)-3\).
The function \(f\) is our toolkit absolute value function. We know that this graph has a V shape, with the point at the origin. The graph of \(h\) has transformed \(f\) in two ways: \(f(x+1)\) is a change on the inside of the function, giving a horizontal shift left by 1 , and the subtraction by 3 in \(f(x+1)-3\) is a change to the outside of the function, giving a vertical shift down by 3 . The transformation of the graph is illustrated below.
Let us follow one point of the graph of \(f(x)=|x|\).
- The point \((0,0)\) is transformed first by shifting left 1 unit: \((0,0) \rightarrow(-1,0)\)
- The point \((-1,0)\) is transformed next by shifting down 3 units: \((-1,0) \rightarrow(-1,-3)\)


Below is the graph of \(h\).


\section*{TRY IT}

Given \(f(x)=|x|\), sketch a graph of \(h(x)=f(x-2)+4\).
Check your work with an online graphing calculator.

Answer


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\section*{EXAMPLE: IDENTIFYING COMBINED VERTICAL AND HORIZONTAL SHIFTS}

Write a formula for the graph shown below, which is a transformation of the toolkit square root function.


\section*{Answer}

The graph of the toolkit function starts at the origin, so this graph has been shifted 1 to the right and up 2. In function notation, we could write that as
\(h(x)=f(x-1)+2\)
Using the formula for the square root function, we can write
\(h(x)=\sqrt{x-1}+2\)

\section*{Analysis of the Solution}

Note that this transformation has changed the domain and range of the function. This new graph has domain \([1, \infty)\) and range \([2, \infty)\).

Write a formula for a transformation of the toolkit reciprocal function \(f(x)=\frac{1}{x}\) that shifts the function's graph one unit to the right and one unit up.
Answer
\(g(x)=\frac{1}{(x+4)^{2}}+2\)
Horizontal transformations are a little trickier to think about. When we write \(g(x)=f(2 x+3)\), for example, we have to think about how the inputs to the function \(g\) relate to the inputs to the function \(f\). Suppose we know \(f(7)=12\). What input to \(g\) would produce that output? In other words, what value of \(x\) will allow \(g(x)=f(2 x+3)=12\) ? We would need \(2 x+3=7\). To solve for \(x\), we would first subtract 3 , resulting in a horizontal shift, and then divide by 2 , causing a horizontal compression.

This format ends up being very difficult to work with, because it is usually much easier to horizontally stretch a graph before shifting. We can work around this by factoring inside the function.
\(f(b x+p)=f\left(b\left(x+\frac{p}{b}\right)\right)\)
Let's work through an example.
\(f(x)=(2 x+4)^{2}\)
We can factor out a 2.
\(f(x)=(2(x+2))^{2}\)
Now we can more clearly observe a horizontal shift to the left 2 units and a horizontal compression.
Factoring in this way allows us to horizontally stretch first and then shift horizontally.

\section*{EXAMPLE: APPLYING A LEARNING MODEL EQUATION}

A common model for learning has an equation similar to \(k(t)=-2^{-t}+1\), where \(k\) is the percentage of mastery that can be achieved after \(t\) practice sessions. This is a transformation of the function \(f(t)=2^{t}\) shown below. Sketch a graph of \(k(t)\).


\section*{Answer}

This equation combines three transformations into one equation.
- A horizontal reflection: \(f(-t)=2^{-t}\)
- A vertical reflection: \(-f(-t)=-2^{-t}\)
- A vertical shift: \(-f(-t)+1=-2^{-t}+1\)

We can sketch a graph by applying these transformations one at a time to the original function. Let us follow two points through each of the three transformations. We will choose the points \((0,1)\) and \((1,2)\).
1. First, we apply a horizontal reflection: \((0,1)(-1,2)\).
2. Then, we apply a vertical reflection: \((0,-1)(1,-2)\).

3 . Finally, we apply a vertical shift: \((0,0)(1,1)\).
This means that the original points, \((0,1)\) and \((1,2)\) become \((0,0)\) and \((1,1)\) after we apply the transformations.
In the graphs below, the first graph results from a horizontal reflection. The second results from a vertical reflection. The third results from a vertical shift up 1 unit.


Analysis of the Solution
As a model for learning, this function would be limited to a domain of \(t \geq 0\), with corresponding range \([0,1)\).

\section*{TRY IT}

Given the toolkit function \(f(x)=x^{2}\), graph \(g(x)=-f(x)\) and \(h(x)=f(-x)\). Take note of any surprising behavior for these functions.
Answer


Notice: \(g(x)=f(-x)\) looks the same as \(f(x)\).

\section*{A GENERAL NOTE: COMBINING TRANSFORMATIONS}

When combining vertical transformations written in the form \(a f(x)+k\), first vertically stretch by \(a\) and then vertically shift by \(k\).
When combining horizontal transformations written in the form \(f(b x+h)\), first horizontally shift by \(h\) and then horizontally stretch by \(\frac{1}{b}\).
When combining horizontal transformations written in the form \(f(b(x+h))\), first horizontally stretch by \(\frac{1}{b}\) and then horizontally shift by \(h\).
Horizontal and vertical transformations are independent. It does not matter whether horizontal or vertical transformations are performed first.

\section*{EXAMPLE: FINDING A TRIPLE TRANSFORMATION OF A TABULAR FUNCTION}

Given the table below for the function \(f(x)\), create a table of values for the function \(g(x)=2 f(3 x)+1\).
\begin{tabular}{l|l|l|l|l}
\(x\) & 6 & 12 & 18 & 24 \\
\hline\(f(x)\) & 10 & 14 & 15 & 17
\end{tabular}

Answer
There are three steps to this transformation, and we will work from the inside out. Starting with the horizontal transformations, \(f(3 x)\) is a horizontal compression by \(\frac{1}{3}\), which means we multiply each \(x\) value by \(\frac{1}{3}\).
\(x\)
\(f(3 x)\)
2
10
4
6
8
17

Looking now to the vertical transformations, we start with the vertical stretch, which will multiply the output values by 2. We apply this to the previous transformation.
\(x\)
\(2 f(3 x)\)
2
20
4
28
6
30
8
34

Finally, we can apply the vertical shift, which will add 1 to all the output values.
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 2 & 4 & 6 & 8 \\
\hline\(g(x)=2 f(3 x)+1\) & 21 & 29 & 31 & 35 \\
\hline
\end{tabular}

\section*{TRY IT}

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\section*{EXAMPLE: FINDING A TRIPLE TRANSFORMATION OF A GRAPH}

Use the graph of \(f(x)\) to sketch a graph of \(k(x)=f\left(\frac{1}{2} x+1\right)-3\).


\section*{Answer}

To simplify, let's start by factoring out the inside of the function.
\(f\left(\frac{1}{2} x+1\right)-3=f\left(\frac{1}{2}(x+2)\right)-3\)
By factoring the inside, we can first horizontally stretch by 2 , as indicated by the \(\frac{1}{2}\) on the inside of the function. Remember that twice the size of 0 is still 0 , so the point \((0,2)\) remains at \((0,2)\) while the point \((2,0)\) will stretch to \((4,0)\).


Next, we horizontally shift left by 2 units, as indicated by \(x+2\).


Last, we vertically shift down by 3 to complete our sketch, as indicated by the -3 on the outside of the function.


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\section*{SUMMARY: TRANSFORMATIONS OF FUNCTIONS}

\section*{Key Equations}
\begin{tabular}{|l|l|}
\hline Vertical shift & \(g(x)=f(x)+k\) (up for \(k>0)\) \\
\hline Horizontal shift & \(g(x)=f(x-h)\) (right for \(h>0)\) \\
\hline Vertical reflection & \(g(x)=-f(x)\) \\
\hline Horizontal reflection & \(g(x)=f(-x)\) \\
\hline Vertical stretch & \(g(x)=a f(x)(a>0)\) \\
\hline Vertical compression & \(g(x)=a f(x)(0<a<1)\) \\
\hline Horizontal stretch & \(g(x)=f(b x)(0<b<1)\) \\
\hline Horizontal compression & \(g(x)=f(b x)(b>1)\) \\
\hline
\end{tabular}

\section*{Key Concepts}
- A function can be shifted vertically by adding a constant to the output.
- A function can be shifted horizontally by adding a constant to the input.
- Relating the shift to the context of a problem makes it possible to compare and interpret vertical and horizontal shifts.
- Vertical and horizontal shifts are often combined.
- A vertical reflection reflects a graph about the \(x\) - axis. A graph can be reflected vertically by multiplying the output by -1 .
- A horizontal reflection reflects a graph about the \(y\) - axis. A graph can be reflected horizontally by multiplying the input by -1 .
- A graph can be reflected both vertically and horizontally. The order in which the reflections are applied does not affect the final graph.
- A function presented in tabular form can also be reflected by multiplying the values in the input and output rows or columns accordingly.
- A function presented as an equation can be reflected by applying transformations one at a time.
- Even functions are symmetric about the \(y\) - axis, whereas odd functions are symmetric about the origin.
- Even functions satisfy the condition \(f(x)=f(-x)\).
- Odd functions satisfy the condition \(f(x)=-f(-x)\).
- A function can be odd, even, or neither.
- A function can be compressed or stretched vertically by multiplying the output by a constant.
- A function can be compressed or stretched horizontally by multiplying the input by a constant.
- The order in which different transformations are applied does affect the final function. Both vertical and horizontal transformations must be applied in the order given. However, a vertical transformation may be combined with a horizontal transformation in any order.

\section*{Glossary}
even function a function whose graph is unchanged by horizontal reflection, \(f(x)=f(-x)\), and is symmetric about the \(y\) - axis
horizontal compression a transformation that compresses a function's graph horizontally, by multiplying the input by a constant \(b>1\)
horizontal reflection a transformation that reflects a function's graph across the \(y\)-axis by multiplying the input by -1
horizontal shift a transformation that shifts a function's graph left or right by adding a positive or negative constant to the input
horizontal stretch a transformation that stretches a function's graph horizontally by multiplying the input by a constant \(0<b<1\)
odd function a function whose graph is unchanged by combined horizontal and vertical reflection, \(f(x)=-f(-x)\), and is symmetric about the origin
vertical compression a function transformation that compresses the function's graph vertically by multiplying the output by a constant \(0<a<1\)
vertical reflection a transformation that reflects a function's graph across the \(x\)-axis by multiplying the output by -1
vertical shift a transformation that shifts a function's graph up or down by adding a positive or negative constant to the output
vertical stretch a transformation that stretches a function's graph vertically by multiplying the output by a constant \(a>1\)

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\section*{INTRODUCTION TO INVERSE FUNCTIONS}

\section*{Learning Outcomes}

By the end of this lesson, you will be able to:
- Verify inverse functions.
- Determine the domain and range of an inverse function, and restrict the domain of a function to make it one-to-one.
- Find or evaluate the inverse of a function.
- Use the graph of a one-to-one function to graph its inverse function on the same axes.

A reversible heat pump is a climate-control system that is an air conditioner and a heater in a single device. Operated in one direction, it pumps heat out of a house to provide cooling. Operating in reverse, it pumps heat into the building from the outside, even in cool weather, to provide heating. As a heater, a heat pump is several times more efficient than conventional electrical resistance heating.

If some physical machines can run in two directions, we might ask whether some of the function "machines" we have been studying can also run backwards. Figure 1 provides a visual representation of this question. In this section, we will consider the reverse nature of functions.


Figure 1. Can a function "machine" operate in reverse?

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\section*{CHARACTERISTICS OF INVERSE FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Identify an inverse for tabular data.
- Test whether two functions are inverses.
- Determine the domain and range of an inverse.

Suppose a fashion designer traveling to Milan for a fashion show wants to know what the temperature will be. He is not familiar with the Celsius scale. To get an idea of how temperature measurements are related, he asks his assistant, Betty, to convert 75 degrees Fahrenheit to degrees Celsius. She finds the formula \(C=\frac{5}{9}(F-32)\) and substitutes 75 for \(F\) to calculate \(\frac{5}{9}(75-32) \approx 24^{\circ} C\).


Knowing that a comfortable 75 degrees Fahrenheit is about 24 degrees Celsius, he sends his assistant the week's weather forecast for Milan, and asks her to convert all of the temperatures to degrees Fahrenheit.

At first, Betty considers using the formula she has already found to complete the conversions. After all, she knows her algebra, and can easily solve the equation for \(F\) after substituting a value for \(C\). For example, to convert 26 degrees Celsius, she could write
\(26=\frac{5}{9}(F-32)\)
\(26 \cdot \frac{9}{5}=F-32\)
\(F=26 \cdot \frac{9}{5}+32 \approx 79\)
After considering this option for a moment, however, she realizes that solving the equation for each of the temperatures will be awfully tedious. She realizes that since evaluation is easier than solving, it would be much more convenient to have a different formula, one that takes the Celsius temperature and outputs the Fahrenheit temperature.

The formula for which Betty is searching corresponds to the idea of an inverse function, which is a function for which the input of the original function becomes the output of the inverse function and the output of the original function becomes the input of the inverse function.

Given a function \(f(x)\), we represent its inverse as \(f^{-1}(x)\), read as " \(f\) inverse of \(x\)." The raised -1 is part of the notation. It is not an exponent; it does not imply a power of -1 . In other words, \(f^{-1}(x)\) does not mean \(\frac{1}{f(x)}\) because \(\frac{1}{f(x)}\) is the reciprocal of \(f\) and not the inverse.

The "exponent-like" notation comes from an analogy between function composition and multiplication: just as \(a^{-1} a=1\) ( 1 is the identity element for multiplication) for any nonzero number \(a\), so \(f^{-1} \circ f\) equals the identity function, that is,
\[
\left(f^{-1} \circ f\right)(x)=f^{-1}(f(x))=f^{-1}(y)=x
\]

This holds for all \(x\) in the domain of \(f\). Informally, this means that inverse functions "undo" each other. However, just as zero does not have a reciprocal, some functions do not have inverses that are also functions.

Given a function \(f(x)\), we can verify whether some other function \(g(x)\) is the inverse of \(f(x)\) by checking whether either \(g(f(x))=x\) or \(f(g(x))=x\) is true. We can test whichever equation is more convenient to work with because they are logically equivalent (that is, if one is true, then so is the other.)

For example, \(y=4 x\) and \(y=\frac{1}{4} x\) are inverse functions.
\(\left(f^{-1} \circ f\right)(x)=f^{-1}(4 x)=\frac{1}{4}(4 x)=x\)
and
\(\left(f \circ f^{-1}\right)(x)=f\left(\frac{1}{4} x\right)=4\left(\frac{1}{4} x\right)=x\)
A few coordinate pairs from the graph of the function \(y=4 x\) are \((-2,-8),(0,0)\), and \((2,8)\). A few coordinate pairs from the graph of the function \(y=\frac{1}{4} x\) are \((-8,-2),(0,0)\), and \((8,2)\). If we interchange the input and output of each coordinate pair of a function, the interchanged coordinate pairs would appear on the graph of the inverse function.

\section*{A GENERAL NOTE: INVERSE FUNCTION}

For any one-to-one function \(f(x)=y\), a function \(f^{-1}(x)\) is an inverse function of \(f\) if \(f^{-1}(y)=x\). This can also be written as \(f^{-1}(f(x))=x\) for all \(x\) in the domain of \(f\). It also follows that \(f\left(f^{-1}(x)\right)=x\) for all \(x\) in the domain of \(f^{-1}\) if \(f^{-1}\) is the inverse of \(f\).
The notation \(f^{-1}\) is read " \(f\) inverse." Like any other function, we can use any variable name as the input for \(f^{-1}\), so we will often write \(f^{-1}(x)\), which we read as " \(f\) inverse of \(x\) ".
Keep in mind that \(f^{-1}(x) \neq \frac{1}{f(x)}\) and not all functions have inverses.

\section*{TIP FOR SUCCESS}

The note in the box above cannot be taken too lightly. It's worth another read to be sure you understand the notation. As with all difficult concepts, it's perfectly natural to take some time before you understand it. Working the examples in the text section on paper will help.

\section*{EXAMPLE: IDENTIFYING AN INVERSE FUNCTION FOR A GIVEN INPUTOUTPUT PAIR}

If for a particular one-to-one function \(f(2)=4\) and \(f(5)=12\), what are the corresponding input and output values for the inverse function?

\section*{Answer}

The inverse function reverses the input and output quantities, so if
\(f(2)=4\), then \(f^{-1}(4)=2\)
and if
\(f(5)=12\), then \(f^{-1}(12)=5\)
Alternatively, if we want to name the inverse function \(g\), then \(g(4)=2\) and \(g(12)=5\).

\section*{Analysis of the Solution}

Notice that if we show the coordinate pairs in a table form, the input and output are clearly reversed.
\begin{tabular}{|l|l|}
\hline\((x, f(x))\) & \((x, g(x))\) \\
\hline\((2,4)\) & \((4,2)\) \\
\hline\((5,12)\) & \((12,5)\) \\
\hline
\end{tabular}

\section*{TRY IT}

Given that \(h^{-1}(6)=2\), what are the corresponding input and output values of the original function \(h\) ?
Answer
\(h(2)=6\)

\section*{Determine the Domain and Range of an Inverse Function}

The outputs of the function \(f\) are the inputs to \(f^{-1}\), so the range of \(f\) is also the domain of \(f^{-1}\). Likewise, because the inputs to \(f\) are the outputs of \(f^{-1}\), the domain of \(f\) is the range of \(f^{-1}\). We can visualize the situation.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=227

\section*{HOW TO: GIVEN TWO FUNCTIONS \(f(x)\) AND \(g(x)\), TEST WHETHER THE FUNCTIONS ARE INVERSES OF EACH OTHER.}
1. Determine whether \(f(g(x))=x\) and \(g(f(x))=x\).
2. If both statements are true, then \(g=f^{-1}\) and \(f=g^{-1}\). If either statement is false, then \(g \neq f^{-1}\) and \(f \neq g^{-1}\).


Domain and range of a function and its inverse
When a function has no inverse function, it is possible to create a new function where that new function on a limited domain does have an inverse function. For example, the inverse of \(f(x)=\sqrt{x}\) is \(f^{-1}(x)=x^{2}\), because a square "undoes" a square root; but the square is only the inverse of the square root on the domain \([0, \infty)\), since that is the range of \(f(x)=\sqrt{x}\).

We can look at this problem from the other side, starting with the square (toolkit quadratic) function \(f(x)=x^{2}\). If we want to construct an inverse to this function, we run into a problem, because for every given output of the quadratic function, there are two corresponding inputs (except when the input is 0 ). For example, the output 9 from the quadratic function corresponds to the inputs 3 and -3 . But an output from a function is an input to its inverse; if this inverse input corresponds to more than one inverse output (input of the original function), then the "inverse" is not a function at all! To put it differently, the quadratic function is not a one-to-one function; it fails the horizontal line test, so it does not have an inverse function. In order for a function to have an inverse, it must be a one-to-one function.

In many cases, if a function is not one-to-one, we can still restrict the function to a part of its domain on

\section*{EXAMPLE: TESTING INVERSE RELATIONSHIPS ALGEBRAICALLY}

If \(f(x)=\frac{1}{x+2}\) and \(g(x)=\frac{1}{x}-2\), is \(g=f^{-1}\) ?
Answer
\[
\begin{aligned}
g(f(x)) & =\frac{1}{\left(\frac{1}{x+2}\right)}-2 \\
& =x+2-2 \\
& =x
\end{aligned}
\]
so
\(g=f^{-1}\) and \(f=g^{-1}\)
This is enough to answer yes to the question, but we can also verify the other formula.
\[
\begin{aligned}
f(g(x)) & =\frac{1}{\frac{1}{x}-2+2} \\
& =\frac{1}{\frac{1}{x}} \\
& =x
\end{aligned}
\]

Analysis of the Solution
Notice the inverse operations are in reverse order of the operations from the original function.

\section*{TRY IT}

If \(f(x)=x^{3}-4\) and \(g(x)=\sqrt[3]{x+4}\), is \(g=f^{-1}\) ?
Answer
Yes

\section*{EXAMPLE: DETERMINING INVERSE RELATIONSHIPS FOR POWER FUNCTIONS}

If \(f(x)=x^{3}\) (the cube function) and \(g(x)=\frac{1}{3} x\), is \(g=f^{-1}\) ?
Answer
\(f(g(x))=\left(\frac{1}{3} x\right)^{3}=\frac{x^{3}}{27} \neq x\)
No, the functions are not inverses.

Analysis of the Solution

The correct inverse to \(x^{3}\) is the cube root \(\sqrt[3]{x}=x^{\frac{1}{3}}\), that is, the one-third is an exponent, not a multiplier.
which it is one-to-one. For example, we can make a restricted version of the square function \(f(x)=x^{2}\) with

\section*{TRY IT}

If \(f(x)=(x-1)^{3}\) and \(g(x)=\sqrt[3]{x}+1\), is \(g=f^{-1}\) ?
Answer
Yes
its range limited to \([0, \infty)\), which is a one-to-one function (it passes the horizontal line test) and which has an inverse (the square-root function).

If \(f(x)=(x-1)^{2}\) on \([1, \infty)\), then the inverse function is \(f^{-1}(x)=\sqrt{x}+1\).
- The domain of \(f=\) range of \(f^{-1}=[1, \infty)\).
- The domain of \(f^{-1}=\) range of \(f=[0, \infty)\).

\section*{Q \& A}

Is it possible for a function to have more than one inverse?
No. If two supposedly different functions, say, \(g\) and \(h\), both meet the definition of being inverses of another function \(f\), then you can prove that \(g=h\). We have just seen that some functions only have inverse functions if we restrict the domain of the original function. In these cases, there may be more than one way to restrict the domain, leading to different inverse functions. However, on any one domain, the original function still has only one unique inverse function.

\section*{A GENERAL NOTE: DOMAIN AND RANGE OF INVERSE FUNCTIONS}

The range of a function \(f(x)\) is the domain of the inverse function \(f^{-1}(x)\).
The domain of \(f(x)\) is the range of \(f^{-1}(x)\).

\section*{HOW TO: GIVEN A FUNCTION, FIND THE DOMAIN AND RANGE OF ITS INVERSE.}
1. If the function is one-to-one, write the range of the original function as the domain of the inverse, and write the domain of the original function as the range of the inverse.
2. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the range of the inverse function.

\section*{EXAMPLE: FINDING THE INVERSES OF TOOLKIT FUNCTIONS}

Identify which of the toolkit functions besides the quadratic function are not one-to-one, and find a restricted domain on which each function is one-to-one, if any. The toolkit functions are reviewed below. We restrict the domain in such a fashion that the function assumes all \(y\)-values exactly once.
\begin{tabular}{|l|l|l|l|l|}
\hline Constant & Identity & Quadratic & Cubic & Reciprocal \\
\hline\(f(x)=c\) & \(f(x)=x\) & \(f(x)=x^{2}\) & \(f(x)=x^{3}\) & \(f(x)=\frac{1}{x}\) \\
\hline Reciprocal squared & Cube root & Square root & Absolute value & \\
\hline\(f(x)=\frac{1}{x^{2}}\) & \(f(x)=\sqrt[3]{x}\) & \(f(x)=\sqrt{x}\) & \(f(x)=|x|\) & \\
\hline
\end{tabular}

\section*{Answer}

The constant function is not one-to-one, and there is no domain (except a single point) on which it could be one-to-one, so the constant function has no meaningful inverse.
The absolute value function can be restricted to the domain \([0, \infty)\), where it is equal to the identity function.
The reciprocal-squared function can be restricted to the domain \((0, \infty)\).

\section*{Analysis of the Solution}

We can see that these functions (if unrestricted) are not one-to-one by looking at their graphs. They both would fail the horizontal line test. However, if a function is restricted to a certain domain so that it passes the horizontal line test, then in that restricted domain, it can have an inverse.

(a)

(b)
(a) Absolute value (b) Reciprocal squared

\section*{TRY IT}

The domain of the function \(f\) is \((1, \infty)\) and the range of the function \(f\) is \((-\infty,-2)\). Find the domain and range of the inverse function.

\section*{Answer}

The domain of the function \(f^{-1}\) is \((-\infty,-2)\) and the range of the function \(f^{-1}\) is \((1, \infty)\).

\section*{DEFINE AND GRAPH AN INVERSE}

\section*{LEARNING OUTCOMES}
- Find and evaluate the inverse of a function in tabular form.
- Find and evaluate the inverse of a function given as an equation.
- Graph a function and its inverse.

Once we have a one-to-one function, we can evaluate its inverse at specific inverse function inputs or construct a complete representation of the inverse function in many cases.

\section*{Inverting Tabular Functions}

Suppose we want to find the inverse of a function represented in table form. Remember that the domain of a function is the range of the inverse and the range of the function is the domain of the inverse. So we need to interchange the domain and range.

Each row (or column) of inputs becomes the row (or column) of outputs for the inverse function. Similarly, each row (or column) of outputs becomes the row (or column) of inputs for the inverse function.

\section*{RECALL READING FUNCTIONS FROM TABLES}

Look to see which row or column of a table is labeled as the input and which is labeled as the output.
The tables below indicate one row represents " \(t\) " and the other, \(f(t)\).
It is common for input to be represented by time when output is a distance.

\section*{Evaluating the Inverse of a Function, Given a Graph of the Original Function}

The domain of a function can be read by observing the horizontal extent of its graph. We find the domain of the inverse function by observing the vertical extent of the graph of the original function, because this corresponds to the horizontal extent of the inverse function. Similarly, we find the range of the inverse function by observing the horizontal extent of the graph of the original function, as this is the vertical extent of the inverse function. If we want to evaluate an inverse function, we find its input within its domain, which is all or part of the vertical axis of the original function's graph.

\section*{EXAMPLE: INTERPRETING THE INVERSE OF A TABULAR FUNCTION}

A function \(f(t)\) is given below, showing distance in miles that a car has traveled in \(t\) minutes. Find and interpret \(f^{-1}(70)\).
\begin{tabular}{|l|l|l|l|l|}
\hline\(t\) (minutes) & 30 & 50 & 70 & 90 \\
\hline\(f(t)\) (miles) & 20 & 40 & 60 & 70 \\
\hline
\end{tabular}

\section*{Answer}

The inverse function takes an output of \(f\) and returns an input for \(f\). So in the expression \(f^{-1}(70), 70\) is an output value of the original function, representing 70 miles. The inverse will return the corresponding input of the original function \(f, 90\) minutes, so \(f^{-1}(70)=90\). The interpretation of this is that, to drive 70 miles, it took 90 minutes.
Alternatively, recall that the definition of the inverse was that if \(f(a)=b\), then \(f^{-1}(b)=a\). By this definition, if we are given \(f^{-1}(70)=a\), then we are looking for a value \(a\) so that \(f(a)=70\). In this case, we are looking for a \(t\) so that \(f(t)=70\), which is when \(t=90\).

\section*{TRY IT}

Using the table below, find and interpret (a) \(f(60)\), and (b) \(f^{-1}(60)\).
\begin{tabular}{|l|l|l|l|l|l|}
\hline\(t\) (minutes) & 30 & 50 & 60 & 70 & 90 \\
\hline\(f(t)\) (miles) & 20 & 40 & 50 & 60 & 70 \\
\hline
\end{tabular}

Answer
a. \(f(60)=50\). In 60 minutes, 50 miles are traveled.
b. \(f^{-1}(60)=70\). To travel 60 miles, it will take 70 minutes.

\section*{HOW TO: GIVEN THE GRAPH OF A FUNCTION, EVALUATE ITS INVERSE AT SPECIFIC POINTS.}
1. Find the desired input of the inverse function on the \(y\)-axis of the given graph.
2. Read the inverse function's output from the \(x\)-axis of the given graph.

\section*{Finding Inverses of Functions Represented by Formulas}

Sometimes we will need to know an inverse function for all elements of its domain, not just a few. If the original function is given as a formula-for example, \(y\) as a function of \(x\) - we can often find the inverse function by solving to obtain \(x\) as a function of \(y\).

\section*{Graph a Function's Inverse}

Now that we can find the inverse of a function, we will explore the graphs of functions and their inverses. Let us return to the quadratic function \(f(x)=x^{2}\) restricted to the domain \([0, \infty)\), on which this function is one-to-one, and graph it as below.

\section*{EXAMPLE: EVALUATING A FUNCTION AND ITS INVERSE FROM A GRAPH AT SPECIFIC POINTS}

A function \(g(x)\) is given below. Find \(g(3)\) and \(g^{-1}(3)\).


\section*{Answer}

To evaluate \(g(3)\), we find 3 on the \(x\)-axis and find the corresponding output value on the \(y\)-axis. The point \((3,1)\) tells us that \(g(3)=1\).
To evaluate \(g^{-1}(3)\), recall that by definition \(g^{-1}(3)\) means the value of \(x\) for which \(g(x)=3\). By looking for the output value 3 on the vertical axis, we find the point \((5,3)\) on the graph, which means \(g(5)=3\), so by definition, \(g^{-1}(3)=5\).


\section*{TRY IT}

Using the graph in the previous example, (a) find \(g^{-1}(1)\), and (b) estimate \(g^{-1}(4)\).
Answer
a. 3; b. 5.6

\section*{HOW TO: GIVEN A FUNCTION REPRESENTED BY A FORMULA, FIND THE INVERSE.}
1. Verify that \(f\) is a one-to-one function.
2. Replace \(f(x)\) with \(y\).
3. Interchange \(x\) and \(y\).
4. Solve for \(y\), and rename the function \(f^{-1}(x)\).

\section*{EXAMPLE: INVERTING THE FAHRENHEIT-TO-CELSIUS FUNCTION}

Find a formula for the inverse function that gives Fahrenheit temperature as a function of Celsius temperature.
\(C=\frac{5}{9}(F-32)\)
Answer
\(C=\frac{5}{9}(F-32)\)
\(C \cdot \frac{9}{5}=F-32\)
\(F=\frac{9}{5} C+32\)
By solving in general, we have uncovered the inverse function. If
\(C=h(F)=\frac{5}{9}(F-32)\),
then
\(F=h^{-1}(C)=\frac{9}{5} C+32\).
In this case, we introduced a function \(h\) to represent the conversion because the input and output variables are descriptive, and writing \(C^{-1}\) could get confusing.

\section*{TRY IT}

Solve for \(x\) in terms of \(y\) given \(y=\frac{1}{3}(x-5)\)

\section*{Answer}
\(x=3 y+5\)


Quadratic function with domain restricted to \([0, \infty)\).
Restricting the domain to \([0, \infty)\) makes the function one-to-one (it will obviously pass the horizontal line test), so it has an inverse on this restricted domain.

We already know that the inverse of the toolkit quadratic function is the square root function, that is, \(f^{-1}(x)=\sqrt{x}\). What happens if we graph both \(f\) and \(f^{-1}\) on the same set of axes, using the \(x\) - axis for the

\section*{EXAMPLE: SOLVING TO FIND AN INVERSE FUNCTION}

Find the inverse of the function \(f(x)=\frac{2}{x-3}+4\).
Answer
\(y=\frac{2}{x-3}+4 \quad\) Change \(f(x)\) to \(y\).
\(x=\frac{2}{y-3}+4 \quad\) Switch \(x\) and \(y\).
\(y-4=\frac{2}{x-3} \quad\) Subtract 4 from both sides.
\(y-3=\frac{2}{x-4} \quad\) Multiply both sides by \(y-3\) and divide by \(x-4\).
\(y=\frac{2}{x-4}+3 \quad\) Add 3 to both sides.
So \(f^{-1}(x)=\frac{2}{x-4}+3\).

\section*{Analysis of the Solution}

The domain and range of \(f\) exclude the values 3 and 4, respectively. \(f\) and \(f^{-1}\) are equal at two points but are not the same function, as we can see by creating the table below.
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 1 & 2 & 5 & \(f^{-1}(y)\) \\
\hline\(f(x)\) & 3 & 2 & 5 & \(y\) \\
\hline
\end{tabular}
input to both \(f\) and \(f^{-1}\) ?
We notice a distinct relationship: The graph of \(f^{-1}(x)\) is the graph of \(f(x)\) reflected about the diagonal line \(y=x\), which we will call the identity line, shown below.

\section*{EXAMPLE: SOLVING TO FIND AN INVERSE WITH RADICALS}

Find the inverse of the function \(f(x)=2+\sqrt{x-4}\).

\section*{Answer}
\(y=2+\sqrt{x-4}\)
\(x=2+\sqrt{y-4}\)
\((x-2)^{2}=y-4\)
\(y=(x-2)^{2}+4\)
So \(f^{-1}(x)=(x-2)^{2}+4\).
The domain of \(f\) is \([4, \infty)\). Notice that the range of \(f\) is \([2, \infty)\), so this means that the domain of the inverse function \(f^{-1}\) is also \([2, \infty)\).

\section*{Analysis of the Solution}

The formula we found for \(f^{-1}(x)\) looks like it would be valid for all real \(x\). However, \(f^{-1}\) itself must have an inverse (namely, \(f\) ) so we have to restrict the domain of \(f^{-1}\) to \(\left[2, \infty\right.\) ) in order to make \(f^{-1}\) a one-toone function. This domain of \(f^{-1}\) is exactly the range of \(f\).

\section*{TRY IT}

What is the inverse of the function \(f(x)=2-\sqrt{x}\) ? State the domains of both the function and the inverse function.
Use an online graphing calculator to graph the function, it's inverse, and \(f(x)=x\) to check whether you are correct.

\section*{Answer}
\(f^{-1}(x)=(2-x)^{2}\); domain of \(f:[0, \infty)\); domain of \(f^{-1}:(-\infty, 2]\)


This relationship will be observed for all one-to-one functions, because it is a result of the function and its inverse swapping inputs and outputs. This is equivalent to interchanging the roles of the vertical and horizontal axes.

\section*{EXAMPLE: FINDING THE INVERSE OF A FUNCTION USING REFLECTION ABOUT THE IDENTITY LINE}

Given the graph of \(f(x)\), sketch a graph of \(f^{-1}(x)\).


\section*{Answer}

This is a one-to-one function, so we will be able to sketch an inverse. Note that the graph shown has an apparent domain of \((0, \infty)\) and range of \((-\infty, \infty)\), so the inverse will have a domain of \((-\infty, \infty)\) and range of \((0, \infty)\).
If we reflect this graph over the line \(y=x\), the point \((1,0)\) reflects to \((0,1)\) and the point \((4,2)\) reflects to \((2,4)\). Sketching the inverse on the same axes as the original graph gives us the result in the graph below.


The function and its inverse, showing reflection about the identity line

\section*{Q \& A}

Is there any function that is equal to its own inverse?
Yes. If \(f=f^{-1}\), then \(f(f(x))=x\), and we can think of several functions that have this property. The identity function does, and so does the reciprocal function, because
\(\frac{1}{\frac{1}{x}}=x\)
Any function \(f(x)=c-x\), where \(c\) is a constant, is also equal to its own inverse.

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\section*{SUMMARY: INVERSE FUNCTIONS}

\section*{Key Concepts}
- If \(g(x)\) is the inverse of \(f(x)\), then \(g(f(x))=f(g(x))=x\).
- Each of the toolkit functions, except \(y=c\) has an inverse. Some need a restricted domain.
- For a function to have an inverse, it must be one-to-one (pass the horizontal line test).
- A function that is not one-to-one over its entire domain may be one-to-one on part of its domain.
- For a tabular function, exchange the input and output rows to obtain the inverse.
- The inverse of a function can be determined at specific points on its graph.
- To find the inverse of a function \(y=f(x)\), switch the variables \(x\) and \(y\). Then solve for \(y\) as a function of \(x\).
- The graph of an inverse function is the reflection of the graph of the original function across the line \(y=x\).

\section*{Glossary}
inverse function for any one-to-one function \(f(x)\), the inverse is a function \(f^{-1}(x)\) such that \(f^{-1}(f(x))=x\) for all \(x\) in the domain of \(f\); this also implies that \(f\left(f^{-1}(x)\right)=x\) for all \(x\) in the domain of \(f^{-1}\)

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\section*{PUTTING IT TOGETHER: ALGEBRAIC OPERATIONS ON FUNCTIONS}

You have been tasked with creating a dazzling new logo for the school newspaper, The Parabola. So of course you start thinking about how you might incorporate parabolas into the design. The standard parabola, defined by the function \(f(x)=x^{2}\), is simply too boring by itself. However by scaling, shifting, and reflecting, perhaps you might come up something more interesting.

First of all, the basic parabola shape seems too steep. You can use a vertical compression to scale it down. Remember, a vertical stretch or compression takes the form \(y=a f(x)\), for a constant \(a\).

Vertical compression by a factor of 2: \(y=\frac{1}{2} x^{2}\)
Vertical compression by a factor of 3: \(y=\frac{1}{3} x^{2}\)
Vertical compression by a factor of 4: \(y=\frac{1}{4} x^{2}\)
Let's see what all of these functions look like when graphed together. We'll graph \(y=x^{2}\) in black, and then overlay the compressions by 2,3 , and 4 , in purple, blue, and green, respectively.


Not bad - in fact it's looking like a pretty decent logo already, but let's add a few more features. The school colors are red and gray, so let's put two upside-down parabolas in the foreground. The transformation that reflects a graph about the \(x\)-axis is: \(y=f-(x)\). But we also want to shift the graphs upwards after reflecting. Therefore we should combine the reflection with a vertical shift. Altogether, the two transformations are accomplished by \(y=-f(x)+k\) for a constant \(k\).

Reflected and shifted up by 2: \(y=-x^{2}+2\)
Reflected and shifted up by 4: \(y=-x^{2}+4\)


The red curve is \(y=-x^{2}+4\), and the gray curve is \(y=-x^{2}+2\).
Finally, let's combine the parabolas all together and create our logo! We'll get rid of the axes, colorize the background, and include the name of the paper for our finished product. Who knew that mathematics could help to create art?


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\title{
MODULE 7: LINEAR AND ABSOLUTE VALUE FUNCTIONS
}

\section*{WHY IT MATTERS: LINEAR AND ABSOLUTE VALUE FUNCTIONS}

\section*{Why Use Linear and Absolute Value Functions?}

You have a great idea for a small business. You and a friend have developed a battery-powered bike. It's perfect for getting around a college campus or even local stops in town. You enjoy making the bikes, but would it be a worthwhile business-one from which you can earn a profit?

The profit your business can earn depends on two main factors. First, it depends on how much it costs
 you to make the bikes. These costs include the parts you buy to make each bike as well as any rent and utilities you pay for the location where you make the bikes. It also includes any salaries you pay people to help you.

Second, profit depends on revenue, which is the amount of money you take in by selling the bikes.
Profit \(=\) Revenue - Costs
Both revenue and costs are linear functions. They depend on the number of bikes you sell. You can then rewrite the profit equation as a function:
\(P(x)=R(x)-C(x)\)
where \(P(x)\) is profit, \(R(x)\) is revenue, \(C(x)\) is cost and \(x\) is equal to the number of bikes produced and sold.

You and your business partner determine that your fixed costs, those you can't change such as the room you rent for the business, are \(\$ 1,600\), and your variable costs, those associated with each bike, are \(\$ 200\). If you sell each bike for \(\$ 600\), the table shows your profits for different numbers of bikes.
\begin{tabular}{|c|c|}
\hline Number of bikes & Profit (\$) \\
\hline 2 & -800 \\
\hline
\end{tabular}

As seen in the table above, if you only sell 2 bikes, you actually lose money. However, if you sell 5 or more bikes, you earn a profit.
- How can you figure out whether you will have a profit or a loss?
- How can you determine how many bikes you need to sell to break even?
- How will shifting your price affect your profits?

In this module you'll find out how to answer all of these questions. Read on to learn how you might get your business up and running. At the end of the module, we'll revisit your bike business to find out the very point at which you'll start to earn a profit.

\section*{LEARNING OUTCOMES}

\section*{Review Topics for Success}
- Graph linear functions using tables
- Define slope for a linear function
- Calculate slope given two points
- Interpret the slope of a linear function that models a real-world situation

\section*{Linear Functions}
- Represent a linear function with an equation, words, table, and a graph
- Determine whether a linear function is increasing, decreasing, or constant
- Calculate slope for a linear function given two points
- Write and interpret a linear function

\section*{Graphing and Writing Equations of Linear Functions}
- Graph linear functions by plotting points, using the slope and y-intercept, and using transformations
- Write the equation of a linear function given its graph
- Match linear functions with their graphs
- Find the \(x\)-intercept of a function given its equation
- Find the equations of vertical and horizontal lines
- Determine whether lines are parallel or perpendicular given their equations
- Find equations of lines that are parallel or perpendicular to a given line
- Graph an absolute value function
- Find the intercepts of an absolute value function

\section*{Modeling with Linear Functions}
- Use prescribed strategies to build linear models
- Use intercepts and data points to build a linear model
- Use a diagram to build a model
- Draw and interpret scatter plots
- Find the line of best fit using a calculator
- Distinguish between linear and nonlinear relations
- Use a linear model to make predictions
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\section*{REVIEW TOPICS FOR SUCCESS}

Learning Outcomes
By the end of this section, you will be able to:
- Graph linear functions using tables
- Define slope for a linear function
- Calculate slope given two points
- Interpret the slope of a linear function that models a real-world situation

When studying linear equations, you learned that if you can identify two points that lie on the line, you can calculate the slope and write the equation that describes the line. Since graphs of linear functions behave exactly like lines graphed in the plane, the same idea applies. Given two pairs of coordinates, inputs and outputs, you can calculate the rate of change of the function, and write its equation.

The reverse process holds as well; given a linear function in equation form, you can evaluate the function for chosen input values and use the resulting ordered pairs to sketch the graph of the function.
\(y=m x+b\) is an equation that describes an infinite set of points in the plane contained in a line.
\(f(x)=m x+b\) is a function that takes a value \(x\) as an input and produces an output, \(f(x)\).
When graphed, a linear function describes a line in the plane.
The formula for slope and equations of lines you've already learned are also applicable to linear functions.
- The slope of a line indicates the direction it slants and its steepness. Slope is defined algebraically as: \(m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\)
- Given the slope and one point on a line, we can find the equation of the line using point-slope form, \(y-y_{1}=m\left(x-x_{1}\right)\)
- When simplified, the point-slope form of the equation of a line reveals the slope-intercept form, \(y=m x+b\).

In this review section, you'll prepare to write and graph linear functions and absolute value functions, build linear models, and fit linear models to data by reviewing and practicing the skills you learned to work with linear and absolute value equations. Just remember the important translation: \(y=f(x)\). This allows us to shift our perception from looking at lines as geometric objects containing an infinite set of points, \((x, y)\) to seeing them as descriptions of functions that take an input \(x\) and produce an output \(f(x)\). Points on the line can be viewed as data points of the form \((x, f(x))\). And slope can be viewed as a rate of change from one input to the next. Keep this idea in mind as you work through the review section to graph a linear function using a table of values, then calculate and interpret the slope of such a graph.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

\section*{THE GRAPH OF A LINEAR FUNCTION}

\section*{Learning Outcome}
- Graph linear functions using tables

A helpful first step in graphing a function is to make a table of values. This is particularly useful when you do not know the general shape the function will have. You probably already know that a linear function will be a straight line, but let us make a table first to see how it can be helpful.

When making a table, it is a good idea to include negative values, positive values, and zero to ensure that you do have a linear function.

Make a table of values for \(f(x)=3 x+2\).
Make a two-column table. Label the columns \(x\) and \(f(x)\).
\begin{tabular}{|cc|}
\hline \(\boldsymbol{x}\) \\
\hline & \(\boldsymbol{f}(\boldsymbol{x})\) \\
\hline
\end{tabular}

Choose several values for \(x\) and put them as separate rows in the \(x\) column. These are YOUR CHOICE there is no "right" or "wrong" values to pick, just go for it.

Tip: It is always good to include 0 , positive values, and negative values, if possible.
\begin{tabular}{|c|c|}
\hline \(\boldsymbol{x}\) & \(\boldsymbol{f}(\boldsymbol{x})\) \\
\hline-2 & \\
\hline-1 & \\
\hline 0 & \\
\hline 1 & \\
\hline 3 & \\
\hline
\end{tabular}

Evaluate the function for each value of \(x\), and write the result in the \(f(x)\) column next to the \(x\) value you used.
When \(x=0, f(0)=3(0)+2=2\),
\(f(1)=3(1)+2=5\),
\(f(-1)=3(-1)+2=-3+2=-1\), and so on.
\begin{tabular}{|c|c|}
\hline-1 & -1 \\
\hline 0 & 2 \\
\hline 1 & 5 \\
\hline 3 & 11 \\
\hline
\end{tabular}
(Note that your table of values may be different from someone else's. You may each choose different numbers for \(x\).)

Now that you have a table of values, you can use them to help you draw both the shape and location of the function. Important: The graph of the function will show all possible values of \(x\) and the corresponding values of \(y\). This is why the graph is a line and not just the dots that make up the points in our table.

Now graph \(f(x)=3 x+2\).
Using the table of values we created above, you can think of \(f(x)\) as \(y\). Each row forms an ordered pair that you can plot on a coordinate grid.
\begin{tabular}{|c|c|}
\hline \(\boldsymbol{x}\) & \(\boldsymbol{f}(\boldsymbol{x})\) \\
\hline-2 & -4 \\
\hline-1 & -1 \\
\hline 0 & 2 \\
\hline 1 & 5 \\
\hline 3 & 11 \\
\hline
\end{tabular}

Plot the points.


Since the points lie on a line, use a straight edge to draw the line. Try to go through each point without moving the straight edge.


Let us try another one. Before you look at the answer, try to make the table yourself and draw the graph on a piece of paper.

\section*{Example}

Graph \(f(x)=-x+1\).
Answer
Start with a table of values. You can choose different values for \(x\), but once again, it is helpful to include 0 , some positive values, and some negative values.

If you think of \(f(x)\) as \(y\), each row forms an ordered pair that you can plot on a coordinate grid.
\(f(-2)=-(-2)+1=2+1=3\)
\(f(-1)=-(-1)+1=1+1=2\)
\(f(0)=-(0)+1=0+1=1\)
\(f(1)=-(1)+1=-1+1=0\)
\(f(2)=-(2)+1=-2+1=-1\)
\begin{tabular}{c|c}
\hline \(\boldsymbol{x}\) & \(\boldsymbol{f}(\boldsymbol{x})\) \\
\hline-2 & 3 \\
\hline-1 & 2 \\
\hline 0 & 1 \\
\hline 1 & 0 \\
\hline 2 & -1 \\
\hline
\end{tabular}

Plot the points.


Since the points lie on a line, use a straight edge to draw the line. Try to go through each point without moving the straight edge.


The following video shows another example of how to graph a linear function on a set of coordinate axes.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=235

\section*{LEARNING OUTCOMES}
- Define slope for a linear function
- Calculate slope given two points
- Interpret the slope of a linear function that models a real-world situation

Writing the equation that describes a linear function follows the same procedure for writing the equation of a line. You may be familiar with the slope-intercept form of a linear equation, where \(m\) stands for the slope of the line and \(b\) represents the \(y\)-intercept, the place where the line crosses the \(y\)-axis of the graph.

When discussing a linear function, ordered pairs of the form \((x, y)\) are written using function notation as \((x, f(x))\). The slope can be thought of as a rate of change in function output over a corresponding interval of input.

Equation form \(\quad y=m x+b\)
Function notation \(\quad f(x)=m x+b\)

We can calculate the slope \(m\) of a line given two points on the line, \(\left(x_{1}, y_{1}\right)\) and \(\left(x_{2}, y_{2}\right)\) by using the formula
\(m=\frac{\text { change in output (rise) }}{\text { change in input (run) }}=\frac{\Delta y}{\Delta x}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\),
where \(\Delta y\) represents the difference between the \(y\) coordinates and \(\Delta x\) represents the corresponding difference between the \(x\) coordinates.

In a linear function, the coordinates of the points on the graph of a function represent input and output values for the function. The ordered pairs are written \(\left(x_{1}, f\left(x_{1}\right)\right)\) and \(\left(x_{2}, f\left(x_{2}\right)\right)\), and the formula for slope may be written equivalently as the formula for average rate of change
\(m=\frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{x_{2}-x_{1}}\).
The units for slope or for a rate of change are expressed as a ratio of output units over input units in the form \(\frac{\text { units for the output }}{\text { units for the input }}\).

Read these units as "change in units of the output value for each unit of change in input value." For example, a rate may be given in miles per hour or a wage given in dollars per day.

Given the equation of a line or a linear function in slope-intercept form, certain behavior of the graph can be

\section*{CALCULATING SLOPE}

The slope, or rate of change, of a function \(m\) can be calculated using the following formula:
\(m=\frac{\text { change in output (rise) }}{\text { change in input (run) }}=\frac{\Delta y}{\Delta x}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\)
where \(x_{1}\) and \(x_{2}\) are input values, \(y_{1}\) and \(y_{2}\) are output values.
seen by examining the value of \(m\) in the equation \(y=m x+b\) or the function \(f(x)=m x+b\).
- When the slope of a line or the rate of change of a linear function is positive, \(m>0\), it will describe an "uphill" line in the plane, rising from left to right as the input increases.
- When the slope of a line or the rate of change of a linear function is negative, \(m<0\), it will describe a "downhill" line in the plan, falling from left to right as the input increases.
- When the slope of a line or the rate of change of a linear function is \(0, m=0\), it will describe a flat, horizontal line in the plane, neither rising nor falling as the input increases.

Watch the video below to see how to find the slope of a line passing through two points and then determine whether the line is increasing, decreasing or neither.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=236

Watch the next video to see an example of an application of slope in determining the increase in cost for producing solar panels given two data points.


The following video provides an example of how to write a function that will give the cost in dollars for a given number of credit hours taken, x .


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\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
- A linear function can be graphed by making a table of inputs and outputs then plotting them as points on a coordinate plane, then drawing a line between them.
- The slope of the graph of a linear function can be calculated given any two ordered pairs ( \(x, f(x)\) ).
- The slope of the graph of a linear function by examining whether the function values rise, fall, or remain constant as the function input increases.
- The units for the rate of change are given in a ratio of input units over output units.
- The slope of the graph of a linear function represents the rate of change in the function values over a given change in input.

\section*{Glossary}
average rate of change the slope of a line between two points on the graph of a function, calculated via a ratio of the change in function output over the corresponding change in function input
ordered pair a coordinate pair of input and output, \((x, f(x))\)
slope a measurement of the steepness of a line graphed in the plane.
slope-intercept form \(y=m x+b\)

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\section*{INTRODUCTION TO LINEAR FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Represent a linear function.
- Determine whether a linear function is increasing, decreasing, or constant.
- Calculate slope for a linear function given two points.
- Write and interpret a linear function.


A bamboo forest in China. (credit: "JFXie"/Flickr)
Imagine placing a plant in the ground one day and finding that it has doubled its height just a few days later. Although it may seem incredible, this can happen with certain species of bamboo. These members of the grass family are the fastest growing plants in the world. One species of bamboo has been observed to grow nearly 1.5 inches every hour. (Note: http://www.guinnessworldrecords.com/records-3000/fastest-growingplant/) In a twenty-four hour period, this bamboo plant grows about 36 inches, or an incredible 3 feet! A constant rate of change, such as the growth cycle of this bamboo plant, is a linear function.

Recall that a function is a relation that assigns to every element in the domain exactly one element in the range. A linear function is a specific type of function that can be used to model many real-world applications such as plant growth over time. In this chapter we will explore linear functions, their graphs and how to relate them to data.

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\section*{CHARACTERISTICS OF LINEAR FUNCTIONS}
- Represent a linear function with an equation, words, table, and a graph
- Determine whether a linear function is increasing, decreasing, or constant

Just as with the growth of a bamboo plant, there are many situations that involve constant change over time. For example, consider the first commercial Maglev train in the world, the Shanghai Maglev Train. It carries passengers comfortably for a 30-kilometer trip from the airport to the subway station in only 8 minutes. (Note: http://www.chinahighlights.com/shanghai/transportation/maglev-train.htm)


A view of the Shanghai Maglev Train. (credit: Rolf Wilhelm Pfennig)
Suppose that a Maglev train were to travel a long distance, and the train maintains a constant speed of 83 meters per second for a period of time once it is 250 meters from the station. How can we analyze the train's distance from the station as a function of time? In this section, we will investigate a type of function that is useful for this purpose and use it to investigate real-world situations such as the train's distance from the station at a given point in time.

The function describing the train's motion is a linear function, which is defined as a function with a constant rate of change, that is, a polynomial of degree 1 . There are several ways to represent a linear function including word form, function notation, tabular form and graphical form. We will describe the train's motion as a function using each method.

\section*{THE LANGUAGE OF FUNCTIONS}

Recall the translation from the language of equations to the language of functions. We say that " \(y\) is a function of \(x\) " and write ordered pairs of data in the form of (input, output). That is, the output \(y\) is a function of the input \(x\).
" \(y\) is a function of \(x " \Rightarrow y=f(x)\)
\((x, y) \Rightarrow\) (input, output) \(\Rightarrow(x, f(x))\)

To say that " \(y\) is a function of \(x^{\prime \prime}\) is to say that the output of a function (the coordinate indicated on the vertical axis) is dependent on the input to the function (the coordinate indicated on the horizontal axis).

\section*{Representing a Linear Function in Word Form}

Let's begin by describing the linear function in words. For the train problem we just considered, the following word sentence may be used to describe the function relationship.
- The train's distance from the station is a function of the time during which the train moves at a constant speed plus its original distance from the station when it began moving at a constant speed.

The speed is the rate of change. Recall that a rate of change is a measure of how quickly the dependent variable changes with respect to the independent variable. The rate of change for this example is constant, which means that it is the same for each input value. As the time (input) increases by 1 second, the corresponding distance (output) increases by 83 meters. The train began moving at this constant speed at a distance of 250 meters from the station.

\section*{Representing a Linear Function in Function Notation}

Another approach to representing linear functions is by using function notation. One example of function notation is an equation written in the form known as slope-intercept form of a line, where \(x\) is the input value, \(m\) is the rate of change, and \(b\) is the initial value of the dependent variable. Slope-intercept form is given below:
\[
\begin{array}{ll}
\text { Equation form } & y=m x+b \\
\text { Function notation } & f(x)=m x+b
\end{array}
\]

In the example of the train, we might use the notation \(D(t)\) in which the total distance \(D\) is a function of the time \(t\). The rate, \(m\), is 83 meters per second. The initial value, \(b\), of the dependent variable is the original distance from the station, 250 meters. We can write a generalized equation to represent the motion of the train.
\[
D(t)=83 t+250
\]

\section*{TIP FOR SUCCESS}

In the example above, the function notation, \(y=f(x)\) is written, descriptively for the situation, as \(y=D(t)\)
The function, \(D(t)=83 t+250\), takes an input \(t\), multiplies it by 83 , then adds 250 . The result is the output, \(D(t)\).
We can choose input and output variables stylistically as desired. It is the form of the function that remains consistent from function to function.

\section*{Representing a Linear Function in Tabular Form}

A third method of representing a linear function is through the use of a table. The relationship between the distance from the station and the time is represented in the table below. From the table, we can see that the distance changes by 83 meters for every 1 second increase in time.


Tabular representation of the function D showing selected input and output values.

\section*{Q \& A}

Can the input in the previous example be any real number?
No. The input represents time, so while nonnegative rational and irrational numbers are possible, negative real numbers are not possible for this example. The input consists of nonnegative real numbers.

\section*{TRY IT}

An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=239

\section*{Representing a Linear Function in Graphical Form}

Another way to represent linear functions is visually by using a graph. We can use the function relationship from above, \(D(t)=83 t+250\), to draw a graph as seen below. Notice the graph is a line. When we plot a linear function, the graph is always a line.

\section*{TIP FOR SUCCESS}

The y-intercept of the line in the graph below represents the initial value of the function.
The term initial value is used in situations where it would be illogical to consider negative input. Even though it would be reasonable to graph the line \(y=83 x+250\) where \(x<0\), in this situation we don't consider negative input values of time.

The rate of change, which is always constant for linear functions, determines the slant or slope of the line. The point at which the input value is zero is the \(y\)-intercept of the line. We can see from the graph that the \(y\)-intercept in the train example we just saw is \((0,250)\) and represents the distance of the train from the station when it began moving at a constant speed.


The graph of \(D(t)=83 t+250\). Graphs of linear functions are lines because the rate of change is constant.
Notice the graph of the train example is restricted since the input value, time, must always be a nonnegative real number. However, this is not always the case for every linear function. Consider the graph of the line \(f(x)=2 x+1\). Ask yourself what input values can be plugged into this function, that is, what is the domain of the function? The domain is comprised of all real numbers because any number may be doubled and then have one added to the product.

\section*{A GENERAL NOTE: LINEAR FUNCTIONS}

A linear function is a function whose graph is a line. Linear functions can be written in slope-intercept form of a line:
\(f(x)=m x+b\)
where \(b\) is the initial or starting value of the function (when input, \(x=0\) ) and \(m\) is the constant rate of change or slope of the function. The \(\boldsymbol{y}\)-intercept is at \((0, b)\).

\section*{TRY IT}

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\section*{Determine Whether a Linear Function is Increasing, Decreasing, or Constant}

The linear functions we used in the two previous examples increased over time, but not every linear function does this. A linear function may be increasing, decreasing, or constant. For an increasing function, as with the train example, the output values increase as the input values increase. The graph of an increasing

\section*{EXAMPLE: USING A LINEAR FUNCTION TO MODEL THE PRESSURE ON A DIVER}

The pressure, \(P\), in pounds per square inch (PSI) on a diver depends on depth below the water surface, \(d\), in feet. This relationship may be modeled by the equation \(P(d)=0.434 d+14.696\). Restate this function in words.


A diver exploring the depths of the ocean. (credit: Ilse Reijs and Jan-Noud Hutten)

\section*{Answer}

To restate the function in words, we need to describe each part of the equation. The pressure as a function of depth equals four hundred thirty-four thousandths times depth plus fourteen and six hundred ninety-six thousandths.

\section*{Analysis of the Solution}

The initial value, 14.696, is the pressure in PSI on the diver at a depth of 0 feet, which is the surface of the water. The rate of change, or slope, is 0.434 PSI per foot. This tells us that the pressure on the diver increases 0.434 PSI for each foot her depth increases.
function has a positive slope. A line with a positive slope slants upward from left to right as in (a). For a decreasing function, the slope is negative. The output values decrease as the input values increase. A line with a negative slope slants downward from left to right as in (b). If the function is constant, the output values are the same for all input values, so the slope is zero. A line with a slope of zero is horizontal as in (c).

(a)

(b)

(c)

\section*{A GENERAL NOTE: INCREASING AND DECREASING FUNCTIONS}

The slope determines if a linear function function is an increasing, decreasing or constant.
- \(f(x)=m x+b\) is an increasing function if \(m>0\)
- \(f(x)=m x+b\) is an decreasing function if \(m<0\)
- \(f(x)=m x+b\) is a constant function if \(m=0\)

\section*{EXAMPLE: DECIDING WHETHER A FUNCTION IS INCREASING, DECREASING, OR CONSTANT}

Some recent studies suggest that a teenager sends an average of 60 text messages per day. (Note: http://www.cbsnews.com/8301-501465_162-57400228-501465/teens-are-sending-60-texts-a-day-studysays/) For each of the following scenarios, find the linear function that describes the relationship between the input value and the output value. Then determine whether the graph of the function is increasing, decreasing, or constant.
1. The total number of texts a teenager sends is considered a function of time in days. The input is the number of days and output is the total number of texts sent.
2. A teenager has a limit of 500 texts per month in his or her data plan. The input is the number of days and output is the total number of texts remaining for the month.
3. A teenager has an unlimited number of texts in his or her data plan for a cost of \(\$ 50\) per month. The input is the number of days and output is the total cost of texting each month.

\section*{Answer}

Analyze each function.
1. The function can be represented as \(f(x)=60 x\) where \(x\) is the number of days. The slope, 60 , is positive so the function is increasing. This makes sense because the total number of texts increases with each day.
2. The function can be represented as \(f(x)=500-60 x\) where \(x\) is the number of days. In this case, the slope is negative so the function is decreasing. This makes sense because the number of texts remaining decreases each day and this function represents the number of texts remaining in the data plan after \(x\) days.
3. The cost function can be represented as \(f(x)=50\) because the number of days does not affect the total cost. The slope is 0 so the function is constant.

\section*{TRY IT}

Use an online graphing calculator to graph the function: \(f(x)=-\frac{2}{3} x-\frac{4}{3}\).
If you are using Desmos, you can add sliders to represent various aspects of your equation. Below is a short tutorial on how to add sliders to your graphs in Desmos. Other online graphing calculators may or may not have this feature.


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Try adding a slider to the function \(f(x)=-\frac{2}{3} x-\frac{4}{3}\) that will let you change the slope. Limit the range of values for the slope such that your function is increasing, then do the same for a function that is decreasing. Finally, write and graph a function whose slope is constant.

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\section*{WRITING A LINEAR FUNCTION}

\section*{LEARNING OUTCOMES}
- Calculate slope for a linear function given two points
- Write the equation of a linear function given two points and a slope

\section*{Calculate and Interpret Slope}

In the examples we have seen so far, we have had the slope provided for us. However, we often need to calculate the slope given input and output values. Given two values for the input, \(x_{1}\) and \(x_{2}\), and two corresponding values for the output, \(y_{1}\) and \(y_{2}\) which can be represented by a set of points, \(\left(x_{1}, y_{1}\right)\) and \(\left(x_{2}, y_{2}\right)\), we can calculate the slope \(m\), as follows:
\(m=\frac{\text { change in output (rise) }}{\text { change in input (run) }}=\frac{\Delta y}{\Delta x}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\)
where \(\Delta y\) is the change in output and \(\Delta x\) is the change in input. In function notation, \(y_{1}=f\left(x_{1}\right)\) and \(y_{2}=f\left(x_{2}\right)\) so we could write:
\(m=\frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{x_{2}-x_{1}}\)
The graph below indicates how the slope of the line between the points, \(\left(x_{1}, y_{1}\right)\) and \(\left(x_{2}, y_{2}\right)\), is calculated. Recall that the slope measures steepness. The greater the absolute value of the slope, the steeper the line is.


The slope of a function is calculated by the change in \(y\) divided by the change in \(x\). It does not matter which coordinate is used as the \(\left(x_{2}, y_{2}\right)\) and which is the \(\left(x_{1}, y_{1}\right)\), as long as each calculation is started with the elements from the same coordinate pair.

\section*{Q \& A}

Are the units for slope always \(\frac{\text { units for the output }}{\text { units for the input }} \boldsymbol{?}\)
Yes. Think of the units as the change of output value for each unit of change in input value. An example of slope could be miles per hour or dollars per day. Notice the units appear as a ratio of units for the output

\section*{A GENERAL NOTE: CALCULATING SLOPE}

The slope, or rate of change, \(m\), of a function can be calculated according to the following:
\(m=\frac{\text { change in output (rise) }}{\text { change in input (run) }}=\frac{\Delta y}{\Delta x}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}} \Rightarrow \frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{x_{2}-x_{1}}\)
where \(x_{1}\) and \(x_{2}\) are input values and \(f\left(x_{1}\right)\) and \(f\left(x_{2}\right)\) are output values.

\section*{HOW TO: GIVEN TWO POINTS FROM A LINEAR FUNCTION, CALCULATE AND INTERPRET THE SLOPE.}
1. Determine the units for output and input values.
2. Calculate the change of output values and change of input values.
3. Interpret the slope as the change in output values per unit of the input value.

\section*{EXAMPLE: FINDING THE SLOPE OF A LINEAR FUNCTION}

If \(f(x)\) is a linear function and \((3,-2)\) and \((8,1)\) are points on the line, find the slope. Is this function increasing or decreasing?
Answer
The coordinate pairs are \((3,-2)\) and \((8,1)\). To find the rate of change, we divide the change in output by the change in input.
\(m=\frac{\text { change in output }}{\text { change in input }}=\frac{1-(-2)}{8-3}=\frac{3}{5}\)
We could also write the slope as \(m=0.6\). The function is increasing because \(m>0\).

\section*{Analysis of the Solution}

As noted earlier, the order in which we write the points does not matter when we compute the slope of the line as long as the first output value or \(y\)-coordinate used corresponds with the first input value or \(x\) coordinate used.

\section*{TRY IT}

If \(f(x)\) is a linear function, and \((2,3)\) and \((0,4)\) are points on the line, find the slope. Is this function increasing or decreasing?
Answer
\(m=\frac{4-3}{0-2}=\frac{1}{-2}=-\frac{1}{2} ;\) decreasing because \(m<0\)
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\section*{INTERPRETING SLOPE}

When interpreting slope as an average rate of change, it is customary to express the calculation as change in output per unit of input.
Ex. A vehicle traveled from mile marker 57 to mile marker 180 between 4:15 pm and 6:21 pm. Can you use the formula for slope to find the vehicle's average rate of change in distance as a function of the time it traveled?

\section*{Answer}

Time is usually considered the input variable. We say that distance traveled depends on the amount of time spent traveling at a constant rate. But we can calculate an average rate traveled over the entire time traveled using the formula for slope. The average rate of speed will be calculated by the total distance over the total time traveled.
\(\frac{\text { change in distance }}{\text { change in time }}=\frac{180 \text { miles }-57 \text { miles }}{6.35 \text { hours }-4.15 \text { hours }}=\frac{123 \text { miles }}{2.1 \text { hours }}=\frac{123}{2.1}\) miles per hour \(\approx 58.6 \mathrm{mph}\)

\section*{EXAMPLE: FINDING THE POPULATION CHANGE FROM A LINEAR FUNCTION}

The population of a city increased from 23,400 to 27,800 between 2008 and 2012. Find the change in population per year if we assume the change was constant from 2008 to 2012.

\section*{Answer}

The rate of change relates the change in population to the change in time. The population increased by \(27,800-23,400=4400\) people over the four-year time interval. To find the rate of change, divide the change in the number of people by the number of years.
\(\frac{4,400 \text { people }}{4 \text { years }}=1,100 \frac{\text { people }}{\text { year }}\)
So the population increased by 1,100 people per year.

\section*{Analysis of the Solution}

Because we are told that the population increased, we would expect the slope to be positive. This positive slope we calculated is therefore reasonable.

\section*{TRY IT}

The population of a small town increased from 1,442 to 1,868 between 2009 and 2012. Find the change in population per year if we assume the change was constant from 2009 to 2012.

\section*{Answer}
\(m=\frac{1,868-1,442}{2,012-2,009}=\frac{426}{3}=142\) people per year
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\section*{Point-Slope Form}

\section*{RECALL EQUATIONS OF LINES}

The following methods of writing the equation of a line apply to writing the equation of a function. Just remember the translation: \(y=f(x)\). That is, graphs of functions behave exactly like lines. Wherever you see a y-coordinate, you can replace it with a function output.

Up until now, we have been using the slope-intercept form of a linear equation to describe linear functions. Now we will learn another way to write a linear function called point-slope form which is given below:
\[
y-y_{1}=m\left(x-x_{1}\right)
\]
where \(m\) is the slope of the linear function and \(\left(x_{1}, y_{1}\right)\) is any point which satisfies the linear function.
The point-slope form is derived from the slope formula.
\(m=\frac{y-y_{1}}{x-x_{1}} \quad\) assuming \(x \neq x_{1}\)
\(m\left(x-x_{1}\right)=\frac{y-y_{1}}{x-x_{1}}\left(x-x_{1}\right) \quad\) Multiply both sides by \(\left(x-x_{1}\right)\)
\(m\left(x-x_{1}\right)=y-y_{1} \quad\) Simplify
\(y-y_{1}=m\left(x-x_{1}\right) \quad\) Rearrange
Keep in mind that slope-intercept form and point-slope form can be used to describe the same linear function. We can move from one form to another using basic algebra. For example, suppose we are given the equation \(y-4=-\frac{1}{2}(x-6)\) which is in point-slope form. We can convert it to slope-intercept form as shown below.
\(y-4=-\frac{1}{2}(x-6)\)
\(y-4=-\frac{1}{2} x+3 \quad\) Distribute the \(-\frac{1}{2}\).
\(y=-\frac{1}{2} x+7 \quad\) Add 4 to each side.
Therefore, the same line can be described in slope-intercept form as \(y=-\frac{1}{2} x+7\).

\section*{A GENERAL NOTE: POINT-SLOPE FORM OF A LINEAR EQUATION}

Point-slope form of a linear equation takes the form
\(y-y_{1}=m\left(x-x_{1}\right)\)
where \(m\) is the slope and \(x_{1}\) and \(y_{1}\) are the \(x\) and \(y\) coordinates of a specific point through which the line passes.

Point-slope form is particularly useful if we know one point and the slope of a line. For example, suppose we are told that a line has a slope of 2 and passes through the point \((4,1)\). We know that \(m=2\) and that \(x_{1}=4\) and \(y_{1}=1\). We can substitute these values into point-slope form.
\[
\begin{aligned}
& y-y_{1}=m\left(x-x_{1}\right) \\
& y-1=2(x-4)
\end{aligned}
\]

If we wanted to rewrite the equation in slope-intercept form, we apply algebraic techniques.
\[
y-1=2(x-4)
\]
\(y-1=2 x-8 \quad\) Distribute the 2.
\(y=2 x-7 \quad\) Add 1 to each side.
Both equations \(y-1=2(x-4)\) and \(y=2 x-7\) describe the same line. The graph of the line can be seen below.


\section*{EXAMPLE: WRITING LINEAR EQUATIONS USING A POINT AND THE SLOPE}

Write the point-slope form of an equation of a line with a slope of 3 that passes through the point \((6,-1)\). Then rewrite the equation in slope-intercept form.

\section*{Answer}

Let's figure out what we know from the given information. The slope is 3 , so \(m=3\). We also know one point, so we know \(x_{1}=6\) and \(y_{1}=-1\). Now we can substitute these values into point-slope form.
\(y-y_{1}=m\left(x-x_{1}\right)\)
\(y-(-1)=3(x-6) \quad\) Substitute known values.
\(y+1=3(x-6) \quad\) Distribute -1 to find point-slope form.
We use algebra to find slope-intercept form of the linear equation.
\(y+1=3(x-6)\)
\(y+1=3 x-18 \quad\) Distribute 3.
\(y=3 x-19 \quad\) Simplify to slope-intercept form.

\section*{TRY IT}

Write the equation of a line in point-slope form with a slope of -2 that passes through the point \((-2,2)\). Then rewrite the equation in the slope-intercept form.

\section*{Answer}
\(y-2=-2(x+2) ; y=-2 x-2\)

\section*{Writing the Equation of a Line Using Two Points}

Point-slope form of an equation is also useful if we know any two points through which a line passes. Suppose, for example, we know that a line passes through the points \((0,1)\) and \((3,2)\). We can use the coordinates of the two points to find the slope.
\(m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\)
\(m=\frac{2-1}{3-0}\)
\(m=\frac{1}{3}\)
Now we can use the slope we found and the coordinates of one of the points to find the equation for the line. Let's use \((0,1)\) for our point.
\(y-y_{1}=m\left(x-x_{1}\right)\)
\(y-1=\frac{1}{3}(x-0)\)
As before, we can use algebra to rewrite the equation in slope-intercept form.
\[
y-1=\frac{1}{3}(x-0)
\]
\(y-1=\frac{1}{3} x \quad\) Distribute the \(\frac{1}{3}\).
\(y=\frac{1}{3} x+1 \quad\) Add 1 to each side.
Both equations describe the line graphed below.


\section*{EXAMPLE: WRITING LINEAR EQUATIONS USING TWO POINTS}

Write the point-slope form of an equation of a line that passes through the points \((5,1)\) and \((8,7)\). Then rewrite the equation in slope-intercept form.
Answer
Let's begin by finding the slope.
\[
\begin{aligned}
& m=\frac{y_{2}-y_{1}}{x_{2}-x_{1}} \\
& m=\frac{7-1}{8-5} \\
& m=\frac{6}{3} \\
& m=2
\end{aligned}
\]

So \(m=2\). Next, we substitute the slope and the coordinates for one of the points into point-slope form. We can choose either point, but we will use \((5,1)\).
\[
\begin{aligned}
& y-y_{1}=m\left(x-x_{1}\right) \\
& y-1=2(x-5)
\end{aligned}
\]

To rewrite the equation in slope-intercept form, we use algebra.
\[
\begin{aligned}
& y-1=2(x-5) \\
& y-1=2 x-10 \\
& y=2 x-9
\end{aligned}
\]

The slope-intercept form of the equation of the line is \(y=2 x-9\).

\section*{TRY IT}

Write the point-slope form of an equation of a line that passes through the points \((-1,3)\) and \((0,0)\). Then rewrite the equation in slope-intercept form.
Answer
\(y-0=-3(x-0) ; y=-3 x\)
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\section*{SUMMARY: CHARACTERISTICS OF LINEAR FUNCTIONS}

\section*{Key Equations}
\begin{tabular}{|l|l|}
\hline slope-intercept form of a line & \(y=m x+b\) \\
\hline slope & \(m=\frac{\text { change in output (rise) }}{\text { change in input (run) }}=\frac{\Delta y}{\Delta x}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\) \\
\hline point-slope form of a line & \(y-y_{1}=m\left(x-x_{1}\right)\) \\
\hline
\end{tabular}

\section*{Key Concepts}
- The ordered pairs given by a linear function represent points on a line.
- Linear functions can be represented in words, function notation, tabular form and graphical form.
- The rate of change of a linear function is also known as the slope.
- An equation in slope-intercept form of a line includes the slope and the initial value of the function.
- The initial value, or \(y\)-intercept, is the output value when the input of a linear function is zero. It is the \(y\) value of the point where the line crosses the \(y\)-axis.
- An increasing linear function results in a graph that slants upward from left to right and has a positive slope.
- A decreasing linear function results in a graph that slants downward from left to right and has a negative slope.
- A constant linear function results in a graph that is a horizontal line.
- Analyzing the slope within the context of a problem indicates whether a linear function is increasing, decreasing, or constant.
- The slope of a linear function can be calculated by dividing the difference between \(y\)-values by the difference in corresponding \(x\)-values of any two points on the line.
- The slope and initial value can be determined given a graph or any two points on the line.
- One form of a linear function is slope-intercept form.
- Point-slope form is useful for finding the equation of a linear function when given the slope of a line and one point.
- Point-slope form is also convenient for finding the equation of a linear function when given two points through which a line passes.
- The equation for a linear function can be written in slope-intercept form if the slope \(m\) and initial value \(b\) are known.
- A linear function can be used to solve real-world problems.
- A linear function can be written from tabular form.

\section*{Glossary}
decreasing linear function a function with a negative slope: If \(m<0\), then \(f(x)=m x+b\) is decreasing.
increasing linear function a function with a positive slope: If \(m>0\), then \(f(x)=m x+b\) is increasing.
linear function a function with a constant rate of change that is a polynomial of degree 1 whosegraph is a straight line
point-slope form the equation of a linear function of the form \(y-y_{1}=m\left(x-x_{1}\right)\)
slope the ratio of the change in output values to the change in input values; a measure of the steepness of a line
slope-intercept form the equation of a linear function of the form \(f(x)=m x+b\)
\(\boldsymbol{y}\)-intercept the value of a function when the input value is zero; also known as initial value

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\section*{INTRODUCTION TO GRAPHING AND WRITING EQUATIONS OF LINEAR FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Graph linear functions by plotting points, using the slope and y-intercept, and using transformations.
- Write the equation of a linear function given its graph.
- Match linear functions with their graphs.
- Find the \(x\)-intercept of a function given its equation.
- Find the equations of vertical and horizontal lines.
- Determine whether lines are parallel or perpendicular given their equations.
- Find equations of lines that are parallel or perpendicular to a given line.
- Graph an absolute value function.
- Find the intercepts of an absolute value function.

We can now describe a variety of characteristics that explain the behavior of linear functions. We will use this information to analyze a graphed line and write an equation based on its observable properties. From evaluating the graph, what can you determine about this linear function?

- initial value (y-intercept)?
- one or two points?
- slope?
- increasing or decreasing?
- vertical or horizontal?

In this section, you will practice writing linear function equations using the information you've gathered. We will also practice graphing linear functions using different methods and predict how the graphs of linear functions will change when parts of the equation are altered.

\section*{GRAPHING LINEAR FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Graph a linear function by plotting points
- Graph a linear function using the slope and y-intercept
- Graph a linear function using transformations

\section*{RECALL GRAPHING FUNCTIONS}

Recall that you can graph a linear function by evaluating it at chosen inputs to obtain a table of points that exist on the line, then plotting those points and drawing a line between them. Since it only takes two points to describe a line, this method works very well for graphing linear functions that yield integer output for integer input.
This section provides this method along with two other good ways to draw a quick, accurate sketch of a linear function.

We previously saw that that the graph of a linear function is a straight line. We were also able to see the points of the function as well as the initial value from a graph.

There are three basic methods of graphing linear functions. The first is by plotting points and then drawing a line through the points. The second is by using the \(y\)-intercept and slope. The third is applying transformations to the identity function \(f(x)=x\).

\section*{Graphing a Function by Plotting Points}

To find points of a function, we can choose input values, evaluate the function at these input values, and calculate output values. The input values and corresponding output values form coordinate pairs. We then plot the coordinate pairs on a grid. In general we should evaluate the function at a minimum of two inputs in order to find at least two points on the graph of the function. For example, given the function \(f(x)=2 x\), we might use the input values 1 and 2 . Evaluating the function for an input value of 1 yields an output value of 2 which is represented by the point \((1,2)\). Evaluating the function for an input value of 2 yields an output value of 4 which is represented by the point \((2,4)\). Choosing three points is often advisable because if all three points do not fall on the same line, we know we made an error.

\section*{HOW TO: GIVEN A LINEAR FUNCTION, GRAPH BY PLOTTING POINTS.}
1. Choose a minimum of two input values.
2. Evaluate the function at each input value.
3. Use the resulting output values to identify coordinate pairs.
4. Plot the coordinate pairs on a grid.
5. Draw a line through the points.

\section*{EXAMPLE: GRAPHING BY PLOTTING POINTS}

Graph \(f(x)=-\frac{2}{3} x+5\) by plotting points.
Answer
Begin by choosing input values. This function includes a fraction with a denominator of 3 so let's choose multiples of 3 as input values. We will choose 0,3 , and 6.
Evaluate the function at each input value and use the output value to identify coordinate pairs.
\[
\begin{array}{ll}
x=0 & f(0)=-\frac{2}{3}(0)+5=5 \Rightarrow(0,5) \\
x=3 & f(3)=-\frac{2}{3}(3)+5=3 \Rightarrow(3,3) \\
x=6 & f(6)=-\frac{2}{3}(6)+5=1 \Rightarrow(6,1)
\end{array}
\]

Plot the coordinate pairs and draw a line through the points. The graph below is of the function \(f(x)=-\frac{2}{3} x+5\).


\section*{Analysis of the Solution}

The graph of the function is a line as expected for a linear function. In addition, the graph has a downward slant which indicates a negative slope. This is also expected from the negative constant rate of change in the equation for the function.

\section*{TRY IT}

Graph \(f(x)=-\frac{3}{4} x+6\) by plotting points.
Answer


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\section*{Graphing a Linear Function Using y-intercept and Slope}

Another way to graph linear functions is by using specific characteristics of the function rather than plotting points. The first characteristic is its \(y\)-intercept which is the point at which the input value is zero. To find the \(y\)-intercept, we can set \(x=0\) in the equation.

\section*{TIP FOR SUCCESS}

Keep in mind that if a function has a y-intercept, we can always find it by setting \(x=0\) and then solving for \(y\).

The other characteristic of the linear function is its slope \(m\), which is a measure of its steepness. Recall that the slope is the rate of change of the function. The slope of a linear function is equal to the ratio of the change in outputs to the change in inputs. Another way to think about the slope is by dividing the vertical difference, or rise, between any two points by the horizontal difference, or run. The slope of a linear function will be the same between any two points. We encountered both the \(y\)-intercept and the slope in Linear Functions.

Let's consider the following function.
\(f(x)=\frac{1}{2} x+1\)
The slope is \(\frac{1}{2}\). Because the slope is positive, we know the graph will slant upward from left to right. The \(y\) intercept is the point on the graph when \(x=0\). The graph crosses the \(y\)-axis at \((0,1)\). Now we know the slope and the \(y\)-intercept. We can begin graphing by plotting the point \((0,1)\) We know that the slope is rise over run, \(m=\frac{\text { rise }}{\text { rum }}\). From our example, we have \(m=\frac{1}{2}\), which means that the rise is 1 and the run is 2 . Starting from our \(y\)-intercept \((0,1)\), we can rise 1 and then run 2 or run 2 and then rise 1 . We repeat until we have multiple points, and then we draw a line through the points as shown below.


\section*{A GENERAL NOTE: GRAPHICAL INTERPRETATION OF A LINEAR FUNCTION}

In the equation \(f(x)=m x+b\)
- \(b\) is the \(y\)-intercept of the graph and indicates the point \((0, b)\) at which the graph crosses the \(y\)-axis.
- \(m\) is the slope of the line and indicates the vertical displacement (rise) and horizontal displacement (run) between each successive pair of points. Recall the formula for the slope:
\(m=\frac{\text { change in output (rise) }}{\text { change in input (run) }}=\frac{\Delta y}{\Delta x}=\frac{y_{2}-y_{1}}{x_{2}-x_{1}}\)

\section*{Q \& A}

Do all linear functions have \(\boldsymbol{y}\)-intercepts?
Yes. All linear functions cross the \(y\)-axis and therefore have \(y\)-intercepts. (Note: A vertical line parallel to the \(y\)-axis does not have a \(y\)-intercept. Keep in mind that a vertical line is the only line that is not a function.)

\section*{HOW TO: GIVEN THE EQUATION FOR A LINEAR FUNCTION, GRAPH THE FUNCTION USING THE \(Y\)-INTERCEPT AND SLOPE.}
1. Evaluate the function at an input value of zero to find the \(y\)-intercept.
2. Identify the slope.
3. Plot the point represented by the \(y\)-intercept.
4. Use rise to determine at least two more points on the line.
5. Draw a line which passes through the points.

\section*{Graphing a Linear Function Using Transformations}

Another option for graphing is to use transformations on the identity function \(f(x)=x\). A function may be transformed by a shift up, down, left, or right. A function may also be transformed using a reflection, stretch, or compression.

\section*{EXAMPLE: GRAPHING BY USING THE \(Y\)-INTERCEPT AND SLOPE}

Graph \(f(x)=-\frac{2}{3} x+5\) using the \(y\)-intercept and slope.
Answer
Evaluate the function at \(x=0\) to find the \(y\)-intercept. The output value when \(x=0\) is 5 , so the graph will cross the \(y\)-axis at \((0,5)\).
According to the equation for the function, the slope of the line is \(-\frac{2}{3}\). This tells us that for each vertical decrease in the "rise" of -2 units, the "run" increases by 3 units in the horizontal direction. We can now graph the function by first plotting the \(y\)-intercept. From the initial value \((0,5)\) we move down 2 units and to the right 3 units. We can extend the line to the left and right by repeating, and then draw a line through the points.


\section*{Analysis of the Solution}

The graph slants downward from left to right which means it has a negative slope as expected.

\section*{TRY IT}

Find a point on the graph we drew in Example: Graphing by Using the \(y\)-intercept and Slope that has a negative \(x\)-value.

\section*{Answer}

Possible answers include \((-3,7),(-6,9)\), or \((-9,11)\).
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In the equation \(f(x)=m x\), the \(m\) is acting as the vertical stretch or compression of the identity function. When \(m\) is negative, there is also a vertical reflection of the graph. Notice that multiplying the equation \(f(x)=x\) by \(m\) stretches the graph of \(f\) by a factor of \(m\) units if \(m>1\) and compresses the graph of \(f\) by a factor of \(m\) units if \(0<m<1\). This means the larger the absolute value of \(m\), the steeper the slope.


Vertical stretches and compressions and reflections on the function \(f(x)=x\).

\section*{Vertical Shift}

In \(f(x)=m x+b\), the \(b\) acts as the vertical shift, moving the graph up and down without affecting the slope of the line. Notice that adding a value of \(b\) to the equation of \(f(x)=x\) shifts the graph of \(f\) a total of \(b\) units up if \(b\) is positive and \(l b l\) units down if \(b\) is negative.


This graph illustrates vertical shifts of the function \(f(x)=x\).
Using vertical stretches or compressions along with vertical shifts is another way to look at identifying different types of linear functions. Although this may not be the easiest way to graph this type of function, it is still important to practice each method.
\[
\begin{aligned}
& \text { HOW TO: GIVEN THE EQUATION OF A LINEAR FUNCTION, USE } \\
& \text { TRANSFORMATIONS TO GRAPH THE LINEAR FUNCTION IN THE FORM } \\
& \qquad f(x)=m x+b .
\end{aligned}
\]
1. Graph \(f(x)=x\).
2. Vertically stretch or compress the graph by a factor \(m\).
3. Shift the graph up or down \(b\) units.

\section*{EXAMPLE: GRAPHING BY USING TRANSFORMATIONS}

Graph \(f(x)=\frac{1}{2} x-3\) using transformations.
Answer
The equation for the function shows that \(m=\frac{1}{2}\) so the identity function is vertically compressed by \(\frac{1}{2}\).
The equation for the function also shows that \(b=-3\), so the identity function is vertically shifted down 3 units.
First, graph the identity function, and show the vertical compression.


The function \(y=x\) compressed by a factor of \(\frac{1}{2}\).
Then, show the vertical shift.


The function \(y=\frac{1}{2} x\) shifted down 3 units.

\section*{TRY IT}

Graph \(f(x)=4+2 x\), using transformations.
Answer


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\section*{Q \& A}

In Example: Graphing by Using Transformations, could we have sketched the graph by reversing the order of the transformations?
No. The order of the transformations follows the order of operations. When the function is evaluated at a given input, the corresponding output is calculated by following the order of operations. This is why we performed the compression first. For example, following order of operations, let the input be 2.
\(f(2)=\frac{1}{2}(2)-3\)
\(=1-3\)
\(=-2\)

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\section*{WRITING EQUATIONS OF LINEAR FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Write the equation of a linear function given its graph
- Match linear functions with their graphs
- Find the x-intercept of a function given its equation
- Find the equations of vertical and horizontal lines

We previously wrote the equation for a linear function from a graph. Now we can extend what we know about graphing linear functions to analyze graphs a little more closely. Begin by taking a look at the graph below. We can see right away that the graph crosses the \(y\)-axis at the point \((0,4)\), so this is the \(y\)-intercept.


Then we can calculate the slope by finding the rise and run. We can choose any two points, but let's look at the point ( \(-2,0\) ). To get from this point to the \(y\)-intercept, we must move up 4 units (rise) and to the right 2 units (run). So the slope must be:
\(m=\frac{\text { rise }}{\text { run }}=\frac{4}{2}=2\)
Substituting the slope and \(y\)-intercept into slope-intercept form of a line gives:
\(y=2 x+4\)

\section*{HOW TO: GIVEN THE GRAPH OF A LINEAR FUNCTION, FIND THE EQUATION TO DESCRIBE THE FUNCTION.}
1. Identify the \(y\)-intercept from the graph.
2. Choose two points to determine the slope.
3. Substitute the \(y\)-intercept and slope into slope-intercept form of a line.

Finding the \(x\)-intercept of a Line

\section*{EXAMPLE: MATCHING LINEAR FUNCTIONS TO THEIR GRAPHS}

Match each equation of a linear function with one of the lines in the graph below.
1. \(f(x)=2 x+3\)
2. \(g(x)=2 x-3\)
3. \(h(x)=-2 x+3\)
4. \(j(x)=\frac{1}{2} x+3\)


\section*{Answer}

Analyze the information for each function.
1. This function has a slope of 2 and a \(y\)-intercept of 3 . It must pass through the point \((0,3)\) and slant upward from left to right. We can use two points to find the slope, or we can compare it with the other functions listed. Function \(g\) has the same slope, but a different \(y\)-intercept. Lines I and III have the same slant because they have the same slope. Line III does not pass through ( 0,3 ) so \(f\) must be represented by line I.
2. This function also has a slope of 2 , but a \(y\)-intercept of -3 . It must pass through the point \((0,-3)\) and slant upward from left to right. It must be represented by line III.
3. This function has a slope of -2 and a \(y\)-intercept of 3 . This is the only function listed with a negative slope, so it must be represented by line IV because it slants downward from left to right.
4. This function has a slope of \(\frac{1}{2}\) and a \(y\)-intercept of 3 . It must pass through the point \((0,3)\) and slant upward from left to right. Lines I and II pass through ( 0,3 ), but the slope of \(j\) is less than the slope of \(f\) so the line for \(j\) must be flatter. This function is represented by Line II.

Now we can re-label the lines.


\section*{TRY IT}

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So far we have been finding the \(y\)-intercepts of functions: the point at which the graph of a function crosses the \(y\)-axis. A function may also have an \(x\)-intercept, which is the \(x\)-coordinate of the point where the graph of a function crosses the \(x\)-axis. In other words, it is the input value when the output value is zero.

To find the \(x\)-intercept, set the function \(f(x)\) equal to zero and solve for the value of \(x\). For example, consider the function shown:
\[
f(x)=3 x-6
\]

Set the function equal to 0 and solve for \(x\).
\(0=3 x-6\)
\(6=3 x\)
\(2=x\)
\(x=2\)
The graph of the function crosses the \(x\)-axis at the point \((2,0)\).

\section*{Q \& A}

\section*{Do all linear functions have \(x\)-intercepts?}

No. However, linear functions of the form \(\mathrm{y}=\mathrm{c}\), where c is a nonzero real number are the only examples of linear functions with no x -intercept. For example, \(\mathrm{y}=5\) is a horizontal line 5 units above the x -axis. This function has no x-intercepts.


\section*{A GENERAL NOTE: \(X\)-INTERCEPT}

The \(x\)-intercept of a function is the value of \(x\) where \(f(x)=0\). It can be found by solving the equation \(0=\) \(m x+b\).

\section*{Describing Horizontal and Vertical Lines}

\section*{EXAMPLE: FINDING AN \(X\)-INTERCEPT}

Find the \(x\)-intercept of \(f(x)=\frac{1}{2} x-3\).

\section*{Answer}

Set the function equal to zero to solve for \(x\).
\[
\begin{aligned}
& 0=\frac{1}{2} x-3 \\
& 3=\frac{1}{2} x \\
& 6=x \\
& x=6
\end{aligned}
\]

The graph crosses the \(x\)-axis at the point \((6,0)\).

Analysis of the Solution

A graph of the function is shown below. We can see that the \(x\)-intercept is \((6,0)\) as expected.


The graph of the linear function \(f(x)=\frac{1}{2} x-3\).

\section*{TRY IT}

Find the \(x\)-intercept of \(f(x)=\frac{1}{4} x-4\).
Answer
\((16,0)\)
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There are two special cases of lines on a graph-horizontal and vertical lines. A horizontal line indicates a constant output or \(y\)-value. In the graph below, we see that the output has a value of 2 for every input value. The change in outputs between any two points is 0 . In the slope formula, the numerator is 0 , so the slope is 0 . If we use \(m=0\) in the equation \(f(x)=m x+b\), the equation simplifies to \(f(x)=b\). In other words, the value of the function is a constant. This graph represents the function \(f(x)=2\).


A horizontal line representing the function \(f(x)=2\).


A vertical line indicates a constant input or \(x\)-value. We can see that the input value for every point on the line is 2 , but the output value varies. Because this input value is mapped to more than one output value, a vertical line does not represent a function. Notice that between any two points, the change in the input values is zero. In the slope formula, the denominator will be zero, so the slope of a vertical line is undefined.

Notice that a vertical line has an \(x\)-intercept but no \(y\)-intercept unless it's the line \(x=0\). This graph represents the line \(x=2\).

\begin{tabular}{|l|r|r|l|l|l|}
\hline\(x\) & 2 & 2 & 2 & 2 & 2 \\
\hline\(y\) & -4 & -2 & 0 & 2 & 4 \\
\hline
\end{tabular}

The vertical line \(x=2\) which does not represent a function.

\section*{A GENERAL NOTE: HORIZONTAL AND VERTICAL LINES}

Lines can be horizontal or vertical.
A horizontal line is a line defined by an equation of the form \(f(x)=b\) where \(b\) is a constant. A vertical line is a line defined by an equation of the form \(x=a\) where \(a\) is a constant.

\section*{EXAMPLE: WRITING THE EQUATION OF A HORIZONTAL LINE}

Write the equation of the line graphed below.


Answer
For any \(x\)-value, the \(y\)-value is -4 , so the equation is \(y=-4\).

\section*{TRY IT}

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\section*{EXAMPLE: WRITING THE EQUATION OF A VERTICAL LINE}

Write the equation of the line graphed below.


Answer
The constant \(x\)-value is 7 , so the equation is \(x=7\).

\section*{TRY IT}

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\section*{TRY IT}

From the information given below, write the related equations. Then walk through the slides to experiment with slope changes.
- Write the equation of the function passing through the points \((2,6)\) and \((4,4)\) in slope-intercept form.
- Write the equation of a function whose slope is 2 and passes through the point \((-1,0)\)
- Write the equation of a function whose slope is undefined.

INTERACTIVE, move the points to see what happens to the slope

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\section*{PARALLEL AND PERPENDICULAR LINES}

\section*{LEARNING OUTCOMES}
- Determine whether lines are parallel or perpendicular given their equations
- Find equations of lines that are parallel or perpendicular to a given line

The two lines in the graph below are parallel lines: they will never intersect. Notice that they have exactly the same steepness which means their slopes are identical. The only difference between the two lines is the \(y\)-intercept. If we shifted one line vertically toward the \(y\)-intercept of the other, they would become the same line.


Parallel lines.
\[
\left.\left.\begin{array}{l}
f(x)=-2 x+6 \\
f(x)=-2 x-4
\end{array}\right\} \text { parallel } \begin{array}{l}
f(x)=3 x+2 \\
f(x)=2 x+2
\end{array}\right\}
\]

We can determine from their equations whether two lines are parallel by comparing their slopes. If the slopes are the same and the \(y\)-intercepts are different, the lines are parallel. If the slopes are different, the lines are not parallel.

Unlike parallel lines, perpendicular lines do intersect. Their intersection forms a right or 90-degree angle. The two lines below are perpendicular.


Perpendicular lines.
Perpendicular lines do not have the same slope. The slopes of perpendicular lines are different from one another in a specific way. The slope of one line is the negative reciprocal of the slope of the other line. The product of a number and its reciprocal is 1 . If \(m_{1}\) and \(m_{2}\) are negative reciprocals of one another, they can be multiplied together to yield -1 .
\(m_{1} * m_{2}=-1\)
To find the reciprocal of a number, divide 1 by the number. So the reciprocal of 8 is \(\frac{1}{8}\), and the reciprocal of \(\frac{1}{8}\) is 8 . To find the negative reciprocal, first find the reciprocal and then change the sign.

As with parallel lines, we can determine whether two lines are perpendicular by comparing their slopes. The slope of each line below is the negative reciprocal of the other so the lines are perpendicular.
\(f(x)=\frac{1}{4} x+2 \quad\) negative reciprocal of \(\frac{1}{4}\) is -4
\(f(x)=-4 x+3\) negative reciprocal of -4 is \(\frac{1}{4}\)
The product of the slopes is -1 .
\(-4\left(\frac{1}{4}\right)=-1\)

\section*{A GENERAL NOTE: PARALLEL AND PERPENDICULAR LINES}

Two lines are parallel lines if they do not intersect. The slopes of the lines are the same.
\(f(x)=m_{1} x+b_{1}\) and \(g(x)=m_{2} x+b_{2}\) are parallel if \(m_{1}=m_{2}\).
If and only if \(b_{1}=b_{2}\) and \(m_{1}=m_{2}\), we say the lines coincide. Coincident lines are the same line.
Two lines are perpendicular lines if they intersect at right angles.
\(f(x)=m_{1} x+b_{1}\) and \(g(x)=m_{2} x+b_{2}\) are perpendicular if \(m_{1} * m_{2}=-1\), and \(m_{2}=-\frac{1}{m_{1}}\).

\section*{TIP FOR SUCCESS}

Be careful when discussing horizontal and vertical lines as being perpendicular by having slopes whose product is -1 . We can use the qualifying statement \(m_{2}=-\frac{1}{m_{1}}\) above, though, to cover the case of horizontal and vertical lines.
Ex. Certainly the lines \(x=c\) and \(y=c\) are perpendicular.
The slope of the horizontal line is \(m_{1}=0\). The slope of the vertical line is \(m_{2}=\frac{1}{m_{1}}=\frac{1}{0}\), which is undefined. But what about the requirement for negative, reciprocal slopes? In this case, the reciprocal characteristic holds. Since negativity is defined in proximity to zero, we may qualify this special case as perpendicular.
But, when discussing function graphs that are perpendicular to each other, it makes no sense to discuss the possibility of horizontal and vertical functions. A vertical line fails to be a function by the vertical line test.

\section*{Writing Equations of Parallel Lines}

If we know the equation of a line, we can use what we know about slope to write the equation of a line that is either parallel or perpendicular to the given line.

Suppose we are given the following function:
\(f(x)=3 x+1\)
We know that the slope of the line is 3 . We also know that the \(y\)-intercept is \((0,1)\). Any other line with a slope of 3 will be parallel to \(f(x)\). The lines formed by all of the following functions will be parallel to \(f(x)\).
\[
\begin{aligned}
& g(x)=3 x+6 \\
& h(x)=3 x+1 \\
& p(x)=3 x+\frac{2}{3}
\end{aligned}
\]

\section*{EXAMPLE: IDENTIFYING PARALLEL AND PERPENDICULAR LINES}

Given the functions below, identify the functions whose graphs are a pair of parallel lines and a pair of perpendicular lines.
\[
\begin{array}{ll}
f(x)=2 x+3 & h(x)=-2 x+2 \\
g(x)=\frac{1}{2} x-4 & j(x)=2 x-6
\end{array}
\]

\section*{Answer}

Parallel lines have the same slope. Because the functions \(f(x)=2 x+3\) and \(j(x)=2 x-6\) each have a slope of 2 , they represent parallel lines. Perpendicular lines have negative reciprocal slopes. Because -2 and \(\frac{1}{2}\) are negative reciprocals, the equations, \(g(x)=\frac{1}{2} x-4\) and \(h(x)=-2 x+2\) represent perpendicular lines.

Analysis of the Solution
A graph of the lines is shown below.


The graph shows that the lines \(f(x)=2 x+3\) and \(j(x)=2 x-6\) are parallel, and the lines \(g(x)=\frac{1}{2} x-4\) and \(h(x)=-2 x+2\) are perpendicular.

Suppose then we want to write the equation of a line that is parallel to \(f\) and passes through the point \((1,7)\). We already know that the slope is 3 . We just need to determine which value for \(b\) will give the correct line. We can begin by using point-slope form of an equation for a line. We can then rewrite it in slope-intercept form.
\[
\begin{aligned}
& y-y_{1}=m\left(x-x_{1}\right) \\
& y-7=3(x-1) \\
& y-7=3 x-3 \\
& y=3 x+4
\end{aligned}
\]

So \(g(x)=3 x+4\) is parallel to \(f(x)=3 x+1\) and passes through the point ( 1,7 ).

\section*{HOW TO: GIVEN THE EQUATION OF A LINEAR FUNCTION, WRITE THE EQUATION OF A LINE WHICH PASSES THROUGH A GIVEN POINT AND IS}
1. Find the slope of the function.
2. Substitute the slope and given point into point-slope or slope-intercept form.
3. Simplify.

\section*{EXAMPLE: FINDING A LINE PARALLEL TO A GIVEN LINE}

Find a line parallel to the graph of \(f(x)=3 x+6\) that passes through the point \((3,0)\).

\section*{Answer}

The slope of the given line is 3 . If we choose slope-intercept form, we can substitute \(m=3, x=3\), and \(f(x)=0\) into slope-intercept form to find the \(y\)-intercept.
\[
\begin{aligned}
& g(x)=3 x+b \\
& 0=3(3)+b \\
& b=-9
\end{aligned}
\]

The line parallel to \(f(x)\) that passes through \((3,0)\) is \(g(x)=3 x-9\).

\section*{Analysis of the Solution}

We can confirm that the two lines are parallel by graphing them. The figure below shows that the two lines will never intersect.


\section*{Writing Equations of Perpendicular Lines}

We can use a very similar process to write the equation of a line perpendicular to a given line. Instead of using the same slope, however, we use the negative reciprocal of the given slope. Suppose we are given the following function:
\(f(x)=2 x+4\)
The slope of the line is 2 , and its negative reciprocal is \(-\frac{1}{2}\). Any function with a slope of \(-\frac{1}{2}\) will be perpendicular to \(f(x)\). The lines formed by all of the following functions will be perpendicular to \(f(x)\).
\[
\begin{aligned}
& g(x)=-\frac{1}{2} x+4 \\
& h(x)=-\frac{1}{2} x+2 \\
& p(x)=-\frac{1}{2} x-\frac{1}{2}
\end{aligned}
\]

As before, we can narrow down our choices for a particular perpendicular line if we know that it passes through a given point. Suppose that we want to write the equation of a line that is perpendicular to \(f(x)\) and passes through the point \((4,0)\). We already know that the slope is \(-\frac{1}{2}\). Now we can use the point to find the \(y\)-intercept by substituting the given values into the slope-intercept form of a line and solving for \(b\).
\[
\begin{aligned}
& g(x)=m x+b \\
& 0=-\frac{1}{2}(4)+b \\
& 0=-2+b \\
& 2=b \\
& b=2
\end{aligned}
\]

The equation for the function with a slope of \(-\frac{1}{2}\) and a \(y\)-intercept of 2 is
\(g(x)=-\frac{1}{2} x+2\).
So \(g(x)=-\frac{1}{2} x+2\) is perpendicular to \(f(x)=2 x+4\) and passes through the point \((4,0)\). Be aware that perpendicular lines may not look obviously perpendicular on a graphing calculator unless we use the square zoom feature.

\section*{Q \& A}

A horizontal line has a slope of zero and a vertical line has an undefined slope. These two lines are perpendicular, but the product of their slopes is not \(\mathbf{- 1}\). Doesn't this fact contradict the definition of perpendicular lines?
No. For two perpendicular linear functions, the product of their slopes is \(\mathbf{- 1}\). However, a vertical line is not a function so the definition is not contradicted.

\section*{HOW TO: GIVEN THE EQUATION OF A LINEAR FUNCTION, WRITE THE EQUATION OF A LINE WHICH PASSES THROUGH A GIVEN POINT AND IS PERPENDICULAR TO THE GIVEN LINE.}
1. Find the slope of the given function.
2. Determine the negative reciprocal of the slope.
3. Substitute the new slope and the values for \(x\) and \(y\) from given point into \(g(x)=m x+b\).
4. Solve for \(b\).
5. Write the equation of the line.

\section*{EXAMPLE: FINDING THE EQUATION OF A PERPENDICULAR LINE}

Find the equation of a line perpendicular to \(f(x)=3 x+3\) that passes through the point \((3,0)\).
Answer
The original line has slope \(m=3\), so the slope of the perpendicular line will be its negative reciprocal, or \(-\frac{1}{3}\). Using this slope and the given point, we can find the equation for the line.
\[
\begin{aligned}
& g(x)=-\frac{1}{3} x+b \\
& 0=-\frac{1}{3}(3)+b \\
& 1=b \\
& b=1
\end{aligned}
\]

The line perpendicular to \(f(x)\) that passes through \((3,0)\) is \(g(x)=-\frac{1}{3} x+1\).

\section*{Analysis of the Solution}

A graph of the two lines is shown below.


\section*{TRY IT}

Using an online graphing calculator, create a line that passes through the points \((0,5)\) and \((4,1)\), call this \(f(x)\). Now create a line that is parallel to that first line, let's call this \(g(x)\).
1. For what y-intercept will the graph of \(g(x)\) pass through the point \((-2,5)\) ?
2. Add a new function, \(j(x)\) that uses the slope \(m\) that will create a line that is perpendicular to the function \(h(x)=m x-2\).
3. For what \(y\)-intercept will the new function pass through the point \((4,1)\), and still be perpendicular to \(f(x)\)

\section*{Answer}
1. When the \(y\)-intercept is \((0,3)\) the function will be \(g(x)=m x+3\) and the function will pass through the point \((-2,5)\).
2. \(j(x)=\frac{-1}{m} x-2\) for example. Any value for the \(y\)-intercept will work.
3. The \(y\)-intercept \((0,-3)\) will give a line perpendicular to \(h(x)\) that passes through the point \((4,1)\).

\section*{HOW TO: GIVEN TWO POINTS ON A LINE AND A THIRD POINT, WRITE THE EQUATION OF THE PERPENDICULAR LINE THAT PASSES THROUGH THE POINT.}
1. Determine the slope of the line passing through the points.
2. Find the negative reciprocal of the slope.
3. Use slope-intercept form or point-slope form to write the equation by substituting the known values.
4. Simplify.

\section*{EXAMPLE: FINDING THE EQUATION OF A LINE GOING THROUGH A POINT AND PERPENDICULAR TO A GIVEN LINE}

A line passes through the points \((-2,6)\) and \((4,5)\). Find the equation of a line that is perpendicular and passes through the point \((4,5)\).
Answer
From the two points of the given line, we can calculate the slope of that line.
\[
\begin{aligned}
& m_{1}=\frac{5-6}{4-(-2)} \\
& m_{1}=\frac{-1}{6} \\
& m_{1}=-\frac{1}{6}
\end{aligned}
\]

Find the negative reciprocal of the slope.
\[
\begin{aligned}
& m_{2}=\frac{-1}{-\frac{1}{6}} \\
& m_{2}=-1\left(-\frac{6}{1}\right) \\
& m_{2}=6
\end{aligned}
\]

We can then solve for the \(y\)-intercept of the line passing through the point \((4,5)\).
\(g(x)=6 x+b\)
\(5=6(4)+b\)
\(5=24+b\)
\(-19=b\)
\(b=-19\)
The equation of the line that passes through the point \((4,5)\) and is perpendicular to the line passing through the two given points is \(y=6 x-19\).

\section*{TRY IT}

A line passes through the points, \((-2,-15)\) and \((2,-3)\). Find the equation of a perpendicular line that passes through the point, (6, 4).
Answer
\(y=-\frac{1}{3} x+6\)

\section*{ABSOLUTE VALUE FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Graph an absolute value function.
- Find the intercepts of an absolute value function.


Distances in deep space can be measured in all directions. As such, it is useful to consider distance in terms of absolute values. (credit: "s58y"/Flickr)

Until the 1920s, the so-called spiral nebulae were believed to be clouds of dust and gas in our own galaxy some tens of thousands of light years away. Then, astronomer Edwin Hubble proved that these objects are galaxies in their own right at distances of millions of light years. Today, astronomers can detect galaxies that are billions of light years away. Distances in the universe can be measured in all directions. As such, it is useful to consider distance as an absolute value function. In this section, we will investigate absolute value functions.

\section*{Understanding Absolute Value}

Recall that in its basic form \(f(x)=|x|\), the absolute value function, is one of our toolkit functions. The absolute value function is commonly thought of as providing the distance a number is from zero on a number line. Algebraically, for whatever the input value is, the output is the value without regard to sign.

\section*{A GENERAL NOTE: ABSOLUTE VALUE FUNCTION}

The absolute value function can be defined as a piecewise function
\(f(x)=\left\{\begin{array}{l}x, x \geq 0 \\ -x, x<0\end{array}\right.\)

\section*{TIP FOR SUCCESS}

It can help to visualize the graph of an absolute value function as the graph of the identity function, \(f(x)=x\) where, for all negative input, the function value is forced to be positive.
When describing absolute value in words, visualize it as a distance. We can see, for example, that the two values that are 3 units away from the number 1 on the number line are 4 and -2 .
We can use an absolute value function statement to describe this by asking the question what values are 3 units away from 1? Let \(x\) be the numbers. That is, the distance from some number and 1 is 3 .
\(|x-1|=3\)
\(x-1=3 \quad \Rightarrow \quad x=4 \quad\) and \(\quad x-1=-3 \quad \Rightarrow \quad x=-2\)

\section*{EXAMPLE: DETERMINE A NUMBER WITHIN A PRESCRIBED DISTANCE}

Describe all values \(x\) within or including a distance of 4 from the number 5 .

\section*{Answer}


We want the distance between \(x\) and 5 to be less than or equal to 4 . We can draw a number line to represent the condition to be satisfied.
The distance from \(x\) to 5 can be represented using \(|x-5|\). We want the values of \(x\) that satisfy the condition \(|x-5| \leq 4\).

Analysis of the Solution
Note that
\(-4 \leq x-5\)
\(1 \leq x\)
And:
\(x-5 \leq 4\)
\(x \leq 9\)
So \(|x-5| \leq 4\) is equal to \(1 \leq x \leq 9\).
However, mathematicians generally prefer absolute value notation.

Describe all values \(x\) within a distance of 3 from the number 2 .
Answer
\(|x-2| \leq 3\)
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\section*{EXAMPLE: RESISTANCE OF A RESISTOR}

Electrical parts, such as resistors and capacitors, come with specified values of their operating parameters: resistance, capacitance, etc. However, due to imprecision in manufacturing, the actual values of these parameters vary somewhat from piece to piece even when they are supposed to be the same. The best that manufacturers can do is to try to guarantee that the variations will stay within a specified range, often \(\pm 1 \%, \pm 5 \%\), or \(\pm 10 \%\).
Suppose we have a resistor rated at 680 ohms, \(\pm 5 \%\). Use the absolute value function to express the range of possible values of the actual resistance.

\section*{Answer}
\(5 \%\) of 680 ohms is 34 ohms. The absolute value of the difference between the actual and nominal resistance should not exceed the stated variability, so, with the resistance \(R\) in ohms, \(|R-680| \leq 34\)

\section*{TRY IT}

Students who score within 20 points of 80 will pass a test. Write this as a distance from 80 using absolute value notation.
Answer
Using the variable \(p\) for passing, \(|p-80| \leq 20\).
The most significant feature of the absolute value graph is the corner point at which the graph changes direction. This point is shown at the origin.


The graph below is of \(y=2|x-3|+4\). The graph of \(y=|x|\) has been shifted right 3 units, vertically stretched by a factor of 2 , and shifted up 4 units. This means that the corner point is located at \((3,4)\) for this transformed function.


\section*{Find the Intercepts of an Absolute Value Function}

Knowing how to solve problems involving absolute value functions is useful. For example, we may need to identify numbers or points on a line that are at a specified distance from a given reference point.

\section*{EXAMPLE: WRITING AN EQUATION FOR AN ABSOLUTE VALUE FUNCTION}

Write an equation for the function graphed below.


\section*{Answer}

The basic absolute value function changes direction at the origin, so this graph has been shifted to the right 3 units and down 2 units from the basic toolkit function.


We also notice that the graph appears vertically stretched, because the width of the final graph on a horizontal line is not equal to 2 times the vertical distance from the corner to this line, as it would be for an unstretched absolute value function. Instead, the width is equal to 1 times the vertical distance.


From this information we can write the equation
\(f(x)=2|x-3|-2, \quad\) treating the stretch as a vertical stretch, or
\(f(x)=|2(x-3)|-2\), treating the stretch as a horizontal compression.

\section*{Analysis of the Solution}

Note that these equations are algebraically the same-the stretch for an absolute value function can be written interchangeably as a vertical or horizontal stretch or compression.

\section*{Q \& A \\ If we couldn't observe the stretch of the function from the graphs, could we algebraically determine it?}

Yes. If we are unable to determine the stretch based on the width of the graph, we can solve for the stretch factor by putting in a known pair of values for \(x\) and \(f(x)\).
\(f(x)=a|x-3|-2\)
Now substituting in the point \((1,2)\)
\(2=a|1-3|-2\)
\(4=2 a\)
\(a=2\)

\section*{TRY IT}

Write the equation for the absolute value function that is horizontally shifted left 2 units, vertically flipped, and vertically shifted up 3 units.
Answer
\(f(x)=-|x+2|+3\)

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\section*{Q \& A}

Do the graphs of absolute value functions always intersect the vertical axis? The horizontal axis? Yes, they always intersect the vertical axis. The graph of an absolute value function will intersect the vertical axis when the input is zero.
No, they do not always intersect the horizontal axis. The graph may or may not intersect the horizontal axis depending on how the graph has been shifted and reflected. It is possible for the absolute value function to intersect the horizontal axis at zero, one, or two points.

(a)

(b)

(c)
(a) The absolute value function does not intersect the horizontal axis. (b) The absolute value function intersects the horizontal axis at one point. (c) The absolute value function intersects the horizontal axis at two points.

\section*{HOW TO: GIVEN THE FORMULA FOR AN ABSOLUTE VALUE FUNCTION, FIND THE HORIZONTAL INTERCEPTS OF ITS GRAPH.}
1. Isolate the absolute value term.
2. Use \(|A|=B\) to write \(A=B\) or \(-\mathrm{A}=B\), assuming \(B>0\).
3. Solve for \(x\).

\section*{TIP FOR SUCCESS}

The places where a graph crosses the horizontal axis are sometimes called horizontal intercepts, \(x\) intercepts, or zeros.
We call them zeros because those are the places where the function value equals zero. To find them, set the function value equal to zero and solve for \(x\). We can do this for many functions using the techniques of college algebra.

\section*{EXAMPLE: FINDING THE ZEROS OF AN ABSOLUTE VALUE FUNCTION}

For the function \(f(x)=|4 x+1|-7\), find the values of \(x\) such that \(f(x)=0\).

\section*{Answer}
\(0=|4 x+1|-7\)
\(7=|4 x+1|\)

Substitute 0 for \(f(x)\).
Isolate the absolute value on one side of the equation.

Break into two separate equations and solve.
\(-8=4 x\)
\(x=\frac{-8}{4}=-2\)


The function outputs 0 when \(x=1.5\) or \(x=-2\).

\section*{TRY IT}

For the function \(f(x)=|2 x-1|-3\), find the values of \(x\) such that \(f(x)=0\).
Answer
\(x=-1\) or \(x=2\)
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\section*{SUMMARY: GRAPHS OF LINEAR FUNCTIONS}

\section*{Key Concepts}
- Linear functions may be graphed by plotting points or by using the \(y\)-intercept and slope.
- Graphs of linear functions may be transformed by shifting the graph up, down, left, or right as well as using stretches, compressions, and reflections.
- The \(y\)-intercept and slope of a line may be used to write the equation of a line.
- The \(x\)-intercept is the point at which the graph of a linear function crosses the \(x\)-axis.
- Horizontal lines are written in the form, \(f(x)=b\).
- Vertical lines are written in the form, \(x=b\).
- Parallel lines have the same slope.
- Perpendicular lines have negative reciprocal slopes, assuming neither is vertical.
- A line parallel to another line, passing through a given point, may be found by substituting the slope value of the line and the \(x\) - and \(y\)-values of the given point into the equation \(f(x)=m x+b\) and using the \(b\) that results. Similarly, point-slope form of an equation can also be used.
- A line perpendicular to another line, passing through a given point, may be found in the same manner, with the exception of using the negative reciprocal slope.
- The absolute value function is commonly used to measure distances between points.
- Applied problems, such as ranges of possible values, can also be solved using the absolute value function.
- The graph of the absolute value function resembles the letter V . It has a corner point at which the graph changes direction.
- In an absolute value equation, an unknown variable is the input of an absolute value function.
- If the absolute value of an expression is set equal to a positive number, expect two solutions for the unknown variable.
- An absolute value equation may have one solution, two solutions, or no solutions.
- An absolute value inequality is similar to an absolute value equation but takes the form \(|A|<B,|A| \leq B,|A|>B\), or \(|A| \geq B\). It can be solved by determining the boundaries of the solution set and then testing which segments are in the set.
- Absolute value inequalities can also be solved graphically.

\section*{Glossary}
absolute value equation an equation of the form \(|A|=B\), with \(B \geq 0\); it will have solutions when \(A=B\) or \(-A=B\)
absolute value inequality a relationship in the form \(|A|<B,|A| \leq B,|A|>B\), or \(|A| \geq B\)
horizontal line a line defined by \(f(x)=b\) where \(b\) is a real number. The slope of a horizontal line is 0 .
parallel lines two or more lines with the same slope
perpendicular lines two lines that intersect at right angles and have slopes that are negative reciprocals of each other
vertical line a line defined by \(x=a\) where \(a\) is a real number. The slope of a vertical line is undefined.
\(\boldsymbol{x}\)-intercept the point on the graph of a linear function when the output value is 0 ; the point at which the graph crosses the horizontal axis

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\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Identify steps for modeling linear functions and solving problems.
- Build linear models from verbal descriptions.
- Draw and interpret scatter plots.
- Find the line of best fit using a calculator.
- Distinguish between linear and nonlinear relations.
- Use a linear model to make predictions.


A view of the Seattle skyline. (credit: EEK Photography/Flickr)
Emily is a college student who plans to spend a summer in Seattle. She has saved \$3,500 for her trip and anticipates spending \(\$ 400\) each week on rent, food, and activities. How can we write a linear model to represent her situation? What would be the \(x\)-intercept, and what can she learn from it? To answer these and related questions, we can create a model using a linear function. Models such as this one can be extremely useful for analyzing relationships and making predictions based on those relationships. In this section, we will explore examples of linear function models.

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\section*{BUILDING LINEAR MODELS}

\section*{LEARNING OUTCOMES}
- Use prescribed strategies to build linear models
- Use intercepts and data points to build a linear model
- Use a diagram to build a model

\section*{TIP FOR SUCCESS}

When following the steps given below for solving real-world situations, it is helpful to keep these tips in mind.
- Read the problem carefully, more than once if needed.
- Look for the question being asked. That's usually where you'll find the unknown.
- Separate the information given in the problem into chunks and look for relationships between the pieces to translate the information into expressions, equations, or inequalities.

When modeling scenarios with linear functions and solving problems involving quantities with a constant rate of change, we typically follow the same problem solving strategies that we would use for any type of function. Let's briefly review them:
1. Identify changing quantities, and then define descriptive variables to represent those quantities. When appropriate, sketch a picture or define a coordinate system.
2. Look for information that provides values for the variables or values for parts of the functional model, such as slope and initial value.
3. Determine what we are trying to find, identify, solve, or interpret.
4. Identify a solution pathway from the provided information to what we are trying to find. Often this will involve checking and tracking units, building a table, or even finding a formula for the function being used to model the problem.
5. When needed, write a formula for the function.
6. Solve or evaluate the function using the formula.
7. Reflect on whether your answer is reasonable for the given situation and whether it makes sense mathematically.
8. Clearly convey your result using appropriate units, and answer in full sentences when necessary.

\section*{Building Linear Models}

Now let's take a look at the student from the introduction who wants to spend the summer in Seattle. In her situation, there are two changing quantities: time and money. The amount of money she has remaining while on vacation depends on how long she stays. We can use this information to define our variables, including units.
- Output: \(M\), money remaining, in dollars
- Input: \(t\), time, in weeks

So, the amount of money remaining depends on the number of weeks. Hence, amount of money remaining is a function of time: \(M(t)\)

We can also identify the initial value and the rate of change.
- Initial Value: She saved \(\$ 3,500\), so \(\$ 3,500\) is the initial value for \(M\).
- Rate of Change: She anticipates spending \(\$ 400\) each week, so - \(\$ 400\) per week is the rate of change, or slope.

Notice that the unit of dollars per week matches the unit of our output variable divided by our input variable. Also, because the slope is negative, the linear function is decreasing. This should make sense because she is spending money each week.


The rate of change is constant, so we can start with the linear model \(M(t)=m t+b\). Then we can substitute the intercept and slope provided.

To find the \(x\)-intercept, we set the output to zero and solve for the input.
\(0=-400 t+3500\)
\(t=\frac{3500}{400}\)
\(t=8.75\)
The \(x\)-intercept is 8.75 weeks. Because this represents the input value when the output will be zero, we could say that Emily will have no money left after 8.75 weeks.

When modeling any real-life scenario with functions, there is typically a limited domain over which that model will be valid-almost no trend continues indefinitely. Here the domain refers to the number of weeks. In this case, it doesn't make sense to talk about input values less than zero. A negative input value could refer to a number of weeks before she saved \(\$ 3,500\), but the scenario discussed poses the question once she saved \(\$ 3,500\) because this is when her trip and subsequent spending starts. It is also likely that this model is not valid after the \(x\)-intercept, unless Emily will use a credit card and goes into debt. The domain represents the set of input values, so the reasonable domain for this function is \(0 \leq t \leq 8.75\).

In the above example, we were given a written description of the situation. We followed the steps of modeling a problem to analyze the information. However, the information provided may not always be the same. Sometimes we might be provided with an intercept. Other times we might be provided with an output value. We must be careful to analyze the information we are given and use it appropriately to build a linear model.

\section*{Using a Given Intercept to Build a Model}

Some real-world problems provide the \(y\)-intercept, which is the constant or initial value. Once the \(y\)-intercept is known, the \(x\)-intercept can be calculated. Suppose, for example, that Hannah plans to pay off a no-interest loan from her parents. Her loan balance is \(\$ 1,000\). She plans to pay \(\$ 250\) per month until her balance is \(\$ 0\). The \(y\)-intercept is the initial amount of her debt, or \(\$ 1,000\). The rate of change, or slope, is \(-\$ 250\) per month. We can then use slope-intercept form and the given information to develop a linear model.
\[
\begin{aligned}
& f(x)=m x+b \\
& f(x)=-250 x+1000
\end{aligned}
\]

Now we can set the function equal to 0 and solve for \(x\) to find the \(x\)-intercept.
\[
\begin{aligned}
& 0=-250 x+1000 \\
& 1000=250 x \\
& 4=x \\
& x=4
\end{aligned}
\]

The \(x\)-intercept is the number of months it takes her to reach a balance of \(\$ 0\). The \(x\)-intercept is 4 months, so it will take Hannah four months to pay off her loan.

\section*{Using a Given Input and Output to Build a Model}

Many real-world applications are not as direct as the ones we just considered. Instead they require us to identify some aspect of a linear function. We might sometimes be asked to evaluate the linear model at a
given input or set the equation of the linear model equal to a specified output.

\section*{HOW TO: GIVEN A WORD PROBLEM THAT INCLUDES TWO PAIRS OF INPUT AND OUTPUT VALUES, USE THE LINEAR FUNCTION TO SOLVE A PROBLEM.}
1. Identify the input and output values.
2. Convert the data to two coordinate pairs.
3. Find the slope.
4. Write the linear model.
5. Use the model to make a prediction by evaluating the function at a given \(x\) value.
6. Use the model to identify an \(x\) value that results in a given \(y\) value.
7. Answer the question posed.

\section*{Using a Diagram to Model a Problem}

It is useful for many real-world applications to draw a picture to gain a sense of how the variables representing the input and output may be used to answer a question. To draw the picture, first consider what the problem is asking for. Then, determine the input and the output. The diagram should relate the variables. Often, geometric shapes or figures are drawn. Distances are often traced out. If a right triangle is sketched, the Pythagorean Theorem relates the sides. If a rectangle is sketched, labeling width and height is helpful.

\section*{EXAMPLE: USING A LINEAR MODEL TO INVESTIGATE A TOWN’S POPULATION}

A town's population has been growing linearly. In 2004 the population was 6,200 . By 2009 the population had grown to 8,100 . Assume this trend continues.
1. Predict the population in 2013.
2. Identify the year in which the population will reach 15,000 .

\section*{Answer}

The two changing quantities are the population size and time. While we could use the actual year value as the input quantity, doing so tends to lead to very cumbersome equations because the \(y\)-intercept would correspond to the year 0 , more than 2000 years ago!
To make computation a little nicer, we will define our input as the number of years since 2004:
- Input: \(t\), years since 2004
- Output: \(P(t)\), the town's population

To predict the population in \(2013(t=9)\), we would first need an equation for the population. Likewise, to find when the population would reach 15,000, we would need to solve for the input that would provide an output of 15,000 . To write an equation, we need the initial value and the rate of change, or slope.
To determine the rate of change, we will use the change in output per change in input.
\(m=\frac{\text { change in output }}{\text { change in input }}\)
The problem gives us two input-output pairs. Converting them to match our defined variables, the year 2004 would correspond to \(t=0\), giving the point \((0,6200)\). Notice that through our clever choice of variable definition, we have "given" ourselves the \(y\)-intercept of the function. The year 2009 would correspond to \(t=5\), giving the point \((5,8100)\).
The two coordinate pairs are \((0,6200)\) and \((5,8100)\). Recall that we encountered examples in which we were provided two points earlier in the module. We can use these values to calculate the slope.
\[
\begin{aligned}
& m=\frac{8100-6200}{5-0} \\
& m=\frac{1900}{5} \\
& m=380 \text { people per year }
\end{aligned}
\]

We already know the \(y\)-intercept of the line, so we can immediately write the equation:
\[
P(t)=380 t+6200
\]

To predict the population in 2013, we evaluate our function at \(t=9\).
\[
P(9)=380(9)+6,200
\]
\[
P(9)=9,620
\]

If the trend continues, our model predicts a population of 9,620 in 2013.
To find when the population will reach 15,000 , we can set \(P(t)=15000\) and solve for \(t\).
\(15000=380 t+6200\)
\(8800=380 t\)
\(t \approx 23.158\)

Our model predicts the population will reach 15,000 in a little more than 23 years after 2004, or somewhere around the year 2027.

\section*{TRY IT}

A company sells doughnuts. They incur a fixed cost of \(\$ 25,000\) for rent, insurance, and other expenses. It costs \(\$ 0.25\) to produce each doughnut.
1. Write a linear model to represent the cost \(C\) of the company as a function of \(x\), the number of doughnuts produced.
2. Find and interpret the \(y\)-intercept.

\section*{Answer}
\(C(x)=0.25 x+25,000\) The \(y\)-intercept is ( \(0,25,000\) ). If the company does not produce a single doughnut, they still incur a cost of \(\$ 25,000\).

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\section*{TRY IT}

A city's population has been growing linearly. In 2008, the population was 28,200. By 2012, the population was 36,800 . Assume this trend continues.
1. Predict the population in 2014.
2. Identify the year in which the population will reach 54,000 .

Answer
41,100; 2020
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=249

\section*{TIP FOR SUCCESS}

It simply cannot be stressed too often; sketching a diagram or trying a few sample equations is an essential part of doing mathematics. Paths that don't pan out and wrong turns are common and natural. We need to make mistakes in math in order to rule out possibilities that don't lead to success. Don't be afraid to sketch a diagram or take a wrong turn. It's all part of the mathematical process.

\section*{EXAMPLE: USING A DIAGRAM TO MODEL DISTANCE WALKED}

Anna and Emanuel start at the same intersection. Anna walks east at 4 miles per hour while Emanuel walks south at 3 miles per hour. They are communicating with a two-way radio that has a range of 2 miles. How long after they start walking will they fall out of radio contact?

\section*{Answer}

In essence, we can partially answer this question by saying they will fall out of radio contact when they are 2 miles apart, which leads us to ask a new question: "How long will it take them to be 2 miles apart?"
In this problem, our changing quantities are time and position, but ultimately we need to know how long will it take for them to be 2 miles apart. We can see that time will be our input variable, so we'll define our input and output variables.
- Input: \(t\), time in hours.
- Output: \(A(t)\), distance in miles, and \(E(t)\), distance in miles

Because it is not obvious how to define our output variable, we'll start by drawing a picture.


Initial Value: They both start at the same intersection so when \(t=0\), the distance traveled by each person should also be 0 . Thus the initial value for each is 0 .
Rate of Change: Anna is walking 4 miles per hour and Emanuel is walking 3 miles per hour, which are both rates of change. The slope for \(A\) is 4 and the slope for \(E\) is 3 .
Using those values, we can write formulas for the distance each person has walked.
\(A(t)=4 t\)
\(E(t)=3 t\)
For this problem, the distances from the starting point are important. To notate these, we can define a coordinate system, identifying the "starting point" at the intersection where they both started. Then we can use the variable, \(A\), which we introduced above, to represent Anna's position and define it to be a measurement from the starting point in the eastward direction. Likewise, can use the variable, \(E\), to represent Emanuel's position measured from the starting point in the southward direction. Note that in defining the coordinate system, we specified both the starting point of the measurement and the direction of measure.
We can then define a third variable, \(D\), to be the measurement of the distance between Anna and Emanuel. Showing the variables on the diagram is often helpful.
Recall that we need to know how long it takes for \(D\), the distance between them, to equal 2 miles. Notice that for any given input \(t\), the outputs \(A(t), E(t)\), and \(D(t)\) represent distances.


This picture shows us that we can use the Pythagorean Theorem because we have drawn a right triangle. Using the Pythagorean Theorem, we get:
\(D(t)^{2}=A(t)^{2}+E(t)^{2}\)
\(D(t)^{2}=(4 t)^{2}+(3 t)^{2}\)
\(D(t)^{2}=16 t^{2}+9 t^{2}\)
\(D(t)^{2}=25 t^{2}\)
\(D(t)= \pm \sqrt{25 t^{2}} \quad\) Solve for \(D(t)\) by taking the square root of each side of the equation
\(D(t)= \pm 5 t\)
In this scenario we are considering only positive values of \(t\), so our distance \(D(t)\) will always be positive. We can simplify this answer to \(D(t)=5 t\). This means that the distance between Anna and Emanuel is also a linear function. Because \(D\) is a linear function, we can now answer the question of when the distance between them will reach 2 miles. We will set the output \(D(t)=2\) and solve for \(t\).
\(D(t)=2\)
\(5 t=2\)
\(t=\frac{2}{5}=0.4\)
They will fall out of radio contact in 0.4 hours, or 24 minutes.

\section*{Q \& A}

Should I draw diagrams when given information based on a geometric shape?
Yes. Sketch the figure and label the quantities and unknowns on the sketch.

\section*{EXAMPLE: USING A DIAGRAM TO MODEL DISTANCE BETWEEN CITIES}

There is a straight road leading from the town of Westborough to Agritown 30 miles east and 10 miles north. A certain distance down this road, it junctions with a second road, perpendicular to the first, leading to the town of Eastborough. If the town of Eastborough is located 20 miles directly east of the town of Westborough, how far is the road junction from Westborough?

\section*{Answer}

It might help here to draw a picture of the situation. It would then be helpful to introduce a coordinate system. While we could place the origin anywhere, placing it at Westborough seems convenient. This puts Agritown at coordinates (30, 10), and Eastborough at (20, 0).


Using this point along with the origin, we can find the slope of the line from Westborough to Agritown:
\(m=\frac{10-0}{30-0}=\frac{1}{3}\)
The equation of the road from Westborough to Agritown would be
\(W(x)=\frac{1}{3} x\)
From this, we can determine the perpendicular road to Eastborough will have slope \(m=-3\). Because the town of Eastborough is at the point \((20,0)\), we can find the equation:
\[
\begin{aligned}
& E(x)=-3 x+b \\
& 0=-3(20)+b \quad \text { Substitute in }(20,0) \\
& b=60 \\
& E(x)=-3 x+60
\end{aligned}
\]

We can now find the coordinates of the junction of the roads by finding the intersection of these lines. Setting them equal,
\[
\begin{aligned}
& \frac{1}{3} x=-3 x+60 \\
& \frac{10}{3} x=60 \\
& 10 x=180 \\
& x=18 \\
& y=W(18) \\
& y=\frac{1}{3}(18) \\
& y=6
\end{aligned} \quad \text { Substituting this back into } W(x)
\]

The roads intersect at the point \((18,6)\). Using the distance formula, we can now find the distance from Westborough to the junction.
\[
\begin{aligned}
& \text { distance }=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}} \\
& =\sqrt{(18-0)^{2}+(6-0)^{2}}
\end{aligned}
\]

\section*{Analysis of the Solution}

One nice use of linear models is to take advantage of the fact that the graphs of these functions are lines. This means real-world applications discussing maps need linear functions to model the distances between reference points.

\section*{TRY IT}

There is a straight road leading from the town of Timpson to Ashburn 60 miles east and 12 miles north. Partway down the road, it junctions with a second road, perpendicular to the first, leading to the town of Garrison. If the town of Garrison is located 22 miles directly east of the town of Timpson, how far is the road junction from Timpson?

\section*{Answer}
21.15 miles

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\section*{FITTING LINEAR MODELS TO DATA}

\section*{LEARNING OUTCOMES}
- Draw and interpret scatter plots
- Find the line of best fit using a calculator
- Distinguish between linear and nonlinear relations
- Use a linear model to make predictions

A professor is attempting to identify trends among final exam scores. His class has a mixture of students, so he wonders if there is any relationship between age and final exam scores. One way for him to analyze the scores is by creating a diagram that relates the age of each student to the exam score received. In this section, we will examine one such diagram known as a scatter plot.

When expressing pairs of inputs and outputs on a graph, they take the form of (input, output). In scatter plots, the two variables relate to create each data point, (variable 1, variable 2), but it is often not necessary to declare that one is dependent on the other. In the example below, each Age coordinate corresponds to a Final Exam Score in the form (age, score). Each corresponding pair is plotted on the graph.

A scatter plot is a graph of plotted points that may show a relationship between two sets of data. If the relationship is from a linear model, or a model that is nearly linear, the professor can draw conclusions using his knowledge of linear functions. Below is a sample scatter plot.

\section*{Final Exam Score Vs Age}


A scatter plot of age and final exam score variables.
Notice this scatter plot does not indicate a linear relationship. The points do not appear to follow a trend. In other words, there does not appear to be a relationship between the age of the student and the score on the final exam.

\section*{Finding the Line of Best Fit}

One way to approximate our linear function is to sketch the line that seems to best fit the data. Then we can extend the line until we can verify the \(y\)-intercept. We can approximate the slope of the line by extending it until we can estimate the \(\frac{\text { rise }}{\text { run }}\).

\section*{Recognizing Interpolation or Extrapolation}

While the data for most examples does not fall perfectly on the line, the equation is our best guess as to how the relationship will behave outside of the values for which we have data. We use a process known as interpolation when we predict a value inside the domain and range of the data. The process of extrapolation is used when we predict a value outside the domain and range of the data.

\section*{EXAMPLE: USING A SCATTER PLOT TO INVESTIGATE CRICKET CHIRPS}

The table below shows the number of cricket chirps in 15 seconds, for several different air temperatures, in degrees Fahrenheit. (Note: Selected data from http://classic.globe.gov/fsl/scientistsblog/2007/10/.
Retrieved Aug 3, 2010) Plot this data, and determine whether the data appears to be linearly related.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|l|}
\hline Chirps & 44 & 35 & 20.4 & 33 & 31 & 35 & 18.5 & 37 & 26 \\
\hline Temperature & 80.5 & 70.5 & 57 & 66 & 68 & 72 & 52 & 73.5 & 53 \\
\hline
\end{tabular}

\section*{Answer}

Plotting this data suggests that there may be a trend. We can see from the trend in the data that the number of chirps increases as the temperature increases. The trend appears to be roughly linear, though

Cricket Chirps vs. Temperature

certainly not perfectly so.
Cricket Chirps in 15 Seconds
The graph below compares the two processes for the cricket-chirp data addressed in the previous example. We can see that interpolation would occur if we used our model to predict temperature when the values for chirps are between 18.5 and 44 . Extrapolation would occur if we used our model to predict temperature when the values for chirps are less than 18.5 or greater than 44.

There is a difference between making predictions inside the domain and range of values for which we have data and outside that domain and range. Predicting a value outside of the domain and range has its limitations. When our model no longer applies after a certain point, it is sometimes called model breakdown. For example, predicting a cost function for a period of two years may involve examining the data where the input is the time in years and the output is the cost. But if we try to extrapolate a cost when \(x=50\), that is, in 50 years, the model would not apply because we could not account for factors fifty years in the future.

\section*{EXAMPLE: FINDING A LINE OF BEST FIT}

Find a linear function that fits the data in the table below by "eyeballing" a line that seems to fit.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|l|}
\hline Chirps & 44 & 35 & 20.4 & 33 & 31 & 35 & 18.5 & 37 & 26 \\
\hline Temperature & 80.5 & 70.5 & 57 & 66 & 68 & 72 & 52 & 73.5 & 53 \\
\hline Answer
\end{tabular}

Answer
On a graph, we could try sketching a line.
Using the starting and ending points of our hand drawn line, points \((0,30)\) and \((50,90)\), this graph has a slope of \(m=\frac{60}{50}=1.2\) and a \(y\)-intercept at 30 . This gives an equation of \(T(c)=1.2 c+30\)
where \(c\) is the number of chirps in 15 seconds, and \(T(c)\) is the temperature in degrees Fahrenheit. The resulting equation is represented in the graph below.

\section*{Cricket Chirps vs. Temperature}


\section*{Analysis of the Solution}

This linear equation can then be used to approximate answers to various questions we might ask about the trend.

\section*{TRY IT}

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\section*{Cricket Chirps vs. Temperature}


Interpolation occurs within the domain and range of the provided data whereas extrapolation occurs outside.

\section*{A GENERAL NOTE: INTERPOLATION AND EXTRAPOLATION}

Different methods of making predictions are used to analyze data.
- The method of interpolation involves predicting a value inside the domain and/or range of the data.
- The method of extrapolation involves predicting a value outside the domain and/or range of the data.
- Model breakdown occurs at the point when the model no longer applies.

\section*{Finding the Line of Best Fit Using a Graphing Utility}

While eyeballing a line works reasonably well, there are statistical techniques for fitting a line to data that minimize the differences between the line and data values. (Note: Technically, the method minimizes the sum of the squared differences in the vertical direction between the line and the data values.) One such technique is called least squares regression and can be computed by many graphing calculators as well as both spreadsheet and statistical software. Least squares regression is also called linear regression, and we can use an online graphing calculator to perform linear regressions.

\section*{EXAMPLE: UNDERSTANDING INTERPOLATION AND EXTRAPOLATION}
\begin{tabular}{|l|l|l|l|l|l|l|l|l|l|}
\hline Chirps & 44 & 35 & 20.4 & 33 & 31 & 35 & 18.5 & 37 & 26 \\
\hline Temperature & 80.5 & 70.5 & 57 & 66 & 68 & 72 & 52 & 73.5 & 53 \\
\hline
\end{tabular}

Use the cricket data above to answer the following questions:
1. Would predicting the temperature when crickets are chirping 30 times in 15 seconds be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.
2. Would predicting the number of chirps crickets will make at 40 degrees be interpolation or extrapolation? Make the prediction, and discuss whether it is reasonable.

\section*{Answer}
1. The number of chirps in the data provided varied from 18.5 to 44 . A prediction at 30 chirps per 15 seconds is inside the domain of our data so would be interpolation. Using our model:
\(T(30)=30+1.2(30)\)
\(T(30)=66\) degrees

Based on the data we have, this value seems reasonable.
2. The temperature values varied from 52 to 80.5 . Predicting the number of chirps at 40 degrees is extrapolation because 40 is outside the range of our data. Using our model:
\[
\begin{aligned}
& 40=30+1.2 c \\
& 10=1.2 c \\
& c \approx 8.33
\end{aligned}
\]

We can compare the regions of interpolation and extrapolation using the graph below.
Cricket Chirps vs. Temperature


\section*{Analysis of the Solution}

Our model predicts the crickets would chirp 8.33 times in 15 seconds. While this might be possible, we have no reason to believe our model is valid outside the domain and range. In fact, generally crickets stop chirping altogether at or below 50 degrees.

\section*{TRY IT}

According to the data from the table in the cricket-chirp example, what temperature can we predict if we counted 20 chirps in 15 seconds?

\section*{Answer}
\(54^{\circ} \mathrm{F}\)

\section*{Distinguish Between Linear and Nonlinear Relations}

As we saw in the cricket-chirp example, some data exhibit strong linear trends, but other data, like the final exam scores plotted by age, are clearly nonlinear. Most calculators and computer software can also provide us with the correlation coefficient, which is a measure of how closely the line fits the data. Many graphing calculators require the user to turn a "diagnostic on" selection to find the correlation coefficient, which mathematicians label as \(r\). The correlation coefficient provides an easy way to get an idea of how close to a line the data falls.

We should compute the correlation coefficient only for data that follows a linear pattern or to determine the degree to which a data set is linear. If the data exhibits a nonlinear pattern, the correlation coefficient for a linear regression is meaningless. To get a sense of the relationship between the value of \(r\) and the graph of the data, the image below shows some large data sets with their correlation coefficients. Remember, for all plots, the horizontal axis shows the input and the vertical axis shows the output.


Plotted data and related correlation coefficients. (credit: "DenisBoigelot," Wikimedia Commons)

\section*{Use a Linear Model to Make Predictions}

Once we determine that a set of data is linear using the correlation coefficient, we can use the regression line to make predictions. As we learned previously, a regression line is a line that is closest to the data in the scatter plot, which means that only one such line is a best fit for the data.

\section*{EXAMPLE: FINDING A LEAST SQUARES REGRESSION LINE}

Find the least squares regression line using the cricket-chirp data in the table below.
Use an online graphing calculator.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|l|}
\hline Chirps & 44 & 35 & 20.4 & 33 & 31 & 35 & 18.5 & 37 & 26 \\
\hline Temperature & 80.5 & 70.5 & 57 & 66 & 68 & 72 & 52 & 73.5 & 53 \\
\hline
\end{tabular}

\section*{Answer}

The following instructions are for Desmos, and other online graphing tools may be slightly different.
1. Click the plus button (add item) in the upper left corner and select table.
2. Enter chirps data in the \(x 1\) column.
3. Enter temperature data in the y1 column.
\begin{tabular}{l|l|l|l|l|l|l|l|l|}
\hline \(\mathbf{x 1}\) & 44 & 35 & 20.4 & 33 & 31 & 35 & 18.5 & 37 \\
\hline \(\mathbf{y 1}\) & 80.5 & 70.5 & 57 & 66 & 68 & 72 & 52 & 73.5 \\
\hline
\end{tabular}
4. If you can't see the points on the grid, use the plus and minus buttons in the upper right hand corner to zoom in or out on the grid, or click on the wrench and change the upper bound of x 1 to 60 and y 1 to 100
5. In the empty cell below the table you created, enter the expression \(\mathrm{y} 1 \sim \mathrm{mx} 1+\mathrm{b}\)
6. You can add labels to your graph by clicking on the wrench in the upper right hand corner and typing them into the cells that say "add a label"

Here is an example of how your graph may look:


Notice that this line is quite similar to the equation we "eyeballed" but should fit the data better. Notice also that using this equation would change our prediction for the temperature when hearing 30 chirps in 15 seconds from 66 degrees to:
\(T(30)=30.281+1.143(30)\)
\(T(30)=64.571\)
\(T(30) \approx 64.6\) degrees

\section*{Q \& A}

Will there ever be a case where two different lines will serve as the best fit for the data?
No. There is only one best fit line.

\section*{A GENERAL NOTE: CORRELATION COEFFICIENT}

The correlation coefficient is a value, \(r\), between -1 and 1 .
- \(r>0\) suggests a positive (increasing) relationship
- \(r<0\) suggests a negative (decreasing) relationship
- The closer the value is to 0 , the more scattered the data.
- The closer the value is to 1 or -1 , the less scattered the data is.

\section*{EXAMPLE: FINDING A CORRELATION COEFFICIENT}

Calculate the correlation coefficient for cricket-chirp data in the table below.
\begin{tabular}{l|l|l|l|l|l|l|l|l|l}
\hline Chirps & 44 & 35 & 20.4 & 33 & 31 & 35 & 18.5 & 37 & 26 \\
\hline Temperature & 80.5 & 70.5 & 57 & 66 & 68 & 72 & 52 & 73.5 & 53
\end{tabular}

Answer
Online graphing calculators provide you with the correlation coefficient when you use it to calculate a linear regression. The correlation coefficients is labeled as \(r=0.951\) for this dataset. This value is very close to 1 which suggests a strong increasing linear relationship. Below is an example of what your graph will look like if you choose to use Desmos.


\section*{EXAMPLE: USING A REGRESSION LINE TO MAKE PREDICTIONS}

Gasoline consumption in the United States has been steadily increasing. Consumption data from 1994 to 2004 is shown in the table below. (Note:
http://www.bts.gov/publications/national_transportation_statistics/2005/html/table_04_10.html) Determine whether the trend is linear, and if so, find a model for the data. Use the model to predict the consumption in 2008.Is this an interpolation or an extrapolation?
\begin{tabular}{|l|c|c|c|c|c|c|c|c|c|c|}
\hline Year & '94 & '95 & '96 & '97 & '98 & '99 & '00 & '01 & '02 & '03 \\
\hline '04 \\
\hline Consumption (billions of gallons) & 113 & 116 & 118 & 119 & 123 & 125 & 126 & 128 & 131 & 133 \\
\hline
\end{tabular}

\section*{Answer}

We can introduce an input variable, \(t\), representing years since 1994. This makes entering the data into online graphing calculator easier.
Read the value for \(b\) and the value for the slope, \(m\), from the online graphing calculator to create the equation for the regression line:
\(C(t)=113.318+2.209 t\)
The correlation coefficient was calculated to be 0.997 , suggesting a very strong increasing linear trend.
Using this to predict consumption in 2008, which is 14 years after \(1994(t=14)\),
\[
\begin{aligned}
& C(14)=113.318+2.209(14) \\
& C(14)=144.244
\end{aligned}
\]

The model predicts 144.244 billion gallons of gasoline consumption in 2008. This is an extrapolation because there is not a datapoint whose x 1 value is 2008 .
The scatter plot of the data, including the least squares regression line, is shown below in Desmos, but you can use any online graphing calculator. Note how we changed the viewing window for the y -axis to \(80<y<150\).


\section*{TRY IT}

Use an online graphing calculator to find a linear regression for the following data, which represents the amount of time a scuba diver can spend underwater as a function of the depth of the water.
\begin{tabular}{|l|l|}
\hline Depth (feet) & Time (minutes) \\
\hline 50 & 80 \\
\hline 60 & 55 \\
\hline 70 & 45 \\
\hline 80 & 35 \\
\hline 90 & 25 \\
\hline 100 & 22 \\
\hline
\end{tabular}
1) Write the equation for the least squares regression line.
2) According to the regression line, how long can a diver spend at a depth of 110 feet?
3)How about 120 feet? Why doesn't this make sense?
4) At what depth would the dive time be zero?

Answer
Here is a sample graph for this dataset.

1) The equation for the regression line is \(y=-1.1143 x+127.24\)
2) A diver can spend \(y=-1.1143(110)+127.24=1.51\) minutes at a depth of 110 feet.
3) A diver can spend \(y=-1.1143(120)+127.24=-6.48\) minutes at a depth of 120 feet. This doesn't make sense because a negative value for time doesn't have any meaning.
4) To find at what depth the dive time would be zero, we need to set the regression equation equal to zero. \(0=-1.1143 x+127.24\)
\(-127.24=-1.1143 x\)
\(114.19=x\)
A diver, at a depth of 114.19 feet, would have a dive time of 0 minutes.

\section*{TRY IT}

Here are more data sets that you can plot using an online graphing calculator. Try to find a linear regression for them then look at the correlation coefficient to determine whether there is a linear relationship.
\begin{tabular}{|c|c|c|}
\hline \multicolumn{2}{|l|}{Depth of the Columbia River} & Water Velocity \\
\hline \multicolumn{2}{|l|}{0.66} & 1.55 \\
\hline \multicolumn{2}{|l|}{1.98} & 1.11 \\
\hline \multicolumn{2}{|l|}{2.64} & 1.42 \\
\hline \multicolumn{2}{|l|}{3.3} & 1.39 \\
\hline \multicolumn{2}{|l|}{4.62} & 1.39 \\
\hline \multicolumn{2}{|l|}{5.94} & 1.14 \\
\hline \multicolumn{2}{|l|}{7.26} & 0.91 \\
\hline \multicolumn{2}{|l|}{8.58} & 0.59 \\
\hline \multicolumn{2}{|l|}{9.9} & 0.59 \\
\hline \multicolumn{2}{|l|}{10.56} & 0.41 \\
\hline \multicolumn{2}{|l|}{11.22} & 0.22 \\
\hline \% of Mississippi River in Crops (By Basin) & Nitrate Concentration (mg/ L) & \\
\hline 2.4 & 0.647 & \\
\hline 1.3 & 1.062 & \\
\hline 14.3 & 1.432 & \\
\hline 0.5 & 0.579 & \\
\hline 45.6 & 3.561 & \\
\hline 46.6 & 3.938 & \\
\hline 1.5 & 0.927 & \\
\hline 53.6 & 2.549 & \\
\hline 4.1 & 0.357 & \\
\hline 3.1 & 0.245 & \\
\hline
\end{tabular}

Dimensions of the Lava Dome in Mt. St. Helens, t = 0 on 18 October 1980 (eruption was 18 May 1980).
\begin{tabular}{|l|l|}
\hline Days & Millions of Cubic Meters \\
\hline 0 & 2.9 \\
\hline 70 & 13 \\
\hline 109 & 28 \\
\hline 173 & 40 \\
\hline 242 & 56 \\
\hline 322 & 64 \\
\hline 376 & 75 \\
\hline 547 & 88 \\
\hline 603 & 100 \\
\hline 699 & 115 \\
\hline 872 & 152 \\
\hline 922 & 154 \\
\hline 1087 & 173 \\
\hline 1343 & 178 \\
\hline 1692 & 212 \\
\hline 1858 & 243 \\
\hline
\end{tabular}

\section*{FYI}

Divers who want or need to descend to depths greater than 100 feet employ different techniques and equipment to help them safely navigate the depth. For example, different gas mixtures or rebreather equipment may be used. Gas mixtures such as oxygen, helium, and nitrogen can help to mitigate the narcotic effects of breathing gas at great depths. (Note: https://en.wikipedia.org/wiki/Trimix_(breathing_gas))


A scuba diver using rebreather with open circuit bailout cylinders returning from a 600-foot ( 180 m ) dive.

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\section*{SUMMARY: MODELING WITH LINEAR FUNCTIONS}

\section*{Key Concepts}
- Scatter plots show the relationship between two sets of data.
- Scatter plots may represent linear or non-linear models.
- The line of best fit may be estimated or calculated using a calculator or statistical software.
- Interpolation can be used to predict values inside the domain and range of the data, whereas extrapolation can be used to predict values outside the domain and range of the data.
- The correlation coefficient, \(r\), indicates the degree of linear relationship between data.
- A regression line best fits the data.
- The least squares regression line is found by minimizing the squares of the distances of points from a line passing through the data and may be used to make predictions regarding either of the variables.

\section*{Glossary}
correlation coefficient a value, \(r\), between -1 and 1 that indicates the degree of linear correlation of variables or how closely a regression line fits a data set.
extrapolation predicting a value outside the domain and range of the data
interpolation predicting a value inside the domain and range of the data
least squares regression a statistical technique for fitting a line to data in a way that minimizes the differences between the line and data values
model breakdown when a model no longer applies after a certain point

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\section*{PUTTING IT TOGETHER: LINEAR AND ABSOLUTE VALUE FUNCTIONS}

At the start of this module, you were wondering whether you could earn a profit by making bikes and were given a profit function.
\(P(x)=R(x)-C(x)\)
where
\(x=\) the number of bikes produced and sold
\(P(x)=\) profit as a function of x
\(R(x)=\) revenue as a function of x
\(C(x)=\) cost as a function of x
Profit, revenue, and cost are all linear functions. They are a function of the number of bikes sold. Remember that you were planning to sell each bike for \(\$ 600\) and it cost you \(\$ 1,600\) for fixed costs plus \(\$ 200\) per bike.

Since you take in \(\$ 600\) for each bike you sell, revenue is:
\(R(x)=600 x\)
Your costs are \(\$ 200\) per bike plus a fixed cost of \(\$ 1600\), so your overall cost is:
\(C(x)=200 x+1600\)
That means that profit becomes:
\(P(x)=600 x-(200 x+1600)\)
\(=400 x-1600\)
With that in mind, let's take another look at the table. The profit is found by subtracting the cost function from the revenue function for each number of bikes.
\begin{tabular}{|l|l|}
\hline Number of bikes \((x)\) & Profit (\$) \\
\hline 2 & \(400(2)-1600=-800 \quad\) (loss) \\
\hline 5 & \(400(5)-1600=400\) \\
\hline 10 & \(400(10)-1600=2400\) \\
\hline
\end{tabular}

That still doesn't explain how to figure out the break-even point, the number of bikes for which the revenue equals the costs.

One way is to set the functions equal to one another and solve for \(x\).
\(R(x)=C(x)\)
\(600 x=200 x+1600\)
\(400 x=1600\)
\(x=4\)
That means that if you sell 4 bikes, you will take in the same amount you spent to make the bikes. So selling less than 4 bikes will result in a loss and selling more than 4 bikes will result in a profit.

Another method for finding the break-even point is by graphing the two functions. You can use whichever method of graphing you find useful.

To plot points, for example, make a list of values for each function. Then use them to determine coordinates that you can plot.
\(R(x)=600 x\)
\begin{tabular}{|l|l|l|}
\hline\(x\) & \(R(x)\) & \((x, R(x))\) \\
\hline 0 & 0 & \((0,0)\) \\
\hline 2 & 1200 & \((2,1200)\) \\
\hline 4 & 2400 & \((4,2400)\) \\
\hline 6 & 3600 & \((6,3600)\) \\
\hline 8 & 4800 & \((8,4800)\) \\
\hline
\end{tabular}
\(C(x)=200 x+1600\)
\begin{tabular}{|l|l|l|}
\hline\(x\) & \(C(x)\) & \((x, C(x))\) \\
\hline 0 & 1600 & \((0,1600)\) \\
\hline 2 & 2000 & \((2,2000)\) \\
\hline 4 & 2400 & \((4,2400)\) \\
\hline 6 & 2800 & \((6,2800)\) \\
\hline 8 & 3200 & \((8,3200)\) \\
\hline
\end{tabular}

To use the slope and \(y\)-intercept method, first find the \(y\)-intercept by setting \(x=0\). Then determine the slope as the coefficient of the variable, which is the \(m\) value.
\begin{tabular}{l|l|}
\hline\(R(x)\) & \(C(x)\) \\
\hline\(y\)-intercept \(=0\) & y-intercept \(=1600\) \\
\hline slope \(=600\) & slope \(=200\) \\
\hline
\end{tabular}

Using this information, the functions can be graphed


Now we can see the break-even point right away. It is the point at which the cost function intersects the revenue function. It occurs when \(x=4\). At this point you neither make a profit nor incur a loss.

The last question to consider is how changing your price affects your profit. If your costs remain the same, increasing the price you charge will shift your break-even point to a lower number of bikes and increase your revenue for every value of \(x\). However, people may not buy as many bikes. Lowering your price will shift your break-even point to a higher number of bikes and decrease your revenue for every value of \(x\). However, more people may buy your bikes.

Let's take a look at the graphs for two other possible prices.
Graphing lets you quickly see a visual representation of the functions and how they are


related to one another. Notice how the break-even point shifts to 8 bikes for a lower price and only 2 bikes for a higher price.

So writing, graphing, and comparing linear functions can be quite useful. As for deciding what price people are willing to pay for your bike, that's a whole different topic!

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\section*{MODULE 8: QUADRATIC FUNCTIONS}

\section*{WHY IT MATTERS: QUADRATIC FUNCTIONS}

\section*{Why study quadratic functions?}

An arrow is shot into the air. How high will it go? How far away will it land? It turns out that you can answer these and related questions with just a little knowledge of quadratic functions. In fact quadratic functions can be used to track to the position of any object that has been thrown, shot, or launched near the surface of the Earth. As long as wind resistance does not play a huge role and the distances are not too great, you can use a quadratic function to model the flight path.

For example, suppose an archer fires an arrow from a height of 2 meters above sea level on a calm day. While the arrow is in the air, someone else tracks and records its
 height precisely at each second. The following table shows the arrow's height (meters) versus time (seconds).
\begin{tabular}{|l|l|}
\hline Time \((t)\) & Height \((h)\) \\
\hline 0 sec. & 2 m \\
\hline 1 sec. & 16.7 m \\
\hline 2 sec. & 21.6 m \\
\hline
\end{tabular}

Let's plot the data on a coordinate plane.


Notice the up-and-down shape? As Isaac Newton would say, what goes up must come down. The smooth arc, first rising and then falling, is a tell-tale clue that there is a quadratic function lurking in the data. The curve that best fits this situation is a parabola, which is what we call the graph of a quadratic function. With a little more work, you can find the equation of this function:
\(h(t)=-4.9 t^{2}+19.6 t+2\)
In the above equation, \(t\) represents time in seconds, and \(h\) represents height in meters. This is an example of a quadratic function. By analyzing the function, you can figure out precisely how high the arrow will go and when it will hit the ground.

In this module you'll learn how to graph and analyze quadratic functions and discover many other useful applications in which they play a role.

\section*{LEARNING OUTCOMES}

\section*{Complex Numbers}
- Express square roots of negative numbers as multiples of \(i\).
- Plot complex numbers on the complex plane.
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

\section*{Graphs of Quadratic Functions}
- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.

Analysis of Quadratic Functions
- Use the quadratic formula and factoring to find both real and complex roots ( \(x\)-intercepts) of quadratic functions.
- Use algebra to find the \(y\)-intercepts of a quadratic function.
- Solve problems involving the roots and intercepts of a quadratic function.
- Use the discriminant to determine the nature (real or complex) and quantity of solutions to quadratic equations.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.
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\section*{REVIEW TOPICS FOR SUCCESS}

\section*{Learning Outcomes}

By the end of this section, you will be able to:
- Graph quadratic functions using tables and transformations
- Identify important features of the graph of a quadratic function of the form \(f(x)=a x^{2}+b x+c\)

Quadratic functions are especially helpful for modeling real-world situations where a minimum or maximum function value would help solve a problem. They are used to answer questions in business, physics, engineering, and more, such as
- What maximum height does a projectile achieve before falling back to the ground and how long after launch will it land?
- What price should be applied to a cup of coffee to maximize revenue?
- What number of units should a factory produce each day to minimize costs?

Before studying problems such as these, though, it is necessary to define the characteristics of the function and learn what they reveal about the model. You'll need an understanding of the vertex and intercepts (real and complex), different algebraic forms of the quadratic function, and some formulas for quickly identifying certain features.

In preparation for studying these functions, take a close look in this review section at the shape of the graph of the function and how it can be used to describe projectile motion in the plane.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

\section*{THE GRAPH OF A QUADRATIC FUNCTION}

\section*{LEARNING OUTCOMES}
- Graph quadratic functions using tables and transformations
- Identify important features of the graph of a quadratic function of the form \(f(x)=a x^{2}+b x+c\)

Quadratic functions form parabolas when graphed in the coordinate plane, so you can be sure fairly quickly when graphing a quadratic if you have the correct shape. As with lines in the plane, creating a table of input and output values then plotting points will reveal the shape. But unlike straight lines between points, the parabola is a smooth curve. Later in this module, you'll learn some other good methods for sketching a quick, accurate graph of a quadratic equation.

It is helpful to have an idea about what the shape of the graph of a quadratic function should be so you can be sure that you have chosen enough points to plot as a guide when sketching the graph. Let's start with the most basic quadratic function, \(f(x)=x^{2}\).
Graph \(f(x)=x^{2}\).
Start with a table of values. Then think of each row of the table as an ordered pair.
\begin{tabular}{|c|c|}
\hline \(\boldsymbol{x}\) & \(\boldsymbol{f}(\boldsymbol{x})\) \\
\hline-2 & 4 \\
\hline-1 & 1 \\
\hline 0 & 0 \\
\hline 1 & 1 \\
\hline 2 & 4 \\
\hline
\end{tabular}

Plot the points \((-2,4),(-1,1),(0,0),(1,1),(2,4)\)


Since the points are not on a line, you cannot use a straight edge. Connect the points as best as you can using a smooth curve (not a series of straight lines). You may want to find and plot additional points (such as the ones in blue here). Placing arrows on the tips of the lines implies that they continue in that direction forever.


Notice that the shape is similar to the letter U . This is called a parabola. One-half of the parabola is a mirror image of the other half. The lowest point on this graph is called the vertex. The vertical line that goes through the vertex is called the line of reflection. In this case, that line is the \(y\)-axis.

The equations for quadratic functions have the form \(f(x)=a x^{2}+b x+c\) where \(a \neq 0\). In the basic graph above, \(a=1, b=0\), and \(c=0\).

In the following video, we show an example of plotting a quadratic function using a table of values.
Changing a changes the width of the parabola and whether it opens up \((a>0)\) or down \((a<0)\). If a is


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positive, the vertex is the lowest point, if a is negative, the vertex is the highest point. In the following example, we show how changing the value of a will affect the graph of the function.

If there is no \(b\) term, changing \(c\) moves the parabola up or down so that the \(y\) intercept is \((0, c)\). In the next example, we show how changes to \(c\) affect the graph of the function.

Changing \(b\) moves the line of reflection, which is the vertical line that passes through the vertex ( the high or low point) of the parabola. It may help to know how to calculate the vertex of a parabola to understand how changing the value of \(b\) in a function will change its graph.

To find the vertex of the parabola, use the formula \(\left(\frac{-b}{2 a}, f\left(\frac{-b}{2 a}\right)\right)\).
For example, if the function being considered is \(f(x)=2 x^{2}-3 x+4\), to find the vertex, first calculate \(\frac{-b}{2 a}\) \(a=2\), and \(b=-3\), therefore \(\frac{-b}{2 a}=\frac{-(-3)}{2(2)}=\frac{3}{4}\).

This is the \(x\) value of the vertex.
Now evaluate the function at \(x=\frac{3}{4}\) to get the corresponding \(y\)-value for the vertex.
\(f\left(\frac{-b}{2 a}\right)=2\left(\frac{3}{4}\right)^{2}-3\left(\frac{3}{4}\right)+4=2\left(\frac{9}{16}\right)-\frac{9}{4}+4=\frac{18}{16}-\frac{9}{4}+4=\frac{9}{8}-\frac{9}{4}+4=\frac{9}{8}-\frac{18}{8}+\frac{32}{8}=\)
The vertex is at the point \(\left(\frac{3}{4}, \frac{23}{8}\right)\). This means that the vertical line of reflection passes through this point as well. It is not easy to tell how changing the values for \(b\) will change the graph of a quadratic function, but if you find the vertex, you can tell how the graph will change.

In the next example, we show how changing \(b\) can change the graph of the quadratic function.
Note that the vertex can change if the value for c changes because the \(y\)-value of the vertex is calculated by substituting the \(x\)-value into the function. Here is a summary of how the changes to the values for \(a, b\), and, c of a quadratic function can change it is graph.

\section*{EXAMPLE}

Match each function with its graph.
a) \(f(x)=3 x^{2}\)
b) \(f(x)=-3 x^{2}\)
c) \(f(x)=\frac{1}{2} x^{2}\)
1)

2)

3)


Answer
Function a) matches graph 2
Function b) matches graph 1
Function c) matches graph 3

Function a) \(f(x)=3 x^{2}\) means that inputs are squared and then multiplied by three, so the outputs will be greater than they would have been for \(f(x)=x^{2}\). This results in a parabola that has been squeezed, so graph 2 is the best match for this function.


Function b) \(f(x)=-3 x^{2}\) means that inputs are squared and then multiplied by negative three, so the outputs will be farther away from the \(x\)-axis than they would have been for \(f(x)=x^{2}\), but negative in value, so graph 1 is the best match for this function.


Function c) \(f(x)=\frac{1}{2} x^{2}\) means that inputs are squared then multiplied by \(\frac{1}{2}\), so the outputs are less than they would be for \(f(x)=x^{2}\). This results in a parabola that has been opened wider than \(f(x)=x^{2}\). Graph 3 is the best match for this function.


For \(f(x)=a x^{2}+b x+c\), where \(a, b\), and \(c\) are real numbers,
- The parabola opens upward if \(a>0\) and downward if \(a<0\).
- a changes the width of the parabola. The parabola gets narrower if \(|a|>1\) and wider if \(|a|<1\).
- The vertex depends on the values of \(\mathrm{a}, \mathrm{b}\), and c . The vertex is \(\left(\frac{-b}{2 a}, f\left(\frac{-b}{2 a}\right)\right)\).

In the last example, we showed how you can use the properties of a parabola to help you make a graph without having to calculate an exhaustive table of values.

\section*{EXAMPLE}

Match each of the following functions with its graph.
a) \(f(x)=x^{2}+3\)
b) \(f(x)=x^{2}-3\)
1)

2)


Answer
Function a) \(f(x)=x^{2}+3\) means square the inputs then add three, so every output will be moved up 3 units. The graph that matches this function best is 2 .


Function b) \(f(x)=x^{2}-3\) means square the inputs then subtract three, so every output will be moved down 3 units. The graph that matches this function best is 1 .


\section*{EXAMPLE}

Match each of the following functions with its graph.
a) \(f(x)=x^{2}+2 x\)
b) \(f(x)=x^{2}-2 x\)
a)

b)


Answer
Find the vertex of function a)
\(f(x)=x^{2}+2 x\).
\(a=1, b=2\)
x -value:
\(\frac{-b}{2 a}=\frac{-2}{2(1)}=-1\)
y -value:
\(f\left(\frac{-b}{2 a}\right)=(-1)^{2}+2(-1)=1-2=-1\).
Vertex \(=(-1,-1)\), which means the graph that best fits this function is a)


Find the vertex of function b)
\(f(x)=x^{2}-2 x\).
\(a=1, b=-2\)
\(x\)-value:
\(\frac{-b}{2 a}=\frac{2}{2(1)}=1\)
y -value:
\(f\left(\frac{-b}{2 a}\right)=(1)^{2}-2(1)=1-2=-1\).
Vertex \(=(1,-1)\), which means the graph that best fits this function is \(b\) )


\section*{Example}

Graph \(f(x)=-2 x^{2}+3 x-3\).

\section*{Answer}

Before making a table of values, look at the values of \(a\) and \(c\) to get a general idea of what the graph should look like.
\(a=-2\), so the graph will open down and be thinner than \(f(x)=x^{2}\).
\(c=-3\), so it will move to intersect the \(y\)-axis at \((0,-3)\).
To find the vertex of the parabola, use the formula \(\left(\frac{-b}{2 a}, f\left(\frac{-b}{2 a}\right)\right)\). Finding the vertex may make graphing the parabola easier.

Vertex formula \(=\left(\frac{-b}{2 a}, f\left(\frac{-b}{2 a}\right)\right)\)
\(x\)-coordinate of vertex:
\[
\frac{-b}{2 a}=\frac{-(3)}{2(-2)}=\frac{-3}{-4}=\frac{3}{4}
\]
\(y\)-coordinate of vertex:
\(f\left(\frac{-b}{2 a}\right)=f\left(\frac{3}{4}\right)\)
\(f\left(\frac{3}{4}\right)=-2\left(\frac{3}{4}\right)^{2}+3\left(\frac{3}{4}\right)-3\)
\(=-2\left(\frac{9}{16}\right)+\frac{9}{4}-3\)
\(=\frac{-18}{16}+\frac{9}{4}-3\)
\(=\frac{-9}{8}+\frac{18}{8}-\frac{24}{8}\)
\(=-\frac{15}{8}\)
Vertex: \(\left(\frac{3}{4},-\frac{15}{8}\right)\)

Use the vertex, \(\left(\frac{3}{4},-\frac{15}{8}\right)\), and the properties you described to get a general idea of the shape of the graph. You can create a table of values to verify your graph. Notice that in this table, the \(x\) values increase. The \(y\) values increase and then start to decrease again. This indicates a parabola.


Connect the points as best you can using a smooth curve. Remember that the parabola is two mirror images, so if your points do not have pairs with the same value, you may want to include additional points (such as the ones in blue shown below). Plot points on either side of the vertex. \(x=\frac{1}{2}\) and \(x=\frac{3}{2}\) are good values to include.


The following video shows another example of plotting a quadratic function using the vertex.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=256

\section*{Summary}

Creating a graph of a function is one way to understand the relationship between the inputs and outputs of that function. Creating a graph can be done by choosing values for \(x\), finding the corresponding \(y\) values, and plotting them. However, it helps to understand the basic shape of the function. Knowing how changes to the basic function equation affect the graph is also helpful.

The shape of a quadratic function is a parabola. Parabolas have the equation \(f(x)=a x^{2}+b x+c\), where \(a, b\), and \(c\) are real numbers and \(a \neq 0\). The value of a determines the width and the direction of the parabola, while the vertex depends on the values of \(a, b\), and \(c\). The vertex is \(\left(\frac{-b}{2 a}, f\left(\frac{-b}{2 a}\right)\right)\).

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\section*{PROJECTILE MOTION}

\section*{LEARNING OUTCOMES}
- Define projectile motion
- Solve a polynomial function that represents projectile motion
- Interpret the solution to a polynomial function that represents projectile motion

Projectile motion happens when you throw a ball into the air and it comes back down because of gravity. A projectile will follow a curved path that behaves in a predictable way. This predictable motion has been studied for centuries, and in simple cases, an object's height from the ground at a given time, \(t\), can be
modeled with a polynomial function of the form \(h(t)=a t^{2}+b t+c\), where \(h(t)\) represents the height of an object at a given time, \(t\). Projectile motion is also called a parabolic trajectory because of the shape of the path of a projectile's motion, as in the water in the image of the fountain below.


Parabolic water trajectory in a fountain.

Parabolic motion and its related functions allow us to launch satellites for telecommunications and rockets for space exploration. Recently, police departments have even begun using projectiles with GPS to track fleeing suspects in vehicles rather than pursuing them by high-speed chase (Note: "Cops' Latest Tool in High-speed Chases: GPS Projectiles." CBSNews. CBS Interactive, n.d. Web. 14 June 2016.).

In this section, we will use polynomial functions to answer questions about the parabolic motion of a projectile. The real mathematical model for the path of a rocket or a police GPS projectile may have different coefficients or more variables, but the concept remains the same. We will also learn to interpret the meaning of the variables in a polynomial function that models projectile motion.

\section*{Example}

A small toy rocket is launched from a 4 -foot pedestal. The height ( \(h\), in feet) of the rocket \(t\) seconds after taking off is given by the function \(h(t)=-2 t^{2}+7 t+4\). How long will it take the rocket to hit the ground?

\section*{Answer}

The rocket will be on the ground when \(h(t)=0\). We want to know how long, \(t\), the rocket is in the air.
\[
h(t)=-2 t^{2}+7 t+4=0
\]
\[
0=-2 t^{2}+7 t+4
\]

We can factor the polynomial \(-2 t^{2}+7 t+4\) more easily by first factoring out a -1
\[
\begin{aligned}
& 0=-1\left(2 t^{2}-7 t-4\right) \\
& 0=-1(2 t+1)(t-4)
\end{aligned}
\]

Use the Zero Product Property. There is no need to set the constant factor -1 to zero, because -1 will never equal zero.
\(2 t+1=0 \quad\) OR \(\quad t-4=0\)
Solve each equation.
\(t=-\frac{1}{2} \quad\) OR \(\quad t=4\)
Interpret the answer. Since \(t\) represents time, it cannot be a negative number; only \(t=4\) makes sense in this context.

We can check our answer when \(t=4\).
\[
\begin{aligned}
& h(4)=-2(4)^{2}+7(4)+4=0 \\
& h(4)=-2(16)+28+4=0 \\
& h(4)-32+32=0 \\
& h(4)=0
\end{aligned}
\]

Therefore, the rocket will hit the ground 4 seconds after being launched.
In the next example, we will solve for the time that the rocket reaches a given height other than zero.
In the following video, we show another example of how to find the time when a object following a parabolic trajectory hits the ground.

\section*{EXAMPLE}

Use the formula for the height of the rocket in the previous example to find the time when the rocket is 4 feet from hitting the ground on its way back down. Refer to the image.
\(h(t)=-2 t^{2}+7 t+4\)


\section*{Answer}

We are given that the height of the rocket is 4 feet from the ground on its way back down. We want to know how long it will take for the rocket to get to that point in its path. We are going to solve for t .
Substitute \(h(t)=4\) into the formula for height, and try to get zero on one side since we know we can use the zero product principle to solve polynomials.

\section*{Write and Solve:}
\(h(t)=4=-2 t^{2}+7 t+4\)
\(4=-2 t^{2}+7 t+4\)
-4 \(\quad \underline{-4}\)
\(0=-2 t^{2}+7 t\)
Now we can factor out a \(t\) from each term:
\(0=t(-2 t+7)\)
Solve each equation for \(t\) using the zero product principle:
\[
\begin{aligned}
& t=0 \mathrm{OR}-2 t+7=0 \\
& \frac{-7}{\frac{-2 t}{-2}}=\frac{-7}{-2} \\
& t=\frac{7}{2}=3.5
\end{aligned}
\]

It does not make sense for us to choose \(t=0\), because we are interested in the amount of time that has passed when the projectile is 4 feet from hitting the ground on its way back down. We will choose \(t=3.5\). Check the answer on your own for practice.
The rocket will be 4 feet from the ground at \(t=3.5\) seconds.


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\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
- Quadratic functions of form \(f(x)=a x^{2}+b x+c\) may be graphed by evaluating the function at various values of the input variable \(x\) to find each coordinating output \(f(x)\). Plot enough points to obtain the shape of the graph, then draw a smooth curve between them.
- The vertex (the turning point) of the graph of a parabola may be obtained using the formula \(\left(-\frac{b}{2 a}, f\left(-\frac{b}{2 a}\right)\right)\)
- The graph of a quadratic function opens up if the leading coefficient \(a\) is positive, and opens down if \(a\) is negative.
- Quadratic functions may be used to model various real-life situations such as projectile motion, and used to determine inputs required to maximize or minimize certain outputs in cost or revenue models.

\section*{Glossary}
projectile motion (also called parabolic trajectory) a projectile launched or thrown into the air will follow a curved path in the shape of a parabola
quadratic function a function of form \(f(x)=a x^{2}+b x+c\) whose graph forms a parabola in the real plane vertex the turning point of the graph of quadratic function

\section*{INTRODUCTION TO COMPLEX NUMBERS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Express square roots of negative numbers as multiples of \(i\).
- Plot complex numbers on the complex plane.
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

The study of mathematics continuously builds upon itself. Negative integers, for example, fill a void left by the set of positive integers. The set of rational numbers, in turn, fills a void left by the set of integers. The set of real numbers fills a void left by the set of rational numbers. The set of real numbers has voids as well. For example, we still have no solution to equations such as
\(x^{2}+4=0\)
Our best guesses might be +2 or -2 . But if we test +2 in this equation, it does not work. If we test -2 , it does not work. If we want to have a solution for this equation, we will have to go farther than we have so far. After all, to this point we have described the square root of a negative number as undefined. Fortunately there is another system of numbers that provides solutions to problems such as these. In this section we will explore this number system and how to work within it.
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\section*{EXPRESS AND PLOT COMPLEX NUMBERS}

\section*{LEARNING OUTCOMES}
- Express square roots of negative numbers as multiples of \(i\).
- Plot complex numbers on the complex plane.

We know how to find the square root of any positive real number. In a similar way we can find the square root of a negative number. The difference is that the root is not real. If the value in the radicand is negative, the root is said to be an imaginary number. The imaginary number \(i\) is defined as the square root of negative 1 .
\(\sqrt{-1}=i\)
So, using properties of radicals,
\[
i^{2}=(\sqrt{-1})^{2}=-1
\]

We can write the square root of any negative number as a multiple of \(i\). Consider the square root of -25 .
\[
\begin{aligned}
\sqrt{-25} & =\sqrt{25 \cdot(-1)} \\
& =\sqrt{25} \cdot \sqrt{-1} \\
& =5 i
\end{aligned}
\]

We use \(5 i\) and not \(-5 i\) because the principal root of 25 is the positive root.
A complex number is the sum of a real number and an imaginary number. A complex number is expressed in standard form when written \(a+b i\) where \(a\) is the real part and \(b i\) is the imaginary part. For example, \(5+2 i\) is a complex number. So, too, is \(3+4 \sqrt{3} i\).


Imaginary numbers are distinguished from real numbers because a squared imaginary number produces a negative real number. Recall, when a positive real number is squared, the result is a positive real number and when a negative real number is squared, again, the result is a positive real number. Complex numbers are a combination of real and imaginary numbers.

\section*{A GENERAL NOTE: IMAGINARY AND COMPLEX NUMBERS}

A complex number is a number of the form \(a+b i\) where
- \(a\) is the real part of the complex number.
- \(b i\) is the imaginary part of the complex number.

If \(b=0\), then \(a+b i\) is a real number. If \(a=0\) and \(b\) is not equal to 0 , the complex number is called an imaginary number. An imaginary number is an even root of a negative number.

\section*{HOW TO: GIVEN AN IMAGINARY NUMBER, EXPRESS IT IN STANDARD FORM.}
1. Write \(\sqrt{-a}\) as \(\sqrt{a} \cdot \sqrt{-1}\).
2. Express \(\sqrt{-1}\) as \(i\).
3. Write \(\sqrt{a} \cdot i\) in simplest form.

\section*{RECALL WRITING SQUARE ROOTS IN SIMPLEST FORM}

Recall that the principal square root of \(a\) is the nonnegative number that, when multiplied by itself, equals \(a\). It is written as a radical expression, with a symbol called a radical over the term called the radicand: \(\sqrt{a}\).


To simplify a square root, we rewrite it such that there are no perfect squares in the radicand. Use the product rule for simplifying square roots, which allows us to separate the square root of a product of two numbers into the product of two separate rational expressions. For instance, we can rewrite \(\sqrt{-75}\) as \(\sqrt{25} \cdot \sqrt{3} \cdot \sqrt{-1}=5 \sqrt{3} i\).

\section*{EXAMPLE: EXPRESSING AN IMAGINARY NUMBER IN STANDARD FORM}

Express \(\sqrt{-9}\) in standard form.
Answer
\(\sqrt{-9}=\sqrt{9} \cdot \sqrt{-1}=3 i\)
In standard form, this is \(0+3 i\).

\section*{TRY IT}

Express \(\sqrt{-24}\) in standard form.
Answer
\(\sqrt{-24}=2 i \sqrt{6}\)
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Plot complex numbers on the complex plane

We cannot plot complex numbers on a number line as we might real numbers. However, we can still represent them graphically. To represent a complex number we need to address the two components of the number. We use the complex plane, which is a coordinate system in which the horizontal axis represents the real component and the vertical axis represents the imaginary component. Complex numbers are the points on the plane, expressed as ordered pairs \((a, b)\), where \(a\) represents the coordinate for the horizontal axis and \(b\) represents the coordinate for the vertical axis.


Let's consider the number \(-2+3 i\). The real part of the complex number is -2 and the imaginary part is \(3 i\). We plot the ordered pair \((-2,3)\) to represent the complex number \(-2+3 i\).

A GENERAL NOTE: COMPLEX PLANE


In the complex plane, the horizontal axis is the real axis, and the vertical axis is the imaginary axis.

\section*{HOW TO: GIVEN A COMPLEX NUMBER, REPRESENT ITS COMPONENTS ON THE COMPLEX PLANE.}
1. Determine the real part and the imaginary part of the complex number.
2. Move along the horizontal axis to show the real part of the number.
3. Move parallel to the vertical axis to show the imaginary part of the number.
4. Plot the point.

\section*{EXAMPLE: PLOTTING A COMPLEX NUMBER ON THE COMPLEX PLANE}

Plot the complex number \(3-4 i\) on the complex plane.
Answer
The real part of the complex number is 3 , and the imaginary part is \(-4 i\). We plot the ordered pair \((3,-4)\).


\section*{TRY IT}

Plot the complex number \(-4-i\) on the complex plane.
Answer


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\section*{ADD, SUBTRACT, AND MULTIPLY COMPLEX NUMBERS}

\section*{LEARNING OUTCOMES}
- Add and subtract complex numbers
- Multiply complex numbers

Just as with real numbers, we can perform arithmetic operations on complex numbers. To add or subtract complex numbers, we combine the real parts and combine the imaginary parts.

\section*{RECALL DOING OPERATIONS ON ALGEBRAIC EXPRESSIONS}

Performing arithmetic on complex numbers is very similar to adding, subtracting, and multiplying algebraic variable expressions. Recall that doing so involves combining like terms, carefully subtracting, and using the distributive property.
Complex numbers of the form \(a+b i\) each contain a real part \(a\) and an imaginary part \(b i\). Real parts are like terms with real parts. Likewise, imaginary parts are like with other imaginary parts.

\section*{A GENERAL NOTE: ADDITION AND SUBTRACTION OF COMPLEX NUMBERS}

Adding complex numbers:
\((a+b i)+(c+d i)=(a+c)+(b+d) i\)
Subtracting complex numbers:
\((a+b i)-(c+d i)=(a-c)+(b-d) i\)

\section*{Multiplying Complex Numbers}

Multiplying complex numbers is much like multiplying binomials. The major difference is that we work with

\section*{HOW TO: GIVEN TWO COMPLEX NUMBERS, FIND THE SUM OR DIFFERENCE.}
1. Identify the real and imaginary parts of each number.
2. Add or subtract the real parts.
3. Add or subtract the imaginary parts.

\section*{EXAMPLE: ADDING COMPLEX NUMBERS}

Add \(3-4 i\) and \(2+5 i\).
Answer
We add the real parts and add the imaginary parts.
\((3-4 i)+(2+5 i)=(3+2)+(-4+5) i=5+i\)

\section*{TRY IT}

Subtract \(2+5 i\) from \(3-4 i\).
Answer
\((3-4 i)-(2+5 i)=1-9 i\)
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the real and imaginary parts separately.

\section*{Multiplying a Complex Number by a Real Number}

Let's begin by multiplying a complex number by a real number. We distribute the real number just as we would with a binomial. So, for example,
\[
\begin{aligned}
3(6+2 i) & =(3 \cdot 6)+(3 \cdot 2 i) & & \text { Distribute. } \\
& =18+6 i & & \text { Simplify } .
\end{aligned}
\]

\section*{HOW TO: GIVEN A COMPLEX NUMBER AND A REAL NUMBER, MULTIPLY TO FIND THE PRODUCT.}
1. Use the distributive property.
2. Simplify.

\section*{EXAMPLE: MULTIPLYING A COMPLEX NUMBER BY A REAL NUMBER}

Find the product \(4(2+5 i)\).
Answer
\(4(2+5 i)=(4 \cdot 2)+(4 \cdot 5 i)=8+20 i\)

\section*{TRY IT}

Find the product \(-4(2+6 i)\).
Answer
\(-8-24 i\)
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\section*{Multiplying Complex Numbers Together}

Now, let's multiply two complex numbers. We can use either the distributive property or the FOIL method. Recall that FOIL is an acronym for multiplying First, Outer, Inner, and Last terms together. Using either the distributive property or the FOIL method, we get
\((a+b i)(c+d i)=a c+a d i+b c i+b d i^{2}\)
Because \(i^{2}=-1\), we have
\((a+b i)(c+d i)=a c+a d i+b c i-b d\)
To simplify, we combine the real parts, and we combine the imaginary parts.
\((a+b i)(c+d i)=(a c-b d)+(a d+b c) i\)

\section*{HOW TO: GIVEN TWO COMPLEX NUMBERS, MULTIPLY TO FIND THE PRODUCT.}
1. Use the distributive property or the FOIL method.
2. Simplify.

\section*{EXAMPLE: MULTIPLYING A COMPLEX NUMBER BY A COMPLEX NUMBER}

Multiply \((4+3 i)(2-5 i)\).

\section*{Answer}
\[
\begin{aligned}
(4+3 i)(2-5 i) & =4 \cdot 2+4 \cdot(-5 i)+3 i \cdot 2+3 i \cdot(-5 i) \\
& =8-20 i+6 i-15 i^{2} \\
& =8+15-20 i+6 i \\
& =23-14 i
\end{aligned}
\]

\section*{TRY IT}

Multiply \((3-4 i)(2+3 i)\).
Answer
\(18+i\)
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\section*{LEARNING OUTCOMES}
- Identify and write the complex conjugate of a complex number.
- Divide complex numbers.
- Simplify powers of \(i\).

\section*{RECALL A TECHNIQUE: RATIONALIZING THE DENOMINATOR}

In mathematics, a technique that works well in one situation often works equally as well in a different, but stylistically similar situation.
Dividing complex numbers uses a technique you used when rewriting a fraction that contains a radical in the denominator. We called it rationalizing a denominator. To accomplish it, you multiplied the fraction in the numerator and denominator by a number that would clear the radical in the denominator. We use the same general idea when dividing complex numbers. This time, instead of multiplying a radical by a specially chosen radical though, we'll multiply a complex number by a specially chosen complex number. When rationalizing the denominator, we took advantage of the fact that \((\sqrt{a})^{2}=a\).
When dividing complex numbers, we'll take advantage of the fact that \(i^{2}=-1\).

\section*{Dividing Complex Numbers}

Division of two complex numbers is more complicated than addition, subtraction, and multiplication because we cannot divide by an imaginary number, meaning that any fraction must have a real-number denominator. We need to find a term by which we can multiply the numerator and the denominator that will eliminate the imaginary portion of the denominator so that we end up with a real number as the denominator. This term is called the complex conjugate of the denominator, which is found by changing the sign of the imaginary part of the complex number. In other words, the complex conjugate of \(a+b i\) is \(a-b i\).

Note that complex conjugates have a reciprocal relationship: The complex conjugate of \(a+b i\) is \(a-b i\), and the complex conjugate of \(a-b i\) is \(a+b i\). Importantly, complex conjugate pairs have a special property. Their product is always real.
\[
\begin{aligned}
(a+b i)(a-b i) & =a^{2}-a b i+a b i-b^{2} i^{2} \\
& =a^{2}-b^{2}(-1) \\
& =a^{2}+b^{2}
\end{aligned}
\]

Suppose we want to divide \(c+d i\) by \(a+b i\), where neither \(a\) nor \(b\) equals zero. We first write the division as a fraction, then find the complex conjugate of the denominator, and multiply.
\(\frac{c+d i}{a+b i}\) where \(a \neq 0\) and \(b \neq 0\).
Multiply the numerator and denominator by the complex conjugate of the denominator.
\(\frac{(c+d i)}{(a+b i)} \cdot \frac{(a-b i)}{(a-b i)}=\frac{(c+d i)(a-b i)}{(a+b i)(a-b i)}\)
Apply the distributive property.
\(=\frac{c a-c b i+a d i-b d i^{2}}{a^{2}-a b i+a b i-b^{2} i^{2}}\)
Simplify, remembering that \(i^{2}=-1\).
\[
\begin{aligned}
& =\frac{c a-c b i+a d i-b d(-1)}{a^{2}-a b i+a b i-b^{2}(-1)} \\
& =\frac{(c a+b d)+(a d-c b) i}{a^{2}+b^{2}}
\end{aligned}
\]

\section*{A GENERAL NOTE: THE COMPLEX CONJUGATE}

The complex conjugate of a complex number \(a+b i\) is \(a-b i\). It is found by changing the sign of the imaginary part of the complex number. The real part of the number is left unchanged.
- When a complex number is multiplied by its complex conjugate, the result is a real number.
- When a complex number is added to its complex conjugate, the result is a real number.

\section*{EXAMPLE: FINDING COMPLEX CONJUGATES}

Find the complex conjugate of each number.
1. \(2+i \sqrt{5}\)
2. \(-\frac{1}{2} i\)

\section*{Answer}
1. The number is already in the form \(a+b i\). The complex conjugate is \(a-b i\), or \(2-i \sqrt{5}\).
2. We can rewrite this number in the form \(a+b i\) as \(0-\frac{1}{2} i\). The complex conjugate is \(a-b i\), or \(0+\frac{1}{2} i\).

This can be written simply as \(\frac{1}{2} i\).

\section*{Analysis of the Solution}

Although we have seen that we can find the complex conjugate of an imaginary number, in practice we generally find the complex conjugates of only complex numbers with both a real and an imaginary component. To obtain a real number from an imaginary number, we can simply multiply by \(i\).

\section*{HOW TO: GIVEN TWO COMPLEX NUMBERS, DIVIDE ONE BY THE OTHER.}
1. Write the division problem as a fraction.
2. Determine the complex conjugate of the denominator.
3. Multiply the numerator and denominator of the fraction by the complex conjugate of the denominator.
4. Simplify.

\section*{Simplifying Powers of \(i\)}

The powers of \(i\) are cyclic. Let's look at what happens when we raise \(i\) to increasing powers.
\(i^{1}=i\)
\(i^{2}=-1\)

\section*{EXAMPLE: DIVIDING COMPLEX NUMBERS}

Divide \((2+5 i)\) by \((4-i)\).
Answer
We begin by writing the problem as a fraction.
\(\frac{(2+5 i)}{(4-i)}\)
Then we multiply the numerator and denominator by the complex conjugate of the denominator.
\(\frac{(2+5 i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)}\)
To multiply two complex numbers, we expand the product as we would with polynomials (the process commonly called FOIL).
\[
\begin{aligned}
\frac{(2+5 i)}{(4-i)} \cdot \frac{(4+i)}{(4+i)} & =\frac{8+2 i+20 i+5 i^{2}}{16+4 i-4 i-i^{2}} \\
& =\frac{8+2 i+20 i+5(-1)}{16+4 i-4 i-(-1)} \quad \text { Because } i^{2}=-1 \\
& =\frac{3+22 i}{17}
\end{aligned}
\]
\[
=\frac{3}{17}+\frac{22}{17} i \quad \text { Separate real and imaginary parts }
\]

Note that this expresses the quotient in standard form.

\section*{TRY IT}

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\section*{TIP FOR SUCCESS}

We have seen that we can evaluate functions for any real number or algebraic expression we choose. For example,
given the function \(f(x)=2 x-7\),
\(f(-1)=2(-1)-7=-9\)
\(f(5+h)=2(5+h)-7=10+2 h-7=3+2 h\).
We may also evaluate a function for a complex number if we wish, as shown in the Example and Try It boxes below.
\(i^{3}=i^{2} \cdot i=-1 \cdot i=-i\)
\(i^{4}=i^{3} \cdot i=-i \cdot i=-i^{2}=-(-1)=1\)
\(i^{5}=i^{4} \cdot i=1 \cdot i=i\)
We can see that when we get to the fifth power of \(i\), it is equal to the first power. As we continue to multiply \(i\) by itself for increasing powers, we will see a cycle of 4 . Let's examine the next 4 powers of \(i\).
\(i^{6}=i^{5} \cdot i=i \cdot i=i^{2}=-1\)

\section*{EXAMPLE: SUBSTITUTING A COMPLEX NUMBER INTO A POLYNOMIAL FUNCTION}

Let \(f(x)=x^{2}-5 x+2\). Evaluate \(f(3+i)\).
Answer
Substitute \(x=3+i\) into the function \(f(x)=x^{2}-5 x+2\) and simplify.
\[
\begin{aligned}
f(3+i) & =(3+i)^{2}-5(3+i)+2 & & \text { Substitute } 3+i \text { for } x \\
& =\left(3+6 i+i^{2}\right)-(15+5 i)+2 & & \text { Multiply } \\
& =9+6 i+(-1)-15-5 i+2 & & \text { Substitute }-1 \text { for } i^{2} \\
& =-5+i & & \text { Combine like terms }
\end{aligned}
\]

Analysis of the Solution
We write \(f(3+i)=-5+i\). Notice that the input is \(3+i\) and the output is \(-5+i\).

\section*{TRY IT}

Let \(f(x)=2 x^{2}-3 x\). Evaluate \(f(8-i)\).
Answer
102-29i
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\(i^{7}=i^{6} \cdot i=i^{2} \cdot i=i^{3}=-i\)
\(i^{8}=i^{7} \cdot i=i^{3} \cdot i=i^{4}=1\)
\(i^{9}=i^{8} \cdot i=i^{4} \cdot i=i^{5}=i\)

\section*{EXAMPLE: SUBSTITUTING AN IMAGINARY NUMBER IN A RATIONAL FUNCTION}

Let \(f(x)=\frac{2+x}{x+3}\). Evaluate \(f(10 i)\).
Answer
Substitute \(x=10 i\) and simplify.
\begin{tabular}{ll}
\(\frac{2+10 i}{10 i+3}\) & Substitute \(10 i\) for \(x\) \\
\(\frac{2+10 i}{3+10 i}\) & Rewrite the denominator in standard form
\end{tabular}
\(\frac{2+10 i}{3+10 i} \cdot \frac{3-10 i}{3-10 i} \quad\) Multiply the numerator and denominator by the complex conjugate of \(t\)
\(\frac{6-20 i+30 i-100 i^{2}}{9-30 i+30 i-100 i^{2}} \quad\) Multiply using the distributive property or the FOIL method
\(\frac{6-20 i+30 i-100(-1)}{9-30 i+30 i-100(-1)} \quad\) Substitute -1 for \(i^{2}\)
\(\frac{106+10 i}{109} \quad\) Simplify
\(\frac{106}{109}+\frac{10}{109} i \quad\) Separate the real and imaginary parts

TRY IT
Let \(f(x)=\frac{x+1}{x-4}\). Evaluate \(f(-i)\).
Answer
\(-\frac{3}{17}+\frac{5}{17} i\)


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\section*{RECALL PROPERTIES OF EXPONENTS}

Recall that \(\left(a^{m}\right)^{n}=a^{m n}\).

\section*{EXAMPLE: SIMPLIFYING POWERS OF \(i\)}

Evaluate \(i^{35}\).

\section*{Answer}

Since \(i^{4}=1\), we can simplify the problem by factoring out as many factors of \(i^{4}\) as possible. To do so, first determine how many times 4 goes into \(35: 35=4 \cdot 8+3\).
\(i^{35}=i^{4 \cdot 8+3}=i^{4.8} \cdot i^{3}=\left(i^{4}\right)^{8} \cdot i^{3}=1^{8} \cdot i^{3}=i^{3}=-i\)

\section*{Q \& A}

Can we write \(i^{35}\) in other helpful ways?
As we saw in Example: Simplifying Powers of \(i\), we reduced \(i^{35}\) to \(i^{3}\) by dividing the exponent by 4 and using the remainder to find the simplified form. But perhaps another factorization of \(i^{35}\) may be more useful. The table below shows some other possible factorizations.
\begin{tabular}{|l|l|l|l|l|}
\hline Factorization of \(i^{35}\) & \(i^{34} \cdot i\) & \(i^{33} \cdot i^{2}\) & \(i^{31} \cdot i^{4}\) & \(i^{19} \cdot i^{16}\) \\
\hline Reduced form & \(\left(i^{2}\right)^{17} \cdot i\) & \(i^{33} \cdot(-1)\) & \(i^{31} \cdot 1\) & \(i^{19} \cdot\left(i^{4}\right)^{4}\) \\
\hline Simplified form & \((-1)^{17} \cdot i\) & \(-i^{33}\) & \(i^{31}\) & \(i^{19}\) \\
\hline
\end{tabular}

Each of these will eventually result in the answer we obtained above but may require several more steps than our earlier method.

\section*{SUMMARY: COMPLEX NUMBERS}

\section*{Key Concepts}
- The square root of any negative number can be written as a multiple of \(i\).
- To plot a complex number, we use two number lines, crossed to form the complex plane. The horizontal axis is the real axis, and the vertical axis is the imaginary axis.
- Complex numbers can be added and subtracted by combining the real parts and combining the imaginary parts.
- Complex numbers can be multiplied and divided.
- To multiply complex numbers, distribute just as with polynomials.
- To divide complex numbers, multiply both the numerator and denominator by the complex conjugate of the denominator to eliminate the complex number from the denominator.
- The powers of \(i\) are cyclic, repeating every fourth one.

\section*{Glossary}
complex conjugate the complex number in which the sign of the imaginary part is changed and the real part of the number is left unchanged; when added to or multiplied by the original complex number, the result is a real number
complex number the sum of a real number and an imaginary number, written in the standard form \(a+b i\), where \(a\) is the real part, and \(b i\) is the imaginary part
complex plane a coordinate system in which the horizontal axis is used to represent the real part of a complex number and the vertical axis is used to represent the imaginary part of a complex number
imaginary number a number in the form \(b i\) where \(i=\sqrt{-1}\)

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\section*{INTRODUCTION TO GRAPHS OF QUADRATIC FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Recognize characteristics of parabolas.
- Understand how the graph of a parabola is related to its quadratic function.


An array of satellite dishes. (credit: Matthew Colvin de Valle, Flickr)
Curved antennas, such as the ones shown in the photo, are commonly used to focus microwaves and radio waves to transmit television and telephone signals, as well as satellite and spacecraft communication. The cross-section of the antenna is in the shape of a parabola, which can be described by a quadratic function.

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\section*{CHARACTERISTICS OF PARABOLAS}

\section*{LEARNING OUTCOMES}
- Identify the vertex, axis of symmetry, \(y\)-intercept, and minimum or maximum value of a parabola from it's graph.
- Identify a quadratic function written in general and vertex form.
- Given a quadratic function in general form, find the vertex.
- Define the domain and range of a quadratic function by identifying the vertex as a maximum or minimum.

The graph of a quadratic function is a U-shaped curve called a parabola. One important feature of the graph is that it has an extreme point, called the vertex. If the parabola opens up, the vertex represents the lowest point on the graph, or the minimum value of the quadratic function. If the parabola opens down, the vertex represents the highest point on the graph, or the maximum value. In either case, the vertex is a turning point on the graph. The graph is also symmetric with a vertical line drawn through the vertex, called the axis of symmetry.


The \(y\)-intercept is the point at which the parabola crosses the \(y\)-axis. The \(x\)-intercepts are the points at which the parabola crosses the \(x\)-axis. If they exist, the \(x\)-intercepts represent the zeros, or roots, of the quadratic function, the values of \(x\) at which \(y=0\).

\section*{TIP FOR SUCCESS}

The places where a function's graph crosses the horizontal axis are the places where the function value equals zero. You've seen that these values are called horizontal intercepts, x-intercepts, and zeros so far. They can also be referred to as the roots of a function.

\section*{Equations of Quadratic Functions}

The general form of a quadratic function presents the function in the form
\(f(x)=a x^{2}+b x+c\)
where \(a, b\), and \(c\) are real numbers and \(a \neq 0\). If \(a>0\), the parabola opens upward. If \(a<0\), the parabola opens downward. We can use the general form of a parabola to find the equation for the axis of symmetry.

The axis of symmetry is defined by \(x=-\frac{b}{2 a}\). If we use the quadratic formula, \(x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}\), to solve \(a x^{2}+b x+c=0\) for the \(x\)-intercepts, or zeros, we find the value of \(x\) halfway between them is always \(x=-\frac{b}{2 a}\), the equation for the axis of symmetry.

The figure below shows the graph of the quadratic function written in general form as \(y=x^{2}+4 x+3\). In this form, \(a=1, b=4\), and \(c=3\). Because \(a>0\), the parabola opens upward. The axis of symmetry is \(x=-\frac{4}{2(1)}=-2\). This also makes sense because we can see from the graph that the vertical line \(x=-2\) divides the graph in half. The vertex always occurs along the axis of symmetry. For a parabola that opens upward, the vertex occurs at the lowest point on the graph, in this instance, \((-2,-1)\). The \(x\)-intercepts, those points where the parabola crosses the \(x\)-axis, occur at \((-3,0)\) and \((-1,0)\).

\section*{EXAMPLE: IDENTIFYING THE CHARACTERISTICS OF A PARABOLA}

Determine the vertex, axis of symmetry, zeros, and \(y\)-intercept of the parabola shown below.


\section*{Answer}

The vertex is the turning point of the graph. We can see that the vertex is at \((3,1)\). The axis of symmetry is the vertical line that intersects the parabola at the vertex. So the axis of symmetry is [latex] \(x=3[/ / a t e x]\). This parabola does not cross the \(x\)-axis, so it has no zeros. It crosses the \(y\)-axis at \((0,7)\) so this is the \(y\) intercept.

\section*{TRY IT}

Use an online graphing calculator to help you solve the question below.
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=265


The standard form of a quadratic function presents the function in the form
\(f(x)=a(x-h)^{2}+k\)
where \((h, k)\) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the vertex form of a quadratic function.

\section*{TRY IT}

Using an online graphing calculator, plot the function \(f(x)=2(x-h)^{2}+k\).
Change the values of \(h\) and \(k\) to examine how changing the location of the vertex ( \(h, k\) ) of a parabola also changes the axis of symmetry. Notice that when you move \(k\) independently of \(h\), you are only moving the vertical location of the vertex. Experiment with values between -10 and 10 .

The vertex of a parabola is the location of either the maximum or minimum value of the parabola. If \(a>0\), the parabola opens upward and the parabola has a minimum value of \(k\) at \(x=h\). If \(a<0\), the parabola opens downward, and the parabola has a maximum value of \(k\) at \(x=h\). In this case, the vertex is the location of the minimum value of the function because \(a=2\).

\section*{Given a quadratic function in general form, find the vertex of the parabola.}

One reason we may want to identify the vertex of the parabola is that this point will inform us where the maximum or minimum value of the output occurs, \(k\), and where it occurs, \(h\). If we are given the general form of a quadratic function:
\(f(x)=a x^{2}+b x+c\)
We can define the vertex, \((h, k)\), by doing the following:
- Identify \(a, b\), and \(c\).
- Find \(h\), the \(x\)-coordinate of the vertex, by substituting \(a\) and \(b\) into \(h=-\frac{b}{2 a}\).
- Find \(k\), the \(y\)-coordinate of the vertex, by evaluating \(k=f(h)=f\left(-\frac{b}{2 a}\right)\)

\section*{EXAMPLE: FINDING THE VERTEX OF A QUADRATIC FUNCTION}

Find the vertex of the quadratic function \(f(x)=2 x^{2}-6 x+7\). Rewrite the quadratic in standard form (vertex form).
Answer
The horizontal coordinate of the vertex will be at
\[
\begin{aligned}
h & =-\frac{b}{2 a} \\
& =-\frac{-6}{2(2)} \\
& =\frac{6}{4} \\
& =\frac{3}{2}
\end{aligned}
\]

The vertical coordinate of the vertex will be at
\(k=f(h)\)
\(=f\left(\frac{3}{2}\right)\)
\(=2\left(\frac{3}{2}\right)^{2}-6\left(\frac{3}{2}\right)+7\)
\(=\frac{5}{2}\)
So the vertex is \(\left(\frac{3}{2}, \frac{5}{2}\right)\)
Rewriting into standard form, the stretch factor will be the same as the \(a\) in the original quadratic.
\(f(x)=2\left(x-\frac{3}{2}\right)^{2}+\frac{5}{2}\)

\section*{TRY IT}

Given the equation \(g(x)=13+x^{2}-6 x\), write the equation in general form and then in standard form.
Answer
\(g(x)=x^{2}-6 x+13\) in general form; \(g(x)=(x-3)^{2}+4\) in standard form

\section*{Finding the Domain and Range of a Quadratic Function}

Any number can be the input value of a quadratic function. Therefore the domain of any quadratic function is all real numbers. Because parabolas have a maximum or a minimum at the vertex, the range is restricted. Since the vertex of a parabola will be either a maximum or a minimum, the range will consist of all \(y\)-values
greater than or equal to the \(y\)-coordinate of the vertex or less than or equal to the \(y\)-coordinate at the turning point, depending on whether the parabola opens up or down.

\section*{A GENERAL NOTE: DOMAIN AND RANGE OF A QUADRATIC FUNCTION}

The domain of any quadratic function is all real numbers.
The range of a quadratic function written in general form \(f(x)=a x^{2}+b x+c\) with a positive \(a\) value is \(f(x) \geq f\left(-\frac{b}{2 a}\right)\), or \(\left[f\left(-\frac{b}{2 a}\right), \infty\right)\); the range of a quadratic function written in general form with a negative \(a\) value is \(f(x) \leq f\left(-\frac{b}{2 a}\right)\), or \(\left(-\infty, f\left(-\frac{b}{2 a}\right)\right]\).
The range of a quadratic function written in standard form \(f(x)=a(x-h)^{2}+k\) with a positive \(a\) value is \(f(x) \geq k\); the range of a quadratic function written in standard form with a negative \(a\) value is \(f(x) \leq k\).

\section*{HOW TO: GIVEN A QUADRATIC FUNCTION, FIND THE DOMAIN AND RANGE.}
1. The domain of any quadratic function as all real numbers.
2. Determine whether \(a\) is positive or negative. If \(a\) is positive, the parabola has a minimum. If \(a\) is negative, the parabola has a maximum.
3. Determine the maximum or minimum value of the parabola, \(k\).
4. If the parabola has a minimum, the range is given by \(f(x) \geq k\), or \([k, \infty)\). If the parabola has a maximum, the range is given by \(f(x) \leq k\), or \((-\infty, k]\).

\section*{EXAMPLE: FINDING THE DOMAIN AND RANGE OF A QUADRATIC FUNCTION}

Find the domain and range of \(f(x)=-5 x^{2}+9 x-1\).
Answer
As with any quadratic function, the domain is all real numbers or \((-\infty, \infty)\).
Because \(a\) is negative, the parabola opens downward and has a maximum value. We need to determine the maximum value. We can begin by finding the \(x\)-value of the vertex.
\(h=-\frac{b}{2 a}=-\frac{9}{2(-5)}=\frac{9}{10}\)
The maximum value is given by \(f(h)\).
\(f\left(\frac{9}{10}\right)=5\left(\frac{9}{10}\right)^{2}+9\left(\frac{9}{10}\right)-1=\frac{61}{20}\)
The range is \(f(x) \leq \frac{61}{20}\), or \(\left(-\infty, \frac{61}{20}\right]\).

\section*{TRY IT}

Find the domain and range of \(f(x)=2\left(x-\frac{4}{7}\right)^{2}+\frac{8}{11}\).
Answer
The domain is all real numbers. The range is \(f(x) \geq \frac{8}{11}\), or \(\left[\frac{8}{11}, \infty\right)\).

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\section*{TRANSFORMATIONS OF QUADRATIC FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Graph vertical and horizontal shifts of quadratic functions
- Graph vertical compressions and stretches of quadratic functions
- Write the equation of a transformed quadratic function using the vertex form
- Identify the vertex and axis of symmetry for a given quadratic function in vertex form

The standard form of a quadratic function presents the function in the form
\(f(x)=a(x-h)^{2}+k\)
where \((h, k)\) is the vertex. Because the vertex appears in the standard form of the quadratic function, this form is also known as the vertex form of a quadratic function.

The standard form is useful for determining how the graph is transformed from the graph of \(y=x^{2}\). The figure below is the graph of this basic function.


\section*{Shift Up and Down by Changing the Value of \(k\)}

You can represent a vertical (up, down) shift of the graph of \(f(x)=x^{2}\) by adding or subtracting a constant, \(k\)
\(f(x)=x^{2}+k\)
If \(k>0\), the graph shifts upward, whereas if \(k<0\), the graph shifts downward.

\section*{EXAMPLE}

Using an online graphing calculator, plot the function \(f(x)=x^{2}+k\). Now change the \(k\) value to shift the graph down 4 units, then up 4 units.
Answer
The equation for the graph of \(f(x)=x^{2}\) that has been shifted up 4 units is
\(f(x)=x^{2}+4\)
The equation for the graph of \(f(x)=x^{2}\) that has been shifted down 4 units is \(f(x)=x^{2}-4\)

\section*{Shift left and right by changing the value of \(h\)}

You can represent a horizontal (left, right) shift of the graph of \(f(x)=x^{2}\) by adding or subtracting a constant, \(h\), to the variable \(x\), before squaring.
\(f(x)=(x-h)^{2}\)
If \(h>0\), the graph shifts toward the right and if \(h<0\), the graph shifts to the left.

Remember that the negative sign inside the argument of the vertex form of a parabola (in the parentheses with the variable \(x\) ) is part of the formula \(f(x)=(x-h)^{2}+k\).
If \(h>0\), we have \(f(x)=(x-h)^{2}+k\). You'll see the negative sign, but the graph will shift right.
If \(h<0\), we have \(f(x)=(x-(-h))^{2}+k \rightarrow f(x)=(x+h)^{2}+k\). You'll see the positive sign, but the graph will shift left.

\section*{EXAMPLE}

Using an online graphing calculator, plot the function \(f(x)=(x-h)^{2}\). Now change the \(h\) value to shift the graph 2 units to the right, then 2 units to the left.

\section*{Answer}

The equation for the graph of \(f(x)=x^{2}\) that has been shifted right 2 units is
\(f(x)=(x-2)^{2}\)
The equation for the graph of \(f(x)={ }^{2}\) that has been shifted left 2 units is
\(f(x)=(x+2)^{2}\)

\section*{Stretch or compress by changing the value of \(a\).}

You can represent a stretch or compression (narrowing, widening) of the graph of \(f(x)=x^{2}\) by multiplying the squared variable by a constant, \(a\).
\(f(x)=a x^{2}\)
The magnitude of \(a\) indicates the stretch of the graph. If \(|a|>1\), the point associated with a particular \(x\) value shifts farther from the \(x\)-axis, so the graph appears to become narrower, and there is a vertical stretch. But if \(|a|<1\), the point associated with a particular \(x\)-value shifts closer to the \(x\)-axis, so the graph appears to become wider, but in fact there is a vertical compression.

\section*{EXAMPLE}

Using an online graphing calculator plot the function \(f(x)=a x^{2}\). Now adjust the \(a\) value to create a graph that has been compressed vertically by a factor of \(\frac{1}{2}\) and another that has been vertically stretched by a factor of 3 . What are the equations of the two graphs?
Answer
The equation for the graph of \(f(x)=x^{2}\) that has been compressed vertically by a factor of \(\frac{1}{2}\) is
\(f(x)=\frac{1}{2} x^{2}\)
The equation for the graph of \(f(x)=x^{2}\) that has been vertically stretched by a factor of 3 is \(f(x)=3 x^{2}\)

The standard form and the general form are equivalent methods of describing the same function. We can see this by expanding out the general form and setting it equal to the standard form.
\[
\begin{aligned}
& a(x-h)^{2}+k=a x^{2}+b x+c \\
& a x^{2}-2 a h x+\left(a h^{2}+k\right)=a x^{2}+b x+c
\end{aligned}
\]

For the two sides to be equal, the corresponding coefficients must be equal. In particular, the coefficients of \(x\) must be equal.
\[
-2 a h=b, \text { so } h=-\frac{b}{2 a} .
\]

This is the \(x\) coordinate of the vertexr and \(x=-\frac{b}{2 a}\) is the axis of symmetry we defined earlier. Setting the constant terms equal gives us:
\[
\begin{aligned}
a h^{2}+k & =c \\
k & =c-a h^{2} \\
& =c-a-\left(\frac{b}{2 a}\right)^{2} \\
& =c-\frac{b^{2}}{4 a}
\end{aligned}
\]

In practice, though, it is usually easier to remember that \(h\) is the output value of the function when the input is \(h\), so \(f(h)=f\left(-\frac{b}{2 a}\right)=k\).

\section*{EXAMPLE}

Using an online graphing calculator plot the function \(f(x)=a(x-h)^{2}+k\). Now adjust the variables \(a, h, k\) to define two quadratic functions whose axis of symmetry is \(x=-3\), and whose vertex is \((-3,2)\).
How many potential values are there for \(h\) in this scenario? How about \(k\) ? How about \(a\)
Answer
There is only one \((h, k)\) pair that will satisfy these conditions, \((-3,2)\). The value of \(a\) does not affect the line of symmetry or the vertex of a quadratic graph, so \(a\) can take on any value.

\section*{Challenge Problem}

Define a function whose axis of symmetry is \(x=-3\), and whose vertex is \((-3,2)\) and has an average rate of change of 2 on the interval \([-2,0]\).

Use the online graphing calculator where you plotted \(f(x)=a(x-h)^{2}+k\) to help with this.

\section*{TRY IT}

A coordinate grid has been superimposed over the quadratic path of a basketball in the picture below. Find an equation for the path of the ball. Does the shooter make the basket?

(credit: modification of work by Dan Meyer)

\section*{Answer}

The path passes through the origin and has vertex at \((-4,7)\), so \((h) x=-\frac{7}{16}(x+4)^{2}+7\). To make the shot, \(h(-7.5)\) would need to be about 4 but \(h(-7.5) \approx 1.64\); he doesn't make it.

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\section*{SUMMARY: GRAPHS OF QUADRATIC FUNCTIONS}

\section*{Key Equations}

> general form of a quadratic function
standard form of a quadratic function
\[
\begin{aligned}
& f(x)=a x^{2}+b x+c \\
& f(x)=a(x-h)^{2}+k
\end{aligned}
\]

\section*{Key Concepts}
- A polynomial function of degree two is called a quadratic function.
- The graph of a quadratic function is a parabola. A parabola is a U-shaped curve that can open either up or down.
- The axis of symmetry is the vertical line passing through the vertex.
- Quadratic functions are often written in general form. Standard or vertex form is useful to easily identify the vertex of a parabola. Either form can be written from a graph.
- The vertex can be found from an equation representing a quadratic function.
- The domain of a quadratic function is all real numbers. The range varies with the function.

\section*{Glossary}
axis of symmetry a vertical line drawn through the vertex of a parabola around which the parabola is symmetric; it is defined by \(x=-\frac{b}{2 a}\).
general form of a quadratic function the function that describes a parabola, written in the form \(f(x)=a x^{2}+b x+c\), where \(a, b\), and \(c\) are real numbers and \(a \neq 0\).
standard form of a quadratic function the function that describes a parabola, written in the form \(f(x)=a(x-h)^{2}+k\), where \((h, k)\) is the vertex.

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\section*{INTRODUCTION TO ANALYSIS OF QUADRATIC FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Use the quadratic formula and factoring to find both real and complex roots ( \(x\)-intercepts) of quadratic functions.
- Use algebra to find the \(y\)-intercepts of a quadratic function.
- Solve problems involving the roots and intercepts of a quadratic function.
- Use the discriminant to determine the nature (real or complex) and quantity of solutions to quadratic equations.
- Determine a quadratic function's minimum or maximum value.
- Solve problems involving a quadratic function's minimum or maximum value.

In this section we will investigate quadratic functions further, including solving problems involving area and projectile motion. Working with quadratic functions can be less complex than working with higher degree polynomial functions, so they provide a good opportunity for a detailed study of function behavior.

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\section*{INTERCEPTS OF QUADRATIC FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Find the \(y\)-intercept of a quadratic function.
- Find the real-number \(x\)-intercepts, or roots of a quadratic function using factoring and the quadratic formula.

Much as we did in the application problems above, we also need to find intercepts of quadratic equations for graphing parabolas. Recall that we find the \(y\)-intercept of a quadratic by evaluating the function at an input of zero, and we find the \(x\)-intercepts at locations where the output is zero. Notice that the number of \(x\) intercepts can vary depending upon the location of the graph.


Number of \(x\)-intercepts of a parabola


One \(x\)-intercept


We've built up several names for the values where a function crosses the horizontal axis:
- horizontal intercepts or x-intercepts
- zeros
- roots

We tend to talk about intercepts with regard to a graph, zeros as solutions to an equation, and roots with regard to a function. Each term refers to the same value, though, for each quadratic function.
Answer
Ex. \(f(x)=x^{2}-x-12\)
- The \(\mathbf{x}\)-intercepts of the graph are located at \((-3,0)\) and \((4,0)\).
- The zeros of the equation \(0=x^{2}-x-12\) are -3 and 4 .
- The function \(f(x)=x^{2}-x-12\) has roots -3 and 4 .

\section*{HOW TO: GIVEN A QUADRATIC FUNCTION \(f(x)\), FIND THE \(Y\) - AND \(X\) INTERCEPTS.}
1. Evaluate \(f(0)\) to find the \(y\)-intercept.
2. Solve the quadratic equation \(f(x)=0\) to find the \(x\)-intercepts.

In the above example the quadratic was easily solved by factoring. However, there are many quadratics that cannot be factored. We can solve these quadratics by first rewriting them in standard form.

\section*{EXAMPLE: FINDING THE \(\gamma\) - AND \(X\)-INTERCEPTS OF A PARABOLA}

Find the \(y\) - and \(x\)-intercepts of the quadratic \(f(x)=3 x^{2}+5 x-2\).

\section*{Answer}

We find the \(y\)-intercept by evaluating \(f(0)\).
\(f(0)=3(0)^{2}+5(0)-2=-2\)
So the \(y\)-intercept is at \((0,-2)\).
For the \(x\)-intercepts, or roots, we find all solutions of \(f(x)=0\).
\(3 x^{2}+5 x-2=0\)
In this case, the quadratic can be factored easily, providing the simplest method for solution.
\((3 x-1)(x+2)=0\)
\(3 x-1=0 \quad x+2=0\)
\(x=\frac{1}{3} \quad\) or \(\quad x=-2\)
So the roots are at \(\left(\frac{1}{3}, 0\right)\) and \((-2,0)\).

\section*{Analysis of the Solution}

By graphing the function, we can confirm that the graph crosses the \(y\)-axis at \((0,-2)\). We can also confirm that the graph crosses the \(x\)-axis at \(\left(\frac{1}{3}, 0\right)\) and \((-2,0)\).


\section*{HOW TO: GIVEN A QUADRATIC FUNCTION, FIND THE \(X\)-INTERCEPTS BY REWRITING IN STANDARD FORM.}
1. Substitute \(a\) and \(b\) into \(h=-\frac{b}{2 a}\).
2. Substitute \(x=h\) into the general form of the quadratic function to find \(k\).
3. Rewrite the quadratic in standard form using \(h\) and \(k\).
4. Solve for when the output of the function will be zero to find the \(x\)-intercepts.

\section*{EXAMPLE: FINDING THE ROOTS OF A PARABOLA}

Find the \(x\)-intercepts of the quadratic function \(f(x)=2 x^{2}+4 x-4\).

\section*{Answer}

We begin by solving for when the output will be zero.
\(0=2 x^{2}+4 x-4\)
Because the quadratic is not easily factorable in this case, we solve for the intercepts by first rewriting the quadratic in standard form.
\(f(x)=a(x-h)^{2}+k\)
We know that \(a=2\). Then we solve for \(h\) and \(k\).
\(h=-\frac{b}{2 a}=-\frac{4}{2(2)}=-1\)
\(k=f(h)\)
\(=f(-1)\)
\(=2(-1)^{2}+4(-1)-4\)
\(=-6\)
So now we can rewrite in standard form.
\(f(x)=2(x+1)^{2}-6\)
We can now solve for when the output will be zero.
\(0=2(x+1)^{2}-6\)
\(6=2(x+1)^{2}\)
\(3=(x+1)^{2}\)
\(x+1= \pm \sqrt{3}\)
\(x=-1 \pm \sqrt{3}\)
The graph has \(x\)-intercepts at \((-1-\sqrt{3}, 0)\) and \((-1+\sqrt{3}, 0)\).

Analysis of the Solution


We can check our work by graphing the given function on a graphing utility and observing the roots.

\section*{TRY IT}

Using an online graphing calculator, plot the function \(g(x)=-7+x^{2}-6 x\). You can use this tool to find the \(x\)-and \(y\)-intercepts by clicking on the graph. Four points will appear. List each point, and what kind of point it is, we got you started with the vertex:
1. Vertex \(=(3,-16)\)
2.
3.
4.

Answer
\(y\)-intercept at \((0,-7)\)
\(x\)-intercept at \((-1,0)\)
\(x\)-intercept at \((7,0)\)
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\section*{TIP FOR SUCCESS}

When attempting to identify the roots of a quadratic function, first look to see if it can be factored when the function value is set equal to zero, and solved by applying the zero-product principle. If so, this is often the quickest method. But most quadratic functions cannot be factored. In this case, you can either write it in vertex form or use the quadratic formula.
Answer
1. The quadratic formula will reveal the \(x\)-intercepts of the graph of any quadratic equation.
\(x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}\)
2. The vertex formula will reveal the vertex of any parabola.
\(\left(\frac{-b}{2 a}, f\left(\frac{-b}{2 a}\right)\right)\)
3. The \(y\)-intercept, often called the initial value, of the function can be found by setting the input variable to 0.

Evaluate the function at zero to find the y-intercept: \(f(0)=y\)
These three pieces of information are of particular value when solving applications of quadratic functions.

\section*{EXAMPLE: SOLVING A QUADRATIC EQUATION WITH THE QUADRATIC FORMULA}

Solve \(x^{2}+x+2=0\).

\section*{Answer}

Let's begin by writing the quadratic formula: \(x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}\).
When applying the quadratic formula, we identify the coefficients \(a, b\), and \(c\). For the equation \(x^{2}+x+2=0\), we have \(a=1, b=1\), and \(c=2\). Substituting these values into the formula we have:
\(x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}\)
\(=\frac{-1 \pm \sqrt{1^{2}-4 \cdot 1 \cdot(2)}}{2 \cdot 1}\)
\(=\frac{-1 \pm \sqrt{1-8}}{2}\)
\(=\frac{-1 \pm \sqrt{-7}}{2}\)
\(=\frac{-1 \pm i \sqrt{7}}{2}\)
The solutions to the equation are \(x=\frac{-1+i \sqrt{7}}{2}\) and \(x=\frac{-1-i \sqrt{7}}{2}\) or \(x=-\frac{1}{2}+\frac{i \sqrt{7}}{2}\) and \(x=-\frac{1}{2}-\frac{i \sqrt{7}}{2}\).

\section*{Analysis of the Solution}

This quadratic equation has only non-real solutions.

\section*{EXAMPLE: APPLYING THE VERTEX AND \(X\)-INTERCEPTS OF A PARABOLA}

A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation \(H(t)=-16 t^{2}+80 t+40\).
a. When does the ball reach the maximum height?
b. What is the maximum height of the ball?
c. When does the ball hit the ground?

\section*{Answer}
a. The ball reaches the maximum height at the vertex of the parabola.
\[
\begin{aligned}
h & =-\frac{80}{2(-16)} \\
& =\frac{80}{32} \\
& =\frac{5}{2} \\
& =2.5
\end{aligned}
\]

The ball reaches a maximum height after 2.5 seconds.
b. To find the maximum height, find the \(y\) coordinate of the vertex of the parabola.
\[
\begin{aligned}
k & =H(2.5) \\
& =-16(2.5)^{2}+80(2.5)+40 \\
& =140
\end{aligned}
\]

The ball reaches a maximum height of 140 feet.
c. To find when the ball hits the ground, we need to determine when the height is zero, \(H(t)=0\).

We use the quadratic formula.
\[
\begin{aligned}
t & =\frac{-80 \pm \sqrt{80^{2}-4(-16)(40)}}{2(-16)} \\
& =\frac{-80 \pm \sqrt{8960}}{-32}
\end{aligned}
\]

Because the square root does not simplify nicely, we can use a calculator to approximate the values of the solutions.
\(t=\frac{-80+\sqrt{8960}}{-32} \approx 5.458\) or \(t=\frac{-80-\sqrt{8960}}{-32} \approx-0.458\)
Since the domain starts at \(t=0\) when the ball is thrown, the second answer is outside the reasonable domain of our model. The ball will hit the ground after about 5.46 seconds.


\section*{TRY IT}

A rock is thrown upward from the top of a 112-foot high cliff overlooking the ocean at a speed of 96 feet per second. The rock's height above ocean can be modeled by the equation \(H(t)=-16 t^{2}+96 t+112\).
a. When does the rock reach the maximum height?
b. What is the maximum height of the rock?
c. When does the rock hit the ocean?

Answer
a. 3 seconds b. 256 feet c. 7 seconds

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\section*{COMPLEX ROOTS}

\section*{LEARNING OUTCOMES}
- Find the complex roots of a quadratic function using the quadratic formula.
- Use the discriminant to determine whether a quadratic function has real or complex roots.

\section*{Complex Roots}

Now you will hopefully begin to understand why we introduced complex numbers at the beginning of this module. Consider the following function: \(f(x)=x^{2}+2 x+3\), and it's graph below:


Does this function have roots? It's probably obvious that this function does not cross the \(x\)-axis, therefore it doesn't have any \(x\)-intercepts. Recall that the \(x\)-intercepts of a function are found by setting the function equal to zero:
\(x^{2}+2 x+3=0\)
In the next example we will solve this equation. You will see that there are roots, but they are not \(x\) intercepts because the function does not contain \((x, y)\) pairs that are on the \(x\)-axis. We call these complex roots.

By setting the function equal to zero and using the quadratic formula to solve, you will see that the roots are complex numbers.

The following video gives another example of how to use the quadratic formula to find complex solutions to a quadratic equation.

\section*{The Discriminant}

The quadratic formula not only generates the solutions to a quadratic equation, it tells us about the nature of the solutions. When we consider the discriminant, or the expression under the radical, \(b^{2}-4 a c\), it tells us whether the solutions are real numbers or complex numbers, and how many solutions of each type to expect. In turn, we can then determine whether a quadratic function has real or complex roots. The table below relates the value of the discriminant to the solutions of a quadratic equation.
\begin{tabular}{|l|l|}
\hline Value of Discriminant & Results \\
\hline\(b^{2}-4 a c=0\) & One repeated rational solution \\
\hline\(b^{2}-4 a c>0\), perfect square & Two rational solutions \\
\hline\(b^{2}-4 a c>0\), not a perfect square & Two irrational solutions \\
\hline\(b^{2}-4 a c<0\) & Two complex solutions \\
\hline
\end{tabular}

We have seen that a quadratic equation may have two real solutions, one real solution, or two complex solutions.

Let's summarize how the discriminant affects the evaluation of \(\sqrt{b^{2}-4 a c}\), and how it helps to determine the solution set.
- If \(b^{2}-4 a c>0\), then the number underneath the radical will be a positive value. You can always find the square root of a positive, so evaluating the Quadratic Formula will result in two real solutions (one by adding the square root, and one by subtracting it).
- If \(b^{2}-4 a c=0\), then you will be taking the square root of 0 , which is 0 . Since adding and subtracting 0 both give the same result, the " \(\pm\) " portion of the formula doesn't matter. There will be one real repeated solution.
- If \(b^{2}-4 a c<0\), then the number underneath the radical will be a negative value. Since you cannot find the square root of a negative number using real numbers, there are no real solutions. However, you can use imaginary numbers. You will then have two complex solutions, one by adding the imaginary square root and one by subtracting it.

\section*{EXAMPLE}

Find the \(x\)-intercepts of the quadratic function. \(f(x)=x^{2}+2 x+3\)
Answer
The \(x\)-intercepts of the function \(f(x)=x^{2}+2 x+3\) are found by setting it equal to zero, and solving for \(x\) since the \(y\) values of the \(x\)-intercepts are zero.
First, identify \(a, b\), and \(c\).
\(x^{2}+2 x+3=0\)
\(a=1, b=2, c=3\)
Substitute these values into the quadratic formula.
\[
\begin{aligned}
x & =\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a} \\
& =\frac{-2 \pm \sqrt{2^{2}-4(1)(3)}}{2(1)} \\
& =\frac{-2 \pm \sqrt{4-12}}{2} \\
& =\frac{-2 \pm \sqrt{-8}}{2} \\
& =\frac{-2 \pm 2 i \sqrt{2}}{2} \\
& =-1 \pm i \sqrt{2} \\
x & =-1+\sqrt{2},-1-\sqrt{2}
\end{aligned}
\]

The solutions to this equation are complex, therefore there are no \(x\)-intercepts for the function \(f(x)=x^{2}+2 x+3\) in the set of real numbers that can be plotted on the Cartesian Coordinate plane. The graph of the function is plotted on the Cartesian Coordinate plane below:


Graph of quadratic function with no \(x\)-intercepts in the real numbers.
Note how the graph does not cross the \(x\)-axis, therefore there are no real \(x\)-intercepts for this function.

\section*{TRY IT}

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\section*{TRY IT}

Use an online graphing calculator to construct a quadratic function that has complex roots.


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\section*{A GENERAL NOTE: THE DISCRIMINANT}

For \(a x^{2}+b x+c=0\), where \(a, b\), and \(c\) are real numbers, the discriminant is the expression under the radical in the quadratic formula: \(b^{2}-4 a c\). It tells us whether the solutions are real numbers or complex numbers and how many solutions of each type to expect.

\section*{EXAMPLE}

Use the discriminant to find the nature of the solutions to the following quadratic equations:
1. \(x^{2}+4 x+4=0\)
2. \(8 x^{2}+14 x+3=0\)
3. \(3 x^{2}-5 x-2=0\)
4. \(3 x^{2}-10 x+15=0\)

\section*{Answer}

Calculate the discriminant \(b^{2}-4 a c\) for each equation and state the expected type of solutions.
1. \(x^{2}+4 x+4=0 \quad b^{2}-4 a c=(4)^{2}-4(1)(4)=0 \quad\) There will be one repeated rational solution.
2. \(8 x^{2}+14 x+3=0 \quad b^{2}-4 a c=(14)^{2}-4(8)(3)=100 \quad\) As 100 is a perfect square, there will be two rational solutions.
3. \(3 x^{2}-5 x-2=0 \quad b^{2}-4 a c=(-5)^{2}-4(3)(-2)=49 \quad\) As 49 is a perfect square, there will be two rational solutions.
4. \(3 x^{2}-10 x+15=0 \quad b^{2}-4 a c=(-10)^{2}-4(3)(15)=-80 \quad\) There will be two complex solutions.

\section*{Example}

Use the discriminant to determine how many and what kind of solutions the quadratic equation \(x^{2}-4 x+10=0\) has.
Answer
Evaluate \(b^{2}-4 a c\). First note that \(a=1, b=-4\), and \(c=10\).
\(c b^{2}-4 a c\)
\((-4)^{2}-4(1)(10)\)
\(16-40=-24\)
The result is a negative number. The discriminant is negative, so \(x^{2}-4 x+10=0\) has two complex solutions.

\section*{TRY IT}

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\section*{APPLICATIONS WITH QUADRATIC FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Find the maximum of a quadratic function that models area, and revenue.

There are many real-world scenarios that involve finding the maximum or minimum value of a quadratic function, such as applications involving area and revenue. In this lesson, we will explore a way to maximize the area of a fenced enclosure, as well as how selling price can affect the number of units sold. In graph (a) below, the parabola has a minimum value at \((2,1)\), and the parabola in graph (b) has a maximum at \((-3,4)\). These values can take on meaning in applications such as the examples we will work through next.


The problem we solved above is called a constrained optimization problem. We can optimize our desired outcome given a constraint, which in this case was a limited amount of fencing materials. Try it yourself in the next problem.

In the example above, we knew the number of subscribers to a newspaper and used that information to find the optimal price for each subscription. What if the price of subscriptions is affected by competition?

\section*{EXAMPLE: FINDING THE MAXIMUM VALUE OF A QUADRATIC FUNCTION}

A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.
1. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length \(L\).
2. What dimensions should she make her garden to maximize the enclosed area?

\section*{Answer}


Let's use a diagram such as the one above to record the given information. It is also helpful to introduce a temporary variable, \(W\), to represent the width of the garden and the length of the fence section parallel to the backyard fence.
1. We know we have only 80 feet of fence available, and \(L+W+L=80\), or more simply,
\(2 L+W=80\). This allows us to represent the width, \(W\), in terms of \(L\).
\(W=80-2 L\) Now we are ready to write an equation for the area the fence encloses. We know the area of a rectangle is length multiplied by width, so \(A=L W=L(80-2 L)=80 L-2 L^{2}\). This formula represents the area of the fence in terms of the variable length \(L\). The function, written in general form, is \(A(L)=-2 L^{2}+80 L\).
2. The quadratic has a negative leading coefficient, so the graph will open downward, and the vertex will be the maximum value for the area. In finding the vertex, we must be careful because the equation is not written in standard polynomial form with decreasing powers. This is why we rewrote the function in general form above. Since \(a\) is the coefficient of the squared term, \(a=-2, b=80\), and \(c=0\).

To find the vertex:
\(h=-\frac{80}{2(-2)}=20\)
and
\(k=A(20)\)
\(=80(20)-2(20)^{2}\)
\(=800\)
The maximum value of the function is an area of 800 square feet, which occurs when \(L=20\) feet. When the shorter sides are 20 feet, there is 40 feet of fencing left for the longer side. To maximize the area, she should enclose the garden so the two shorter sides have length 20 feet and the longer side parallel to the existing fence has length 40 feet.

\section*{Analysis of the Solution}

This problem also could be solved by graphing the quadratic function. We can see where the maximum area occurs on a graph of the quadratic function below.


\section*{TRY IT}

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\section*{HOW TO: GIVEN AN APPLICATION INVOLVING REVENUE, USE A QUADRATIC EQUATION TO FIND THE MAXIMUM.}
1. Write a quadratic equation for revenue.
2. Find the vertex of the quadratic equation.

3 . Determine the \(y\)-value of the vertex.

\section*{EXAMPLE: FINDING MAXIMUM REVENUE}

The unit price of an item affects its supply and demand. That is, if the unit price goes up, the demand for the item will usually decrease. For example, a local newspaper currently has 84,000 subscribers at a quarterly charge of \(\$ 30\). Market research has suggested that if the owners raise the price to \(\$ 32\), they would lose 5,000 subscribers. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?

\section*{Answer}

Revenue is the amount of money a company brings in. In this case, the revenue can be found by multiplying the price per subscription times the number of subscribers, or quantity. We can introduce variables, \(p\) for price per subscription and \(Q\) for quantity, giving us the equation Revenue \(=p Q\).
Because the number of subscribers changes with the price, we need to find a relationship between the variables. We know that currently \(p=30\) and \(Q=84,000\). We also know that if the price rises to \(\$ 32\), the newspaper would lose 5,000 subscribers, giving a second pair of values, \(p=32\) and \(Q=79,000\). From this we can find a linear equation relating the two quantities. The slope will be
\[
\begin{aligned}
m & =\frac{79,000-84,000}{32-30} \\
& =\frac{-5,000}{2} \\
& =-2,500
\end{aligned}
\]

This tells us the paper will lose 2,500 subscribers for each dollar they raise the price. We can then solve for the \(y\)-intercept.
\(Q=-2500 p+b \quad\) Substitute in the point \(Q=84,000\) and \(p=30\)
\(84,000=-2500(30)+b \quad\) Solve for \(b\)
\(b=159,000\)
This gives us the linear equation \(Q=-2,500 p+159,000\) relating cost and subscribers. We now return to our revenue equation.
Revenue \(=p Q\)
Revenue \(=p(-2,500 p+159,000)\)
Revenue \(=-2,500 p^{2}+159,000 p\)
We now have a quadratic function for revenue as a function of the subscription charge. To find the price that will maximize revenue for the newspaper, we can find the vertex.
\[
\begin{aligned}
h & =-\frac{159,000}{2(-2,500)} \\
& =31.8
\end{aligned}
\]

The model tells us that the maximum revenue will occur if the newspaper charges \(\$ 31.80\) for a subscription. To find what the maximum revenue is, we evaluate the revenue function.
maximum revenue \(=-2,500(31.8)^{2}+159,000(31.8)\)
\[
=\$ 2,528,100
\]

\section*{Analysis of the Solution}

This could also be solved by graphing the quadratic. We can see the maximum revenue on a graph of the quadratic function.


\section*{TRY IT}

Previously, we found a quadratic function that modeled revenue as a function of price.
Revenue \(-2,500 p^{2}+159,000 p\)
We found that selling the paper at \(\$ 31.80\) per subscription would maximize revenue. What if your closest competitor sells their paper for \(\$ 25.00\) per subscription? What is the maximum revenue you can make you sell your paper for the same?
Answer
Evaluating the function for \(p=25\) gives \(\$ 2,412,500\).

\section*{TRY IT}

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\section*{SUMMARY: ANALYSIS OF QUADRATIC FUNCTIONS}

\section*{Key Equations}
the quadratic formula \(x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}\)
The discriminant is defined as \(b^{2}-4 a c\)

\section*{Key Concepts}
- The zeros, or \(x\)-intercepts, are the points at which the parabola crosses the \(x\)-axis. The \(y\)-intercept is the point at which the parabola crosses the \(y\)-axis.
- The vertex can be found from an equation representing a quadratic function.
- A quadratic function's minimum or maximum value is given by the \(y\)-value of the vertex.
- The minium or maximum value of a quadratic function can be used to determine the range of the function and to solve many kinds of real-world problems, including problems involving area and revenue.
- Some quadratic equations must be solved by using the quadratic formula.
- The vertex and the intercepts can be identified and interpreted to solve real-world problems.
- Some quadratic functions have complex roots.

\section*{Glossary}
discriminant the value under the radical in the quadratic formula, \(b^{2}-4 a c\), which tells whether the quadratic has real or complex roots
vertex the point at which a parabola changes direction, corresponding to the minimum or maximum value of the quadratic function
vertex form of a quadratic function another name for the standard form of a quadratic function
zeros in a given function, the values of \(x\) at which \(y=0\), also called roots

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\section*{PUTTING IT TOGETHER: QUADRATIC FUNCTIONS}

Now that you have seen how to work with quadratic functions in this module, let's apply our skills to a problem involving projectile motion.

A trebuchet is a medieval weapon used to hurl large stones at enemy forts or castles in order to smash through the thick walls. Basically the trebuchet works by putting the stone in a sling at the end of a long pole

which is allowed to rotate around a fixed axis. The shorter end of the pole is attached to a very heavy weight to counterbalance the projectile. When the weight is released at the short end, the pole rotates, flinging the stone at high speed upward and toward the enemy.

Suppose a particular trebuchet can launch a 10 kilogram stone with an initial upward speed of 24.5 meters per second. Furthermore, suppose that the stone is 8 meters above the ground when it leaves the sling - that is, its initial height is 8 m . We can use what we know about quadratic functions to answer some interesting questions about the stone that is launched from the trebuchet:
1. What is the maximum height achieved by the stone during its flight to the enemy castle?
2. When does that maximum occur?
3. How long would it take before the stone comes crashing down to Earth (in case it missed the castle altogether)?

In order to answer these questions, we need to know about a particular quadratic function from physics called the ballistics equation:
\(h(t)=-\frac{g}{2} t^{2}+v_{0} t+h_{0}\)

Here, \(h(t)\) stands for the height at time \(t, v_{0}\) is the initial upward velocity (speed), \(h_{0}\) is the initial height of the projectile, and \(g\) is a constant called the acceleration due to gravity. Near the surface of the Earth, \(g \approx 9.8\) meters per second squared, so \(-\frac{g}{2} \approx-4.9\). Now putting the given values into their proper places, we have:
\(h(t)=-4.9 t^{2}+24.5 t+8\)
This is a quadratic function with \(a=-4.9, b=24.5, c=8\). The graph is a parabola, and it will have a maximum because \(a<0\). The maximum function value in this case represents the maximum height of the stone.

Now we can answer the first two questions, What is the maximum height achieved by the stone during its flight to the enemy castle, and at what time does this occur? To find the maximum height and time at which it occurs, we use the vertex formula which will give \(\left(t_{\max }, h\left(t_{\max }\right)\right.\) ). the time at maximum point and height at the time of the maximum point.
\(t_{\max }=-\frac{b}{2 a}=-\frac{24.5}{2(-4.9)}=2.5\)
\(h\left(t_{\max }\right)=h\left(-\frac{b}{2 a}\right)=h(2.5)=-4.9(2.5)^{2}+24.5(2.5)+8 \approx 38.6\)
Therefore, the maximum height will be 38.6 meters, which occurs 2.5 seconds after launch. That's a highflying projectile!

Finally, to determine when the stone hits the ground, we just have to find the \(x\)-intercept of the function. In other words, we have to solve \(h(t)=0\). That's a job for the quadratic formula.
\(t_{\text {final }}=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}=\frac{-24.5 \pm \sqrt{24.5^{2}-4(4.9)(8)}}{2 a} \approx-0.3,5.3\)
As expected, the quadratic formula gives us two answers, but only the positive one makes sense in this context. The stone lands on the ground about 5.3 seconds after it was launched.

Quadratic functions can be used to model the behavior of objects in free fall, amongst other things. We can use algebra to analyze this behavior for interesting features.
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\section*{MODULE 9: POWER AND \\ POLYNOMIAL FUNCTIONS}

\section*{WHY IT MATTERS: POWER AND POLYNOMIAL FUNCTIONS}

You're suddenly rethinking your decision as you find yourself 456 feet above the ground on the world's tallest roller coaster-Kingda Ka at Six Flags Great Adventure. Perhaps you don't have time to think about it though since before you can catch your breath, you're already plummeting down in a 270-degree spiral. Within your 50 -second ride, you travel over 3,000 feet and reach a top speed of 128 mph . This roller coaster is not for the faint of heart!

So how does a thrilling roller coaster experience come to be? It all begins with advanced planning. Roller coaster designers use the knowledge of math, specifically polynomials, to create an experience that meets specific requirements.


Imagine you're a roller coaster designer entrusted with the task of designing the next big attraction for a nearby theme park. Some criteria for your coaster includes a starting height of 200 ft , dive below ground level at 3 seconds into the ride, return to ground level at 5 seconds, and another dive beneath the ground 10 seconds later.

Where do you begin? Read on because you'll find the answer to this question and many more in this module. At the end of the module, we'll return to your design problem so you can come up with a plan.

\section*{Review Topics for Success}
- Determine if a given function is a polynomial function.
- Determine the degree and leading coefficient of a polynomial function.
- Identify graphs of polynomial functions.
- Identify general characteristics of a polynomial function from its graph.

\section*{Characteristics of Power and Polynomial Functions}
- Identify power functions.
- Describe end behavior of power functions given its equation or graph.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.
- Describe the end behavior of a polynomial function.
- Identify turning points of a polynomial function from its graph.
- Identify the number of turning points and intercepts of a polynomial function from its degree.
- Determine x and y -intercepts of a polynomial function given its equation in factored form.

\section*{Graphs of Polynomial Functions}
- Identify zeros of polynomial functions with even and odd multiplicity.
- Draw the graph of a polynomial function using end behavior, turning points, intercepts, and the Intermediate Value Theorem.
- Write the equation of a polynomial function given its graph.

\section*{Dividing Polynomials}
- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.

\section*{Methods for Finding Zeros of Polynomials}
- Evaluate a polynomial using the Remainder Theorem.
- Use the Rational Zero Theorem to find rational zeros.
- Use the Factor Theorem to solve a polynomial equation.
- Use synthetic division to find the zeros of a polynomial function.
- Use the Fundamental Theorem of Algebra to find complex zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs to determine the maximum number of possible real zeros of a polynomial function.
- Solve real-world applications of polynomial equations.

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\section*{REVIEW TOPICS FOR SUCCESS}

\section*{Learning Outcomes}

By the end of this section, you will be able to:
- Determine if a given function is a polynomial function
- Determine the degree and leading coefficient of a polynomial function
- Identify graphs of polynomial functions
- Identify general characteristics of a polynomial function from its graph

Polynomial functions are both powerful and versatile. In this module, you'll learn about the basic characteristics of polynomial functions and the behavior of their graphs in the plane as well as techniques for working with them algebraically. You've seen already that polynomial functions such as quadratics and linear functions can be combined arithmetically. In this module, you'll learn to divide them, too. Before beginning the study of these interesting functions, though, it will be helpful to refresh your knowledge of their basic characteristics and get some practice identifying them.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

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\section*{BASIC CHARACTERISTICS OF POLYNOMIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Determine if a given function is a polynomial function
- Determine the degree and leading coefficient of a polynomial function

\section*{Recognize Polynomial Functions}

We have introduced polynomials and functions, so now we will combine these ideas to describe polynomial functions. Polynomials are algebraic expressions that are created by adding or subtracting monomial terms, such as \(-3 x^{2}\), where the exponents are only non-negative integers. Functions are a specific type of relation in which each input value has one and only one output value. Polynomial functions have all of these characteristics as well as a domain and range, and corresponding graphs. In this section, we will identify and evaluate polynomial functions. Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the powers of the variables.

When we introduced polynomials, we presented the following: \(4 x^{3}-9 x^{2}+6 x\). We can turn this into a polynomial function by using function notation:
\(f(x)=4 x^{3}-9 x^{2}+6 x\)
Polynomial functions are written with the leading term first and all other terms in descending order as a matter of convention. In the first example, we will identify some basic characteristics of polynomial functions.

\section*{EXAMPLE}

Which of the following are polynomial functions?
\[
\begin{aligned}
& f(x)=5 x^{7}+4 \\
& g(x)=-x^{2}\left(x-\frac{2}{5}\right) \\
& h(x)=\frac{1}{2} x^{2}+\sqrt{x}+2
\end{aligned}
\]

\section*{Answer}

The first two functions are examples of polynomial functions because they contain powers that are nonnegative integers and the coefficients are real numbers. Note that the second function can be written as \(g(x)=-x^{3}+\frac{2}{5} x\) after applying the distributive property.
The third function is not a polynomial function because the variable is under a square root in the middle term, therefore the function contains an exponent that is not a non-negative integer.

In the following video, you will see additional examples of how to identify a polynomial function using the definition.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=277

\section*{Determine the Degree and Leading Coefficient of a Polynomial Function}

Just as we identified the degree of a polynomial, we can identify the degree of a polynomial function. To review: the degree of the polynomial is the highest power of the variable that occurs in the polynomial; the leading term is the term containing the highest power of the variable or the term with the highest degree. The leading coefficient is the coefficient of the leading term.

Watch the next video for more examples of how to identify the degree, leading term and leading coefficient of a polynomial function.

\section*{EXAMPLE}

Identify the degree, leading term, and leading coefficient of the following polynomial functions.
\[
\begin{aligned}
& f(x)=5 x^{2}+7-4 x^{3} \\
& g(x)=9 x-x^{6}-3 x^{4} \\
& h(x)=6\left(x^{2}-x\right)+11
\end{aligned}
\]

\section*{Answer}

For the function \(f(x)\), the highest power of \(x\) is 3 , so the degree is 3 . The leading term is the term containing that degree, \(-4 x^{3}\). The leading coefficient is the coefficient of that term, -4 .
For the function \(g(x)\), the highest power of \(x\) is 6 , so the degree is 6 . The leading term is the term containing that degree, \(-x^{6}\). The leading coefficient is the coefficient of that term, -1 .
For the function \(h(x)\), first rewrite the polynomial using the distributive property to identify the terms. \(h(x)=6 x^{2}-6 x+11\). The highest power of \(x\) is 2 , so the degree is 2 . The leading term is the term containing that degree, \(6 x^{2}\). The leading coefficient is the coefficient of that term, 6 .


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\section*{Summary}

Polynomial functions contain powers that are non-negative integers and the coefficients are real numbers. It is often helpful to know how to identify the degree and leading coefficient of a polynomial function. To do this, follow these suggestions:
1. Find the highest power of \(x\) to determine the degree of the function.
2. Identify the term containing the highest power of \(x\) to find the leading term.
3. Identify the coefficient of the leading term.
```

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\section*{GRAPHS OF POLYNOMIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Identify graphs of polynomial functions
- Identify general characteristics of a polynomial function from its graph

Plotting polynomial functions using tables of values can be misleading because of some of the inherent characteristics of polynomials. Additionally, the algebra of finding points like x-intercepts for higher degree polynomials can get very messy and oftentimes be impossible to find by hand. We have therefore developed some techniques for describing the general behavior of polynomial graphs.

Polynomial functions of degree 2 or more have graphs that do not have sharp corners. These types of graphs are called smooth curves. Polynomial functions also display graphs that have no breaks. Curves with no breaks are called continuous. The figure below shows a graph that represents a polynomial function and a graph that represents a function that is not a polynomial.



\section*{Identifying the Shape of the Graph of a Polynomial Function}

Knowing the degree of a polynomial function is useful in helping us predict what its graph will look like. Because the power of the leading term is the highest, that term will grow significantly faster than the other terms as \(x\) gets very large or very small, so its behavior will dominate the graph. For any polynomial, the graph of the polynomial will match the end behavior of the term of highest degree.

As an example, we compare the outputs of a degree 2 polynomial and a degree 5 polynomial in the following table.

\section*{EXAMPLE}

Which of the graphs below represents a polynomial function?





Answer
The graphs of \(f\) and \(h\) are graphs of polynomial functions. They are smooth and continuous.
The graphs of \(g\) and \(k\) are graphs of functions that are not polynomials. The graph of function \(g\) has a sharp corner. The graph of function \(k\) is not continuous.

\section*{Q \& A}

Do all polynomial functions have "all real numbers" as their domain?
Yes. Any real number is a valid input for a polynomial function.
\begin{tabular}{l|l|l|}
\hline \(\mathbf{x}\) & \(f(x)=2 x^{2}-2 x+4\) & \(g(x)=x^{5}+2 x^{3}-12 x+3\) \\
\hline 1 & 4 & 8 \\
\hline 10 & 184 & 98117 \\
\hline 100 & 19804 & 9998001197 \\
\hline 1000 & 1998004 & 9999980000000000 \\
\hline
\end{tabular}

As the inputs for both functions get larger, the degree 5 polynomial outputs get much larger than the degree 2 polynomial outputs. This is why we use the leading term to get a rough idea of the behavior of polynomial graphs.

There are two other important features of polynomials that influence the shape of its graph. The first is whether the degree is even or odd, and the second is whether the leading term is negative.

\section*{Even Degree Polynomials}

In the figure below, we show the graphs of \(f(x)=x^{2}, g(x)=x^{4}\), and \(h(x)=x^{6}\) which all have even degrees. Notice that these graphs have similar shapes, very much like that of a quadratic function. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.


\section*{Odd Degree Polynomials}

The next figure shows the graphs of \(f(x)=x^{3}, g(x)=x^{5}\), and \(h(x)=x^{7}\) which all have odd degrees.


Notice that one arm of the graph points down and the other points up. This is because when your input is negative, you will get a negative output if the degree is odd. The following table of values shows this.
\begin{tabular}{|l|l|l|}
\hline x & \(f(x)=x^{4}\) & \(h(x)=x^{5}\) \\
\hline-1 & 1 & -1 \\
\hline-2 & 16 & -32 \\
\hline-3 & 81 & -243 \\
\hline
\end{tabular}

Now you try it.

\section*{EXAMPLE}

Identify whether each graph represents a polynomial function that has a degree that is even or odd.
a)

b)


Answer
a) Both arms of this polynomial point upward, similar to a quadratic polynomial, therefore the degree must be even. If you apply negative inputs to an even degree polynomial, you will get positive outputs back.
b) As the inputs of this polynomial become more negative the outputs also become negative. The only way this is possible is with an odd degree polynomial. Therefore, this polynomial must have an odd degree.

\section*{The Sign of the Leading Term}

What would happen if we change the sign of the leading term of an even degree polynomial? For example, let us say that the leading term of a polynomial is \(-3 x^{4}\). We will use a table of values to compare the outputs for a polynomial with leading term \(-3 x^{4}\) and \(3 x^{4}\).
\begin{tabular}{|l|l|l|}
\hline x & \(-3 x^{4}\) & \(3 x^{4}\) \\
\hline-2 & -48 & 48 \\
\hline-1 & -3 & 3 \\
\hline 0 & 0 & 0 \\
\hline 1 & -3 & 3 \\
\hline 2 & -48 & 48 \\
\hline
\end{tabular}

The grid below shows a plot with these points. The red points indicate a negative leading coefficient, and the blue points indicate a positive leading coefficient:


The negative sign creates a reflection of \(3 x^{4}\) across the x -axis. The arms of a polynomial with a leading term of \(-3 x^{4}\) will point down, whereas the arms of a polynomial with leading term \(3 x^{4}\) will point up.

Now you try it.

\section*{EXAMPLE}

Identify whether the leading term is positive or negative and whether the degree is even or odd for the following graphs of polynomial functions.
a)

b)


\section*{Answer}
a) Both arms of this polynomial point in the same direction so it must have an even degree. The leading term of the polynomial must be negative since the arms are pointing downward.
b) The arms of this polynomial point in different directions, so the degree must be odd. As the inputs get really big and positive, the outputs get really big and negative, so the leading coefficient must be negative.

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\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
- A polynomial function is one whose equation contains only non-negative integer powers on the variable.
- The polynomial term containing the highest power on the variable is the leading term, and its degree is the number of the power. The leading coefficient of a polynomial is the coefficient of the leading term.
- The graph of a polynomial function describes a smooth, continuous curve.
- The domain of all polynomial functions is all real numbers.
- Even degree polynomial functions describe graphs whose ends both point up or both point down.
- Odd degree polynomial functions describe graphs whose ends points in opposite directions.
- The sign of the leading term will determine the direction of the ends of the graph:
- even degree and positive coefficient: both ends point up
- even degree and negative coefficient: both ends point down
- odd degree and positive coefficient: the left-most end points down and the right-most end points up.
- odd degree and negative coefficient: the left-most end points up and the right-most end points down.

\section*{Glossary}
degree the highest power of the polynomial
leading coefficient the coefficient of the leading term
leading term the term containing the highest power

\section*{INTRODUCTION TO CHARACTERISTICS OF POWER AND POLYNOMIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Identify power functions.
- Describe end behavior of power functions given its equation or graph.
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.
- Describe the end behavior of a polynomial function.
- Identify turning points of a polynomial function from its graph.
- Identify the number of turning points and intercepts of a polynomial function from its degree.
- Determine x and y -intercepts of a polynomial function given its equation in factored form.

In this section we will examine functions that are used to estimate and predict things like changes in animal and bird populations or fluctuations in financial markets.

We will also continue to learn how to analyze the behavior of functions by looking at their graphs. We will introduce and describe a new term called end behavior and show which parts of the function equation determine end behavior. We will also identify intercepts of polynomial functions.

\section*{END BEHAVIOR OF POWER FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Identify a power function.
- Describe the end behavior of a power function given its equation or graph.


Three birds on a cliff with the sun rising in the background. Functions discussed in this module can be used to model populations of various animals, including birds. (credit: Jason Bay, Flickr)

Suppose a certain species of bird thrives on a small island. Its population over the last few years is shown below.
\begin{tabular}{|l|l|l|l|l|l|}
\hline Year & 2009 & 2010 & 2011 & 2012 & 2013 \\
\hline Bird Population & 800 & 897 & 992 & 1,083 & 1,169 \\
\hline
\end{tabular}

The population can be estimated using the function \(P(t)=-0.3 t^{3}+97 t+800\), where \(P(t)\) represents the bird population on the island \(t\) years after 2009. We can use this model to estimate the maximum bird
population and when it will occur. We can also use this model to predict when the bird population will disappear from the island.

\section*{Identifying Power Functions}

In order to better understand the bird problem, we need to understand a specific type of function. A power function is a function with a single term that is the product of a real number, coefficient, and variable raised to a fixed real number power. Keep in mind a number that multiplies a variable raised to an exponent is known as a coefficient.

As an example, consider functions for area or volume. The function for the area of a circle with radius \(r\) is:
\(A(r)=\pi r^{2}\)
and the function for the volume of a sphere with radius \(r\) is:
\(V(r)=\frac{4}{3} \pi r^{3}\)
Both of these are examples of power functions because they consist of a coefficient, \(\pi\) or \(\frac{4}{3} \pi\), multiplied by a variable \(r\) raised to a power.

\section*{A GENERAL NOTE: POWER FUNCTIONS}

A power function is a function that can be represented in the form
\(f(x)=a x^{n}\)
where \(a\) and \(n\) are real numbers and \(a\) is known as the coefficient.

\section*{Q \& A}

Is \(f(x)=2^{x}\) a power function?
No. A power function contains a variable base raised to a fixed power. This function has a constant base raised to a variable power. This is called an exponential function, not a power function.

\section*{TIP FOR SUCCESS}

Unlike a polynomial function, in which all the variable powers must be non-negative integers, a power function only requires the power on the exponent be a real number.

\section*{Identifying End Behavior of Power Functions}

The graph below shows the graphs of \(f(x)=x^{2}, g(x)=x^{4}, h(x)=x^{6}, k(x)=x^{8}\), and \(p(x)=x^{10}\) which are all power functions with even, whole-number powers. Notice that these graphs have similar shapes, very much like that of the quadratic function. However, as the power increases, the graphs flatten somewhat near the origin and become steeper away from the origin.

\section*{EXAMPLE: IDENTIFYING POWER FUNCTIONS}

Which of the following functions are power functions?
\(f(x)=1 \quad\) Constant function
\(f(x)=x \quad\) Identity function
\(f(x)=x^{2} \quad\) Quadratic function
\(f(x)=x^{3} \quad\) Cubic function
\(f(x)=\frac{1}{x} \quad\) Reciprocal function
\(f(x)=\frac{1}{x^{2}} \quad\) Reciprocal squared function
\(f(x)=\sqrt{x}\) Square root function
\(f(x)=\sqrt[3]{x}\) Cube root function

\section*{Answer}

All of the listed functions are power functions.
The constant and identity functions are power functions because they can be written as \(f(x)=x^{0}\) and \(f(x)=x^{1}\) respectively.
The quadratic and cubic functions are power functions with whole number powers \(f(x)=x^{2}\) and \(f(x)=x^{3}\).
The reciprocal and reciprocal squared functions are power functions with negative whole number powers because they can be written as \(f(x)=x^{-1}\) and \(f(x)=x^{-2}\).
The square and cube root functions are power functions with fractional powers because they can be written as \(f(x)=x^{1 / 2}\) or \(f(x)=x^{1 / 3}\).

\section*{TRY IT}

Which functions are power functions?
\[
\begin{aligned}
& f(x)=2 x^{2} \cdot 4 x^{3} \\
& g(x)=-x^{5}+5 x^{3}-4 x \\
& h(x)=\frac{2 x^{5}-1}{3 x^{2}+4}
\end{aligned}
\]

\section*{Answer}
\(f(x)\) is a power function because it can be written as \(f(x)=8 x^{5}\). The other functions are not power functions.
\[
\begin{aligned}
& f(x)=x^{2} \\
& g(x)=x^{4} \\
& h(x)=x^{6} \\
& k(x)=x^{8} \\
& p(x)=x^{10}
\end{aligned}
\]


To describe the behavior as numbers become larger and larger, we use the idea of infinity. We use the symbol \(\infty\) for positive infinity and \(-\infty\) for negative infinity. When we say that " \(x\) approaches infinity," which can be symbolically written as \(x \rightarrow \infty\), we are describing a behavior; we are saying that \(x\) is increasing without bound.

With even-powered power functions, as the input increases or decreases without bound, the output values become very large, positive numbers. Equivalently, we could describe this behavior by saying that as \(x\) approaches positive or negative infinity, the \(f(x)\) values increase without bound. In symbolic form, we could write
as \(x \rightarrow \pm \infty, f(x) \rightarrow \infty\)
The graph below shows \(f(x)=x^{3}, g(x)=x^{5}, h(x)=x^{7}, k(x)=x^{9}\), and \(p(x)=x^{11}\), which are all power functions with odd, whole-number powers. Notice that these graphs look similar to the cubic function. As the power increases, the graphs flatten near the origin and become steeper away from the origin.
\[
\begin{aligned}
& f(x)=x^{3} \\
& g(x)=x^{5} \\
& h(x)=x^{7} \\
& k(x)=x^{9} \\
& p(x)=x^{11}
\end{aligned}
\]


These examples illustrate that functions of the form \(f(x)=x^{n}\) reveal symmetry of one kind or another. First, in the even-powered power functions, we see that even functions of the form \(f(x)=x^{n}, n\) even, are symmetric about the \(y\)-axis. In the odd-powered power functions, we see that odd functions of the form \(f(x)=x^{n}, n\) odd, are symmetric about the origin.

For these odd power functions, as \(x\) approaches negative infinity, \(f(x)\) decreases without bound. As \(x\) approaches positive infinity, \(f(x)\) increases without bound. In symbolic form we write
\[
\begin{aligned}
& \text { as } x \rightarrow-\infty, f(x) \rightarrow-\infty \\
& \text { as } x \rightarrow \infty, f(x) \rightarrow \infty
\end{aligned}
\]

The behavior of the graph of a function as the input values get very small ( \(x \rightarrow-\infty\) ) and get very large ( \(x \rightarrow \infty)\) is referred to as the end behavior of the function. We can use words or symbols to describe end behavior.

The table below shows the end behavior of power functions of the form \(f(x)=a x^{n}\) where \(n\) is a nonnegative integer depending on the power and the constant.


HOW TO: GIVEN A POWER FUNCTION \(f(x)=a x^{n}\) WHERE \(n\) IS A NONNEGATIVE INTEGER, IDENTIFY THE END BEHAVIOR.
1. Determine whether the power is even or odd.
2. Determine whether the constant is positive or negative.
3. Use the above graphs to identify the end behavior.

\section*{EXAMPLE: IDENTIFYING THE END BEHAVIOR OF A POWER FUNCTION}

Describe the end behavior of the graph of \(f(x)=x^{8}\).
Answer
The coefficient is 1 (positive) and the exponent of the power function is 8 (an even number). As \(x\) (input) approaches infinity, \(f(x)\) (output) increases without bound. We write as \(x \rightarrow \infty, f(x) \rightarrow \infty\). As \(x\) approaches negative infinity, the output increases without bound. In symbolic form, as \(x \rightarrow-\infty, f(x) \rightarrow \infty\). We can graphically represent the function.


\section*{EXAMPLE: IDENTIFYING THE END BEHAVIOR OF A POWER FUNCTION}

Describe the end behavior of the graph of \(f(x)=-x^{9}\).

\section*{Answer}

The exponent of the power function is 9 (an odd number). Because the coefficient is -1 (negative), the graph is the reflection about the \(x\)-axis of the graph of \(f(x)=x^{9}\). The graph shows that as \(x\) approaches infinity, the output decreases without bound. As \(x\) approaches negative infinity, the output increases without bound. In symbolic form, we would write as \(x \rightarrow-\infty, f(x) \rightarrow \infty\) and as \(x \rightarrow \infty, f(x) \rightarrow-\infty\).


\section*{Analysis of the Solution}

We can check our work by using the table feature on an online graphing calculator.
1. Enter the function \(f(x)=-x^{9}\) into an online graphing calculator
2. Create a table with the following x values, and observe the sign of the outputs. \(-10,-5,0,5,10\)
3. Now, enter the function \(g(x)=x^{9}\), and create a similar table. Compare the signs of the outputs for both functions.

\section*{TRY IT}

Describe in words and symbols the end behavior of \(f(x)=-5 x^{4}\).
Answer
As \(x\) approaches positive or negative infinity, \(f(x)\) decreases without bound: as \(x \rightarrow \pm \infty, f(x) \rightarrow-\infty\) because of the negative coefficient.

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\section*{END BEHAVIOR OF POLYNOMIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Identify polynomial functions.
- Identify the degree and leading coefficient of polynomial functions.
- Describe the end behavior of a polynomial function.

\section*{Identifying Polynomial Functions}

An oil pipeline bursts in the Gulf of Mexico causing an oil slick in a roughly circular shape. The slick is currently 24 miles in radius, but that radius is increasing by 8 miles each week. We want to write a formula for the area covered by the oil slick by combining two functions. The radius \(r\) of the spill depends on the number of weeks \(w\) that have passed. This relationship is linear.
\(r(w)=24+8 w\)
We can combine this with the formula for the area \(A\) of a circle.
\[
A(r)=\pi r^{2}
\]

Composing these functions gives a formula for the area in terms of weeks.
\[
\begin{aligned}
& A(w)=A(r(w)) \\
& A(w)=A(24+8 w) \\
& A(w)=\pi(24+8 w)^{2}
\end{aligned}
\]

Multiplying gives the formula below.
\(A(w)=576 \pi+384 \pi w+64 \pi w^{2}\)

This formula is an example of a polynomial function. A polynomial function consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a non-negative integer power.

\section*{A GENERAL NOTE: POLYNOMIAL FUNCTIONS}

Let \(n\) be a non-negative integer. A polynomial function is a function that can be written in the form \(f(x)=a_{n} x^{n}+\cdots+a_{2} x^{2}+a_{1} x+a_{0}\)
This is called the general form of a polynomial function. Each \(a_{i}\) is a coefficient and can be any real number. Each product \(a_{i} x^{i}\) is a term of a polynomial function.

\section*{EXAMPLE: IDENTIFYING POLYNOMIAL FUNCTIONS}

Which of the following are polynomial functions?
\(f(x)=2 x^{3} \cdot 3 x+4\)
\(g(x)=-x\left(x^{2}-4\right)\)
\(h(x)=5 \sqrt{x}+2\)

\section*{Answer}

The first two functions are examples of polynomial functions because they can be written in the form \(f(x)=a_{n} x^{n}+\cdots+a_{2} x^{2}+a_{1} x+a_{0}\), where the powers are non-negative integers and the coefficients are real numbers.
- \(f(x)\) can be written as \(f(x)=6 x^{4}+4\).
- \(g(x)\) can be written as \(g(x)=-x^{3}+4 x\).
- \(h(x)\) cannot be written in this form and is therefore not a polynomial function.

\section*{TRY IT}

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\section*{Defining the Degree and Leading Coefficient of a Polynomial Function}

Because of the form of a polynomial function, we can see an infinite variety in the number of terms and the power of the variable. Although the order of the terms in the polynomial function is not important for performing operations, we typically arrange the terms in descending order based on the power on the variable. This is called writing a polynomial in general or standard form. The degree of the polynomial is the highest power of the variable that occurs in the polynomial; it is the power of the first variable if the function is in general form. The leading term is the term containing the variable with the highest power, also called the term with the highest degree. The leading coefficient is the coefficient of the leading term.

\section*{A GENERAL NOTE: TERMINOLOGY OF POLYNOMIAL FUNCTIONS}


We often rearrange polynomials so that the powers on the variable are descending. When a polynomial is written in this way, we say that it is in general form.

\section*{HOW TO: GIVEN A POLYNOMIAL FUNCTION, IDENTIFY THE DEGREE AND LEADING COEFFICIENT}
1. Find the highest power of \(x\) to determine the degree of the function.
2. Identify the term containing the highest power of \(x\) to find the leading term.
3. The leading coefficient is the coefficient of the leading term.

\section*{EXAMPLE: IDENTIFYING THE DEGREE AND LEADING COEFFICIENT OF A POLYNOMIAL FUNCTION}

Identify the degree, leading term, and leading coefficient of the following polynomial functions.
\[
\begin{aligned}
& f(x)=3+2 x^{2}-4 x^{3} \\
& g(t)=5 t^{5}-2 t^{3}+7 t \\
& h(p)=6 p-p^{3}-2
\end{aligned}
\]

\section*{Answer}

For the function \(f(x)\), the highest power of \(x\) is 3 , so the degree is 3 . The leading term is the term containing that degree, \(-4 x^{3}\). The leading coefficient is the coefficient of that term, -4 .
For the function \(g(t)\), the highest power of \(t\) is 5 , so the degree is 5 . The leading term is the term containing that degree, \(5 t^{5}\). The leading coefficient is the coefficient of that term, 5 .
For the function \(h(p)\), the highest power of \(p\) is 3 , so the degree is 3 . The leading term is the term containing that degree, \(-p^{3}\); the leading coefficient is the coefficient of that term, -1 .

\section*{TRY IT}

Identify the degree, leading term, and leading coefficient of the polynomial \(f(x)=4 x^{2}-x^{6}+2 x-6\).

\section*{Answer}

The degree is 6 . The leading term is \(-x^{6}\). The leading coefficient is -1 .
Watch the following video for more examples of how to determine the degree, leading term, and leading coefficient of a polynomial.

\section*{Identifying End Behavior of Polynomial Functions}

Knowing the leading coefficient and degree of a polynomial function is useful when predicting its end behavior. To determine its end behavior, look at the leading term of the polynomial function. Because the


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power of the leading term is the highest, that term will grow significantly faster than the other terms as \(x\) gets very large or very small, so its behavior will dominate the graph. For any polynomial, the end behavior of the polynomial will match the end behavior of the term of highest degree.
\begin{tabular}{|c|c|c|}
\hline Polynomial Function & Leading Term & Graph of Polynomial Function \\
\hline \(f(x)=5 x^{4}+2 x^{3}-x-4\) & \(5 x^{4}\) &  \\
\hline \(f(x)=-2 x^{6}-x^{5}+3 x^{4}+x^{3}\) & \(-2 x^{6}\) &  \\
\hline \(f(x)=3 x^{5}-4 x^{4}+2 x^{2}+1\) & \(3 x^{5}\) &  \\
\hline & & \\
\hline \(f(x)=-6 x^{3}+7 x^{2}+3 x+1\) & \(-6 x^{3}\) &  \\
\hline
\end{tabular}

In the following video, you'll see more examples that summarize the end behavior of polynomial functions and which components of the function contribute to it.

\section*{EXAMPLE: IDENTIFYING END BEHAVIOR AND DEGREE OF A POLYNOMIAL FUNCTION}

Describe the end behavior and determine a possible degree of the polynomial function in the graph below.


\section*{Answer}

As the input values \(x\) get very large, the output values \(f(x)\) increase without bound. As the input values \(x\) get very small, the output values \(f(x)\) decrease without bound. We can describe the end behavior symbolically by writing
as \(x \rightarrow-\infty, f(x) \rightarrow-\infty\)
as \(x \rightarrow \infty, f(x) \rightarrow \infty\)
In words, we could say that as \(x\) values approach infinity, the function values approach infinity, and as \(x\) values approach negative infinity, the function values approach negative infinity.
We can tell this graph has the shape of an odd degree power function that has not been reflected, so the degree of the polynomial creating this graph must be odd and the leading coefficient must be positive.


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\section*{TRY IT}

Describe the end behavior of the polynomial function in the graph below.


Answer
As \(x \rightarrow \infty, f(x) \rightarrow-\infty\) and as \(x \rightarrow-\infty, f(x) \rightarrow-\infty\). It has the shape of an even degree power function with a negative coefficient.

\section*{TIP FOR SUCCESS}

To identify the end behavior and degree of a polynomial function, it must be in expanded (general) form. If the function is given to you in factored form, expand it first, then you can identify the leading term.
Pro-tip: You do not have to fully expand the factored form to find the leading term. Note that each of the first terms of the factors multiplied together will give you the leading term.
Ex: given \(f(x)=-2 x(x+3)(x-3)\), we know that this will expand as
\(f(x)=-2 x(x+3)(x-3)\)
\(=-2 x\left(x^{2}-9\right)\)
\(=-2 x^{3}+18 x\)
But, multiplying just the first terms together will also reveal the leading term of \(-2 x^{3}\) \(-2 x \cdot x \cdot x=-2 x^{3}\).

\section*{EXAMPLE: IDENTIFYING END BEHAVIOR AND DEGREE OF A POLYNOMIAL FUNCTION}

Given the function \(f(x)=-3 x^{2}(x-1)(x+4)\), express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

\section*{Answer}

Obtain the general form by expanding the given expression \(f(x)\).
\(f(x)=-3 x^{2}(x-1)(x+4)\)
\(f(x)=-3 x^{2}\left(x^{2}+3 x-4\right)\)
\(f(x)=-3 x^{4}-9 x^{3}+12 x^{2}\)
The general form is \(f(x)=-3 x^{4}-9 x^{3}+12 x^{2}\). The leading term is \(-3 x^{4}\); therefore, the degree of the polynomial is 4 . The degree is even (4) and the leading coefficient is negative \((-3)\), so the end behavior is
as \(x \rightarrow-\infty, f(x) \rightarrow-\infty\)
as \(x \rightarrow \infty, f(x) \rightarrow-\infty\)

\section*{TRY IT}

Given the function \(f(x)=0.2(x-2)(x+1)(x-5)\), express the function as a polynomial in general form and determine the leading term, degree, and end behavior of the function.

\section*{Answer}

The leading term is \(0.2 x^{3}\), so it is a degree 3 polynomial. As x approaches positive infinity, \(f(x)\) increases without bound; as x approaches negative infinity, \(f(x)\) decreases without bound.

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\section*{LOCAL BEHAVIOR OF POLYNOMIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Identify turning points of a polynomial function from its graph.
- Identify the number of turning points and intercepts of a polynomial function from its degree.

\section*{Identifying Local Behavior of Polynomial Functions}

In addition to the end behavior of polynomial functions, we are also interested in what happens in the "middle" of the function. In particular, we are interested in locations where graph behavior changes. A turning point is a point at which the function values change from increasing to decreasing or decreasing to increasing.


We are also interested in the intercepts. As with all functions, the \(y\)-intercept is the point at which the graph intersects the vertical axis. The point corresponds to the coordinate pair in which the input value is zero. Because a polynomial is a function, only one output value corresponds to each input value so there can be only one \(y\)-intercept \(\left(0, a_{0}\right)\). The \(x\)-intercepts occur at the input values that correspond to an output value of zero. It is possible to have more than one \(x\)-intercept.

\section*{A GENERAL NOTE: INTERCEPTS AND TURNING POINTS OF POLYNOMIAL FUNCTIONS}
- A turning point of a graph is a point where the graph changes from increasing to decreasing or decreasing to increasing.
- The \(y\)-intercept is the point where the function has an input value of zero.
- The \(x\)-intercepts are the points where the output value is zero.
- A polynomial of degree \(n\) will have, at most, \(n x\)-intercepts and \(n-1\) turning points.

\title{
Determining the Number of Turning Points and Intercepts from the Degree of the Polynomial
}

A continuous function has no breaks in its graph: the graph can be drawn without lifting the pen from the paper. A smooth curve is a graph that has no sharp corners. The turning points of a smooth graph must always occur at rounded curves. The graphs of polynomial functions are both continuous and smooth.

The degree of a polynomial function helps us to determine the number of \(x\)-intercepts and the number of turning points. A polynomial function of \(n\)th degree is the product of \(n\) factors, so it will have at most \(n\) roots or zeros, or \(x\)-intercepts. The graph of the polynomial function of degree \(n\) must have at most \(n-1\) turning points. This means the graph has at most one fewer turning point than the degree of the polynomial or one fewer than the number of factors.

\section*{TIP FOR SUCCESS}

Why do we use the phrase "at most \(n\) " when describing the number of real roots (x-intercepts) of the graph of an \(n^{\text {th }}\) degree polynomial? Can it have fewer?
Answer
Ex. Consider the graph of the polynomial function \(f(x)=x^{2}-x+1\). The function is a \(2^{\text {nd }}\) degree polynomial, so it must have at most \(n\) roots and \(n-1\) turning points.
We know this function has non-real roots since the discriminant of the quadratic formula is negative. This means that this \(2^{\text {nd }}\) polynomial has no real roots (apply the quadratic formula to prove this to yourself if needed). That is, it has no x-intercepts. But it does have two distinct complex roots.
Can you picture the graph of a quadratic function with one distinct real root? Two? But you can also see that there will never be more than two x-intercepts. Since a parabola (the graph of a \(2^{\text {nd }}\) degree polynomial) has only one turning point, it can't cross the \(x\)-axis more than twice.

\section*{EXAMPLE: DETERMINING THE NUMBER OF INTERCEPTS AND TURNING POINTS OF A POLYNOMIAL}

Without graphing the function, determine the local behavior of the function by finding the maximum number of \(x\)-intercepts and turning points for \(f(x)=-3 x^{10}+4 x^{7}-x^{4}+2 x^{3}\).

\section*{Answer}

The polynomial has a degree of 10 , so there are at most \(10 x\)-intercepts and at most \(10-1=9\) turning points.

The following video gives a 5 minute lesson on how to determine the number of intercepts and turning points of a polynomial function given its degree.

\section*{Using the Principle of Zero Products to Find the Roots of a Polynomial in Factored Form}

The Principle of Zero Products states that if the product of \(n\) numbers is 0 , then at least one of the factors is 0 . If \(a b=0\), then either \(a=0\) or \(b=0\), or both a and b are 0 . We will use this idea to find the zeros of a polynomial that is either in factored form or can be written in factored form. For example, the polynomial
\(P(x)=(x-4)^{2}(x+1)(x-7)\)
is in factored form. In the following examples, we will show the process of factoring a polynomial and calculating its x and y -intercepts.


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\section*{TRY IT}

Without graphing the function, determine the maximum number of \(x\)-intercepts and turning points for \(f(x)=108-13 x^{9}-8 x^{4}+14 x^{12}+2 x^{3}\)
Answer
There are at most \(12 x\)-intercepts and at most 11 turning points.
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\section*{HOW TO: GIVEN A POLYNOMIAL FUNCTION, DETERMINE THE INTERCEPTS}
1. Determine the \(y\)-intercept by setting \(x=0\) and finding the corresponding output value.
2. Determine the \(x\)-intercepts by setting the function equal to zero and solving for the input values.

\section*{The Whole Picture}

Now we can bring the two concepts of turning points and intercepts together to get a general picture of the behavior of polynomial functions. These types of analyses on polynomials developed before the advent of mass computing as a way to quickly understand the general behavior of a polynomial function. We now have a quick way, with computers, to graph and calculate important characteristics of polynomials that once took a lot of algebra.

In the first example, we will determine the least degree of a polynomial based on the number of turning points and intercepts.

Now you try to determine the least possible degree of a polynomial given its graph.
Now we will show that you can also determine the least possible degree and number of turning points of a polynomial function given in factored form.

Now it is your turn to determine the local behavior of a polynomial given in factored form.

\section*{EXAMPLE: DETERMINING THE INTERCEPTS OF A POLYNOMIAL FUNCTION}

Given the polynomial function \(f(x)=(x-2)(x+1)(x-4)\), written in factored form for your convenience, determine the \(y\) and \(x\)-intercepts.
Answer
The \(y\)-intercept occurs when the input is zero, so substitute 0 for \(x\).
\[
\begin{aligned}
& f(0)=(0-2)(0+1)(0-4) \\
& f(0)=(-2)(1)(-4) \\
& f(0)=8
\end{aligned}
\]

The \(y\)-intercept is \((0,8)\).
The \(x\)-intercepts occur when the output \(f(x)\) is zero.
\begin{tabular}{lllll}
\(0=(x-2)\) & \((x+1)(x-4)\) & & \\
\(x-2=0\) & or & \(x+1=0\) & or & \(x-4=0\) \\
\(x=2\) & or & \(x=-1\) & or & \(x=4\)
\end{tabular}

The \(x\)-intercepts are \((2,0),(-1,0)\), and \((4,0)\).
We can see these intercepts on the graph of the function shown below.


\section*{EXAMPLE: DETERMINING THE INTERCEPTS OF A POLYNOMIAL FUNCTION BY FACTORING}

Given the polynomial function \(f(x)=x^{4}-4 x^{2}-45\), determine the \(y\) and \(x\)-intercepts.

\section*{Answer}

The \(y\)-intercept occurs when the input is zero.
\[
\begin{aligned}
& f(0)=(0)^{4}-4(0)^{2}-45 \\
& f(0)=-45
\end{aligned}
\]

The \(y\)-intercept is \((0,-45)\).
The \(x\)-intercepts occur when the output is zero. To determine when the output is zero, we will need to factor the polynomial.
\[
\begin{aligned}
& f(x)=x^{4}-4 x^{2}-45 \\
& f(x)=\left(x^{2}-9\right)\left(x^{2}+5\right) \\
& f(x)=(x-3)(x+3)\left(x^{2}+5\right)
\end{aligned}
\]

Then set the polynomial function equal to 0 .
\[
\begin{aligned}
& 0=(x-3)(x+3)\left(x^{2}+5\right) \\
& x-3=0 \text { or } x+3=0 \text { or } x^{2}+5=0 \\
& x=3 \quad \text { or } x=-3 \quad \text { or (no real solution) }
\end{aligned}
\]

The \(x\)-intercepts are \((3,0)\) and \((-3,0)\).
We can see these intercepts on the graph of the function shown below. We can see that the function has y -axis symmetry or is even because \(f(x)=f(-x)\).


In the example above, you needed to factor a \(4^{\text {th }}\) polynomial in order to use the zero-product principle. Some, but not all, polynomials of degree higher than 2 are factorable using special techniques you learned earlier in the course. What technique was used to factor \(x^{4}-4 x^{2}-45\) ?

\section*{Answer}

First, we can recognize that the smallest power in this trinomial is exactly half the larger power. This allows us to make a substitution. We can let \(u=x^{2}\). Then \(u^{2}=x^{4}\).
We now have the factorable quadratic equation
\(0=u^{2}-4 u-45\)
\(0=(u-9)(u+5)\).
Back substituting \(u=x^{2}\) into the function yields the factored form of the equation to which we can apply the zero-product principle.
\(0=\left(x^{2}-9\right)\left(x^{2}+5\right)\)
\(0=(x+3)(x-3)\left(x^{2}+5\right)\)
So, \(x=-3,3\), and \(\pm \sqrt{5} i\).

\section*{TRY IT}

Given the polynomial function \(f(x)=2 x^{3}-6 x^{2}-20 x\), determine the \(y\) and \(x\)-intercepts.
Answer
\(y\)-intercept \((0,0) ; x\)-intercepts \((0,0),(-2,0)\), and \((5,0)\)

\section*{EXAMPLE: DRAWING CONCLUSIONS ABOUT A POLYNOMIAL FUNCTION FROM ITS GRAPH}

Given the graph of the polynomial function below, determine the least possible degree of the polynomial and whether it is even or odd. Use end behavior, the number of intercepts, and the number of turning points to help you.


Answer


The end behavior of the graph tells us this is the graph of an even-degree polynomial.
The graph has \(2 x\)-intercepts, suggesting a degree of 2 or greater, and 3 turning points, suggesting a degree of 4 or greater. Based on this, it would be reasonable to conclude that the degree is even and at least 4.

\section*{TRY IT}

Given the graph of the polynomial function below, determine the least possible degree of the polynomial and whether it is even or odd. Use end behavior, the number of intercepts, and the number of turning points to help you.


\section*{Answer}

The end behavior indicates an odd-degree polynomial function; there are \(3 x\)-intercepts and 2 turning points, so the degree is odd and at least 3 . Because of the end behavior, we know that the leading coefficient must be negative.

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\section*{EXAMPLE: DRAWING CONCLUSIONS ABOUT A POLYNOMIAL FUNCTION FROM ITS FACTORS}

Given the function \(f(x)=-4 x(x+3)(x-4)\), determine the local behavior.

\section*{Answer}

The \(y\)-intercept is found by evaluating \(f(0)\).
\[
\begin{aligned}
& f(0)=-4(0)(0+3)(0-4) \\
& f(0)=0
\end{aligned}
\]

The \(y\)-intercept is \((0,0)\).
The \(x\)-intercepts are found by setting the function equal to 0 .
\(0=-4 x(x+3)(x-4)\)
\[
\begin{array}{lllll}
-4 x=0 & \text { or } & x+3=0 & \text { or } & x-4=0 \\
x=0 & \text { or } & x=-3 & \text { or } & x=4
\end{array}
\]

The \(x\)-intercepts are \((0,0),(-3,0)\), and \((4,0)\).
The degree is 3 so the graph has at most 2 turning points.

\section*{TRY IT}

Given the function \(f(x)=0.2(x-2)(x+1)(x-5)\), determine the local behavior.

Answer
The \(x\)-intercepts are \((2,0),(-1,0)\), and \((5,0)\), the \(y\)-intercept is \((0,2)\), and the graph has at most 2 turning points.

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\section*{SUMMARY: CHARACTERISTICS OF POWER AND POLYNOMIAL FUNCTIONS}

\section*{Key Equations}
general form of a polynomial function
\[
f(x)=a_{n} x^{n}+\cdots+a_{2} x^{2}+a_{1} x+a_{0}
\]

\section*{Key Concepts}
- A power function is a variable base raised to a number power.
- The behavior of a graph as the input decreases beyond bound and increases beyond bound is called the end behavior.
- The end behavior depends on whether the power is even or odd.
- A polynomial function is the sum of terms, each of which consists of a transformed power function with non-negative integer powers.
- The degree of a polynomial function is the highest power of the variable that occurs in a polynomial. The term containing the highest power of the variable is called the leading term. The coefficient of the leading term is called the leading coefficient.
- The end behavior of a polynomial function is the same as the end behavior of the power function represented by the leading term of the function.
- A polynomial of degree \(n\) will have at most \(n x\)-intercepts and at most \(n-1\) turning points.

\section*{Glossary}
coefficient a nonzero real number multiplied by a variable raised to an exponent
continuous function a function whose graph can be drawn without lifting the pen from the paper because there are no breaks in the graph
degree the highest power of the variable that occurs in a polynomial
end behavior the behavior of the graph of a function as the input decreases without bound and increases without bound
leading coefficient the coefficient of the leading term
leading term the term containing the highest power of the variable
polynomial function a function that consists of either zero or the sum of a finite number of non-zero terms, each of which is a product of a number, called the coefficient of the term, and a variable raised to a nonnegative integer power.
power function a function that can be represented in the form \(f(x)=a x^{n}\) where \(a\) is a constant, the base is a variable, and the exponent is \(n\), is a smooth curve represented by a graph with no sharp corners
term of a polynomial function any \(a_{i} x^{i}\) of a polynomial function in the form \(f(x)=a_{n} x^{n}+\cdots+a_{2} x^{2}+a_{1} x+a_{0}\)
turning point the location where the graph of a function changes direction

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\section*{INTRODUCTION TO GRAPHS OF POLYNOMIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Identify zeros of polynomial functions with even and odd multiplicity.
- Draw the graph of a polynomial function using end behavior, turning points, intercepts, and the Intermediate Value Theorem.
- Write the equation of a polynomial function given its graph.

The revenue in millions of dollars for a fictional cable company from 2006 through 2013 is shown in the table below.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|}
\hline Year & 2006 & 2007 & 2008 & 2009 & 2010 & 2011 & 2012 & 2013 \\
\hline Revenue & 52.4 & 52.8 & 51.2 & 49.5 & 48.6 & 48.6 & 48.7 & 47.1 \\
\hline
\end{tabular}

The revenue can be modeled by the polynomial function
\[
R(t)=-0.037 t^{4}+1.414 t^{3}-19.777 t^{2}+118.696 t-205.332
\]
where \(R\) represents the revenue in millions of dollars and \(t\) represents the year, with \(t=6\) corresponding to 2006. Over which intervals is the revenue for the company increasing? Over which intervals is the revenue for the company decreasing? These questions, along with many others, can be answered by examining the graph of the polynomial function. We have already explored the local behavior of quadratics, a special case of polynomials. In this section we will explore the local behavior of polynomials in general.

\section*{ZEROS AND MULTIPLICITY}

\section*{LEARNING OUTCOMES}
- Identify zeros of polynomial functions with even and odd multiplicity.

Graphs behave differently at various \(x\)-intercepts. Sometimes the graph will cross over the \(x\)-axis at an intercept. Other times the graph will touch the x -axis and bounce off.

Suppose, for example, we graph the function \(f(x)=(x+3)(x-2)^{2}(x+1)^{3}\).
Notice in the figure below that the behavior of the function at each of the \(x\)-intercepts is different.


The behavior of a graph at an x-intercept can be determined by examining the multiplicity of the zero.

The \(x\)-intercept \(x=-3\) is the solution to the equation \((x+3)=0\). The graph passes directly through the \(x\) intercept at \(x=-3\). The factor is linear (has a degree of 1 ), so the behavior near the intercept is like that of a line; it passes directly through the intercept. We call this a single zero because the zero corresponds to a single factor of the function.

The \(x\)-intercept \(x=2\) is the repeated solution to the equation \((x-2)^{2}=0\). The graph touches the axis at the intercept and changes direction. The factor is quadratic (degree 2), so the behavior near the intercept is like that of a quadratic-it bounces off of the horizontal axis at the intercept.
\((x-2)^{2}=(x-2)(x-2)\)
The factor is repeated, that is, the factor \((x-2)\) appears twice. The number of times a given factor appears in the factored form of the equation of a polynomial is called the multiplicity. The zero associated with this factor, \(x=2\), has multiplicity 2 because the factor \((x-2)\) occurs twice.

The \(x\)-intercept \(x=-1\) is the repeated solution of factor \((x+1)^{3}=0\). The graph passes through the axis at the intercept but flattens out a bit first. This factor is cubic (degree 3), so the behavior near the intercept is like that of a cubic with the same S-shape near the intercept as the function \(f(x)=x^{3}\). We call this a triple zero, or a zero with multiplicity 3 .

For zeros with even multiplicities, the graphs touch or are tangent to the \(x\)-axis at these \(x\)-values. For zeros with odd multiplicities, the graphs cross or intersect the \(x\)-axis at these \(x\)-values. See the graphs below for examples of graphs of polynomial functions with multiplicity 1,2 , and 3 .


Single zero


Zero with multiplicity 2


Zero with multiplicity 3

For higher even powers, such as 4,6 , and 8 , the graph will still touch and bounce off of the x -axis, but for each increasing even power the graph will appear flatter as it approaches and leaves the \(x\)-axis.

For higher odd powers, such as 5,7 , and 9 , the graph will still cross through the \(x\)-axis, but for each increasing odd power, the graph will appear flatter as it approaches and leaves the \(x\)-axis.

\section*{A GENERAL NOTE: GRAPHICAL BEHAVIOR OF POLYNOMIALS AT \(X\) INTERCEPTS}

If a polynomial contains a factor of the form \((x-h)^{p}\), the behavior near the \(x\)-intercept \(h\) is determined by the power \(p\). We say that \(x=h\) is a zero of multiplicity \(p\).
The graph of a polynomial function will touch the \(x\)-axis at zeros with even multiplicities. The graph will cross the \(x\)-axis at zeros with odd multiplicities.
The sum of the multiplicities is the degree of the polynomial function.

\section*{Determining End Behavior}

\section*{HOW TO: GIVEN A GRAPH OF A POLYNOMIAL FUNCTION OF DEGREE \(n\), IDENTIFY THE ZEROS AND THEIR MULTIPLICITIES.}
1. If the graph crosses the \(x\)-axis and appears almost linear at the intercept, it is a single zero.
2. If the graph touches the \(x\)-axis and bounces off of the axis, it is a zero with even multiplicity.
3. If the graph crosses the \(x\)-axis at a zero, it is a zero with odd multiplicity.
4. The sum of the multiplicities is the degree \(n\).

\section*{EXAMPLE: IDENTIFYING ZEROS AND THEIR MULTIPLICITIES}

Use the graph of the function of degree 6 to identify the zeros of the function and their possible multiplicities.


\section*{Answer}

The polynomial function is of degree \(n\) which is 6 . The sum of the multiplicities must be 6 .
Starting from the left, the first zero occurs at \(x=-3\). The graph touches the \(x\)-axis, so the multiplicity of the zero must be even. The zero of -3 has multiplicity 2 .
The next zero occurs at \(x=-1\). The graph looks almost linear at this point. This is a single zero of multiplicity 1.
The last zero occurs at \(x=4\). The graph crosses the \(x\)-axis, so the multiplicity of the zero must be odd. We know that the multiplicity is 3 and that the sum of the multiplicities must be 6 .

\section*{TRY IT}

Use the graph of the function of degree 5 to identify the zeros of the function and their multiplicities.


\section*{Answer}

The graph has a zero of -5 with multiplicity 1 , a zero of -1 with multiplicity 2 , and a zero of 3 with multiplicity 2.

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\section*{TRY IT}

Use the graph below to find the zeros of the degree 6 function and their multiplicities.


\section*{Answer}
\begin{tabular}{|l|l|}
\hline Zero & Multiplicity \\
\hline\((-3,0)\) & 1 \\
\hline\((-2,0)\) & 2 \\
\hline\((1,0)\) & 3 \\
\hline
\end{tabular}

As we have already learned, the behavior of a graph of a polynomial function of the form
\(f(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\ldots+a_{1} x+a_{0}\)
will either ultimately rise or fall as \(x\) increases without bound and will either rise or fall as \(x\) decreases without bound. This is because for very large inputs, say 100 or 1,000 , the leading term dominates the size of the output. The same is true for very small inputs, say -100 or \(-1,000\).

Recall that we call this behavior the end behavior of a function. As we pointed out when discussing quadratic equations, when the leading term of a polynomial function, \(a_{n} x^{n}\), is an even power function, as \(x\) increases or decreases without bound, \(f(x)\) increases without bound. When the leading term is an odd power function, as \(x\) decreases without bound, \(f(x)\) also decreases without bound; as \(x\) increases without bound, \(f(x)\) also increases without bound. If the leading term is negative, it will change the direction of the end behavior. The table below summarizes all four cases.

\section*{Positive Leading}

Coefficient, \(a_{n}>0\)

\section*{Positive Leading} Coefficient, \(a_{n}>0\)


End Behavior:
\(x \rightarrow \infty, f(x) \rightarrow \infty\)
\(x \rightarrow-\infty, f(x) \rightarrow-\infty\)

Negative Leading
Coefficient, \(a_{n}<0\)

Negative Leading
Coefficient, \(a_{n}<0\)


End Behavior:
\(x \rightarrow \infty, f(x) \rightarrow-\infty\)
\(x \rightarrow-\infty, f(x) \rightarrow \infty\)

\section*{TRY IT}

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\section*{GRAPHING POLYNOMIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Draw the graph of a polynomial function using end behavior, turning points, intercepts, and the Intermediate Value Theorem.

We can use what we have learned about multiplicities, end behavior, and turning points to sketch graphs of polynomial functions. Let us put this all together and look at the steps required to graph polynomial functions.

\section*{HOW TO: GIVEN A POLYNOMIAL FUNCTION, SKETCH THE GRAPH}
1. Find the intercepts.
2. Check for symmetry. If the function is an even function, its graph is symmetric with respect to the \(y\) axis, that is, \(f(-x)=f(x)\).
If a function is an odd function, its graph is symmetric with respect to the origin, that is, \(f(-x)=-f(x)\).
3. Use the multiplicities of the zeros to determine the behavior of the polynomial at the \(x\)-intercepts.
4. Determine the end behavior by examining the leading term.
5. Use the end behavior and the behavior at the intercepts to sketch the graph.
6. Ensure that the number of turning points does not exceed one less than the degree of the polynomial.
7. Optionally, use technology to check the graph.

\section*{The Intermediate Value Theorem}

In some situations, we may know two points on a graph but not the zeros. If those two points are on opposite sides of the \(x\)-axis, we can confirm that there is a zero between them. Consider a polynomial function \(f\) whose graph is smooth and continuous. The Intermediate Value Theorem states that for two numbers \(a\) and \(b\) in the domain of \(f\), if \(a<b\) and \(f(a) \neq f(b)\), then the function \(f\) takes on every value between \(f(a)\) and \(f(b)\).

We can apply this theorem to a special case that is useful for graphing polynomial functions. If a point on the graph of a continuous function \(f\) at \(x=a\) lies above the \(x\)-axis and another point at \(x=b\) lies below the \(x\) axis, there must exist a third point between \(x=a\) and \(x=b\) where the graph crosses the \(x\)-axis. Call this point \((c, f(c)\) ). This means that we are assured there is a value \(c\) where \(f(c)=0\).

In other words, the Intermediate Value Theorem tells us that when a polynomial function changes from a negative value to a positive value, the function must cross the \(x\)-axis. The figure below shows that there is a

\section*{EXAMPLE: SKETCHING THE GRAPH OF A POLYNOMIAL FUNCTION}

Sketch a possible graph for \(f(x)=-2(x+3)^{2}(x-5)\).

\section*{Answer}

This graph has two \(x\)-intercepts. At \(x=-3\), the factor is squared, indicating a multiplicity of 2 . The graph will bounce off the \(x\)-intercept at this value. At \(x=5\), the function has a multiplicity of one, indicating the graph will cross through the axis at this intercept.
The \(y\)-intercept is found by evaluating \(f(0)\).
\[
\begin{aligned}
& f(0)=-2(0+3)^{2}(0-5) \\
& f(0)=-2 \cdot 9 \cdot(-5) \\
& f(0)=90
\end{aligned}
\]

The \(y\)-intercept is \((0,90)\).
Additionally, we can see the leading term, if this polynomial were multiplied out, would be \(-2 x^{3}\), so the end behavior, as seen in the following graph, is that of a vertically reflected cubic with the outputs decreasing as the inputs approach infinity and the outputs increasing as the inputs approach negative infinity.


To sketch the graph, we consider the following:
- As \(x \rightarrow-\infty\) the function \(f(x) \rightarrow \infty\), so we know the graph starts in the second quadrant and is decreasing toward the \(x\)-axis.
- Since \(f(-x)=-2(-x+3)^{2}(-x-5)\) is not equal to \(f(x)\), the graph does not have any symmetry.
- At \((-3,0)\) the graph bounces off of the \(x\)-axis, so the function must start increasing after this point.
- At \((0,90)\), the graph crosses the \(y\)-axis.


Somewhere after this point, the graph must turn back down or start decreasing toward the horizontal axis because the graph passes through the next intercept at \((5,0)\).


As \(x \rightarrow \infty\) the function \(f(x) \rightarrow-\infty\), so we know the graph continues to decrease, and we can stop drawing the graph in the fourth quadrant.
The complete graph of the polynomial function \(f(x)=-2(x+3)^{2}(x-5)\) is as follows:


Sketch a possible graph for \(f(x)=\frac{1}{4} x(x-1)^{4}(x+3)^{3}\).
Check yourself with an online graphing calculator when you are done.
Answer


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\section*{TRY IT}

Use an online graphing calculator to find an odd degree function with one zero at ( \(-3,0\) ) whose multiplicity is 3 and another zero at \((2,0)\) with multiplicity 2 . The end behavior of the graph is:
as \(x \rightarrow-\infty, f(x) \rightarrow \infty\) and as \(x \rightarrow \infty, f(x) \rightarrow-\infty\)
zero between \(a\) and \(b\).


The Intermediate Value Theorem can be used to show there exists a zero.

\section*{A GENERAL NOTE: INTERMEDIATE VALUE THEOREM}

Let \(f\) be a polynomial function. The Intermediate Value Theorem states that if \(f(a)\) and \(f(b)\) have opposite signs, then there exists at least one value \(c\) between \(a\) and \(b\) for which \(f(c)=0\).

\section*{EXAMPLE: USING THE INTERMEDIATE VALUE THEOREM}

Show that the function \(f(x)=x^{3}-5 x^{2}+3 x+6\) has at least two real zeros between \(x=1\) and \(x=4\).
Answer
To start, evaluate \(f(x)\) at the integer values \(x=1,2,3\), and 4 .
\begin{tabular}{|l|l|l|l|l|}
\hline \(\boldsymbol{x}\) & 1 & 2 & 3 & 4 \\
\hline \(\boldsymbol{f}(\boldsymbol{x})\) & 5 & 0 & -3 & 2 \\
\hline
\end{tabular}

We see that one zero occurs at \(x=2\). Also, since \(f(3)\) is negative and \(f(4)\) is positive, by the Intermediate Value Theorem, there must be at least one real zero between 3 and 4.
We have shown that there are at least two real zeros between \(x=1\) and \(x=4\).

\section*{Analysis of the Solution}

We can also graphically see that there are two real zeros between \(x=1\) and \(x=4\).


\section*{TRY IT}

Show that the function \(f(x)=7 x^{5}-9 x^{4}-x^{2}\) has at least one real zero between \(x=1\) and \(x=2\).
Answer
Because \(f\) is a polynomial function and since \(f(1)\) is negative and \(f(2)\) is positive, there is at least one real zero between \(x=1\) and \(x=2\).

\title{
WRITING FORMULAS FOR POLYNOMIAL FUNCTIONS
}

\section*{LEARNING OUTCOMES}
- Write the equation of a polynomial function given its graph.

Now that we know how to find zeros of polynomial functions, we can use them to write formulas based on graphs. Because a polynomial function written in factored form will have an \(x\)-intercept where each factor is equal to zero, we can form a function that will pass through a set of \(x\)-intercepts by introducing a corresponding set of factors.

\section*{A GENERAL NOTE: FACTORED FORM OF POLYNOMIALS}

If a polynomial of lowest degree \(p\) has zeros at \(x=x_{1}, x_{2}, \ldots, x_{n}\), then the polynomial can be written in the factored form: \(f(x)=a\left(x-x_{1}\right)^{p_{1}}\left(x-x_{2}\right)^{p_{2}} \cdots\left(x-x_{n}\right)^{p_{n}}\) where the powers \(p_{i}\) on each factor can be determined by the behavior of the graph at the corresponding intercept, and the stretch factor a can be determined given a value of the function other than the \(x\)-intercept.

\section*{HOW TO: GIVEN A GRAPH OF A POLYNOMIAL FUNCTION, WRITE A FORMULA FOR THE FUNCTION}
1. Identify the \(x\)-intercepts of the graph to find the factors of the polynomial.
2. Examine the behavior of the graph at the \(x\)-intercepts to determine the multiplicity of each factor.
3. Find the polynomial of least degree containing all of the factors found in the previous step.
4. Use any other point on the graph (the \(y\)-intercept may be easiest) to determine the stretch factor.

\section*{Using Local and Global Extrema}

With quadratics, we were able to algebraically find the maximum or minimum value of the function by finding the vertex. For general polynomials, finding these turning points is not possible without more advanced techniques from calculus. Even then, finding where extrema occur can still be algebraically challenging. For now, we will estimate the locations of turning points using technology to generate a graph.

Each turning point represents a local minimum or maximum. Sometimes, a turning point is the highest or lowest point on the entire graph. In these cases, we say that the turning point is a global maximum or a global minimum. These are also referred to as the absolute maximum and absolute minimum values of the

\section*{EXAMPLE: WRITING A FORMULA FOR A POLYNOMIAL FUNCTION FROM ITS GRAPH}

Write a formula for the polynomial function.


\section*{Answer}

This graph has three \(x\)-intercepts: \(x=-3,2\), and 5 . The \(y\)-intercept is located at ( 0,2 ). At \(x=-3\) and \(x=\) 5 , the graph passes through the axis linearly, suggesting the corresponding factors of the polynomial will be linear. At \(x=2\), the graph bounces off the x -axis at the intercept suggesting the corresponding factor of the polynomial will be second degree (quadratic). Together, this gives us
\(f(x)=a(x+3)(x-2)^{2}(x-5)\)
To determine the stretch factor, we utilize another point on the graph. We will use the \(y\)-intercept \((0,-2)\), to solve for \(a\).
\(f(0)=a(0+3)(0-2)^{2}(0-5)\)
\(-2=a(0+3)(0-2)^{2}(0-5)\)
\(-2=-60 a\)
\(a=\frac{1}{30}\)
The graphed polynomial appears to represent the function \(f(x)=\frac{1}{30}(x+3)(x-2)^{2}(x-5)\).

\section*{TRY IT}

Given the graph below, write a formula for the function shown.


Answer
\(f(x)=-\frac{1}{8}(x-2)^{3}(x+1)^{2}(x-4)\)
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\section*{TRY IT}

Use an online graphing calculator to help you write the equation of a degree 5 polynomial function with roots at \((-1,0),(0,2)\), and,\((0,3)\) with multiplicities 3,1 , and 1 respectively, that passes through the point (1, -32 ).
Answer
\(f(x)=-2(x-2)(x+1)^{3}(x-2)\)
function.
A local maximum or local minimum at \(x=a\) (sometimes called the relative maximum or minimum, respectively) is the output at the highest or lowest point on the graph in an open interval around \(x=a\). If a function has a local maximum at \(a\), then \(f(a) \geq f(x)\) for all \(x\) in an open interval around \(x=a\). If a function has a local minimum at \(a\), then \(f(a) \leq f(x)\) for all \(x\) in an open interval around \(x=a\).

A global maximum or global minimum is the output at the highest or lowest point of the function. If a function has a global maximum at \(a\), then \(f(a) \geq f(x)\) for all \(x\). If a function has a global minimum at \(a\), then \(f(a) \leq f(x)\) for all \(x\).

We can see the difference between local and global extrema below.


\section*{Q \& A}

Do all polynomial functions have a global minimum or maximum?
No. Only polynomial functions of even degree have a global minimum or maximum. For example, \(f(x)=x\) has neither a global maximum nor a global minimum.

\section*{EXAMPLE: USING LOCAL EXTREMA TO SOLVE APPLICATIONS}

An open-top box is to be constructed by cutting out squares from each corner of a 14 cm by 20 cm sheet of plastic then folding up the sides. Find the size of squares that should be cut out to maximize the volume enclosed by the box.

\section*{Answer}

We will start this problem by drawing a picture like the one below, labeling the width of the cut-out squares with a variable, \(w\).


Notice that after a square is cut out from each end, it leaves a \((14-2 w) \mathrm{cm}\) by \((20-2 w) \mathrm{cm}\) rectangle for the base of the box, and the box will be \(w \mathrm{~cm}\) tall. This gives the volume
\(V(w)=(20-2 w)(14-2 w) w\)
\(V(w)=280 w-68 w^{2}+4 w^{3}\)


Notice, since the factors are \(w, 20-2 w\) and \(14-2 w\), the three zeros are 10,7 , and 0 , respectively. Because a height of 0 cm is not reasonable, we consider only the zeros 10 and 7 . The shortest side is 14 and we are cutting off two squares, so values \(w\) may take on are greater than zero or less than 7 . This means we will restrict the domain of this function to \(0<w<7\). Using technology to sketch the graph of \(V(w)\) on this reasonable domain, we get a graph like the one above. We can use this graph to estimate the maximum value for the volume, restricted to values for \(w\) that are reasonable for this problem, values from 0 to 7 .
On this graph, we turn our focus to only the portion on the reasonable domain, \([0,7]\). We can estimate the maximum value to be around 340 cubic cm , which occurs when the squares are about 2.75 cm on each
side. To improve this estimate, we could use advanced features of our technology, if available, or simply change our window to zoom in on our graph to produce the graph below.


From this zoomed-in view, we can refine our estimate for the maximum volume to about 339 cubic cm which occurs when the squares measure approximately 2.7 cm on each side.

\section*{TRY IT}

Use an online graphing tool to find the maximum and minimum values on the interval \([-2,7]\) of the function \(f(x)=0.1\left(x-\frac{5}{3}\right)^{3}(x+1)^{2}(x-7)\).
Now change the value of the leading coefficient (a) to see how it affects the end behavior and y-intercept of the graph.
\(f(x)=a\left(x-\frac{5}{3}\right)^{3}(x+1)^{2}(x-7)\).
Make sure to observe both positive and negative \(a\)-values, and large and small \(a\)-values.
Answer
The minimum occurs at approximately the point \((5.98,-398.8)\), and the maximum occurs at approximately the point \((0.02,3.24)\).

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\section*{SUMMARY: GRAPHS OF POLYNOMIAL FUNCTIONS}

\section*{Key Concepts}
- Polynomial functions of degree 2 or more are smooth, continuous functions.
- To find the zeros of a polynomial function, if it can be factored, factor the function and set each factor equal to zero.
- Another way to find the \(x\)-intercepts of a polynomial function is to graph the function and identify the points where the graph crosses the \(x\)-axis.
- The multiplicity of a zero determines how the graph behaves at the \(x\)-intercept.
- The graph of a polynomial will cross the x -axis at a zero with odd multiplicity.
- The graph of a polynomial will touch and bounce off the x-axis at a zero with even multiplicity.
- The end behavior of a polynomial function depends on the leading term.
- The graph of a polynomial function changes direction at its turning points.
- A polynomial function of degree \(n\) has at most \(n-1\) turning points.
- To graph polynomial functions, find the zeros and their multiplicities, determine the end behavior, and ensure that the final graph has at most \(n-1\) turning points.
- Graphing a polynomial function helps to estimate local and global extremas.
- The Intermediate Value Theorem tells us that if \(f(a)\) and \(f(b)\) have opposite signs, then there exists at least one value \(c\) between \(a\) and \(b\) for which \(f(c)=0\).

\section*{Glossary}
global maximum highest turning point on a graph; \(f(a)\) where \(f(a) \geq f(x)\) for all \(\boldsymbol{x}\).
global minimum lowest turning point on a graph; \(f(a)\) where \(f(a) \leq f(x)\) for all \(\boldsymbol{x}\).
Intermediate Value Theorem for two numbers \(a\) and \(b\) in the domain of \(f\), if \(a<b\) and \(f(a) \neq f(b)\), then the function \(f\) takes on every value between \(f(a)\) and \(f(b)\); specifically, when a polynomial function changes from a negative value to a positive value, the function must cross the \(x\)-axis
multiplicity the number of times a given factor appears in the factored form of the equation of a polynomial; if a polynomial contains a factor of the form \((x-h)^{p}, x=h\) is a zero of multiplicity \(p\).

\section*{INTRODUCTION TO DIVIDING POLYNOMIALS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Use long division to divide polynomials.
- Use synthetic division to divide polynomials.


The Lincoln Memorial, Washington, D.C. (credit: Ron Cogswell, Flickr)
The exterior of the Lincoln Memorial in Washington, D.C., is a large rectangular solid with length 61.5 meters (m), width 40 m , and height 30 m . (Note: National Park Service. "Lincoln Memorial Building Statistics." http://www.nps.gov/linc/historyculture/lincoln-memorial-building-statistics.htm. Accessed 4/3/2014) We can easily find the volume using elementary geometry.
\(V=l \cdot w \cdot h\)
\(V=61.5 \cdot 40 \cdot 30\)
\(V=73,800\)
So the volume is 73,800 cubic meters \(\left(\mathrm{m}^{3}\right)\). Suppose we knew the volume, length, and width. We could divide to find the height.
\(h=\frac{V}{l \cdot w}\)
\(h=\frac{73,800}{61.5 \cdot 40}\)
\(h=30\)
As we can confirm from the dimensions above, the height is 30 m . We can use similar methods to find any of the missing dimensions. We can also use the same method if any or all of the measurements contain variable expressions. For example, suppose the volume of a rectangular solid is given by the polynomial \(3 x^{4}-3 x^{3}-33 x^{2}+54 x\). The length of the solid is given by \(3 x\); the width is given by \(x-2\). To find the height of the solid, we can use polynomial division, which is the focus of this section.

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\section*{POLYNOMIAL LONG DIVISION}

\section*{LEARNING OUTCOMES}
- Use long division to divide polynomials.

In the next two sections, we will be learning two ways to divide polynomials. These techniques can help you find the zeros of a polynomial that is not factorable over the integers.

\section*{RECALL LONG DIVISION}

A handy algebraic technique involves rewriting a quotient of polynomials as a polynomial of reduced degree plus a remainder quotient
\(\frac{p(x)}{q(x)}=h(x)+\frac{r(x)}{q(x)}\).
Or equivalently by multiplying both sides by \(q(x)\),
\(p(x)=q(x) h(x)+r(x)\)
This method is akin to rewriting an "improper" fraction such as
\(\frac{17}{3}=5+\frac{2}{3}\).
Or equivalently,
\(17=3 \cdot 5+2\).
The process used to accomplish this is similar to long division of whole numbers, so if you haven't performed long division for awhile, take a moment to refresh before reading this section.

\section*{Answer}

Recall how you can use long division to divide two whole numbers, say 900 divided by 37 .
\[
3 7 \longdiv { 9 0 0 }
\]

First, you would think about how many \(37 s\) are in 90 , as 9 is too small. (Note: you could also think, how many \(40 s\) are there in 90 .)
There are two \(37 s\) in 90 , so write 2 above the last digit of 90 . Two \(37 s\) is 74 ; write that product below the 90.
\[
3 7 \longdiv { 9 0 0 }
\]

Subtract: \(90-74\) is 16 . (If the result is larger than the divisor, 37 , then you need to use a larger number for the quotient.)


Bring down the next digit (0) and consider how many \(37 s\) are in 160 .

There are four \(37 s\) in 160 , so write the 4 next to the two in the quotient. Four \(37 s\) is 148 ; write that product below the 160 .


Subtract: 160-148 is 12 . This is less than 37 so the 4 is correct. Since there are no more digits in the dividend to bring down, you are done.


The final answer is 24 R12, or \(24 \frac{12}{37}\). You can check this by multiplying the quotient (without the remainder) by the divisor, and then adding in the remainder. The result should be the dividend:
\(24 \cdot 37+12=888+12=900\)

We are familiar with the long division algorithm for ordinary arithmetic. We begin by dividing into the digits of the dividend that have the greatest place value. We divide, multiply, subtract, include the digit in the next place value position, and repeat. For example, let's divide 178 by 3 using long division.

\section*{Long Division}
\begin{tabular}{rl}
59 & Step 1: \(5 \times 3=15\) and \(17-15=2\) \\
\(3 \lcm{178}\) & Step 2: Bring down the 8 \\
\(\frac{-15}{28}\) & Step 3: \(9 \times 3=27\) and \(28-27=1\) \\
\(\frac{-27}{1}\) & Answer: \(59 R 1\) or \(59 \frac{1}{3}\)
\end{tabular}

Another way to look at the solution is as a sum of parts. This should look familiar, since it is the same method used to check division in elementary arithmetic.
(divisor \(\cdot\) quotient) + remainder \(=\) dividend
\((3 \cdot 59)+1=177+1=178\)
We call this the Division Algorithm and will discuss it more formally after looking at an example.
Division of polynomials that contain more than one term has similarities to long division of whole numbers.
We can write a polynomial dividend as the product of the divisor and the quotient added to the remainder.
The terms of the polynomial division correspond to the digits (and place values) of the whole number division. This method allows us to divide two polynomials. For example, if we were to divide \(2 x^{3}-3 x^{2}+4 x+5\) by \(x+2\) using the long division algorithm, it would look like this:
\[
\begin{aligned}
& x + 2 \longdiv { 2 x ^ { 3 } - 3 x ^ { 2 } + 4 x + 5 } \quad \text { Set up the division problem. } \\
& x + 2 \longdiv { 2 x ^ { 3 } - 3 x ^ { 2 } + 4 x + 5 } \quad 2 x ^ { 3 } \text { divided by } x \text { is } 2 x^{2} \text {. } \\
& x + 2 \longdiv { 2 x ^ { 3 } - 3 x ^ { 2 } + 4 x + 5 } \\
& \frac{-\left(2 x^{3}+4 x^{2}\right)}{-7 x^{2}+4 x} \\
& x + 2 \longdiv { 2 x ^ { 3 } - 3 x ^ { 2 } + 4 x + 5 } \\
& \frac{-\left(2 x^{3}+4 x^{2}\right)}{-7 x^{2}+4 x} \\
& \frac{-\left(-7 x^{2}+14 x\right)}{18 x+5} \\
& x + 2 \longdiv { 2 x ^ { 2 } - 7 x + 1 8 } \\
& \frac{-\left(2 x^{3}+4 x^{2}\right)}{-7 x^{2}+4 x} \\
& \frac{-\left(-7 x^{2}+14 x\right)}{18 x+5} \\
& \frac{-18 x+36}{-31} \\
& \text { Multiply } x+2 \text { by } 2 x^{2} \text {. } \\
& \text { Subtract. } \\
& \text { Bring down the next term. } \\
& -7 x^{2} \text { divided by } x \text { is }-7 x \text {. } \\
& \text { Multiply } x+2 \text { by }-7 x \text {. } \\
& \text { Subtract. Bring down the next term. } \\
& 18 x \text { divided by } x \text { is } 18 . \\
& \text { Multiply } x+2 \text { by } 18 \text {. } \\
& \text { Subtract. }
\end{aligned}
\]

We have found
\[
\frac{2 x^{3}-3 x^{2}+4 x+5}{x+2}=2 x^{2}-7 x+18-\frac{31}{x+2}
\]
or
\[
2 x^{3}-3 x^{2}+4 x+5=(x+2)\left(2 x^{2}-7 x+18\right)-31
\]

We can identify the dividend, divisor, quotient, and remainder.


Writing the result in this manner illustrates the Division Algorithm.

\section*{A GENERAL NOTE: THE DIVISION ALGORITHM}

The Division Algorithm states that given a polynomial dividend \(f(x)\) and a non-zero polynomial divisor \(d(x)\) where the degree of \(d(x)\) is less than or equal to the degree of \(f(x)\), there exist unique polynomials \(q(x)\) and \(r(x)\) such that
\(f(x)=d(x) q(x)+r(x)\)
\(q(x)\) is the quotient and \(r(x)\) is the remainder. The remainder is either equal to zero or has degree strictly less than \(d(x)\).
If \(r(x)=0\), then \(d(x)\) divides evenly into \(f(x)\). This means that both \(d(x)\) and \(q(x)\) are factors of \(f(x)\).

\section*{HOW TO: GIVEN A POLYNOMIAL AND A BINOMIAL, USE LONG DIVISION TO DIVIDE THE POLYNOMIAL BY THE BINOMIAL}
1. Set up the division problem.
2. Determine the first term of the quotient by dividing the leading term of the dividend by the leading term of the divisor.
3. Multiply the answer by the divisor and write it below the like terms of the dividend.
4. Subtract the bottom binomial from the terms above it.
5. Bring down the next term of the dividend.
6. Repeat steps \(2-5\) until reaching the last term of the dividend.
7. If the remainder is non-zero, express as a fraction using the divisor as the denominator.

\section*{EXAMPLE: USING LONG DIVISION TO DIVIDE A SECOND-DEGREE POLYNOMIAL}

Divide \(5 x^{2}+3 x-2\) by \(x+1\).
Answer
\[
\begin{aligned}
& x + 1 \longdiv { 5 x ^ { 2 } + 3 x - 2 } \quad \text { Set up division problem. } \\
& x + 1 \longdiv { 5 x } \quad 5 x ^ { 2 } \text { divided by } x \text { is } 5 x . \\
& x + 1 \longdiv { 5 x } \quad \quad \text { Multiply } x+1 \text { by } 5 x \text {. } \\
& -\left(5 x^{2}+5 x\right) \\
& -2 x-2 \\
& x + 1 \longdiv { 5 x ^ { 2 } + 3 x - 2 } \\
& \frac{-\left(5 x^{2}+5 x\right)}{-2 x-2} \\
& -(-2 x-2) \quad \text { Multiply } x+1 \text { by }-2 . \\
& \text { Subtract. }
\end{aligned}
\]

The quotient is \(5 x-2\). The remainder is 0 . We write the result as
\[
\frac{5 x^{2}+3 x-2}{x+1}=5 x-2
\]
or
\(5 x^{2}+3 x-2=(x+1)(5 x-2)\)

\section*{Analysis of the Solution}

This division problem had a remainder of 0 . This tells us that the dividend is divided evenly by the divisor and that the divisor is a factor of the dividend.

\section*{EXAMPLE: USING LONG DIVISION TO DIVIDE A THIRD-DEGREE POLYNOMIAL}

Divide \(6 x^{3}+11 x^{2}-31 x+15\) by \(3 x-2\).
Answer
\[
\begin{aligned}
& 2 x^{2}+5 x-7 \\
& 3 x-2 \begin{array}{l}
6 x^{3}+11 x^{2}-31 x+1 \\
-\left(6 x^{3}-4 x^{2}\right) \\
15 x^{2}-31 x \\
\frac{-\left(15 x^{2}+10 x\right)}{21 x+15} \\
\frac{-(-21 x+14)}{1}
\end{array} \text { Multiply } x+2 \text { by } 2 x^{2} . \\
& \text { Subtract. Bring down the next term. } 15 x^{2} \text { divided by } 3 x \text { is } 5 x . \\
& \text { Multiply } 3 x-2 \text { by } 5 x . \\
& \text { Subtract. Bring down the next term. }-21 x \text { divided by } 3 x \text { is }-7 . \\
& \text { Multiply } 3 x-2 \text { by }-7 . \\
& \text { Subtract. The remainder is } 1 .
\end{aligned}
\]

There is a remainder of 1 . We can express the result as:
\(\frac{6 x^{3}+11 x^{2}-31 x+15}{3 x-2}=2 x^{2}+5 x-7+\frac{1}{3 x-2}\)

\section*{Analysis of the Solution}

We can check our work by using the Division Algorithm to rewrite the solution then multiplying.
\((3 x-2)\left(2 x^{2}+5 x-7\right)+1=6 x^{3}+11 x^{2}-31 x+15\)
Notice, as we write our result,
- the dividend is \(6 x^{3}+11 x^{2}-31 x+15\)
- the divisor is \(3 x-2\)
- the quotient is \(2 x^{2}+5 x-7\)
- the remainder is 1

\section*{TRY IT}

Divide \(16 x^{3}-12 x^{2}+20 x-3\) by \(4 x+5\).
Answer
\(4 x^{2}-8 x+15-\frac{78}{4 x+5}\)
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=291

\section*{LEARNING OUTCOMES}
- Use synthetic division to divide polynomials.

As we've seen, long division with polynomials can involve many steps and be quite cumbersome. Synthetic division is a shorthand method of dividing polynomials for the special case of dividing by a linear factor whose leading coefficient is 1 .

To illustrate the process, recall the example at the beginning of the section.
Divide \(2 x^{3}-3 x^{2}+4 x+5\) by \(x+2\) using the long division algorithm.
The final form of the process looked like this:
\[
\begin{gathered}
\frac{2 x^{2}+x+18}{x+2 \sqrt{2 x^{3}-3 x^{2}+4 x+5}} \\
\frac{-\left(2 x^{3}+4 x^{2}\right)}{-7 x^{2}+4 x} \\
\frac{-\left(-7 x^{2}-14 x\right)}{18 x+5} \\
\frac{-(18 x+36)}{-31}
\end{gathered}
\]

There is a lot of repetition in the table. If we don't write the variables but instead line up their coefficients in columns under the division sign and also eliminate the partial products, we already have a simpler version of the entire problem.
\begin{tabular}{rrrr}
\(2 \lcm{2}\) & -3 & 4 & 5 \\
-2 & -4 & & \\
\hline & \(\frac{-7}{}\) & 14 & \\
\hline & \(\frac{18}{}\) & -36 \\
& & & -31
\end{tabular}

Synthetic division carries this simplification even a few more steps. Collapse the table by moving each of the rows up to fill any vacant spots. Also, instead of dividing by 2 , as we would in division of whole numbers, and then multiplying and subtracting the middle product, we change the sign of the "divisor" to -2 , multiply, and add. The process starts by bringing down the leading coefficient.
\[
\begin{aligned}
-2 & \begin{array}{rrrr}
2 & -3 & 4 & 5 \\
& -4 & 14 & -36 \\
\hline 2 & -7 & 18 & -31
\end{array}
\end{aligned}
\]

We then multiply it by the "divisor" and add, repeating this process column by column until there are no entries left. The bottom row represents the coefficients of the quotient; the last entry of the bottom row is the remainder. In this case, the quotient is \(2 x^{2}-7 x+18\) and the remainder is -31 . The process will be made more clear in the examples that follow.

\section*{HOW TO: GIVEN TWO POLYNOMIALS, USE SYNTHETIC DIVISION TO DIVIDE}
1. Write \(k\) for the divisor.
2. Write the coefficients of the dividend.
3. Bring the leading coefficient down.
4. Multiply the leading coefficient by \(k\). Write the product in the next column.
5. Add the terms of the second column.
6. Multiply the result by \(k\). Write the product in the next column.
7. Repeat steps 5 and 6 for the remaining columns.
8. Use the bottom numbers to write the quotient. The number in the last column is the remainder and has degree 0 , the next number from the right has degree 1 , the next number from the right has degree 2 , and so on.

Polynomial division can be used to solve a variety of application problems involving expressions for area and volume. We looked at an application at the beginning of this section. Now we will solve that problem in the following example.

\section*{EXAMPLE: USING SYNTHETIC DIVISION TO DIVIDE A SECOND-DEGREE POLYNOMIAL}

Use synthetic division to divide \(5 x^{2}-3 x-36\) by \(x-3\).

\section*{Answer}

Begin by setting up the synthetic division. Write \(k\) and the coefficients.
\[
\begin{aligned}
-2 & \begin{array}{rrrr}
2 & -3 & 4 & 5 \\
& -4 & 14 & -36 \\
\hline
\end{array} \begin{array}{rrrr} 
& -7 & 18 & -31
\end{array}
\end{aligned}
\]

Bring down the leading coefficient. Multiply the leading coefficient by \(k\).


Continue by adding the numbers in the second column. Multiply the resulting number by \(k\). Write the result in the next column. Then add the numbers in the third column.
3|rrr \begin{tabular}{rrr}
5 & -3 & -36 \\
& 15 & 36 \\
5 & 12 & 0
\end{tabular}

The result is \(5 x+12\). The remainder is 0 . So \(x-3\) is a factor of the original polynomial.

\section*{Analysis of the Solution}

Just as with long division, we can check our work by multiplying the quotient by the divisor and adding the remainder.
\((x-3)(5 x+12)+0=5 x^{2}-3 x-36\)

\section*{EXAMPLE: USING SYNTHETIC DIVISION TO DIVIDE A THIRD-DEGREE POLYNOMIAL}

Use synthetic division to divide \(4 x^{3}+10 x^{2}-6 x-20\) by \(x+2\).

\section*{Answer}

The binomial divisor is \(x+2\), so \(k=-2\). Add each column, multiply the result by -2 , and repeat until the last column is reached.
\(-2 |\)\begin{tabular}{rrrr}
4 & 10 & -6 & -20 \\
& -8 & -4 & 20 \\
4 & 2 & -10 & 0
\end{tabular}

The result is \(4 x^{2}+2 x-10\). The remainder is 0 . Thus, \(x+2\) is a factor of \(4 x^{3}+10 x^{2}-6 x-20\).

Analysis of the Solution
The graph of the polynomial function \(f(x)=4 x^{3}+10 x^{2}-6 x-20\) shows a zero at \(x=-2\). This confirms that \(x+2\) is a factor of \(4 x^{3}+10 x^{2}-6 x-20\).


\section*{EXAMPLE: USING SYNTHETIC DIVISION TO DIVIDE A FOURTH-DEGREE POLYNOMIAL}

Use synthetic division to divide \(-9 x^{4}+10 x^{3}+7 x^{2}-6\) by \(x-1\).
Answer
Notice there is no \(x\) term. We will use a zero as the coefficient for that term.
1 \begin{tabular}{ccccc}
-9 & 10 & 7 & 0 & -6 \\
& -9 & 1 & 8 & 8 \\
\hline-9 & 1 & 8 & 8 & 2
\end{tabular},\(~\)

The result is \(-9 x^{3}+x^{2}+8 x+8+\frac{2}{x-1}\).

\section*{TRY IT}

Use synthetic division to divide \(3 x^{4}+18 x^{3}-3 x+40\) by \(x+7\).
Answer
\(3 x^{3}-3 x^{2}+21 x-150+\frac{1,090}{x+7}\)
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\section*{EXAMPLE: USING POLYNOMIAL DIVISION IN AN APPLICATION PROBLEM}

The volume of a rectangular solid is given by the polynomial \(3 x^{4}-3 x^{3}-33 x^{2}+54 x\). The length of the solid is given by \(3 x\) and the width is given by \(x-2\). Find the height of the solid.

\section*{Answer}

There are a few ways to approach this problem. We need to divide the expression for the volume of the solid by the expressions for the length and width. Let us create a sketch.


We can now write an equation by substituting the known values into the formula for the volume of a rectangular solid.
\(V=l \cdot w \cdot h\)
\(3 x^{4}-3 x^{3}-33 x^{2}+54 x=3 x \cdot(x-2) \cdot h\)
To solve for \(h\), first divide both sides by \(3 x\).
\[
\begin{aligned}
& \frac{3 x \cdot(x-2) \cdot h}{3 x}=\frac{3 x^{4}-3 x^{3}-33 x^{2}+54 x}{3 x} \\
& (x-2) h=x^{3}-x^{2}-11 x+18
\end{aligned}
\]

Now solve for \(h\) using synthetic division.
\(h=\frac{x^{3}-x^{2}-11 x+18}{x-2}\)


The quotient is \(x^{2}+x-9\) and the remainder is 0 . The height of the solid is \(x^{2}+x-9\).

\section*{TRY IT}

The area of a rectangle is given by \(3 x^{3}+14 x^{2}-23 x+6\). The width of the rectangle is given by \(x+\) 6 . Find an expression for the length of the rectangle.
Answer
\(3 x^{2}-4 x+1\)

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\section*{SUMMARY: DIVIDING POLYNOMIALS}

\section*{Key Equations}

> \begin{tabular}{l|l}  Division Algorithm & \(f(x)=d(x) q(x)+r(x)\) where \(q(x) \neq 0\) \end{tabular}

\section*{Key Concepts}
- Polynomial long division can be used to divide a polynomial by any polynomial with equal or lower degree.
- The Division Algorithm tells us that a polynomial dividend can be written as the product of the divisor and the quotient added to the remainder.
- Synthetic division is a shortcut that can be used to divide a polynomial by a binomial of the form \(x-k\).
- Polynomial division can be used to solve application problems, including area and volume.

\section*{Glossary}

Division Algorithm given a polynomial dividend \(f(x)\) and a non-zero polynomial divisor \(d(x)\) where the degree of \(d(x)\) is less than or equal to the degree of \(f(x)\), there exist unique polynomials \(q(x)\) and \(r(x)\) such that \(f(x)=d(x) q(x)+r(x)\) where \(q(x)\) is the quotient and \(r(x)\) is the remainder. The remainder is either equal to zero or has degree strictly less than \(d(x)\).
synthetic division a shortcut method that can be used to divide a polynomial by a binomial of the form \(x-k\)

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\title{
INTRODUCTION TO METHODS FOR FINDING ZEROS OF POLYNOMIALS
}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Evaluate a polynomial using the Remainder Theorem.
- Use the Rational Zero Theorem to find rational zeros.
- Use the Factor Theorem to solve a polynomial equation.
- Use synthetic division to find the zeros of a polynomial function.
- Use the Fundamental Theorem of Algebra to find complex zeros of a polynomial function.
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes' Rule of Signs to determine the maximum number of possible real zeros of a polynomial function.
- Solve real-world applications of polynomial equations.

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

This problem can be solved by writing a cubic function and solving a cubic equation for the volume of the cake. In this section, we will discuss a variety of tools for writing polynomial functions and solving polynomial equations.

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\section*{THEOREMS USED TO ANALYZE POLYNOMIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Evaluate a polynomial using the Remainder Theorem.
- Use the rational zeros theorem to find rational zeros of a polynomial.
- Use the Factor Theorem to solve a polynomial equation.

In the last section, we learned how to divide polynomials. We can now use polynomial division to evaluate polynomials using the Remainder Theorem. If the polynomial is divided by \(x-k\), the remainder may be found quickly by evaluating the polynomial function at \(k\), that is, \(f(k)\). Let's walk through the proof of the theorem.

Recall that the Division Algorithm states that given a polynomial dividend \(f(x)\) and a non-zero polynomial divisor \(d(x)\) where the degree of \(d(x)\) is less than or equal to the degree of \(f(x)\), there exist unique polynomials \(q(x)\) and \(r(x)\) such that
\(f(x)=d(x) q(x)+r(x)\)
If the divisor, \(d(x)\), is \(x-k\), this takes the form
\(f(x)=(x-k) q(x)+r\)
Since the divisor \(x-k\) is linear, the remainder will be a constant, \(r\). And, if we evaluate this for \(x=k\), we have
\(f(k)=(k-k) q(k)+r\)
\(f(k)=0 \cdot q(k)+r\)
\(f(k)=r\)
In other words, \(f(k)\) is the remainder obtained by dividing \(f(x)\) by \(x-k\).

\section*{A GENERAL NOTE: THE REMAINDER THEOREM}

If a polynomial \(f(x)\) is divided by \(x-k\), then the remainder is the value \(f(k)\).

\section*{HOW TO: GIVEN A POLYNOMIAL FUNCTION \(f\), EVALUATE \(f(x)\) AT \(x=k\) USING THE REMAINDER THEOREM}
1. Use synthetic division to divide the polynomial by \(x-k\).
2. The remainder is the value \(f(k)\).

\section*{EXAMPLE: USING THE REMAINDER THEOREM TO EVALUATE A POLYNOMIAL}

Use the Remainder Theorem to evaluate \(f(x)=6 x^{4}-x^{3}-15 x^{2}+2 x-7\) at \(x=2\).
Answer
To find the remainder using the Remainder Theorem, use synthetic division to divide the polynomial by \(x-2\).

2loverline\{)6 \(\quad-1 \quad-15 \quad 2 \quad-7\)
\begin{tabular}{lllll}
12 & 22 & 14 & \(32\}\) \\
\hline
\end{tabular}
\(\begin{array}{lllll}6 & 11 & 7 & 16 & 25\end{array}\)
The remainder is 25 . Therefore, \(f(2)=25\).

Analysis of the Solution
We can check our answer by evaluating \(f(2)\).
\(f(x)=6 x^{4}-x^{3}-15 x^{2}+2 x-7\)
\(f(2)=6(2)^{4}-(2)^{3}-15(2)^{2}+2(2)-7\)
\(f(2)=25\)

Use the Remainder Theorem to evaluate \(f(x)=2 x^{5}-3 x^{4}-9 x^{3}+8 x^{2}+2\)
at \(x=-3\).

\section*{Answer}
\(4 x^{2}-8 x+15-\frac{78}{4 x+5}\)
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\section*{Using the Rational Zero Theorem to Find Rational Zeros}

Another use for the Remainder Theorem is to test whether a rational number is a zero for a given polynomial, but first we need a pool of rational numbers to test. The Rational Zero Theorem helps us to narrow down the number of possible rational zeros using the ratio of the factors of the constant term and factors of the leading coefficient of the polynomial

Consider a quadratic function with two zeros, \(x=\frac{2}{5}\) and \(x=\frac{3}{4}\).
These zeros have factors associated with them. Let us set each factor equal to 0 and then construct the original quadratic function.
\[
\begin{array}{ll}
x-\frac{2}{5}=0 \text { or } x-\frac{3}{4}=0 & \text { Set each factor equal to } 0 . \\
5 x-2=0 \text { or } 4 x-3=0 & \text { Multiply both sides of the equation to eliminate fractions. } \\
f(x)=(5 x-2)(4 x-3) & \text { Create the quadratic function, multiplying the factors. } \\
f(x)=20 x^{2}-23 x+6 & \text { Expand the polynomial. } \\
f(x)=(5 \cdot 4) x^{2}-23 x+(2 \cdot 3) &
\end{array}
\]

Notice that two of the factors of the constant term, 6, are the two numerators from the original rational roots: 2 and 3 . Similarly, two of the factors from the leading coefficient, 20 , are the two denominators from the original rational roots: 5 and 4.

We can infer that the numerators of the rational roots will always be factors of the constant term and the denominators will be factors of the leading coefficient. This is the essence of the Rational Zero Theorem; it is a means to give us a pool of possible rational zeros.

\section*{A GENERAL NOTE: THE RATIONAL ZERO THEOREM}

The Rational Zero Theorem states that if the polynomial \(f(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\ldots+a_{1} x+a_{0}\) has integer coefficients, then every rational zero of \(f(x)\) has the form \(\frac{p}{q}\) where \(p\) is a factor of the constant term \(a_{0}\) and \(q\) is a factor of the leading coefficient \(a_{n}\).
When the leading coefficient is 1 , the possible rational zeros are the factors of the constant term.

\section*{HOW TO: GIVEN A POLYNOMIAL FUNCTION \(f(x)\), USE THE RATIONAL ZERO THEOREM TO FIND RATIONAL ZEROS}
1. Determine all factors of the constant term and all factors of the leading coefficient.
2. Determine all possible values of \(\frac{p}{q}\), where \(p\) is a factor of the constant term and \(q\) is a factor of the leading coefficient. Be sure to include both positive and negative candidates.
3. Determine which possible zeros are actual zeros by evaluating each case of \(f\left(\frac{p}{q}\right)\).

\section*{EXAMPLE: LISTING ALL POSSIBLE RATIONAL ZEROS}

List all possible rational zeros of \(f(x)=2 x^{4}-5 x^{3}+x^{2}-4\).

\section*{Answer}

The only possible rational zeros of \(f(x)\) are the quotients of the factors of the last term, -4 , and the factors of the leading coefficient, 2.
The constant term is -4 ; the factors of -4 are \(p= \pm 1, \pm 2, \pm 4\).
The leading coefficient is 2 ; the factors of 2 are \(q= \pm 1, \pm 2\).
If any of the four real zeros are rational zeros, then they will be of one of the following factors of -4 divided by one of the factors of 2 .
\[
\frac{p}{q}= \pm \frac{1}{1}, \pm \frac{1}{2} \quad \frac{p}{q}= \pm \frac{2}{1}, \pm \frac{2}{2} \quad \frac{p}{q}= \pm \frac{4}{1}, \pm \frac{4}{2}
\]

Note that \(\frac{2}{2}=1\) and \(\frac{4}{2}=2\), which have already been listed, so we can shorten our list.
\[
\frac{p}{q}=\frac{\text { Factors of the constant term }}{\text { Factors of the leading coefficient }}= \pm 1, \pm 2, \pm 4, \pm \frac{1}{2}
\]

\section*{EXAMPLE: USING THE RATIONAL ZERO THEOREM TO FIND RATIONAL ZEROS}

Use the Rational Zero Theorem to find the rational zeros of \(f(x)=2 x^{3}+x^{2}-4 x+1\).

\section*{Answer}

The Rational Zero Theorem tells us that if \(\frac{p}{q}\) is a zero of \(f(x)\), then \(p\) is a factor of 1 and \(q\) is a factor of 2.
\[
\begin{aligned}
& \frac{p}{q}=\frac{\text { Factors of the constant term }}{\text { Factors of the leading coefficient }} \\
& \frac{p}{q}=\frac{\text { Factors of } 1}{\text { Factors of } 2}
\end{aligned}
\]

The factors of 1 are \(\pm 1\) and the factors of 2 are \(\pm 1\) and \(\pm 2\). The possible values for \(\frac{p}{q}\) are \(\pm 1\) and \(\pm \frac{1}{2}\). These are the possible rational zeros for the function. We can determine which of the possible zeros are actual zeros by substituting these values for \(x\) in \(f(x)\).
\[
\begin{aligned}
& f(-1)=2(-1)^{3}+(-1)^{2}-4(-1)+1=4 \\
& f(1)=2(1)^{3}+(1)^{2}-4(1)+1=0 \\
& f\left(-\frac{1}{2}\right)=2\left(-\frac{1}{2}\right)^{3}+\left(-\frac{1}{2}\right)^{2}-4\left(-\frac{1}{2}\right)+1=3 \\
& f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^{3}+\left(\frac{1}{2}\right)^{2}-4\left(\frac{1}{2}\right)+1=-\frac{1}{2}
\end{aligned}
\]

Of those, \(-1,-\frac{1}{2}\), and \(\frac{1}{2}\) are not zeros of \(f(x)\). 1 is the only rational zero of \(f(x)\).

\section*{TRY IT}

Use the Rational Zero Theorem to find the rational zeros of \(f(x)=x^{3}-3 x^{2}-6 x+8\).

\section*{Answer}
\(-2,1\), and 4 are zeros of the polynomial.

\section*{Using the Factor Theorem to Solve a Polynomial Equation}

The Factor Theorem is another theorem that helps us analyze polynomial equations. It tells us how the zeros of a polynomial are related to the factors. Recall that the Division Algorithm tells us \(f(x)=(x-k) q(x)+r\).

If \(k\) is a zero, then the remainder \(r\) is \(f(k)=0\) and \(f(x)=(x-k) q(x)+0\) or \(f(x)=(x-k) q(x)\).
Notice, written in this form, \(x-k\) is a factor of \(f(x)\). We can conclude if \(k\) is a zero of \(f(x)\), then \(x-k\) is a factor of \(f(x)\).

Similarly, if \(x-k\) is a factor of \(f(x)\), then the remainder of the Division Algorithm \(f(x)=(x-k) q(x)+r\) is 0 . This tells us that \(k\) is a zero.

This pair of implications is the Factor Theorem. As we will soon see, a polynomial of degree \(n\) in the complex number system will have \(n\) zeros. We can use the Factor Theorem to completely factor a polynomial into the product of \(n\) factors. Once the polynomial has been completely factored, we can easily determine the zeros of the polynomial.

\section*{A GENERAL NOTE: THE FACTOR THEOREM}

According to the Factor Theorem, \(k\) is a zero of \(f(x)\) if and only if \((x-k)\) is a factor of \(f(x)\).

\section*{HOW TO: GIVEN A FACTOR AND A THIRD-DEGREE POLYNOMIAL, USE THE FACTOR THEOREM TO FACTOR THE POLYNOMIAL}
1. Use synthetic division to divide the polynomial by \((x-k)\).
2. Confirm that the remainder is 0 .
3. Write the polynomial as the product of \((x-k)\) and the quadratic quotient.
4. If possible, factor the quadratic.
5. Write the polynomial as the product of factors.

\section*{EXAMPLE: USING THE FACTOR THEOREM TO SOLVE A POLYNOMIAL EQUATION}

Show that \((x+2)\) is a factor of \(x^{3}-6 x^{2}-x+30\). Find the remaining factors. Use the factors to determine the zeros of the polynomial.
Answer
We can use synthetic division to show that \((x+2)\) is a factor of the polynomial.
\begin{tabular}{rl}
-2 & \begin{tabular}{rrrr}
1 & -6 & -1 & 30 \\
& -2 & 16 & -30 \\
\hline
\end{tabular} \\
\begin{tabular}{llll}
1 & -8 & 15 & 0
\end{tabular}
\end{tabular}

The remainder is zero, so \((x+2)\) is a factor of the polynomial. We can use the Division Algorithm to write the polynomial as the product of the divisor and the quotient:
\((x+2)\left(x^{2}-8 x+15\right)\)
We can factor the quadratic factor to write the polynomial as
\((x+2)(x-3)(x-5)\)
By the Factor Theorem, the zeros of \(x^{3}-6 x^{2}-x+30\) are \(-2,3\), and 5 .

\section*{TRY IT}

Use the Factor Theorem to find the zeros of \(f(x)=x^{3}+4 x^{2}-4 x-16\) given that \((x-2)\) is a factor of the polynomial.

\section*{Answer}

The zeros are 2, -2 , and -4 .
Now use an online graphing calculator to graph \(f(x)\).
On the next line, enter \(g(x)=\frac{f(x)}{(x-2)}\).
Where does \(g(x)\) cross the x-axis? How do those roots compare to the solution we found using the Factor Theorem?

\section*{FINDING ZEROS OF A POLYNOMIAL FUNCTION}

\section*{LEARNING OUTCOMES}
- Use synthetic division to find the zeros of a polynomial function.
- Use the Fundamental Theorem of Algebra to find complex zeros of a polynomial function.

The Rational Zero Theorem helps us to narrow down the list of possible rational zeros for a polynomial function. Once we have done this, we can use synthetic division repeatedly to determine all of the zeros of a polynomial function.

\section*{HOW TO: GIVEN A POLYNOMIAL FUNCTION \(f\), USE SYNTHETIC DIVISION TO FIND ITS ZEROS}
1. Use the Rational Zero Theorem to list all possible rational zeros of the function.
2. Use synthetic division to evaluate a given possible zero by synthetically dividing the candidate into the polynomial. If the remainder is 0 , the candidate is a zero. If the remainder is not zero, discard the candidate.
3. Repeat step two using the quotient found from synthetic division. If possible, continue until the quotient is a quadratic.
4. Find the zeros of the quadratic function. Two possible methods for solving quadratics are factoring and using the quadratic formula.

\section*{The Fundamental Theorem of Algebra}

Now that we can find rational zeros for a polynomial function, we will look at a theorem that discusses the number of complex zeros of a polynomial function. The Fundamental Theorem of Algebra tells us that every polynomial function has at least one complex zero. This theorem forms the foundation for solving polynomial equations.

Suppose \(f\) is a polynomial function of degree four and \(f(x)=0\). The Fundamental Theorem of Algebra states that there is at least one complex solution, call it \(c_{1}\). By the Factor Theorem, we can write \(f(x)\) as a product of \(x-c_{1}\) and a polynomial quotient. Since \(x-c_{1}\) is linear, the polynomial quotient will be of degree three. Now we apply the Fundamental Theorem of Algebra to the third-degree polynomial quotient. It will have at least one complex zero, call it \(c_{2}\). We can write the polynomial quotient as a product of \(x-c_{2}\) and a new polynomial quotient of degree two. Continue to apply the Fundamental Theorem of Algebra until all of the zeros are found. There will be four of them and each one will yield a factor of \(f(x)\).

\section*{EXAMPLE: FINDING THE ZEROS OF A POLYNOMIAL FUNCTION WITH REPEATED REAL ZEROS}

Find the zeros of \(f(x)=4 x^{3}-3 x-1\).

\section*{Answer}

The Rational Zero Theorem tells us that if \(\frac{p}{q}\) is a zero of \(f(x)\), then \(p\) is a factor of -1 and \(q\) is a factor of 4.
\(\frac{p}{q}=\frac{\text { Factors of the constant term }}{\text { Factors of the leading coefficient }}\)
\(\frac{p}{q}=\frac{\text { Factors of }-1}{\text { Factors of } 4}\)
The factors of -1 are \(\pm 1\) and the factors of 4 are \(\pm 1, \pm 2\), and \(\pm 4\). The possible values for \(\frac{p}{q}\) are \(\pm 1, \pm \frac{1}{2}\), and \(\pm \frac{1}{4}\).
These are the possible rational zeros for the function. We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0 . Let's begin with 1.


Dividing by \((x-1)\) gives a remainder of 0 , so 1 is a zero of the function. The polynomial can be written as \((x-1)\left(4 x^{2}+4 x+1\right)\).
The quadratic is a perfect square. \(f(x)\) can be written as \((x-1)(2 x+1)^{2}\).
We already know that 1 is a zero. The other zero will have a multiplicity of 2 because the factor is squared. To find the other zero, we can set the factor equal to 0 .
\[
\begin{aligned}
& 2 x+1=0 \\
& x=-\frac{1}{2}
\end{aligned}
\]

The zeros of the function are 1 and \(-\frac{1}{2}\) with multiplicity 2.

\section*{Analysis of the Solution}

Look at the graph of the function \(f\). Notice, at \(x=-0.5\), the graph bounces off the \(x\)-axis, indicating the even multiplicity \((2,4,6 \ldots)\) for the zero -0.5 . At \(x=1\), the graph crosses the \(x\)-axis, indicating the odd multiplicity \((1,3,5 \ldots)\) for the zero \(x=1\).


\section*{A GENERAL NOTE: THE FUNDAMENTAL THEOREM OF ALGEBRA}

The Fundamental Theorem of Algebra states that, if \(f(x)\) is a polynomial of degree \(n>0\), then \(f(x)\) has at least one complex zero.
We can use this theorem to argue that, if \(f(x)\) is a polynomial of degree \(n>0\), and \(a\) is a non-zero real number, then \(f(x)\) has exactly \(n\) linear factors.
The polynomial can be written as
\(f(x)=a\left(x-c_{1}\right)\left(x-c_{2}\right) \ldots\left(x-c_{n}\right)\)
where \(c_{1}, c_{2}, \ldots, c_{n}\) are complex numbers. Therefore, \(f(x)\) has \(n\) roots if we allow for multiplicities.

\section*{RECALL COMPLEX NUMBERS}

Recall that we defined complex numbers as numbers of the form \(a+b i\). To graph such numbers required the complex plane, made up of a real axis and an imaginary axis. This plane defined the real numbers as a subset of the complex numbers, just as the rational and irrational numbers are subsets of the real numbers.
In other words, each real number is also a complex number of the form \(a+b i\), where \(b=0\).

\section*{Q \& A}

Does every polynomial have at least one imaginary zero?
No. A complex number is not necessarily imaginary. Real numbers are also complex numbers.

\section*{EXAMPLE: FINDING THE ZEROS OF A POLYNOMIAL FUNCTION WITH COMPLEX ZEROS}

Find the zeros of \(f(x)=3 x^{3}+9 x^{2}+x+3\).

\section*{Answer}

The Rational Zero Theorem tells us that if \(\frac{p}{q}\) is a zero of \(f(x)\), then \(p\) is a factor of 3 and \(q\) is a factor of 3 .
\(\frac{p}{q}=\frac{\text { Factors of the constant term }}{\text { Factor of the leading coefficient }}\)
\(\frac{p}{q}=\frac{\text { Factors of } 3}{\text { Factors of } 3}\)
The factors of 3 are \(\pm 1\) and \(\pm 3\). The possible values for \(\frac{p}{q}\), and therefore the possible rational zeros for the function, are \(\pm 3, \pm 1\), and \(\pm \frac{1}{3}\). We will use synthetic division to evaluate each possible zero until we find one that gives a remainder of 0 . Let's begin with -3 .


Dividing by \((x+3)\) gives a remainder of 0 , so -3 is a zero of the function. The polynomial can be written as \((x+3)\left(3 x^{2}+1\right)\).
We can then set the quadratic equal to 0 and solve to find the other zeros of the function.
\[
\begin{aligned}
& 3 x^{2}+1=0 \\
& x^{2}=-\frac{1}{3} \\
& x= \pm \sqrt{-\frac{1}{3}}= \pm \frac{i \sqrt{3}}{3}
\end{aligned}
\]

The zeros of \(f(x)\) are -3 and \(\pm \frac{i \sqrt{3}}{3}\).

\section*{Analysis of the Solution}

Look at the graph of the function \(f\). Notice that, at \(x=-3\), the graph crosses the \(x\)-axis, indicating an odd multiplicity (1) for the zero \(x=-3\). Also note the presence of the two turning points. This means that, since there is a \(3^{\text {rd }}\) degree polynomial, we are looking at the maximum number of turning points. So, the end behavior of increasing without bound to the right and decreasing without bound to the left will continue. Thus, all the \(x\)-intercepts for the function are shown. So either the multiplicity of \(x=-3\) is 1 and there are two complex solutions, which is what we found, or the multiplicity at \(x=-3\) is three. Either way, our result is correct.


\section*{TRY IT}

Find the zeros of \(f(x)=2 x^{3}+5 x^{2}-11 x+4\).
Answer
The zeros are \(-4, \frac{1}{2}\), and 1 .
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\section*{LINEAR FACTORIZATION AND DESCARTES RULE OF SIGNS}

\section*{LEARNING OUTCOMES}
- Use the Linear Factorization Theorem to find polynomials with given zeros.
- Use Descartes rule of signs to determine the maximum number of possible real zeros of a polynomial function
- Solve real-world applications of polynomial equations.

A vital implication of the Fundamental Theorem of Algebra is that a polynomial function of degree \(n\) will have \(n\) zeros in the set of complex numbers if we allow for multiplicities. This means that we can factor the polynomial function into \(n\) factors. The Linear Factorization Theorem tells us that a polynomial function will have the same number of factors as its degree, and each factor will be of the form \((x-c)\) where \(c\) is a complex number.

\section*{RECALL MULTIPLYING COMPLEX CONJUGATES}

When you learned to divide complex numbers, you multiplied the top and bottom of the quotient of complex numbers deliberately by the conjugate of the denominator so that the imaginary part would eliminate from the denominator. We'll use the same idea in the paragraph below.
Ex. \((a+b i)(a-b i) \Rightarrow a^{2}-a b i+a b i-b^{2} i^{2} \Rightarrow a^{2}-b^{2}(-1) \Rightarrow a^{2}+b^{2}\).
That is, multiplying complex conjugates eliminates the imaginary part.
Let \(f\) be a polynomial function with real coefficients and suppose \(a+b i, b \neq 0\), is a zero of \(f(x)\). Then, by the Factor Theorem, \(x-(a+b i)\) is a factor of \(f(x)\). For \(f\) to have real coefficients, \(x-(a-b i)\) must also be a factor of \(f(x)\). This is true because any factor other than \(x-(a-b i)\), when multiplied by \(x-(a+b i)\), will leave imaginary components in the product. Only multiplication with conjugate pairs will eliminate the imaginary parts and result in real coefficients. In other words, if a polynomial function \(f\) with real coefficients has a complex zero \(a+b i\), then the complex conjugate \(a-b i\) must also be a zero of \(f(x)\). This is called the Complex Conjugate Theorem.

\section*{A GENERAL NOTE: COMPLEX CONJUGATE THEOREM}

According to the Linear Factorization Theorem, a polynomial function will have the same number of factors as its degree, and each factor will be of the form \((x-c)\) where \(c\) is a complex number.
If the polynomial function \(f\) has real coefficients and a complex zero of the form \(a+b i\), then the complex conjugate of the zero, \(a-b i\), is also a zero.

\section*{HOW TO: GIVEN THE ZEROS OF A POLYNOMIAL FUNCTION \(f\) AND A POINT \((c, f(c))\) ON THE GRAPH OF \(f\), USE THE LINEAR FACTORIZATION THEOREM TO FIND THE POLYNOMIAL FUNCTION}
1. Use the zeros to construct the linear factors of the polynomial.
2. Multiply the linear factors to expand the polynomial.
3. Substitute ( \(c, f(c)\) ) into the function to determine the leading coefficient.
4. Simplify.

\section*{Descartes' Rule of Signs}

There is a straightforward way to determine the possible numbers of positive and negative real zeros for any polynomial function. If the polynomial is written in descending order, Descartes' Rule of Signs tells us of a relationship between the number of sign changes in \(f(x)\) and the number of positive real zeros.

There is a similar relationship between the number of sign changes in \(f(-x)\) and the number of negative real zeros.

\section*{Solving Real-world Applications of Polynomial Equations}

We have now introduced a variety of tools for solving polynomial equations. Let's use these tools to solve the bakery problem from the beginning of the section.

\section*{EXAMPLE: USING THE LINEAR FACTORIZATION THEOREM TO FIND A POLYNOMIAL WITH GIVEN ZEROS}

Find a fourth degree polynomial with real coefficients that has zeros of \(-3,2, i\), such that \(f(-2)=100\).

\section*{Answer}

Because \(x=i\) is a zero, by the Complex Conjugate Theorem \(x=-i\) is also a zero. The polynomial must have factors of \((x+3),(x-2),(x-i)\), and \((x+i)\). Since we are looking for a degree 4 polynomial and now have four zeros, we have all four factors. Let's begin by multiplying these factors.
\[
\begin{aligned}
& f(x)=a(x+3)(x-2)(x-i)(x+i) \\
& f(x)=a\left(x^{2}+x-6\right)\left(x^{2}+1\right) \\
& f(x)=a\left(x^{4}+x^{3}-5 x^{2}+x-6\right)
\end{aligned}
\]

We need to find a to ensure \(f(-2)=100\). Substitute \(x=-2\) and \(f(2)=100\) into \(f(x)\).
\[
\begin{aligned}
& 100=a\left((-2)^{4}+(-2)^{3}-5(-2)^{2}+(-2)-6\right) \\
& 100=a(-20) \\
& -5=a
\end{aligned}
\]

So the polynomial function is:
\(f(x)=-5\left(x^{4}+x^{3}-5 x^{2}+x-6\right)\)
or
\(f(x)=-5 x^{4}-5 x^{3}+25 x^{2}-5 x+30\)

\section*{Analysis of the Solution}

We found that both \(i\) and \(-i\) were zeros, but only one of these zeros needed to be given. If \(i\) is a zero of a polynomial with real coefficients, then -i must also be a zero of the polynomial because \(-i\) is the complex conjugate of \(i\).

\section*{Q \& A}

If \(2+3 i\) were given as a zero of a polynomial with real coefficients, would \(2-3 i\) also need to be a zero?
Yes. When any complex number with an imaginary component is given as a zero of a polynomial with real coefficients, the conjugate must also be a zero of the polynomial.

\section*{TRY IT}

Find a third degree polynomial with real coefficients that has zeros of 5 and \(-2 i\) such that \(f(1)=10\).
Answer
\(f(x)=-\frac{1}{2} x^{3}+\frac{5}{2} x^{2}-2 x+10\)

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\section*{A GENERAL NOTE: DESCARTES' RULE OF SIGNS}

According to Descartes' Rule of Signs, if we let \(f(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\ldots+a_{1} x+a_{0}\) be a polynomial function with real coefficients:
- The number of positive real zeros is either equal to the number of sign changes of \(f(x)\) or is less than the number of sign changes by an even integer.
- The number of negative real zeros is either equal to the number of sign changes of \(f(-x)\) or is less than the number of sign changes by an even integer.

\section*{EXAMPLE: USING DESCARTES' RULE OF SIGNS}

Use Descartes' Rule of Signs to determine the possible numbers of positive and negative real zeros for \(f(x)=-x^{4}-3 x^{3}+6 x^{2}-4 x-12\).
Answer
Begin by determining the number of sign changes.
\[
f(x)=-x^{4}-3 x^{3}+6 x^{2}-4 x-12
\]

There are two sign changes, so there are either 2 or 0 positive real roots. Next, we examine \(f(-x)\) to determine the number of negative real roots.
\[
\begin{aligned}
& f(-x)=-(-x)^{4}-3(-x)^{3}+6(-x)^{2}-4(-x)-12 \\
& f(-x)=-x^{4}+3 x^{3}+6 x^{2}+4 x-12
\end{aligned}
\]
\[
f(-x)=-\underbrace{4}+3 x^{3}+6 x^{2}+4 x-12
\]

Again, there are two sign changes, so there are either 2 or 0 negative real roots.
There are four possibilities, as we can see below.
\begin{tabular}{|l|l|l|l|}
\hline Positive Real Zeros & Negative Real Zeros & \multicolumn{1}{c|}{ Complex Zeros } & Total Zeros \\
\hline 2 & 2 & 0 & 4 \\
\hline 2 & 0 & 2 & 4 \\
\hline 0 & 2 & 2 & 4 \\
\hline 0 & 0 & 4 & 4 \\
\hline
\end{tabular}

\section*{Analysis of the Solution}

We can confirm the numbers of positive and negative real roots by examining a graph of the function. We can see from the graph that the function has 0 positive real roots and 2 negative real roots.


\section*{TRY IT}

Use Descartes' Rule of Signs to determine the maximum possible number of positive and negative real zeros for \(f(x)=2 x^{4}-10 x^{3}+11 x^{2}-15 x+12\). Use a graph to verify the number of positive and negative real zeros for the function.
Answer
There must be 4,2 , or 0 positive real roots and 0 negative real roots. The graph shows that there are 2 positive real zeros and 0 negative real zeros.

\section*{EXAMPLE: SOLVING POLYNOMIAL EQUATIONS}

A new bakery offers decorated sheet cakes for children's birthday parties and other special occasions. The bakery wants the volume of a small cake to be 351 cubic inches. The cake is in the shape of a rectangular solid. They want the length of the cake to be four inches longer than the width of the cake and the height of the cake to be one-third of the width. What should the dimensions of the cake pan be?

\section*{Answer}

Begin by writing an equation for the volume of the cake. The volume of a rectangular solid is given by \(V=l w h\). We were given that the length must be four inches longer than the width, so we can express the length of the cake as \(l=w+4\). We were given that the height of the cake is one-third of the width, so we can express the height of the cake as \(h=\frac{1}{3} w\). Let's write the volume of the cake in terms of width of the cake.
\[
\begin{aligned}
& V=(w+4)(w)\left(\frac{1}{3} w\right) \\
& V=\frac{1}{3} w^{3}+\frac{4}{3} w^{2}
\end{aligned}
\]

Substitute the given volume into this equation.
\[
\begin{array}{ll}
351=\frac{1}{3} w^{3}+\frac{4}{3} w^{2} & \text { Substitute } 351 \text { for } V . \\
1053=w^{3}+4 w^{2} & \text { Multiply both sides by } 3 . \\
0=w^{3}+4 w^{2}-1053 & \text { Subtract } 1053 \text { from both sides. }
\end{array}
\]

Descartes' rule of signs tells us there is one positive solution. The Rational Zero Theorem tells us that the possible rational zeros are \(\pm 3, \pm 9, \pm 13, \pm 27, \pm 39, \pm 81, \pm 117, \pm 351\), and \(\pm 1053\). We can use synthetic division to test these possible zeros. Only positive numbers make sense as dimensions for a cake, so we need not test any negative values. Let's begin by testing values that make the most sense as dimensions for a small sheet cake. Use synthetic division to check \(x=1\).
\[
\begin{array}{r|rrrr}
1 & 1 & 4 & 0 & -1053 \\
& & 1 & 5 & 5
\end{array}
\]
\(\begin{array}{llll}1 & 5 & 5 & -1048\end{array}\)
Since 1 is not a solution, we will check \(x=3\).
\[
\begin{aligned}
3 & \left.\begin{array}{rrrr}
1 & 4 & 0 & -1053 \\
& 3 & 21 & 63 \\
\hline & 7 & 21 & -990
\end{array} . \begin{array}{rl} 
&
\end{array}\right)
\end{aligned}
\]

Since 3 is not a solution either, we will test \(x=9\).
\[
\begin{array}{rl|rrr}
9 & 1 & 4 & 0 & -1053 \\
& 9 & 117 & 1053 \\
\hline & 1 & 13 & 117 & 0
\end{array}
\]

Synthetic division gives a remainder of 0 , so 9 is a solution to the equation. We can use the relationships between the width and the other dimensions to determine the length and height of the sheet cake pan.
\(l=w+4=9+4=13\) and \(h=\frac{1}{3} w=\frac{1}{3}(9)=3\)
The sheet cake pan should have dimensions 13 inches by 9 inches by 3 inches.

\section*{TRY IT}

A shipping container in the shape of a rectangular solid must have a volume of 84 cubic meters. The client tells the manufacturer that, because of the contents, the length of the container must be one meter longer
than the width, and the height must be one meter greater than twice the width. What should the dimensions of the container be?
Answer
3 meters by 4 meters by 7 meters

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\section*{SUMMARY: METHODS FOR FINDING ZEROS OF POLYNOMIAL FUNCTIONS}

\section*{Key Concepts}
- To find \(f(k)\), determine the remainder of the polynomial \(f(x)\) when it is divided by \(x-k\).
- \(k\) is a zero of \(f(x)\) if and only if \((x-k)\) is a factor of \(f(x)\).
- Each rational zero of a polynomial function with integer coefficients will be equal to a factor of the constant term divided by a factor of the leading coefficient.
- When the leading coefficient is 1 , the possible rational zeros are the factors of the constant term.
- Synthetic division can be used to find the zeros of a polynomial function.
- According to the Fundamental Theorem of Algebra, every polynomial function has at least one complex zero.
- Every polynomial function with degree greater than 0 has at least one complex zero.
- Allowing for multiplicities, a polynomial function will have the same number of factors as its degree. Each factor will be in the form \((x-c)\) where \(c\) is a complex number.
- The number of positive real zeros of a polynomial function is either the number of sign changes of the function or less than the number of sign changes by an even integer.
- The number of negative real zeros of a polynomial function is either the number of sign changes of \(f(-x)\) or less than the number of sign changes by an even integer.
- Polynomial equations model many real-world scenarios. Solving the equations is easiest done by synthetic division.

\section*{Glossary}

Descartes' Rule of Signs a rule that determines the maximum possible numbers of positive and negative real zeros based on the number of sign changes of \(f(x)\) and \(f(-x)\)

Factor Theorem \(k\) is a zero of polynomial function \(f(x)\) if and only if \((x-k)\) is a factor of \(f(x)\)
Fundamental Theorem of Algebra a polynomial function with degree greater than 0 has at least one complex zero

Linear Factorization Theorem allowing for multiplicities, a polynomial function will have the same number of factors as its degree, and each factor will be in the form \((x-c)\) where \(c\) is a complex number

Rational Zero Theorem the possible rational zeros of a polynomial function have the form \(\frac{p}{q}\) where \(p\) is a factor of the constant term and \(q\) is a factor of the leading coefficient

Remainder Theorem if a polynomial \(f(x)\) is divided by \(x-k\), then the remainder is equal to the value \(f(k)\)

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\section*{PUTTING IT TOGETHER: POWER AND POLYNOMIAL FUNCTIONS}

At the start of this module, you were given the challenge of designing a roller coaster given a set of criteria. Now that you know a bit about polynomials, let's take another look at the coaster in terms of a polynomial as a function of time, \(t\).
- The starting height of the coaster must be 200 ft , so \(f(0)=200\).
- At \(t=3\), the roller coaster goes below ground level, \((3,0)\).
- At \(t=5\), the roller coaster returns to ground level, (5, 0).
- At \(t=10\), the roller coaster goes below ground level again, \((10,0)\).

So you can begin to write the polynomial function to represent the criteria. The height as a function of time can be written as follows:
\[
\begin{aligned}
& f(t)=a(t-3)(t-5)(t-10) \\
& f(t)=a\left(t 3-18 t^{2}+95 t-150\right) \\
& f(0)=-150 a-200 \\
& a=-4 / 3
\end{aligned}
\]
\(f(t)=-4 / 3\left(t 3-18 t^{2}+95 t-150\right)\)
Graphing this polynomial will help you analyze the design better. One way to graph a polynomial function is to first find the intercepts. Earlier you found the intercepts \((0,200),(3,0),(5,0)\), and (10, 0).


In this particular case, the graph helps you visualize the shape of your coaster. The starting point on the graph shows the starting point of the coaster. The negative slope shows where the coaster will drop. The \(x\) intercepts show the times at which the coaster is at ground level. You can quickly see how the coaster will rise and fall for this portion of the ride.


So this is turning out to be a scary roller coaster already, and you have only designed 10 seconds of it. Imagine what you can do in the other 40 seconds of the roller coaster ride!

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\title{
MODULE 10: RATIONAL AND RADICAL FUNCTIONS
}

\section*{WHY IT MATTERS: RATIONAL AND RADICAL FUNCTIONS}

\section*{Why study rational and radical functions?}

When a patient undergoes major surgery, the anesthesiologist must prepare a very specific mixture of drugs that will put the patient under for a specific time period. If the drugs are too weak, the patient might wake up in the middle of the procedure; if they are too powerful, there is a risk that the patient may not wake up at all.

The surgery patient's health and well-being rest on the ability of the anesthesiologist to accurately and effectively administer just the right amount of medicine at the right time. How does he or she do this? It turns out that there is a mathematical model that describes how medicine concentration varies over time in the human body. The formula is:
\(C(t)=\frac{A t}{t^{2}+B}\)
in which \(A\) and \(B\) are constants determined by various factors including the body weight of the individual, types of drugs used, and quantity of the drugs administered.


This is an example of a rational function. The graph shown here illustrates the case in which \(A=30\) and \(B=4\). The black curve is the graph of \(C(t)\).

Suppose that a certain drug will only be effective if its concentration in the bloodstream is at least \(5 \%\). However if the drug concentration exceeds \(7.5 \%\), then the patient may experience adverse side effects, while any amount less than \(7.5 \%\) is safe. According to the graph, the drug becomes effective at a little less than an hour from the time it was first administered and remains above the effective level until about 5 hours. So the duration of effect is at least 4 hours total (within the time window between the two dotted blue lines). Furthermore, this graph shows that the concentration peaks at \(7.5 \%\) and then decreases, indicating the drug is safe for the patient.

In this module, you will explore how rational functions can be graphed, analyzed, and used in applications. For example, you will learn how you can use the formula for \(C(t)\) to precisely identify when the drug concentration reaches the effective level. You will also be able to show that over time the drug concentration becomes insignificant. That is, the values of \(C(t)\) get closer to 0 as \(t\) increases. This is good news for the patient who probably wants to get back to his or her daily routine as soon as possible after the surgery.

\section*{LEARNING OUTCOMES}

\section*{Review Topics for Success}
- Solve a rational formula for a specified variable.
- Solve an application using a formula that must be solved for a specified variable.
- Solve applications by defining and solving rational equations.
- Solve an application using a formula containing a radical expression.
- Graph radical functions using tables and transformations.
- Solve an application by writing and solving a proportion.

\section*{Rational Functions}
- Use arrow notation to describe end behavior of rational functions.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical and horizontal asymptotes of graphs of rational functions.
- Graph rational functions.

\section*{Radical Functions}
- Find the inverse of a polynomial function.
- Restrict the domain to find the inverse of a polynomial function.

\section*{Variation}
- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

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\section*{REVIEW TOPICS FOR SUCCESS}

\section*{Learning Outcomes}

By the end of this section, you will be able to:
- Solve a rational formula for a specified variable
- Solve an application using a formula that must be solved for a specified variable
- Solve applications by defining and solving rational equations.
- Solve an application using a formula containing a radical expression
- Graph radical functions using tables and transformations
- Solve an application by writing and solving a proportion

You've studied how to identify and manipulate rational expressions and radical expressions such as
\(\frac{5}{x+2}, \quad\) or \(\quad \frac{x}{x^{2}-9}, \quad\) and \(\sqrt{2 x+1}\).
You've also seen how to solve rational and radical equations (equations in which the variable is contained within one or more rational or radical expressions). This module will extend those concepts to the language of functions, where you'll learn about graphing and applying rational and radical functions to real-world situations.

To prepare, it will be important to practice applications of rational and radical equations and see how radical functions are graphed using a few key points. This module includes a problem-solving technique known as variation, which involves writing relationships as proportions in order to solve for an unknown, so it will be helpful to review proportions as well.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

\section*{APPLICATIONS WITH RATIONAL EQUATIONS}

\section*{Learning Outcomes}
- Solve a rational formula for a specified variable
- Solve an application using a formula that must be solved for a specified variable.
- Solve applications by defining and solving rational equations.

\section*{Rational Formulas}

Rational formulas can be useful tools for representing real-life situations and for finding answers to real problems. You'll see later in this module that certain equations representing relationships called direct, inverse, and joint variation are examples of rational formulas that can model many real-life situations.

When solving problems using rational formulas, after identifying the particular formula that represents the relationship between the known and unknown quantities, it is often helpful to then solve the formula for a specified variable. This is sometimes called solving a literal equation.

\section*{Example}

The formula for finding the density of an object is \(D=\frac{m}{v}\), where \(D\) is the density, \(m\) is the mass of the object, and \(v\) is the volume of the object. Rearrange the formula to solve for the mass \((m)\) and then for the volume ( \(v\) ).

\section*{Answer}

Start with the formula for density.
\(D=\frac{m}{v}\)
Multiply both side of the equation by \(v\) to isolate \(m\).
\(v \cdot D=\frac{m}{v} \cdot v\)
Simplify and rewrite the equation, solving for \(m\).
\(v \cdot D=m \cdot \frac{v}{v}\)
\(v \cdot D=m \cdot 1\)
\(v \cdot D=m\)
To solve the equation \(D=\frac{m}{v}\) in terms of \(v\), you will need do the same steps to this point and then divide both sides by \(D\).
\[
\begin{aligned}
& \frac{v \cdot D}{D}=\frac{m}{D} \\
& \frac{D}{D} \cdot v=\frac{m}{D} \\
& 1 \cdot v=\frac{m}{D} \\
& v=\frac{m}{D}
\end{aligned}
\]

Therefore, \(m=D \cdot v\) and \(v=\frac{m}{D}\).

\section*{Example}

The formula for finding the volume of a cylinder is \(V=\pi r^{2} h\), where \(V\) is the volume, \(r\) is the radius, and \(h\) is the height of the cylinder. Rearrange the formula to solve for the height \((h)\).

\section*{Answer}

Start with the formula for the volume of a cylinder.
\(V=\pi r^{2} h\)
Divide both sides by \(\pi r^{2}\) to isolate \(h\).
\(\frac{V}{\pi r^{2}}=\frac{\pi r^{2} h}{\pi r^{2}}\)

Simplify. You find the height, \(h\), is equal to \(\frac{V}{\pi r^{2}}\).
\(\frac{V}{\pi r^{2}}=h\)
Therefore, \(h=\frac{V}{\pi r^{2}}\).
Watch the following video for more examples of solving for a particular variable in a formula, a literal equation.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=303

\section*{Work}

A work problem is a useful real-world application involving literal equations. Let's say you'd like to calculate how long it will take different people working at different speeds to finish a task. You may recall the formula that relates distance, rate and time, \(d=r t\). A similar formula relates work performed to a work-rate and time spent working: \(W=r t\). The amount of work done \(W\) is the product of the rate of work \(r\) and the time spent working \(t\). Using algebra, you can write the work formula 3 ways:
\(W=r t\)
Solved for time \(t\) the formula is \(t=\frac{W}{r}\) (divide both sides by \(r\) ).
Solved for rate \(r\) the formula is \(r=\frac{W}{t}\) (divide both sides by \(t\) ).
Rational equations can be used to solve a variety of problems that involve rates, times, and work. Using rational expressions and equations can help you answer questions about how to combine workers or machines to complete a job on schedule.

Some work problems include multiple machines or people working on a project together for the same amount of time but at different rates. In this case, you can add their individual work rates together to get a total work rate.

\section*{Example}

Myra takes 2 hours to plant 50 flower bulbs. Francis takes 3 hours to plant 45 flower bulbs. Working together, how long should it take them to plant 150 bulbs?

\section*{Answer}

Think about how many bulbs each person can plant in one hour. This is their planting rate.

Myra: \(\frac{50 \text { bulbs }}{2 \text { hours }}\) or \(\frac{25 \text { bulbs }}{1 \text { hour }}\)
Francis: \(\frac{45 \text { bulbs }}{3 \text { hours }}\) or \(\frac{15 \text { bulbs }}{1 \text { hour }}\)
Combine their hourly rates to determine the rate they work together.
Myra and Francis together:
\(\frac{25 \text { bulbs }}{1 \text { hour }}+\frac{15 \text { bulbs }}{1 \text { hour }}=\frac{40 \text { bulbs }}{1 \text { hour }}\)
Use one of the work formulas to write a rational equation, for example, \(r=\frac{W}{t}\). You know \(r\), the combined work rate, and you know \(W\), the amount of work that must be done. What you do not know is how much time it will take to do the required work at the designated rate.
\(\frac{40}{1}=\frac{150}{t}\)
Solve the equation by multiplying both sides by the common denominator and then isolating \(t\).
\(1 t \cdot \frac{40}{1}=\frac{150}{t} \cdot 1 t\)
\(40 t=150\)
\[
t=\frac{150}{40}=\frac{15}{4}
\]
\(t=3 \frac{3}{4}\) hours
It should take 3 hours 45 minutes for Myra and Francis to plant 150 bulbs together.
Other work problems take a different perspective. You can calculate how long it will take one person to do a job alone when you know how long it takes people working together to complete the job.

\section*{Example}

Joe and John are planning to paint a house together. John thinks that if he worked alone, it would take him 3 times as long as it would take Joe to paint the entire house. Working together, they can complete the job in 24 hours. How long would it take each of them, working alone, to complete the job?

\section*{Answer}

Choose variables to represent the unknowns. Since it takes John 3 times as long as Joe to paint the house, his time is represented as \(3 x\).

Let \(x=\) time it takes Joe to complete the job
\(3 x=\) time it takes John to complete the job
The work is painting 1 house or 1 . Write an expression to represent each person's rate using the formula \(r=\frac{W}{t}\).

Joe's rate: \(\frac{1}{x}\)
John's rate: \(\frac{1}{3 x}\)
Their combined rate is the sum of their individual rates. Use this rate to write a new equation using the formula \(W=r t\).
combined rate: \(\frac{1}{x}+\frac{1}{3 x}\)

The problem states that it takes them 24 hours together to paint a house, so if you multiply their combined hourly rate \(\left(\frac{1}{x}+\frac{1}{3 x}\right)\) by 24 , you will get 1 , which is the number of houses they can paint in 24 hours.
\(1=\left(\frac{1}{x}+\frac{1}{3 x}\right) 24\)
\(1=\frac{24}{x}+\frac{24}{3 x}\)
Now solve the equation for \(x\). (Remember that \(x\) represents the number of hours it will take Joe to finish the job.)
\[
\begin{aligned}
& 1=\frac{3}{3} \cdot \frac{24}{x}+\frac{24}{3 x} \\
& 1=\frac{3 \cdot 24}{3 x}+\frac{24}{3 x} \\
& 1=\frac{72}{3 x}+\frac{24}{3 x} \\
& 1=\frac{72+24}{3 x} \\
& 1=\frac{96}{3 x} \\
& 3 x=96 \\
& x=32
\end{aligned}
\]

Check the solution in the original equation.
\[
\begin{aligned}
& 1=\left(\frac{1}{x}+\frac{1}{3 x}\right) 24 \\
& 1=\left[\frac{1}{32}+\frac{1}{3(32)}\right] 24 \\
& 1=\frac{24}{32}+\frac{24}{3(32)} \\
& 1=\frac{24}{32}+\frac{24}{96} \\
& 1=\frac{3}{3} \cdot \frac{24}{32}+\frac{24}{96} \\
& 1=\frac{72}{96}+\frac{24}{96}
\end{aligned}
\]

The solution checks. Since \(x=32\), it takes Joe 32 hours to paint the house by himself. John's time is \(3 x\), so it would take him 96 hours to do the same amount of work.

As shown above, many work problems can be represented by the equation \(\frac{t}{a}+\frac{t}{b}=1\), where \(t\) represents the quantity of time two people, \(A\) and \(B\), complete the job working together, \(a\) is the amount of time it takes person \(A\) to do the job, and \(b\) is the amount of time it takes person \(B\) to do the job. The 1 on the right hand side represents 1 job, the total work done-in this case, the work was to paint a house.

The key idea here is to figure out each worker's individual rate of work. Then, once those rates are identified, add them together, multiply by the time \(t\), set it equal to the amount of work done, and solve the
rational equation.
If person \(A\) works at a rate of 1 job every \(a\) hours, and person \(B\) works at a rate of 1 job every \(b\) hours, and if \(t\) represents the total amount of time it takes to paint 1 house, we have
\[
\begin{aligned}
W & =\left(r_{1}+r_{2}\right) t \\
1 & =\left(\frac{1}{a}+\frac{1}{b}\right) t \\
1 & =\frac{t}{a}+\frac{t}{b}
\end{aligned}
\]

Watch the following video for an example of finding the total time given two people working together at known rates.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=303

The following video shows an example of finding one person's work rate given a known combined work rate.


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\section*{Mixing}

Mixtures are made of ratios of different substances that may include chemicals, foods, water, or gases. There are many different situations where mixtures may occur both in nature and as a means to produce a desired product or outcome. For example, chemical spills, manufacturing, and even biochemical reactions
involve mixtures. Mixture problems become mathematically interesting when components of the mixture are added at different rates and concentrations. The next example shows how to define an equation that models the concentration (a ratio) of sugar to water in a large mixing tank over time.

\section*{EXAMPLE}

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

\section*{Answer}

Let \(t\) be the number of minutes since the tap opened. Since the water increase at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is a linear equation, as is the amount of sugar in the tank. We can write an equation independently for each:
\(\{\) water: \(W(t)=100+10 t\) in gallons
sugar: \(S(t)=5+1 t\) in pounds
The concentration, \(C\), will be the ratio of pounds of sugar to gallons of water
\(C(t)=\frac{5+t}{100+10 t}\)
The concentration after 12 minutes is given by evaluating \(C(t)\) at \(t=12\).
\(C(12)=\frac{5+12}{100+10(12)}=\frac{17}{220}\)
This means the concentration is 17 pounds of sugar to 220 gallons of water.
At the beginning, the concentration is
\(C(0)=\frac{5+0}{100+10(0)}=\frac{5}{100}=\frac{1}{20}\)
Since \(\frac{17}{220} \approx 0.08>\frac{1}{20}=0.05\), the concentration is greater after 12 minutes than at the beginning.
The following video gives another example of how to use rational functions to model mixing.


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- Rational Function Application - Concentration of a Mixture. Authored by: James Sousa (Mathispower4u.com) for Lumen Learning. Located at: https://youtu.be/GD6H7BE_OEI. License: CC BY: Attribution

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\section*{APPLICATIONS WITH RADICAL EQUATIONS}

\section*{LEARNING OUTCOMES}

Applications with radical equations
- Solve an application using a formula containing a radical expression

\section*{Radical Equations}

Radical equations in real-life applications may contain two or more variables. Solving such an equation involves isolating the unknown variable, which may be contained within a radical such as a square or cube root, while substituting known values for the remaining variables.
\[
S=\sqrt{\frac{x}{32}} \quad T=2 \sqrt{L} \quad r=\sqrt{\frac{V}{h \pi}}
\]

Each of the above equations is an example of a radical equation. Use the examples given in the situations below to practice solving applications involving radical equations.

\section*{Kinetic Energy}

One way to measure the amount of energy that a moving object (such as a car or roller coaster) possesses is by finding its Kinetic Energy. The Kinetic Energy ( \(E_{k}\), measured in Joules) of an object depends on the object's mass ( \(m\), measured in kg ) and velocity ( \(v\), measured in meters per second), and can be written as \(v=\sqrt{\frac{2 E_{k}}{m}}\).

\section*{Example}

What is the Kinetic Energy of an object with a mass of 1, 000 kilograms that is traveling at 30 meters per second?

\section*{Answer}

Identify variables and known values.
\(E_{k}=\) unknown
\(m=1000\)
\[
v=30
\]

Substitute values into the formula.
\(30=\sqrt{\frac{2 E_{k}}{1,000}}\)
Solve the radical equation for \(E_{k}\).
\[
\begin{aligned}
(30)^{2} & =\left(\sqrt{\frac{2 E_{k}}{1,000}}\right)^{2} \\
900 & =\frac{2 E_{k}}{1,000} \\
900 \cdot 1,000 & =\frac{2 E_{k}}{1,000} \cdot 1,000 \\
900,000 & =2 E_{k} \\
\frac{900,000}{2} & =\frac{2 E_{k}}{2} \\
450,000 & =E_{k}
\end{aligned}
\]

Now check the solution by substituting it into the original equation.
\[
\begin{aligned}
& 30=\sqrt{\frac{2 \cdot 450,000}{1,000}} \\
& 30=\sqrt{\frac{900,000}{1,000}} \\
& 30=\sqrt{900} \\
& 30=30
\end{aligned}
\]

\section*{Answer}

The Kinetic Energy is 450,000 Joules.
Here is another example of finding the kinetic energy of an object in motion.


\section*{Volume}

Harvester ants found in the southwest of the U.S. create a vast interlocking network of tunnels for their nests. As a result of all this excavation, a very common above-ground hallmark of a harvester ant nest is a conical mound of small gravel or sand (Note: Taber, Stephen Welton. The World of the Harvester Ants. College Station: Texas A \& M University Press, 1998.)

The radius of a cone whose height is is equal to twice it's radius is given as: \(r=\sqrt[3]{\frac{3 V}{2 \pi}}\).

\section*{EXAMPLE}

A mound of gravel is in the shape of a cone with the height equal to twice the radius. Calculate the volume of such a mound of gravel whose radius is 3.63 ft . Use \(\pi=3.14\).

\section*{Answer}

The radius of a cone given it's volume can be found with the following formula: \(r=\sqrt[3]{\frac{3 V}{2 \pi}}, r \geq 0\) Cube both sides to eliminate the cube root.
\[
\left\{\begin{array}{l}
3.63 \\
(3.63)^{3}=\left(\sqrt[3]{\frac{3 \cdot V}{2 \cdot 3.14}}\right)^{3}=\sqrt[3]{\frac{3 V}{2 \pi}} \\
47.83=\frac{3 V}{2 \pi} \\
47.83 \cdot 2 \cdot 3.14=3 V \\
100 \approx V
\end{array}\right.
\]

Therefore, the volume is about 100 cubic feet.

Answer
\(V \approx 100 \frac{\mathrm{cu}}{f t}\)
Here is another example of finding Volume given the radius of a cone.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=304

\section*{Free-Fall}

When you drop an object from a height, the only force acting on it is gravity (and some air friction) and it is said to be in free-fall. We can use math to describe the height of an object in free fall after a given time because we know how to quantify the force of the earth pulling on us - the force of gravity.

An object dropped from a height of 600 feet has a height, h , in feet after \(t\) seconds have elapsed, such that \(h=600-16 t^{2}\). In our next example we will find the time at which the object is at a given height by first solving for t .

\section*{EXAMPLE}

Find the time is takes to reach a height of 400 feet by first finding an expression for t .

\section*{Answer}

We are asked two things. First, we will solve the height equation for \(t\), then we will find how long it takes for the object's height above the ground to be 400 feet.
Solve for t :
\(h=600-16 t^{2}\)
\(h-600=-16 t^{2}\)
\(\frac{h-600}{-16}=t^{2}\)
\(\pm \sqrt{\frac{h-600}{-16}}=t\)
At this point, let's stop and talk about whether it makes sense to include both \(\pm \sqrt{\frac{h-600}{-16}}\). We want time to be only positive since we are talking about a measurable quantity, so we will restrict our answers to just \(+\sqrt{\frac{h-600}{-16}}\)
We want to know at what time the height will be 400 ft ., so we can substitute 400 for h .
\(\sqrt{\frac{400-600}{-16}}=t\)
\(\sqrt{\frac{-200}{-16}}=t\)
\(\sqrt{12.5}=t\)
\(3.54=t\)

Answer

It takes 3.54 seconds for the object to be at a height of 400 ft .

\section*{Analysis of the solution}

We have made a point of restricting the radicand of radical expressions to non-negative numbers. In the previous example, we divided by a negative number, then took the square root to solve for t . In this example is it possible to get a negative number in the radicand? In other words, for what values for height would we have an issue where we may be taking the square root of a negative number? We can use algebra to answer this question.

Let's translate our question into an inequality. Again, for what values of \(h\) would we get a negative quantity under the radical?

The radicand is \(\frac{h-600}{-16}\) so if we set up an inequality we can solve it for \(h\) :
\[
\begin{aligned}
& \frac{h-600}{-16}<0 \\
& -16 \cdot \frac{h-600}{-16}<0 \cdot-16 \\
& h-600>0 \\
& h>600
\end{aligned}
\]

We can interpret this as "when the height is greater than 600 ft . the radicand will be negative and therefore not a real number." If you re-read the question, you will see that heights greater than 600 don't even make sense, because the object starts at a height of 600 feet and is falling toward the ground, so height is decreasing. Understanding what domain our variables have is important in applications problems so we can get answers that make sense.

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\section*{GRAPH RADICAL FUNCTIONS}

\section*{LEARNING OUTCOME}
- Graph radical functions using tables and transformations

You can also graph radical functions (such as square root functions) by choosing values for \(x\) and finding points that will be on the graph. Again, it is helpful to have some idea about what the graph will look like.

Think about the basic square root function, \(f(x)=\sqrt{x}\). Take a look at a table of values for \(x\) and \(y\) and then graph the function. Notice that all the values for \(x\) in the table are perfect squares. Since you are taking the square root of \(x\), using perfect squares makes more sense than just finding the square roots of \(0,1,2,3,4\), etc.
\begin{tabular}{|c|c|}
\hline \(\boldsymbol{x}\) & \(\boldsymbol{f}(\boldsymbol{x})\) \\
\hline 0 & 0 \\
\hline 1 & 1 \\
\hline 4 & 2 \\
\hline 9 & 3 \\
\hline
\end{tabular}
\begin{tabular}{|c|c|}
\hline \(\boldsymbol{x}\) & \(\boldsymbol{f}(\boldsymbol{x})\) \\
\hline 16 & 4 \\
\hline
\end{tabular}

Recall that \(x\) can never be negative because when you square a real number, the result is always positive.
For example, \(\sqrt{49}\), this means "find the number whose square is 49 ." Since there is no real number that we can square and get a negative, the function \(f(x)=\sqrt{x}\) will be defined for \(x>0\).

Take a look at the graph.


As with parabolas, multiplying and adding numbers makes some changes, but the basic shape is still the same. Here are some examples.

Multiplying \(\sqrt{x}\) by a positive value changes the width of the half-parabola. Multiplying \(\sqrt{x}\) by a negative number gives you the other half of a horizontal parabola.

In the following example, we will show how multiplying a radical function by a constant can change the shape of the graph.

Adding a value outside the radical moves the graph up or down. Think about it as adding the value to the basic \(y\) value of \(\sqrt{x}\), so a positive value added moves the graph up.

Adding a value inside the radical moves the graph left or right. Think about it as adding a value to \(x\) before you take the square root-so the \(y\) value gets moved to a different \(x\) value. For example, for \(f(x)=\sqrt{x}\), the square root is 3 if \(x=9\). For \(f(x)=\sqrt{x+1}\), the square root is 3 when \(x+1\) is 9 , so \(x\) is 8 . Changing \(x\) to \(x+1\) shifts the graph to the left by 1 unit (from 9 to 8 ). Changing \(x\) to \(x-2\) shifts the graph to the right by 2 units.

Notice that as \(x\) gets greater, adding or subtracting a number inside the square root has less of an effect on the value of \(y\).

In the next example, we will combine some of the changes that we have seen into one function.

\section*{EXAMPLE}

Match each of the following functions to the graph that it represents.
a) \(f(x)=-\sqrt{x}\)
b) \(f(x)=2 \sqrt{x}\)
c) \(f(x)=\frac{1}{2} \sqrt{x}\)
1)

2)

3)


Answer
Function a) goes with graph 2)
Function b) goes with graph 1)
Function c) goes with graph 3)
Function a) \(f(x)=-\sqrt{x}\) means that all the outputs will be negative - the function is the negative of the square roots of the input. This will give the other half of the parabola on its side. Therefore graph 2) goes with the function \(f(x)=-\sqrt{x}\).


Function b) \(f(x)=2 \sqrt{x}\) means take the square root of all the inputs, then multiply by two, so the outputs will be larger than the outputs for \(\sqrt{x}\). Graph 1) goes with the function \(f(x)=2 \sqrt{x}\).


Function c) \(f(x)=\frac{1}{2} \sqrt{x}\) means take the squat=re root of the inputs then multiply by \(\frac{1}{2}\). The outputs will be smaller than the outputs for \(\sqrt{x}\). Graph 3) goes with the function \(f(x)=\frac{1}{2} \sqrt{x}\).


\section*{EXAMPLE}

Match each of the following functions to the graph that it represents.
a) \(f(x)=\sqrt{x}+3\)
b) \(f(x)=\sqrt{x}-2\)
1)

2)


\section*{Answer}

Function a) goes with graph 2)
Function b) goes with graph 1)
Function a) \(f(x)=\sqrt{x}+3\) means take the square root of all the inputs and add three, so the out puts will be greater than those for \(\sqrt{x}\), therefore graph 2) goes with this function.


Function b) \(f(x)=\sqrt{x}-2\) means take the square root of the input then subtract two. The outputs will be less than those for \(\sqrt{x}\), therefore graph 1) goes with this function.


\section*{EXAMPLE}

Match each of the following functions to the graph that it represents.
a) \(f(x)=\sqrt{x+1}\)
b) \(f(x)=\sqrt{x-2}\)
1)

2)


Answer
Function a) matches graph 2)
Function b) matches graph 1)
Function a) \(f(x)=\sqrt{x+1}\) adds one to the inputs before the square root is taken. The outputs will be greater, so it ends up looking like a shift to the left. Graph 2) matches this function.


Function b) \(f(x)=\sqrt{x-2}\) means subtract before the square root is taken. This makes the outputs less than they would be for the standard \(\sqrt{x}\), and looks like a shift to the right. Graph 1) matches this function.


\section*{EXAMPLE}

Graph \(f(x)=-2+\sqrt{x-1}\).
Answer
Before making a table of values, look at the function equation to get a general idea about what the graph should look like.
Inside the square root, you are subtracting 1 , so the graph will move to the right 1 from the basic \(f(x)=\sqrt{x}\) graph.
You are also adding -2 outside the square root, so the graph will move down two from the basic \(f(x)=\sqrt{x}\) graph.
Create a table of values. Choose values that will make your calculations easy. You want \(x-1\) to be a perfect square ( \(0,1,4,9\), and so on) so you can take the square root.
\begin{tabular}{|c|c|}
\hline \(\mathbf{x}\) & \(\mathbf{f}(\mathbf{x})\) \\
\hline 1 & -2 \\
\hline 2 & -1 \\
\hline 5 & 0 \\
\hline 10 & 1 \\
\hline
\end{tabular}

Since values of \(x\) less than 1 makes the value inside the square root negative, there will be no points on the coordinate graph to the left of \(x=1\). There is no need to choose x values less than 1 for your table! Use ordered pairs from each row of the table to plot points.


Connect the points as best you can using a smooth curve.


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\section*{PROPORTIONS}

\section*{LEARNING OUTCOME}
- Solve an application by writing and solving a proportion

A proportion is a statement that two ratios are equal to each other. You probably use proportional reasoning in real-life situations quite frequently. For example, say you have volunteered to provide drinks for a community event. You are asked to bring enough drinks for \(35-40\) people. At the store you see that drinks come in packages of 12 . So you multiply 12 by 3 to estimate how many packages you'll need and get 36 . This may not be enough if 40 people show up, so you decide to buy 4 packages of drinks just to be sure everyone who wants one will get a drink.

This process can also be expressed as a proportional equation and solved using mathematical principles. First, we can express the number of drinks in a package as a ratio:
\(\frac{12 \text { drinks }}{1 \text { package }}\)
Then we express the number of people for whom we are buying drinks as a ratio with the unknown number of packages we need. We will use the maximum so we'll have enough.
\[
\frac{40 \text { people }}{x \text { packages }}
\]

We can find out how many packages to purchase by setting the expressions equal to each other:
\[
\frac{12 \text { drinks }}{1 \text { package }}=\frac{40 \text { people }}{x \text { packages }}
\]

To solve for x , we can use techniques for solving linear equations, or we can cross multiply as a shortcut.
\[
\begin{gathered}
\frac{12 \text { drinks }}{1 \text { package }}=\frac{40 \text { people }}{x \text { packages }} \\
x \cdot \frac{12 \text { drinks }}{1 \text { package }}=\frac{40 \text { people }}{x \text { packages }} \cdot x \\
12 x=40 \\
x=\frac{40}{12}=\frac{10}{3}=3.33
\end{gathered}
\]

We can round up to 4 from 3.33 since it doesn't make sense to buy 0.33 of a package of drinks. Of course, you don't usually write out your thinking this way when you are trying to estimate how many drinks to buy at the store. But doing so helps to apply the concepts involved to more challenging problems. The following example shows how to use a proportion in an application of a serious social issue, access to sanitary waste management systems and clean water.

In the next example, we'll see how we can use the length of a person's femur to estimate their height. This process is used in forensic science and anthropology. It has been found in scientific studies to be a good estimate.

Another way to describe the ratio of femur length to height that we found in the last example is to say there

\section*{EXAMPLE}

As of March, 2016 the world's population was estimated at 7.4 billion. (Note: "Current World Population." World Population Clock: 7.4 Billion People (2016). Accessed June 21, 2016.
http://www.worldometers.info/world-population/.). According to water.org, 1 out of every 3 people on the planet lives without access to a toilet. Find the number of people on the planet that do not have access to a toilet.
Answer
We can use a proportion to find the unknown number of people who live without a toilet since we are given that 1 in 3 do not have access, and we are given the population of the planet.
We know that 1 out of every 3 people do not have access, so we can write that as a ratio (fraction).
\(\frac{1}{3}\)
Let the number of people without access to a toilet be \(x\). The ratio of people with and without toilets is then \(\frac{x}{7.4 \text { billion }}\)
Equate the two ratios since they are representing the same fractional amount of the population.
\(\frac{1}{3}=\frac{x}{7.4 \text { billion }}\)
Solve:
\(\frac{1}{3}=\frac{x}{7.4}\)
\(7.4 \cdot \frac{1}{3}=\frac{x}{7.4} \cdot 7.4\)
\(2.46=x\)
The original units were billions of people, so our answer is 2.46 billion people do not have access to a toilet. Wow, that is a lot of people.
is a \(1: 4\) ratio between femur length and height, or 1 to 4 .
Ratios are also used in scale drawings. Scale drawings are enlarged or reduced drawings of objects, buildings, roads, and maps. Maps are smaller than what they represent, while a drawing of dendritic branching (Note: Urbanska M, Blazeczyk M, Jaworski J, "Molecular basis of dendritic arborization" Acta Neurobiol Exp (Wars). 2008:68(2):264-88. Accessed May 19, 2019 https://www.ncbi.nlm.nih.gov/pubmed/18511961) in your brain would most likely be larger than what it represents. The scale of the drawing is a ratio that represents a comparison of the length of the actual object and its representation in the drawing. The image below shows a map of the United States with a scale of 1 inch representing 557 miles. We could write the scale factor as a fraction \(\frac{1}{557}\) or as we did with the femurheight relationship, \(1: 557\).

\section*{EXAMPLE}

It has been shown that a person's height is proportional to the length of their femur (Note: Obialor, Ambrose, Churchill Ihentuge, and Frank Akapuaka. "Determination of Height Using Femur Length in Adult Population of Oguta Local Government Area of Imo State Nigeria." Federation of American Societies for Experimental Biology, April 2015. Accessed June 22, 2016.
http://www.fasebj.org/content/29/1_Supplement/LB19.short.). Given that a person who is 71 inches tall has a femur length of 17.75 inches, how tall is someone with a femur length of 16 inches?
Answer
Height and femur length are proportional for everyone, so we can define a ratio with the given height and femur length. We can then use this to write a proportion to find the unknown height.
Let \(x\) be the unknown height. Define the ratio of femur length and height for both people using the given measurements.
Person 1: \(\frac{\text { femur length }}{\text { height }}=\frac{17.75 \text { inches }}{71 \text { inches }}\)
Person 2: \(\frac{\text { femur length }}{\text { height }}=\frac{16 \text { inches }}{x \text { inches }}\)
Equate the ratios since we are assuming height and femur length are proportional for everyone.
\(\frac{17.75 \text { inches }}{71 \text { inches }}=\frac{16 \text { inches }}{x \text { inches }}\)
Solve by using the common denominator to clear fractions. The common denominator is \(71 x\).
\(\frac{17.75}{71}=\frac{16}{x}\)
\(71 x \cdot \frac{17.75}{71}=\frac{16}{x} \cdot 71 x\)
\(17.75 \cdot x=16 \cdot 71\)
\(17.75 \cdot x=1136\)
\[
x=\frac{1136}{17.75}=64
\]

The unknown height of person 2 is 64 inches. In general, we can simplify the fraction \(\frac{17.75}{71}=0.25=\frac{1}{4}\) to find a general rule for everyone. This would translate to saying a person's height is 4 times the length of their femur.


Map of the United States with scale factor 1:577
In the next example, we will use the scale factor given in the image above to find the distance between Seattle, Washington and San Jose, California.

\section*{EXAMPLE}

Given a scale factor of \(1: 557\) on a map of the US, if the distance from Seattle, WA to San Jose, CA is 1.5 inches on the map, define a proportion to find the actual distance between them.

\section*{Answer}

We need to define a proportion to solve for the unknown distance between Seattle and San Jose.
The scale factor is \(1: 557\), and we will call the unknown distance x . The ratio of inches to miles is \(\frac{1}{557}\).
We know the inches between the two cities, but we do not know miles, so the ratio that describes the distance between them is \(\frac{1.5}{x}\).
The proportion that will help us solve this problem is \(\frac{1}{557}=\frac{1.5}{x}\).
Solve using the common denominator \(557 x\) to clear fractions.
\[
\begin{aligned}
& \frac{1}{557}=\frac{1.5}{x} \\
& 557 x \cdot \frac{1}{557}=\frac{1.5}{x} \cdot 557 x \\
& x=1.5 \cdot 557=835.5
\end{aligned}
\]

We used the scale factor \(1: 557\) to find an unknown distance between Seattle and San Jose. We also checked our answer of 835.5 miles with Google maps and found that the distance is 839.9 miles, so we did pretty well!

In the next example, we will find a scale factor given the length between two cities on a map and their actual distance from each other.

Watch the following video for an example of using proportions to obtain the correct amount of medication for

\section*{EXAMPLE}

Two cities are 2.5 inches apart on a map. Their actual distance from each other is 325 miles. Write a proportion to represent and solve for the scale factor for one inch of the map.

\section*{Answer}

We know that for each 2.5 inches on the map, it represents 325 actual miles. We are looking for the scale factor for one inch of the map.
The ratio we want is \(\frac{1}{x}\) where x is the actual distance represented by one inch on the map. We know that for every 2.5 inches, there are 325 actual miles, so we can define that relationship as \(\frac{2.5}{325}\)
We can use a proportion to equate the two ratios and solve for the unknown distance.
\(\frac{1}{x}=\frac{2.5}{325}\)
\(325 x \cdot \frac{1}{x}=\frac{2.5}{325} \cdot 325 x\)
\(325=2.5 x\)
\(x=130\)

The scale factor for one inch on the map is \(1: 130\) or for every inch of map there are 130 actual miles.
a patient as well as to find a desired mixture of coffees.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=306

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\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
- Formulas for specific types of applications containing more than one variable may be solved for an unknown variable by substituting known values and solving the formula for the unknown variable.
- Functions containing radicals may be graphed by substituting input values and solving for the coordinating output value to create a graph of points existing on the graph. After plotting enough points to obtain the shape of the graph, draw a smooth curve between them.
- Work \(W\), rate \(r\), and time \(t\) may be related by the formulas \(W=r t, t=\frac{W}{r}\), and \(r=\frac{W}{t}\)
- Proportions, equations of the form \(\frac{a}{b}=\frac{c}{d}\), are helpful for setting up relationships in applications. Solve by cross-multiplying or by multiplying both sides of the equation by the reciprocal of one.

\section*{Glossary}
proportion a statement of the equality of two ratios
radical formula an equation used in certain applications that contains at least one radical expression (a variable expression using a radical such as a square or cube root)
rational formula an equation used in certain applications that contains at least one rational expression (a variable expression in fraction form)

\section*{INTRODUCTION TO RATIONAL FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Use arrow notation.
- Solve applied problems involving rational functions.
- Find the domains of rational functions.
- Identify vertical asymptotes.
- Identify horizontal asymptotes.
- Graph rational functions.

Suppose we know that the cost of making a product is dependent on the number of items, \(x\), produced. This is given by the equation \(C(x)=15,000 x-0.1 x^{2}+1000\). If we want to know the average cost for producing \(x\) items, we would divide the cost function by the number of items, \(x\).

The average cost function, which yields the average cost per item for \(x\) items produced, is
\(f(x)=\frac{15,000 x-0.1 x^{2}+1000}{x}\)

Many other application problems require finding an average value in a similar way, giving us variables in the denominator. Written without a variable in the denominator, this function will contain a negative integer power.

In the last few sections, we have worked with polynomial functions, which are functions with non-negative integers for exponents. In this section, we explore rational functions, which have variables in the denominator.

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\section*{CHARACTERISTICS OF RATIONAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Use arrow notation to describe local and end behavior of rational functions.
- Identify horizontal and vertical asymptotes of rational functions from graphs.
- Graph a rational function given horizontal and vertical shifts.
- Solve applied problems involving rational functions.

\section*{Using Arrow Notation}

\section*{RECALL CHARACTERISTICS OF RATIONAL EXPRESSIONS}
- We've seen that rational expressions are fractions that may contain a polynomial in the numerator, denominator, or both.
- A rational function is a function whose argument is defined using a rational expression.
- When a variable is present in the denominator of a rational expression, certain values of the variable may cause the denominator to equal zero. A rational expression with a zero in the denominator is not defined since we cannot divide by zero.
- We'll see in this section that the values of the input to a rational function causing the denominator to equal zero will have an impact on the shape of the function's graph. The graph appears to "bend around" these input values, and the function value increases or decreases without bound near those inputs.
- See the graphs below for how to use a notation called arrow notation to describe this and other behaviors sometimes present in the graphs of functions.

We have seen the graphs of the basic reciprocal function and the squared reciprocal function from our study of toolkit functions. Examine these graphs and notice some of their features.

\section*{Graphs of Toolkit Functions}


Several things are apparent if we examine the graph of \(f(x)=\frac{1}{x}\).
1. On the left branch of the graph, the curve approaches the \(x\)-axis \((y=0)\) as \(x \rightarrow-\infty\).
2. As the graph approaches \(x=0\) from the left, the curve drops, but as we approach zero from the right, the curve rises.
3. Finally, on the right branch of the graph, the curves approaches the \(x\)-axis \((y=0)\) as \(x \rightarrow \infty\).

To summarize, we use arrow notation to show that \(x\) or \(f(x)\) is approaching a particular value.
\begin{tabular}{|l|l|}
\hline \multicolumn{1}{|c|}{ Symbol } & \multicolumn{1}{c|}{ Mrrow Notation } \\
\hline\(x \rightarrow a^{-}\) & \(x\) approaches \(a\) from the left \((x<a\) but close to \(a)\) \\
\hline\(x \rightarrow a^{+}\) & \(x\) approaches \(a\) from the right \((x<a\) but close to \(a)\) \\
\hline\(x \rightarrow \infty\) & \(x\) approaches infinity ( \(x\) increases without bound) \\
\hline\(x \rightarrow-\infty\) & \(x\) approaches negative infinity ( \(x\) decreases without bound) \\
\hline\(f(x) \rightarrow \infty\) & the output approaches infinity (the output increases without bound) \\
\hline\(f(x) \rightarrow-\infty\) & the output approaches negative infinity (the output decreases without bound) \\
\hline\(f(x) \rightarrow a\) & the output approaches \(a\) \\
\hline
\end{tabular}

\section*{Local Behavior of \(f(x)=\frac{1}{x}\)}

Let's begin by looking at the reciprocal function, \(f(x)=\frac{1}{x}\). We cannot divide by zero, which means the function is undefined at \(x=0\); so zero is not in the domain. As the input values approach zero from the left side (becoming very small, negative values), the function values decrease without bound (in other words, they approach negative infinity). We can see this behavior in the table below.
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & -0.1 & -0.01 & -0.001 & -0.0001 \\
\hline & -10 & -100 & -1000 & \(-10,000\) \\
\hline
\end{tabular}
\(f(x)=\frac{1}{x}\)
We write in arrow notation
as \(x \rightarrow 0^{-}, f(x) \rightarrow-\infty\)
As the input values approach zero from the right side (becoming very small, positive values), the function values increase without bound (approaching infinity). We can see this behavior in the table below.
\begin{tabular}{|l|l|l|l|l|}
\hline\(x\) & 0.1 & 0.01 & 0.001 & 0.0001 \\
\hline\(f(x)=\frac{1}{x}\) & 10 & 100 & 1000 & 10,000 \\
\hline
\end{tabular}

We write in arrow notation
As \(x \rightarrow 0^{+}, f(x) \rightarrow \infty\).


This behavior creates a vertical asymptote, which is a vertical line that the graph approaches but never crosses. In this case, as the input nears zero from the left, the function value decreases without bound. As the input nears zero from the right, the function value increases without bound. The line \(x=0\) is a vertical asymptote for the function.


\section*{A GENERAL NOTE: VERTICAL ASYMPTOTE}

A vertical asymptote of a graph is a vertical line \(x=a\) where the graph tends toward positive or negative infinity as the inputs approach \(x\). We write
As \(x \rightarrow a, f(x) \rightarrow \infty\), or as \(x \rightarrow a, f(x) \rightarrow-\infty\).

\section*{ASYMPTOTE. WHAT DOES THAT WORD MEAN?}

The term asymptote has its origins from three Greek roots.
a, meaning not
sun, meaning together
ptotos, meaning to fall
These give us asymptotos, meaning not falling together, which leads to the modern term describing a line that a curve (the graph of a function) approaches but never meets.

\section*{End Behavior of \(f(x)=\frac{1}{x}\)}

As the values of \(x\) approach infinity, the function values approach 0 . As the values of \(x\) approach negative infinity, the function values approach 0 . Symbolically, using arrow notation

As \(x \rightarrow \infty, f(x) \rightarrow 0\), and as \(x \rightarrow-\infty, f(x) \rightarrow 0\).


Based on this overall behavior and the graph, we can see that the function approaches 0 but never actually reaches 0 ; it seems to level off as the inputs become large. This behavior creates a horizontal asymptote, a horizontal line that the graph approaches as the input increases or decreases without bound. In this case, the graph is approaching the horizontal line \(y=0\).


\section*{A GENERAL NOTE: HORIZONTAL ASYMPTOTE}

A horizontal asymptote of a graph is a horizontal line \(y=b\) where the graph approaches the line as the inputs increase or decrease without bound. We write
As \(x \rightarrow \infty\) or \(x \rightarrow-\infty, f(x) \rightarrow b\).

\section*{Solve Applied Problems Involving Rational Functions}

In the previous example, we shifted a toolkit function in a way that resulted in the function \(f(x)=\frac{3 x+7}{x+2}\).
This is an example of a rational function. A rational function is a function that can be written as the quotient of two polynomial functions. Many real-world problems require us to find the ratio of two polynomial functions. Problems involving rates and concentrations often involve rational functions.

\section*{EXAMPLE: USING ARROW NOTATION}

Use arrow notation to describe the end behavior and local behavior of the function below.


\section*{Answer}

Notice that the graph is showing a vertical asymptote at \(x=2\), which tells us that the function is undefined at \(x=2\).
As \(x \rightarrow 2^{-}, f(x) \rightarrow-\infty\), and as \(x \rightarrow 2^{+}, f(x) \rightarrow \infty\)
And as the inputs decrease without bound, the graph appears to be leveling off at output values of 4 , indicating a horizontal asymptote at \(y=4\). As the inputs increase without bound, the graph levels off at 4 .
As \(x \rightarrow \infty, f(x) \rightarrow 4\), and as \(x \rightarrow-\infty, f(x) \rightarrow 4\)

\section*{TRY IT}

Use arrow notation to describe the end behavior and local behavior for the reciprocal squared function.
Answer
End behavior: as \(x \rightarrow \pm \infty, f(x) \rightarrow 0\); Local behavior: as \(x \rightarrow 0, f(x) \rightarrow \infty\) (there are no \(x\) - or \(y\) intercepts)

An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=309

\section*{EXAMPLE: USING TRANSFORMATIONS TO GRAPH A RATIONAL FUNCTION}

Sketch a graph of the reciprocal function shifted two units to the left and up three units. Identify the horizontal and vertical asymptotes of the graph, if any.
Answer
Shifting the graph left 2 and up 3 would result in the function
\(f(x)=\frac{1}{x+2}+3\)
or equivalently, by giving the terms a common denominator,
\(f(x)=\frac{3 x+7}{x+2}\)
The graph of the shifted function is displayed below.


Notice that this function is undefined at \(x=-2\), and the graph also is showing a vertical asymptote at \(x=-2\).
As \(x \rightarrow-2^{-}, f(x) \rightarrow-\infty\), and as \(x \rightarrow-2^{+}, f(x) \rightarrow \infty\)
As the inputs increase and decrease without bound, the graph appears to be leveling off at output values of 3 , indicating a horizontal asymptote at \(y=3\).
As \(x \rightarrow \pm \infty, f(x) \rightarrow 3\)

Analysis of the Solution

Notice that horizontal and vertical asymptotes are shifted left 2 and up 3 along with the function.

\section*{TRY IT}

Sketch the graph, and find the horizontal and vertical asymptotes of the reciprocal squared function that has been shifted right 3 units and down 4 units.
Answer

The function and the asymptotes are shifted 3 units right and 4 units down. As \(x \rightarrow 3, f(x) \rightarrow \infty\), and as \(x \rightarrow \pm \infty, f(x) \rightarrow-4\).
The function is \(f(x)=\frac{1}{(x-3)^{2}}-4\).


\section*{A GENERAL NOTE: RATIONAL FUNCTION}

A rational function is a function that can be written as the quotient of two polynomial functions \(P(x)\) and \(Q(x)\).
\(f(x)=\frac{P(x)}{Q(x)}=\frac{a_{p} x^{p}+a_{p-1} x^{p-1}+\ldots+a_{1} x+a_{0}}{b_{q} x^{q}+b_{q-1} x^{q-1}+\ldots+b_{1} x+b_{0}}, Q(x) \neq 0\)

\section*{SUMMARY: HORIZONTAL AND VERTICAL ASYMPTOTES}

\section*{Horizontal Asymptote}
- A horizontal line of the form \(y=c\)
- The constant \(c\) represents a number that the function value (output) approaches in the long run, either as the input grows very small or very large.
- Horizontal asymptotes represent the long-run behavior (the end behavior) of the graph of the funtion.
- A function's graph may cross a horizontal asymptote briefly, even more than once, but will eventually settle down near it, as the value of the function approaches the constant [latex]c[latex].

\section*{Vertical Asymptote}
- A vertical line of the form \(x=a\)
- The constant \(a\) represents an input for which the function value (output) is undefined.
- Substituting the value of \(a\) into the function will result in a zero in the function's denominator.
- The graph of the function "bends around", either increasing or decreasing without bound as the input nears \(a\)
- A function's graph will never cross a vertical asymptote.

\section*{EXAMPLE: SOLVING AN APPLIED PROBLEM INVOLVING A RATIONAL FUNCTION}

A large mixing tank currently contains 100 gallons of water into which 5 pounds of sugar have been mixed. A tap will open pouring 10 gallons per minute of water into the tank at the same time sugar is poured into the tank at a rate of 1 pound per minute. Find the concentration (pounds per gallon) of sugar in the tank after 12 minutes. Is that a greater concentration than at the beginning?

\section*{Answer}

Let \(t\) be the number of minutes since the tap opened. Since the water increases at 10 gallons per minute, and the sugar increases at 1 pound per minute, these are constant rates of change. This tells us the amount of water in the tank is changing linearly, as is the amount of sugar in the tank. We can write an equation independently for each:
water: \(W(t)=100+10 t\) in gallons
sugar: \(S(t)=5+1 t\) in pounds
The concentration, \(C\), will be the ratio of pounds of sugar to gallons of water
\(C(t)=\frac{5+t}{100+10 t}\)
The concentration after 12 minutes is given by evaluating \(C(t)\) at \(t=12\).
\[
\begin{aligned}
C(12) & =\frac{5+12}{100+10(12)} \\
& =\frac{17}{220}
\end{aligned}
\]

This means the concentration is 17 pounds of sugar to 220 gallons of water.
At the beginning the concentration is
\[
\begin{aligned}
C(0) & =\frac{5+0}{100+10(0)} \\
& =\frac{1}{20}
\end{aligned}
\]

Since \(\frac{17}{220} \approx 0.08>\frac{1}{20}=0.05\), the concentration is greater after 12 minutes than at the beginning.

\section*{Analysis of the Solution}

To find the horizontal asymptote, divide the leading coefficient in the numerator by the leading coefficient in the denominator:
\(\frac{1}{10}=0.1\)
Notice the horizontal asymptote is \(y=0.1\). This means the concentration, \(C\), the ratio of pounds of sugar to gallons of water, will approach 0.1 in the long term.

\section*{TRY IT}

There are 1,200 freshmen and 1,500 sophomores at a prep rally at noon. After 12 p.m., 20 freshmen arrive at the rally every five minutes while 15 sophomores leave the rally. Find the ratio of freshmen to sophomores at 1 p.m.

\section*{Answer}

12
\(\overline{11}\)

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\section*{DOMAIN AND ITS EFFECT ON VERTICAL ASYMPTOTES}

\section*{LEARNING OUTCOMES}
- Define the domain of a rational function given an equation.
- Use the domain of a rational function to define vertical asymptotes.

A vertical asymptote represents a value at which a rational function is undefined, so that value is not in the domain of the function. A reciprocal function (a special case of a rational function) cannot have values in its domain that cause the denominator to equal zero. In general, to find the domain of a rational function, we need to determine which inputs would cause division by zero.

\section*{A GENERAL NOTE: DOMAIN OF A RATIONAL FUNCTION}

The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.

\section*{HOW TO: GIVEN A RATIONAL FUNCTION, FIND THE DOMAIN.}
1. Set the denominator equal to zero.
2. Solve to find the values of the variable that cause the denominator to equal zero.
3. The domain contains all real numbers except those found in Step 2.

By looking at the graph of a rational function, we can investigate its local behavior and easily see whether there are asymptotes. We may even be able to approximate their location. Even without the graph, however, we can still determine whether a given rational function has any asymptotes, and calculate their location.

Watch the following video to see more examples of finding the domain of a rational function.

\section*{Vertical Asymptotes}

The vertical asymptotes of a rational function may be found by examining the factors of the denominator that are not common to the factors in the numerator. Vertical asymptotes occur at the zeros of such factors.

\section*{EXAMPLE: FINDING THE DOMAIN OF A RATIONAL FUNCTION}

Find the domain of \(f(x)=\frac{x+3}{x^{2}-9}\).

\section*{Answer}

Begin by setting the denominator equal to zero and solving.
\[
\begin{aligned}
x^{2}-9 & =0 \\
x^{2} & =9 \\
x & = \pm 3
\end{aligned}
\]

The denominator is equal to zero when \(x= \pm 3\). The domain of the function is all real numbers except \(x= \pm 3\).

\section*{Analysis of the Solution}

A graph of this function confirms that the function is not defined when \(x= \pm 3\).


There is a vertical asymptote at \(x=3\) and a hole in the graph at \(x=-3\). We will discuss these types of holes in greater detail later in this section.

\section*{RECALL EQUATION SOLVING TECHNIQUES}

We have learned several techniques that will enable us to solve for values of the denominator that cause it to equal zero. Techniques for solving linear and quadratic equations will help you find the domain of most of the rational functions you'll encounter.
In the example above, solving for the values that cause \(x^{2}-9=0\) can be found either by factoring or by the square root property. The denominator may also be presented in factored form, as in the example below. In this case, part of the work has already been done, and you can use the zero-product principle to find the zeros of the equation. For example,

If \(\quad(x-a)(x-c)=0\),
then \(\quad x-a=0 \quad\) or \(\quad x-c=0\)
and \(\quad x=a \quad\) or \(\quad x=c\).
You may also use the quadratic formula to solve equations in the form of \(a x^{2}+b x+c=0\).
Rational functions can also have more complicated denominators, containing radicals or degrees greater than 2 . In each case, equation solving techniques you've seen in the course will apply.

\section*{TRY IT}

Find the domain of \(f(x)=\frac{4 x}{5(x-1)(x-5)}\).
Answer
The domain is all real numbers except \(x=1\) and \(x=5\).

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\section*{HOW TO: GIVEN A RATIONAL FUNCTION, IDENTIFY ANY VERTICAL ASYMPTOTES OF ITS GRAPH.}
1. Factor the numerator and denominator.
2. Note any restrictions in the domain of the function.
3. Reduce the expression by canceling common factors in the numerator and the denominator.
4. Note any values that cause the denominator to be zero in this simplified version. These are where the vertical asymptotes occur.
5. Note any restrictions in the domain where asymptotes do not occur. These are removable discontinuities.

\section*{Removable Discontinuities}

Occasionally, a graph will contain a hole: a single point where the graph is not defined, indicated by an open circle. We call such a hole a removable discontinuity.

For example, the function \(f(x)=\frac{x^{2}-1}{x^{2}-2 x-3}\) may be re-written by factoring the numerator and the denominator.
\(f(x)=\frac{(x+1)(x-1)}{(x+1)(x-3)}\)

\section*{EXAMPLE: IDENTIFYING VERTICAL ASYMPTOTES}

Find the vertical asymptotes of the graph of \(k(x)=\frac{5+2 x^{2}}{2-x-x^{2}}\).
Answer
First, factor the numerator and denominator.
\[
\begin{aligned}
k(x) & =\frac{5+2 x^{2}}{2-x-x^{2}} \\
& =\frac{5+2 x^{2}}{(2+x)(1-x)}
\end{aligned}
\]

To find the vertical asymptotes, we determine where this function will be undefined by setting the denominator equal to zero:
\((2+x)(1-x)=0\)
\(x=-2,1\)
Neither \(x=-2\) nor \(x=1\) are zeros of the numerator, so the two values indicate two vertical asymptotes. The graph below confirms the location of the two vertical asymptotes.


\section*{TRY IT}

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Notice that \(x+1\) is a common factor to the numerator and the denominator. The zero of this factor, \(x=-1\), is the location of the removable discontinuity. Notice also that \(x-3\) is not a factor in both the numerator and denominator. The zero of this factor, \(x=3\), is the vertical asymptote.


\section*{A GENERAL NOTE: REMOVABLE DISCONTINUITIES OF RATIONAL FUNCTIONS}

A removable discontinuity occurs in the graph of a rational function at \(x=a\) if \(a\) is a zero for a factor in the denominator that is common with a factor in the numerator. We factor the numerator and denominator and check for common factors. If we find any, we set the common factor equal to 0 and solve. This is the location of the removable discontinuity. This is true if the multiplicity of this factor is greater than or equal to that in the denominator. If the multiplicity of this factor is greater in the denominator, then there is still an asymptote at that value.

\section*{EXAMPLE: IDENTIFYING VERTICAL ASYMPTOTES AND REMOVABLE DISCONTINUITIES FOR A GRAPH}

Find the vertical asymptotes and removable discontinuities of the graph of \(k(x)=\frac{x-2}{x^{2}-4}\).

\section*{Answer}

Factor the numerator and the denominator.
\(k(x)=\frac{x-2}{(x-2)(x+2)}\)
Notice that there is a common factor in the numerator and the denominator, \(x-2\). The zero for this factor is \(x=2\). This is the location of the removable discontinuity.
Notice that there is a factor in the denominator that is not in the numerator, \(x+2\). The zero for this factor is \(x=-2\). The vertical asymptote is \(x=-2\).


The graph of this function will have the vertical asymptote at \(x=-2\), but at \(x=2\) the graph will have a hole.

\section*{TRY IT}

The rational function \(f(x)=\frac{\left(x^{2}-a^{2}\right)}{\left(x^{3}-(a+1) x^{2}+a x\right)}\) needs to be graphed using an online graphing calculator. Investigate the changes in the graph for values of \(a\) on the interval \([-5,5]\), and answer the following questions.
1. For what values of \(a\) does \(f(a)\) represent a removable discontinuity?
2. For what values of \(a\) does the line \(x=a\) represent a vertical asymptote?

\section*{Answer}

Removable discontinuity for values of \(a\) on the interval \([2,5]\) and on the interval \([-5,0]\), because the vertical line crosses the graph. Vertical asymptote for \(a=1\), because the vertical line \(x=1\) coincides with a vertical asymptote.

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\section*{HORIZONTAL ASYMPTOTES AND INTERCEPTS}

\section*{LEARNING OUTCOMES}
- Use the degree of the numerator and denominator of a rational function to determine what kind of horizontal (or slant) asymptote it will have.
- Calculate slant asymptotes.
- Determine the intercepts of a rational function in factored form.

While vertical asymptotes describe the behavior of a graph as the output gets very large or very small, horizontal asymptotes help describe the behavior of a graph as the input gets very large or very small. Recall that a polynomial's end behavior will mirror that of the leading term. Likewise, a rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.

\section*{RECALL END BEHAVIOR OF A GRAPH}

We have learned that knowing the leading coefficient and degree of a polynomial function is useful when predicting its end behavior.
If the leading term is positive or negative, and has even or odd degree, it will tell us the toolkit function's graph behavior it will mimic: \(f(x)=x^{2}, \quad f(x)=-x^{2}, \quad f(x)=x^{3}, \quad\) or \(\quad f(x)=-x^{3}\).
The same idea applies to the ratio of leading terms of a rational function. See below for the three cases to check when determining horizontal (or slant) asymptotes of a rational function.

There are three distinct outcomes when checking for horizontal asymptotes:
Case 1: If the degree of the denominator \(>\) degree of the numerator, there is a horizontal asymptote at \(y=0\).

Example: \(f(x)=\frac{4 x+2}{x^{2}+4 x-5}\)
In this case the end behavior is \(f(x) \approx \frac{4 x}{x^{2}}=\frac{4}{x}\). This tells us that, as the inputs increase or decrease without bound, this function will behave similarly to the function \(g(x)=\frac{4}{x}\), and the outputs will approach zero, resulting in a horizontal asymptote at \(y=0\). Note that this graph crosses the horizontal asymptote.


Horizontal Asymptote \(y=0\) when \(f(x)=\frac{p(x)}{q(x)}, q(x) \neq 0\) where degree of \(p<\) degree of q .
Case 2: If the degree of the denominator < degree of the numerator by one, we get a slant asymptote.
Example: \(f(x)=\frac{3 x^{2}-2 x+1}{x-1}\)
In this case the end behavior is \(f(x) \approx \frac{3 x^{2}}{x}=3 x\). This tells us that as the inputs increase or decrease without bound, this function will behave similarly to the function \(g(x)=3 x\). As the inputs grow large, the outputs will grow and not level off, so this graph has no horizontal asymptote. However, the graph of \(g(x)=3 x\) looks like a diagonal line, and since \(f\) will behave similarly to \(g\), it will approach a line close to \(y=3 x\). This line is a slant asymptote.


Slant Asymptote when \(f(x)=\frac{p(x)}{q(x)}, q(x) \neq 0\) where degree of \(p>\) degree of \(q\) by 1 .
To find the equation of the slant asymptote, divide \(\frac{3 x^{2}-2 x+1}{x-1}\). The quotient is \(3 x+1\), and the remainder is 2 . The slant asymptote is the graph of the line \(g(x)=3 x+1\).

Case 3: If the degree of the denominator = degree of the numerator, there is a horizontal asymptote at \(y=\frac{a_{n}}{b_{n}}\), where \(a_{n}\) and \(b_{n}\) are the leading coefficients of \(p(x)\) and \(q(x)\) for \(f(x)=\frac{p(x)}{q(x)}, q(x) \neq 0\).

Example: \(f(x)=\frac{3 x^{2}+2}{x^{2}+4 x-5}\)
In this case the end behavior is \(f(x) \approx \frac{3 x^{2}}{x^{2}}=3\). This tells us that as the inputs grow large, this function will behave like the function \(g(x)=3\), which is a horizontal line. As \(x \rightarrow \pm \infty, f(x) \rightarrow 3\), resulting in a horizontal asymptote at \(y=3\). Note that this graph crosses the horizontal asymptote.


Horizontal Asymptote when \(f(x)=\frac{p(x)}{q(x)}, q(x) \neq 0\) where degree of \(p=\) degree of \(q\).
Notice that, while the graph of a rational function will never cross a vertical asymptote, the graph may or may not cross a horizontal or slant asymptote. Also, although the graph of a rational function may have many vertical asymptotes, the graph will have at most one horizontal (or slant) asymptote.

It should be noted that, if the degree of the numerator is larger than the degree of the denominator by more than one, the end behavior of the graph will mimic the behavior of the reduced end behavior fraction. For instance, if we had the function
\(f(x)=\frac{3 x^{5}-x^{2}}{x+3}\)
with end behavior
\(f(x) \approx \frac{3 x^{5}}{x}=3 x^{4}\),
the end behavior of the graph would look similar to that of an even polynomial with a positive leading coefficient.

As \(x \rightarrow \pm \infty, f(x) \rightarrow \infty\)

\section*{TRY IT}

You'll need to graph the rational function \(f(x)=\frac{\left(4 x^{a}+2\right)}{\left(x^{b}+2 x-3\right)}\) using an online graphing calculator.
Adjust the \(a\) and the \(b\) values to obtain graphs with the following horizontal asymptotes:
1. Find values of \(a\) and \(b\) that will give a graph with a slant asymptote.
2. Find values of \(a\) and \(b\) that will give a graph with a horizontal asymptote at \(y=0\)
3. Find values of \(a\) and \(b\) that will give a graph with a horizontal asymptote that is the ratio of the leading coefficients of \(f(x)\).

\section*{Answer}

Answers will vary.
1. Find values of \(a\) and \(b\) that will give a graph with a slant asymptote. For example \(b=3, a=4\)
2. Find values of \(a\) and \(b\) that will give a graph with a horizontal asymptote at \(y=0\). For example, \(b=4, a=2\)
3. Find values of \(a\) and \(b\) that will give a graph with a horizontal asymptote that is the ratio of the leading coefficients of \(f(x)\).For example, \(b=4, a=4\)

\section*{A GENERAL NOTE: HORIZONTAL ASYMPTOTES OF RATIONAL FUNCTIONS}

The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and denominator.
- Case 1: Degree of numerator is less than degree of denominator: horizontal asymptote at \(y=0\)
- Case 2: Degree of numerator is greater than degree of denominator by one: no horizontal asymptote; slant asymptote.
- If the degree of the numerator is greater than the degree of the denominator by more than one, the end behavior of the function's graph will mimic that of the graph of the reduced ratio of leading terms.
- Case 3: Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

Watch this video to see more worked examples of determining which kind of horizontal asymptote a rational function will have.

\section*{Finding Intercepts of Rational Functions}

Watch the following video to see more worked examples of finding asymptotes, intercepts and holes of rational functions.

\section*{EXAMPLE: IDENTIFYING HORIZONTAL AND SLANT ASYMPTOTES}

For the functions below, identify the horizontal or slant asymptote.
1. \(g(x)=\frac{6 x^{3}-10 x}{2 x^{3}+5 x^{2}}\)
2. \(h(x)=\frac{x^{2}-4 x+1}{x+2}\)
3. \(k(x)=\frac{x^{2}+4 x}{x^{3}-8}\)

\section*{Answer}

For these solutions, we will use \(f(x)=\frac{p(x)}{q(x)}, q(x) \neq 0\).
1. \(g(x)=\frac{6 x^{3}-10 x}{2 x^{3}+5 x^{2}}\) : The degree of \(p\) and the degree of \(q\) are both equal to 3 , so we can find the horizontal asymptote by taking the ratio of the leading terms. There is a horizontal asymptote at \(y=\frac{6}{2}\) or \(y=3\).
2. \(h(x)=\frac{x^{2}-4 x+1}{x+2}\) : The degree of \(p=2\) and degree of \(q=1\). Since \(p>q\) by 1 , there is a slant asymptote found at \(\frac{x^{2}-4 x+1}{x+2}\).


The quotient is \(x-6\) and the remainder is 13. There is a slant asymptote at \(y=-x-6\).
3. \(k(x)=\frac{x^{2}+4 x}{x^{3}-8}\) : The degree of \(p=2<\) degree of \(q=3\), so there is a horizontal asymptote \(y=0\).

\section*{TRY IT}

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\section*{EXAMPLE: IDENTIFYING HORIZONTAL ASYMPTOTES}

In the sugar concentration problem earlier, we created the equation \(C(t)=\frac{5+t}{100+10 t}\).
Find the horizontal asymptote and interpret it in context of the problem.

\section*{Answer}

Both the numerator and denominator are linear (degree 1). Because the degrees are equal, there will be a horizontal asymptote at the ratio of the leading coefficients. In the numerator, the leading term is \(t\), with coefficient 1 . In the denominator, the leading term is \(10 t\), with coefficient 10 . The horizontal asymptote will be at the ratio of these values:
\(t \rightarrow \infty, C(t) \rightarrow \frac{1}{10}\)
This function will have a horizontal asymptote at \(y=\frac{1}{10}\).
This tells us that as the values of \(t\) increase, the values of \(C\) will approach \(\frac{1}{10}\). In context, this means that, as more time goes by, the concentration of sugar in the tank will approach one-tenth of a pound of sugar per gallon of water or \(\frac{1}{10}\) pounds per gallon.

\section*{EXAMPLE: IDENTIFYING HORIZONTAL AND VERTICAL ASYMPTOTES}

Find the horizontal and vertical asymptotes of the function
\(f(x)=\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}\)
Answer
First, note that this function has no common factors, so there are no potential removable discontinuities.
The function will have vertical asymptotes when the denominator is zero, causing the function to be undefined. The denominator will be zero at \(x=1,-2\), and 5 , indicating vertical asymptotes at these values.
The numerator has degree 2, while the denominator has degree 3 . Since the degree of the denominator is greater than the degree of the numerator, the denominator will grow faster than the numerator, causing the outputs to tend towards zero as the inputs get large, and so as \(x \rightarrow \pm \infty, f(x) \rightarrow 0\). This function will have a horizontal asymptote at \(y=0\).


\section*{TRY IT}

Find the vertical and horizontal asymptotes of the function:
\(f(x)=\frac{(2 x-1)(2 x+1)}{(x-2)(x+3)}\)
Answer
Vertical asymptotes at \(x=2\) and \(x=-3\); horizontal asymptote at \(y=4\).
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Recall that we may always find any horizontal intercepts of the graph of a function by setting the output equal to zero. And we may always find any vertical intercept by setting the input equal to zero.
In a rational function of the form \(r(x)=\frac{P(x)}{Q(x)}\),
- Find the vertical intercept (the y-intercept) by evaluating \(r(0)\). That is, replace all the input variables with 0 and calculate the result.
- Find the horizontal intercept(s) (the x-intercepts) by solving \(r(x)=0\). Since the function is undefined where the denominator equals zero], set the numerator equal to zero to find the horizontal intercepts of the function.
- Note that the graph of a rational function may not possess a vertical- or horizontal-intercept.

\section*{A GENERAL NOTE: INTERCEPTS OF RATIONAL FUNCTIONS}

A rational function will have a \(y\)-intercept when the input is zero, if the function is defined at zero. A rational function will not have a \(y\)-intercept if the function is not defined at zero.
Likewise, a rational function will have \(x\)-intercepts at the inputs that cause the output to be zero. Since a fraction is only equal to zero when the numerator is zero, \(x\)-intercepts can only occur when the numerator of the rational function is equal to zero.

\section*{EXAMPLE: FINDING THE INTERCEPTS OF A RATIONAL FUNCTION}

Find the intercepts of \(f(x)=\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}\).

\section*{Answer}

We can find the \(y\)-intercept by evaluating the function at zero
\[
\begin{aligned}
f(0) & =\frac{(0-2)(0+3)}{(0-1)(0+2)(0-5)} \\
& =\frac{-6}{10} \\
& =-\frac{3}{5} \\
& =-0.6
\end{aligned}
\]

The \(x\)-intercepts will occur when the function is equal to zero. A rational function is equal to 0 when the numerator is 0 , as long as the denominator is not also 0 .
\(0=\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}\)
\(0=(x-2)(x+3)\)
\(x=2,-3\)
The \(y\)-intercept is \((0,-0.6)\), the \(x\)-intercepts are \((2,0)\) and \((-3,0)\).


\section*{TRY IT}

Given the reciprocal squared function that is shifted right 3 units and down 4 units, write this as a rational function. Then, find the \(x\) - and \(y\)-intercepts and the horizontal and vertical asymptotes.
Answer
For the transformed reciprocal squared function, we find the rational form.
\[
\begin{aligned}
f(x) & =\frac{1}{(x-3)^{2}}-4 \\
& =\frac{1-4(x-3)^{2}}{(x-3)^{2}} \\
& =\frac{1-4\left(x^{2}-6 x+9\right)}{(x-3)(x-3)} \\
& =\frac{-4 x^{2}+24 x-35}{x^{2}-6 x+9}
\end{aligned}
\]

Because the numerator is the same degree as the denominator we know that as \(x \rightarrow \pm \infty, f(x) \rightarrow-4\); so \(y=-4\) is the horizontal asymptote. Next, we set the denominator equal to zero, and find that the vertical asymptote is \(x=3\), because as \(x \rightarrow 3, f(x) \rightarrow \infty\). We then set the numerator equal to 0 and find the \(x\)-intercepts are at \((2.5,0)\) and \((3.5,0)\). Finally, we evaluate the function at 0 and find the \(y\)-intercept to be at \(\left(0, \frac{-35}{9}\right)\).


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\section*{GRAPH RATIONAL FUNCTIONS}
- Graph a rational function using intercepts, asymptotes, and end behavior.
- Write a rational function given intercepts and asymptotes.

Previously we saw that the numerator of a rational function reveals the \(x\)-intercepts of the graph, whereas the denominator reveals the vertical asymptotes of the graph. As with polynomials, factors of the numerator may have integer powers greater than one. Fortunately, the effect on the shape of the graph at those intercepts is the same as we saw with polynomials.

The vertical asymptotes associated with the factors of the denominator will mirror one of the two toolkit reciprocal functions. When the degree of the factor in the denominator is odd, the distinguishing characteristic is that on one side of the vertical asymptote the graph heads towards positive infinity, and on the other side the graph heads towards negative infinity.


When the degree of the factor in the denominator is even, the distinguishing characteristic is that the graph either heads toward positive infinity on both sides of the vertical asymptote or heads toward negative infinity on both sides.


For example the graph of \(f(x)=\frac{(x+1)^{2}(x-3)}{(x+3)^{2}(x-2)}\).

- At the \(x\)-intercept \(x=-1\) corresponding to the \((x+1)^{2}\) factor of the numerator, the graph bounces, consistent with the quadratic nature of the factor.
- At the \(x\)-intercept \(x=3\) corresponding to the \((x-3)\) factor of the numerator, the graph passes through the axis as we would expect from a linear factor.
- At the vertical asymptote \(x=-3\) corresponding to the \((x+3)^{2}\) factor of the denominator, the graph heads towards positive infinity on both sides of the asymptote, consistent with the behavior of the function \(f(x)=\frac{1}{x^{2}}\).
- At the vertical asymptote \(x=2\), corresponding to the \((x-2)\) factor of the denominator, the graph heads towards positive infinity on the left side of the asymptote and towards negative infinity on the right side, consistent with the behavior of the function \(f(x)=\frac{1}{x}\).

\section*{HOW TO: GIVEN A RATIONAL FUNCTION, SKETCH A GRAPH.}
1. Evaluate the function at 0 to find the \(y\)-intercept.
2. Factor the numerator and denominator.
3. For factors in the numerator not common to the denominator, determine where each factor of the numerator is zero to find the \(x\)-intercepts.
4. Find the multiplicities of the \(x\)-intercepts to determine the behavior of the graph at those points.
5. For factors in the denominator, note the multiplicities of the zeros to determine the local behavior. For those factors not common to the numerator, find the vertical asymptotes by setting those factors equal to zero and then solve.
6. For factors in the denominator common to factors in the numerator, find the removable discontinuities by setting those factors equal to 0 and then solve.
7. Compare the degrees of the numerator and the denominator to determine the horizontal or slant asymptotes.
8. Sketch the graph.

\section*{EXAMPLE: GRAPHING A RATIONAL FUNCTION}

Sketch a graph of \(f(x)=\frac{(x+2)(x-3)}{(x+1)^{2}(x-2)}\).

\section*{Answer}

We can start by noting that the function is already factored, saving us a step.
Next, we will find the intercepts. Evaluating the function at zero gives the \(y\)-intercept:
\(f(0)=\frac{(0+2)(0-3)}{(0+1)^{2}(0-2)}=3\)
To find the \(x\)-intercepts, we determine when the numerator of the function is zero. Setting each factor equal to zero, we find \(x\)-intercepts at \(x=-2\) and \(x=3\). At each, the behavior will be linear (multiplicity 1 ), with the graph passing through the intercept.
We have a \(y\)-intercept at \((0,3)\) and \(x\)-intercepts at \((-2,0)\) and \((3,0)\).
To find the vertical asymptotes, we determine when the denominator is equal to zero. This occurs when \(x+1=0\) and when \(x-2=0\), giving us vertical asymptotes at \(x=-1\) and \(x=2\).
There are no common factors in the numerator and denominator. This means there are no removable discontinuities.
Finally, the degree of denominator is larger than the degree of the numerator, telling us this graph has a horizontal asymptote at \(y=0\).
To sketch the graph, we might start by plotting the three intercepts. Since the graph has no \(x\)-intercepts between the vertical asymptotes, and the \(y\)-intercept is positive, we know the function must remain positive between the asymptotes, letting us fill in the middle portion of the graph.


The factor associated with the vertical asymptote at \(x=-1\) was squared, so we know the behavior will be the same on both sides of the asymptote. The graph heads toward positive infinity as the inputs approach the asymptote on the right, so the graph will head toward positive infinity on the left as well.
For the vertical asymptote at \(x=2\), the factor was not squared, so the graph will have opposite behavior on either side of the asymptote. After passing through the \(x\)-intercepts, the graph will then level off toward an output of zero, as indicated by the horizontal asymptote.


\section*{TRY IT}

Given the function \(f(x)=\frac{(x+2)^{2}(x-2)}{2(x-1)^{2}(x-3)}\), use the characteristics of polynomials and rational functions to describe its behavior and sketch the function.
Answer
Horizontal asymptote at \(y=\frac{1}{2}\). Vertical asymptotes at \(x=1\) and \(x=3\). \(y\)-intercept at \(\left(0, \frac{4}{3}\right.\).) \(x\)-intercepts at \((2,0)\) and \((-2,0) .(-2,0)\) is a zero with multiplicity 2 , and the graph bounces off the \(x\) axis at this point. \((2,0)\) is a single zero and the graph crosses the axis at this point.


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\section*{WHAT CHARACTERISTICS WERE USED IN THE SECOND EXAMPLE ABOVE?}

In the "Try another version" of the above question, try discerning the correct graphs by examining possible horizontal (or slant) asymptotes of each type, then by checking the behavior of each graph around the vertical asymptotes.
The horizontal (or slant) asymptote depends on the ratio of leading terms in the numerator and denominator.
- If the degree of the numerator is less than the degree of the denominator, the horizontal asymptote is the line \(y=0\).
- If the degree of the numerator is exactly one greater than the degree of the denominator, the graph has a slant asymptote.
- If the degree of the numerator is the same as the degree of the denominator, the horizontal asymptote will be the line \(y=c\), where \(c\) represents the value of the ratio of the leading coefficients of the numerator and denominator.

\section*{The behavior of the graph around a vertical asymptote depends on the multiplicity (the degree) of} the factor containing it.
- A zero in the denominator of even multiplicity will cause the function value to increase or decrease in the same manner on either side of the vertical asymptote.
- A zero in the denominator of odd multiplicity will cause the function value to increase or decrease in opposite manner on either side of the vertical asymptote.

Watch the following video to see another worked example of how to match different kinds of rational functions with their graphs.


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\section*{Writing Rational Functions}

Now that we have analyzed the equations for rational functions and how they relate to a graph of the function, we can use information given by a graph to write the function. A rational function written in factored form will have an \(x\)-intercept where each factor of the numerator is equal to zero. (An exception occurs in the case of a removable discontinuity.) As a result, we can form a numerator of a function whose graph will pass through a set of \(x\)-intercepts by introducing a corresponding set of factors. Likewise, because the function will have a vertical asymptote where each factor of the denominator is equal to zero, we can form a denominator that will produce the vertical asymptotes by introducing a corresponding set of factors.

\section*{A GENERAL NOTE: WRITING RATIONAL FUNCTIONS FROM INTERCEPTS AND ASYMPTOTES}

If a rational function has \(x\)-intercepts at \(x=x_{1}, x_{2}, \ldots, x_{n}\), vertical asymptotes at \(x=v_{1}, v_{2}, \ldots, v_{m}\), and no \(x_{i}=\) any \(v_{j}\), then the function can be written in the form:
\(f(x)=a \frac{\left(x-x_{1}\right)^{p_{1}}\left(x-x_{2}\right)^{p_{2}} \ldots\left(x-x_{n}\right)^{p_{n}}}{\left(x-v_{1}\right)^{q_{1}}\left(x-v_{2}\right)^{q_{2}} \ldots\left(x-v_{m}\right)^{q_{n}}}\)
where the powers \(p_{i}\) or \(q_{i}\) on each factor can be determined by the behavior of the graph at the corresponding intercept or asymptote, and the stretch factor \(a\) can be determined given a value of the function other than the \(x\)-intercept or by the horizontal asymptote if it is nonzero.

\section*{WHAT DO THE SUBSCRIPTS IN THE EXAMPLE ABOVE INDICATE?}

The above example uses an abstract form to help define a rational function. We've seen these before. A linear function can be written in the form \(f(x)=m x+b\), and a quadratic function can be written in the form \(f(x)=a x^{2}+b x+c\).
A quadratic can also be written in the form of a polynomial as \(p(x)=a\left(x-r_{1}\right)\left(x-r_{2}\right)\), where \(a\) represents the leading coefficient and \(r_{1}\) and \(r_{2}\) are the roots, or zeros of the function. Instead of writing " \(r_{1}\) and \(r_{2}\)," we could use some notational shorthand and simply write, "the \(r_{n}\) " to indicate them both at once. That's the same as saying that the number of roots present include \(r_{1}, r_{2}, \ldots, r_{n}\) for \(n\) total roots. In the definition above, to say that "no \(x_{i}=\) any \(v_{j}\) " is to say that there are no common factors in the numerator and denominator. In other words, the roots in the numerator and the vertical asymptotes in the denominator can only be counted in factored and reduced form.

\section*{HOW TO: GIVEN A GRAPH OF A RATIONAL FUNCTION, WRITE THE FUNCTION.}
1. Determine the factors of the numerator. Examine the behavior of the graph at the \(x\)-intercepts to determine the zeroes and their multiplicities. (This is easy to do when finding the "simplest" function with small multiplicities-such as 1 or 3-but may be difficult for larger multiplicities-such as 5 or 7, for example.)
2. Determine the factors of the denominator. Examine the behavior on both sides of each vertical asymptote to determine the factors and their powers.
3. Use any clear point on the graph to find the stretch factor.

\section*{EXAMPLE: WRITING A RATIONAL FUNCTION FROM INTERCEPTS AND ASYMPTOTES}

Write an equation for the rational function below.


\section*{Answer}

The graph appears to have \(x\)-intercepts at \(x=-2\) and \(x=3\). At both, the graph passes through the intercept, suggesting linear factors. The graph has two vertical asymptotes. The one at \(x=-1\) seems to exhibit the basic behavior similar to \(\frac{1}{x}\), with the graph heading toward positive infinity on one side and heading toward negative infinity on the other. The asymptote at \(x=2\) is exhibiting a behavior similar to \(\frac{1}{x^{2}}\), with the graph heading toward negative infinity on both sides of the asymptote.


We can use this information to write a function of the form
\(f(x)=a \frac{(x+2)(x-3)}{(x+1)(x-2)^{2}}\).
To find the stretch factor, we can use another clear point on the graph, such as the \(y\)-intercept \((0,-2)\).
\[
\begin{aligned}
-2 & =a \frac{(0+2)(0-3)}{(0+1)(0-2)^{2}} \\
& -2=a \frac{-6}{4} \\
a=\frac{-8}{-6}= & \frac{4}{3}
\end{aligned}
\]

This gives us a final function of \(f(x)=\frac{4(x+2)(x-3)}{3(x+1)(x-2)^{2}}\).

\section*{USING A POINT ON THE GRAPH TO WRITE THE EQUATION OF THE GRAPH}

Note in the example above that we used a point on the graph to help write the equation of the graph. In doing so, we used the important concept that any point contained on the graph of a function satisfies the equation of the function (that is, substituting both input and output values into the function equation will result in a true statement).
This concept applies to any function we study. If the function can be graphed in the real plane, it has a set of input \(x\) values and a set of corresponding output \(f(x)\) values, and each corresponding pair is given by \((x, f(x))\).
This correspondence allows us to substitute any \(x\) to find its corresponding \(f(x)\) and vice versa. It also allows us to substitute both input and output into any function equation in order to solve for a missing third value, such as the leading coefficient \(a\) in the example above.

\section*{TRY IT}

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\section*{SUMMARY: RATIONAL FUNCTIONS}

\section*{Key Equations}
\[
f(x)=\frac{P(x)}{Q(x)}=\frac{a_{p} x^{p}+a_{p-1} x^{p-1}+\ldots+a_{1} x+a_{0}}{b_{q} x^{q}+b_{q-1} x^{q-1}+\ldots+b_{1} x+b_{0}}, Q(x) \neq 0
\]

\section*{Key Concepts}
- We can use arrow notation to describe local behavior and end behavior of the toolkit functions \(f(x)=\frac{1}{x}\) and \(f(x)=\frac{1}{x^{2}}\).
- A function that levels off at a horizontal value has a horizontal asymptote. A function can have more than one vertical asymptote.
- Application problems involving rates and concentrations often involve rational functions.
- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.
- The vertical asymptotes of a rational function will occur where the denominator of the function is equal to zero and the numerator is not zero.
- A removable discontinuity might occur in the graph of a rational function if an input causes both numerator and denominator to be zero.
- A rational function's end behavior will mirror that of the ratio of the leading terms of the numerator and denominator functions.
- Graph rational functions by finding the intercepts, behavior at the intercepts and asymptotes, and end behavior.
- If a rational function has \(x\)-intercepts at \(x=x_{1}, x_{2}, \ldots, x_{n}\), vertical asymptotes at \(x=v_{1}, v_{2}, \ldots, v_{m}\), and no \(x_{i}=\) any \(v_{j}\), then the function can be written in the form
\(f(x)=a \frac{\left(x-x_{1}\right)^{p_{1}}\left(x-x_{2}\right)^{p_{2}} \cdots\left(x-x_{n}\right)^{p_{n}}}{\left(x-v_{1}\right)^{q_{1}}\left(x-v_{2}\right)^{q_{2}} \cdots\left(x-v_{m}\right)^{q_{n}}}\)

\section*{Glossary}
arrow notation a way to symbolically represent the local and end behavior of a function by using arrows to indicate that an input or output approaches a value
horizontal asymptote a horizontal line \(y=b\) where the graph approaches the line as the inputs increase or decrease without bound.
rational function a function that can be written as the ratio of two polynomials
removable discontinuity a single point at which a function is undefined that, if filled in, would make the function continuous; it appears as a hole on the graph of a function
vertical asymptote a vertical line \(x=a\) where the graph tends toward positive or negative infinity as the inputs approach \(a\)
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\section*{INTRODUCTION TO RADICAL FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Find the inverse of a polynomial function.
- Restrict the domain to find the inverse of a polynomial function.

A mound of gravel is in the shape of a cone with the height equal to twice the radius.


The volume is found using a formula from geometry.
\[
\begin{aligned}
V & =\frac{1}{3} \pi r^{2} h \\
& =\frac{1}{3} \pi r^{2}(2 r) \\
& =\frac{2}{3} \pi r^{3}
\end{aligned}
\]

We have written the volume \(V\) in terms of the radius \(r\). However, in some cases, we may start out with the volume and want to find the radius. For example: A customer purchases 100 cubic feet of gravel to construct a cone shape mound with a height twice the radius. What are the radius and height of the new cone? To answer this question, we use the formula
\(r=\sqrt[3]{\frac{3 V}{2 \pi}}\)
This function is the inverse of the formula for \(V\) in terms of \(r\).
In this section, we will explore the inverses of polynomial and rational functions and in particular the radical functions we encounter in the process.

\section*{RADICALS AS INVERSE POLYNOMIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Verify that a radical and a polynomial function are inverses of each other
- Find the inverse of a polynomial function

Recall that two functions \(f\) and \(g\) are inverse functions if for every coordinate pair in \(f,(a, b)\), there exists a corresponding coordinate pair in the inverse function, \(g,(b, a)\). In other words, the coordinate pairs of the inverse functions have the input and output interchanged.

Only one-to-one functions have inverses that are also functions. Recall that a one-to-one function has a unique output value for each input value and passes the horizontal line test.

For a function that is not one-to-one to have an inverse function, the function must be restricted on its domain to create a new function that is one-to-one and thus would have an inverse function.

\section*{TIP FOR SUCCESS}

The examples in this section will probably seem confusing at first. The best way to use such examples to study math is to write them out by hand slowly and carefully, perhaps more than once or twice, trying to explain each step to yourself as you work through it. Repeated practice will help you understand the concepts that have been applied in each example.

For example, suppose a water runoff collector is built in the shape of a parabolic trough as shown below. We can use the information in the figure to find the surface area of the water in the trough as a function of the depth of the water.


Because it will be helpful to have an equation for the parabolic cross-sectional shape, we will impose a coordinate system at the cross section, with \(x\) measured horizontally and \(y\) measured vertically, with the origin at the vertex of the parabola.


From this we find an equation for the parabolic shape. We placed the origin at the vertex of the parabola, so we know the equation will have form \(y(x)=a x^{2}\). Our equation will need to pass through the point \((6,18)\), from which we can solve for the stretch factor \(a\).
\[
\begin{aligned}
18 & =a 6^{2} \\
a & =\frac{18}{36} \\
a & =\frac{1}{2}
\end{aligned}
\]

Our parabolic cross section has the equation
\(y(x)=\frac{1}{2} x^{2}\)
We are interested in the surface area of the water, so we must determine the width at the top of the water as a function of the water depth. For any depth \(y\) the width will be given by \(2 x\), so we need to solve the equation above for \(x\) and find the inverse function. However, notice that the original function is not one-toone, and indeed, given any output there are two inputs that produce the same output, one positive and one negative.

To find an inverse, we can restrict our original function to a limited domain on which it is one-to-one. In this case, it makes sense to restrict ourselves to positive \(x\) values. On this domain, we can find an inverse by solving for the input variable:
\[
\begin{aligned}
y & =\frac{1}{2} x^{2} \\
2 y & =x^{2} \\
x & = \pm \sqrt{2 y}
\end{aligned}
\]

This is not a function as written. We are limiting ourselves to positive \(x\) values, so we eliminate the negative solution, giving us the inverse function we're looking for.
\(y=\frac{x^{2}}{2}, x>0\)

Because \(x\) is the distance from the center of the parabola to either side, the entire width of the water at the top will be \(2 x\). The trough is 3 feet ( 36 inches) long, so the surface area will then be:
\[
\begin{aligned}
\text { Area } & =l \cdot w \\
& =36 \cdot 2 x \\
& =72 x \\
& =72 \sqrt{2 y}
\end{aligned}
\]

This example illustrates two important points:
1. When finding the inverse function of a quadratic, we have to limit ourselves to a domain on which the function is one-to-one.
2. The inverse function of a quadratic function is a square root function. Both are toolkit functions and different types of power functions.

Functions involving roots are often called radical functions. While it is not possible to find an inverse function of most polynomial functions, some basic polynomials do have inverses that are functions. Such functions are called invertible functions, and we use the notation \(f^{-1}(x)\).

Warning: \(f^{-1}(x)\) is not the same as the reciprocal of the function \(f(x)\). This use of -1 is reserved to denote inverse functions. To denote the reciprocal of a function \(f(x)\), we would need to write \((f(x))^{-1}=\frac{1}{f(x)}\).

An important relationship between inverse functions is that they "undo" each other. If \(f^{-1}\) is the inverse of a function \(f\), then \(f\) is the inverse of the function \(f^{-1}\). In other words, whatever the function \(f\) does to \(x, f^{-1}\) undoes it-and vice-versa. More formally, we write
\(f^{-1}(f(x))=x\), for all \(x\) in the domain of \(f\)
and
\(f\left(f^{-1}(x)\right)=x\), for all \(x\) in the domain of \(f^{-1}\)

\section*{A GENERAL NOTE: VERIFYING TWO FUNCTIONS ARE INVERSES OF ONE ANOTHER}

Two functions, \(f\) and \(g\), are inverses of one another if for all \(x\) in the domain of \(f\) and \(g\). \(g(f(x))=f(g(x))=x\)

\section*{HOW TO: GIVEN A POLYNOMIAL FUNCTION, FIND THE INVERSE OF THE FUNCTION BY RESTRICTING THE DOMAIN IN SUCH A WAY THAT THE NEW FUNCTION IS ONE-TO-ONE.}
1. Verify that \(f\) is a one-to-one function.
2. Replace \(f(x)\) with \(y\).
3. Interchange \(x\) and \(y\).
4. Solve for \(y\), and rename the function \(f^{-1}(x)\).

\section*{EXAMPLE: VERIFYING INVERSE FUNCTIONS}

Show that \(f(x)=\frac{1}{x+1}\) and \(f^{-1}(x)=\frac{1}{x}-1\) are inverses, for \(x \neq 0,-1\).
Answer
We must show that \(f^{-1}(f(x))=x\) and \(f\left(f^{-1}(x)\right)=x\).
\[
\begin{aligned}
f^{-1}(f(x)) & =f^{-1}\left(\frac{1}{x+1}\right) \\
& =\frac{1}{\frac{1}{x+1}}-1 \\
& =(x+1)-1 \\
& =x
\end{aligned}
\]
\[
\begin{aligned}
f\left(f^{-1}(x)\right) & =f\left(\frac{1}{x}-1\right) \\
& =\frac{1}{\left(\frac{1}{x}-1\right)+1} \\
& =\frac{1}{\frac{1}{x}} \\
& =x
\end{aligned}
\]

Therefore, \(f(x)=\frac{1}{x+1}\) and \(f^{-1}(x)=\frac{1}{x}-1\) are inverses.

\section*{TRY IT}

Show that \(f(x)=\frac{x+5}{3}\) and \(f^{-1}(x)=3 x-5\) are inverses.
Answer
\[
\begin{aligned}
f^{-1}(f(x)) & =f^{-1}\left(\frac{x+5}{3}\right) \\
& =3\left(\frac{x+5}{3}\right)-5 \\
& =(x-5)+5 \\
& =x
\end{aligned}
\]
and
\(f\left(f^{-1}(x)\right)=f(3 x-5)\)
\[
=\frac{(3 x-5)+5}{3}
\]
\[
=\frac{3 x}{3}
\]
\[
=x
\]

\section*{EXAMPLE: FINDING THE INVERSE OF A CUBIC FUNCTION}

Find the inverse of the function \(f(x)=5 x^{3}+1\).

\section*{Answer}

This is a transformation of the basic cubic toolkit function, and based on our knowledge of that function, we know it is one-to-one. Solving for the inverse by solving for \(x\).
\[
\begin{aligned}
y & =5 x^{3}+1 \\
x & =5 y^{3}+1 \\
x-1 & =5 y^{3} \\
\frac{x-1}{5} & =y^{3} \\
f^{-1}(x) & =\sqrt[3]{\frac{x-1}{5}}
\end{aligned}
\]

\section*{Analysis of the Solution}

Look at the graph of \(f\) and \(f^{-1}\). Notice that the two graphs are symmetrical about the line \(y=x\). This is always the case when graphing a function and its inverse function.
Also, since the method involved interchanging \(x\) and \(y\), notice corresponding points. If \((a, b)\) is on the graph of \(f\), then \((b, a)\) is on the graph of \(f^{-1}\). Since \((0,1)\) is on the graph of \(f\), then \((1,0)\) is on the graph of \(f^{-1}\). Similarly, since \((1,6)\) is on the graph of \(f\), then \((6,1)\) is on the graph of \(f^{-1}\).


Find the inverse function of \(f(x)=\sqrt[3]{x+4}\).
Answer
\(f^{-1}(x)=x^{3}-4\)
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\section*{DOMAINS OF RADICAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Restrict the domain of a quadratic function, then find its inverse.
- Find the inverse of a radical function.
- Determine the domain of a radical function composed with other functions.
- Find the inverse of a rational function.

So far we have been able to find the inverse functions of cubic functions without having to restrict their domains. However, as we know, not all cubic polynomials are one-to-one. Some functions that are not one-to-one may have their domain restricted so that they are one-to-one, but only over that domain. The function over the restricted domain would then have an inverse function. Since quadratic functions are not one-toone, we must restrict their domain in order to find their inverses.

\section*{A GENERAL NOTE: RESTRICTING THE DOMAIN}

If a function is not one-to-one, it cannot have an inverse function. If we restrict the domain of the function so that it becomes one-to-one, thus creating a new function, this new function will have an inverse function.

\section*{HOW TO: GIVEN A POLYNOMIAL FUNCTION, RESTRICT THE DOMAIN OF A FUNCTION THAT IS NOT ONE-TO-ONE AND THEN FIND THE INVERSE.}
1. Restrict the domain by determining a domain on which the original function is one-to-one.
2. Replace \(f(x)\) with \(y\).
3. Interchange \(x\) and \(y\).
4. Solve for \(y\), and rename the function or pair of function \(f^{-1}(x)\).
5. Revise the formula for \(f^{-1}(x)\) by ensuring that the outputs of the inverse function correspond to the restricted domain of the original function.

\section*{RECALL TRANSFORMING A QUADRATIC EQUATION}

In the examples below, it will be helpful to recall how to transform the graph of a quadratic function. Recall that a quadratic function in the form \(f(x)=a x^{2}+b x+c\) can be written in vertex form, \(f(x)=a(x-h)^{2}+k\), where the vertex \((h, k)\) can be found by evaluating \(\left(-\frac{b}{2 a}, f\left(-\frac{b}{2 a}\right)\right)\).
The vertex form of a quadratic also represents the toolkit function \(f(x)=x^{2}\) that has undergone a vertical stretch or compression by a factor of \(a\), a horizontal shift \(h\) units, and a vertical shift \(k\) units.
Ex. The function \(f(x)=3(x-2)^{2}-5\) has been stretched vertically by a factor of 3 , shifted 2 units to the right, and shifted 5 units down.

Watch the following video to see more examples of how to restrict the domain of a quadratic function to find it's inverse.

Notice that the functions from previous examples were all polynomials, and their inverses were radical functions. If we want to find the inverse of a radical function, we will need to restrict the domain of the answer because the range of the original function is limited.

\section*{Solving Applications of Radical Functions}

Radical functions are common in physical models, as we saw in the section opener. We now have enough tools to be able to solve the problem posed at the start of the section.

\section*{Determining the Domain of a Radical Function Composed with Other Functions}

When radical functions are composed with other functions, determining domain can become more complicated.

\section*{Finding Inverses of Rational Functions}

As with finding inverses of quadratic functions, it is sometimes desirable to find the inverse of a rational function, particularly of rational functions that are the ratio of linear functions, such as in concentration applications.

Watch this video to see another worked example of how to find the inverse of a rational function.

\section*{EXAMPLE: RESTRICTING THE DOMAIN TO FIND THE INVERSE OF A POLYNOMIAL FUNCTION}

Find the inverse function of \(f\) :
1. \(f(x)=(x-4)^{2}, x \geq 4\)
2. \(f(x)=(x-4)^{2}, x \leq 4\)

\section*{Answer}

The original function \(f(x)=(x-4)^{2}\) is not one-to-one, but the function is restricted to a domain of \(x \geq 4\) or \(x \leq 4\) on which it is one-to-one.


\(=4\) and the second is when \(x\)
To find the inverse, start by replacing \(f(x)\) with the simple variable \(y\).
\[
\begin{aligned}
y & =(x-4)^{2} & & \text { Interchange } x \text { and } y . \\
x & =(y-4)^{2} & & \text { Take the square root. } \\
\pm \sqrt{x} & =y-4 & & \text { Add } 4 \text { to both sides. } \\
4 \pm \sqrt{x} & =y & &
\end{aligned}
\]

This is not a function as written. We need to examine the restrictions on the domain of the original function to determine the inverse. Since we reversed the roles of \(x\) and \(y\) for the original \(f(x)\), we looked at the domain: the values \(x\) could assume. When we reversed the roles of \(x\) and \(y\), this gave us the values \(y\) could assume. For this function, \(x \geq 4\), so for the inverse, we should have \(y \geq 4\), which is what our inverse function gives.
1. The domain of the original function was restricted to \(x \geq 4\), so the outputs of the inverse need to be the same, \(f(x) \geq 4\), and we must use the + case:
\(f^{-1}(x)=4+\sqrt{x}\)
2. The domain of the original function was restricted to \(x \leq 4\), so the outputs of the inverse need to be the same, \(f(x) \leq 4\), and we must use the - case:
\(f^{-1}(x)=4-\sqrt{x}\)

\section*{Analysis of the Solution}

On the graphs below, we see the original function graphed on the same set of axes as its inverse function. Notice that together the graphs show symmetry about the line \(y=x\). The coordinate pair \((4,0)\) is on the
graph of \(f\) and the coordinate pair \((0,4)\) is on the graph of \(f^{-1}\). For any coordinate pair, if \((a, b)\) is on the graph of \(f\), then \((b, a)\) is on the graph of \(f^{-1}\). Finally, observe that the graph of \(f\) intersects the graph of \(f^{-1}\) on the line \(y=x\). Points of intersection for the graphs of \(f\) and \(f^{-1}\) will always lie on the line \(y=x\).



\section*{EXAMPLE: FINDING THE INVERSE OF A QUADRATIC FUNCTION WHEN THE RESTRICTION IS NOT SPECIFIED}

Restrict the domain and then find the inverse of
\(f(x)=(x-2)^{2}-3\).
Answer
We can see this is a parabola with vertex at \((2,-3)\) that opens upward. Because the graph will be decreasing on one side of the vertex and increasing on the other side, we can restrict this function to a domain on which it will be one-to-one by limiting the domain to \(x \geq 2\).
To find the inverse, we will use the vertex form of the quadratic. We start by replacing \(f(x)\) with a simple variable, \(y\), then solve for \(x\).
\[
\begin{aligned}
y & =(x-2)^{2}-3 & & \text { Interchange } x \text { and } y . \\
x & =(y-2)^{2}-3 & & \text { Add } 3 \text { to both sides. } \\
x+3 & =(y-2)^{2} & & \text { Take the square root. } \\
\pm \sqrt{x+3} & =y-2 & & \text { Add } 2 \text { to both sides. } \\
2 \pm \sqrt{x+3} & =y & & \text { Rename the function. } \\
f^{-1}(x) & =2 \pm \sqrt{x+3} & &
\end{aligned}
\]

Now we need to determine which case to use. Because we restricted our original function to a domain of \(x \geq 2\), the outputs of the inverse should be the same, telling us to utilize the + case
\(f^{-1}(x)=2+\sqrt{x+3}\)
If the quadratic had not been given in vertex form, rewriting it into vertex form would be the first step. This way we may easily observe the coordinates of the vertex to help us restrict the domain.

\section*{Analysis of the Solution}

Notice that we arbitrarily decided to restrict the domain on \(x \geq 2\). We could just have easily opted to restrict the domain on \(x \leq 2\), in which case \(f^{-1}(x)=2-\sqrt{x+3}\). Observe the original function graphed on the same set of axes as its inverse function in the graph below. Notice that both graphs show symmetry about the line \(y=x\). The coordinate pair \((2,-3)\) is on the graph of \(f\) and the coordinate pair \((-3,2)\) is on the graph of \(f^{-1}\). Observe from the graph of both functions on the same set of axes that domain of \(f=\) range of \(f^{-1}=[2, \infty)\)
and
domain of \(f^{-1}=\) range of \(f=[-3, \infty)\)
Finally, observe that the graph of \(f\) intersects the graph of \(f^{-1}\) along the line \(y=x\).


\section*{TRY IT}

Find the inverse of the function \(f(x)=x^{2}+1\), on the domain \(x \geq 0\).

\section*{Answer}
\(f^{-1}(x)=\sqrt{x-1}\)
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\section*{HOW TO: GIVEN A RADICAL FUNCTION, FIND THE INVERSE.}
1. Determine the range of the original function.
2. Replace \(f(x)\) with \(y\), then solve for \(x\).
3. If necessary, restrict the domain of the inverse function to the range of the original function.

\section*{EXAMPLE: FINDING THE INVERSE OF A RADICAL FUNCTION}

Determine the range of the function \(f(x)=\sqrt{x-4}\). Then, find its inverse and restrict the domain of the inverse as necessary.
Answer
Note that the original function has range \(f(x) \geq 0\). Replace \(f(x)\) with \(y\), then solve for \(x\).
\[
\begin{aligned}
y & =\sqrt{x-4} & & \text { Replace } f(x) \text { with } y . \\
x & =\sqrt{y-4} & & \text { Interchange } x \text { and } y . \\
x & =\sqrt{y-4} & & \text { Square each side. } \\
x^{2} & =y-4 & & \text { Add } 4 . \\
x^{2}+4 & =y & & \text { Rename the function } f^{-1}(x) . \\
f^{-1}(x) & =x^{2}+4 & &
\end{aligned}
\]

Recall that the domain of this function must be limited to the range of the original function.
\(f^{-1}(x)=x^{2}+4, x \geq 0\)

\section*{Analysis of the Solution}

Notice in the graph below that the inverse is a reflection of the original function over the line \(y=x\). Because the original function has only positive outputs, the inverse function has only positive inputs.


\section*{TRY IT}

Determine the range of the function and then find its inverse and appropriate domain. \(f(x)=\sqrt{2 x+3}\).
Answer
\(f^{-1}(x)=\frac{x^{2}-3}{2}, x \geq 0\)
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\section*{EXAMPLE: SOLVING AN APPLICATION WITH A CUBIC FUNCTION}

A mound of gravel is in the shape of a cone with the height equal to twice the radius. The volume of the cone in terms of the radius is given by
\(V=\frac{2}{3} \pi r^{3}\)
Find the inverse of the function \(V=\frac{2}{3} \pi r^{3}\) that determines the volume \(V\) of a cone and is a function of the radius \(r\). Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use \(\pi=3.14\).

\section*{Answer}

Start with the given function for \(V\). Notice that the meaningful domain for the function is \(r \geq 0\) since negative radii would not make sense in this context. Also note the range of the function (hence, the domain of the inverse function) is \(V \geq 0\). Solve for \(r\) in terms of \(V\), using the method outlined previously.
\(V=\frac{2}{3} \pi r^{3}\)
\(r^{3}=\frac{3 V}{2 \pi} \quad\) Solve for \(r^{3}\).
\(r=\sqrt[3]{\frac{3 V}{2 \pi}} \quad\) Solve for \(r\).
This is the result stated in the section opener. Now evaluate this for \(V=100\) and \(\pi=3.14\).
\(r=\sqrt[3]{\frac{3 V}{2 \pi}}\)
\(=\sqrt[3]{\frac{3 \cdot 100}{2 \cdot 3.14}}\)
\(\approx \sqrt[3]{47.7707}\)
\(\approx 3.63\)
Therefore, the radius is about 3.63 ft .

\section*{EXAMPLE: FINDING THE DOMAIN OF A RADICAL FUNCTION COMPOSED WITH A RATIONAL FUNCTION}

Find the domain of the function \(f(x)=\sqrt{\frac{(x+2)(x-3)}{(x-1)}}\).

\section*{Answer}

Because a square root is only defined when the quantity under the radical is non-negative, we need to determine where \(\frac{(x+2)(x-3)}{(x-1)} \geq 0\). The output of a rational function can change signs (change from positive to negative or vice versa) at \(x\)-intercepts and at vertical asymptotes. For this equation the graph could change signs at \(x=-2,1\), and 3 .
To determine the intervals on which the rational expression is positive, we could test some values in the expression or sketch a graph. While both approaches work equally well, for this example we will use a graph.


This function has two \(x\)-intercepts, both of which exhibit linear behavior near the \(x\)-intercepts. There is one vertical asymptote, corresponding to a linear factor; this behavior is similar to the basic reciprocal toolkit function, and there is no horizontal asymptote because the degree of the numerator is larger than the degree of the denominator. There is a \(y\)-intercept at \((0,6)\).
From the \(y\)-intercept and \(x\)-intercept at \(x=-2\), we can sketch the left side of the graph. From the behavior at the asymptote, we can sketch the right side of the graph.
From the graph, we can now tell on which intervals the outputs will be non-negative, so that we can be sure that the original function \(f(x)\) will be defined. \(f(x)\) has domain \(-2 \leq x<1\) or \(x \geq 3\), or in interval notation, \([-2,1) \cup[3, \infty)\).

\section*{TRY IT}

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\section*{EXAMPLE: FINDING THE INVERSE OF A RATIONAL FUNCTION}

The function \(C=\frac{20+0.4 n}{100+n}\) represents the concentration \(C\) of an acid solution after \(n \mathrm{~mL}\) of \(40 \%\) solution has been added to 100 mL of a \(20 \%\) solution. First, find the inverse of the function; that is, find an expression for \(n\) in terms of \(C\). Then use your result to determine how much of the \(40 \%\) solution should be added so that the final mixture is a \(35 \%\) solution.

\section*{Answer}

We first want the inverse of the function. We will solve for \(n\) in terms of \(C\).
\[
\begin{aligned}
C & =\frac{20+0.4 n}{100+n} \\
C(100+n) & =20+0.4 n \\
100 C+C n & =20+0.4 n \\
100 C-20 & =0.4 n-C n \\
100 C-20 & =(0.4-C) n \\
n & =\frac{100 C-20}{0.4-C}
\end{aligned}
\]

Now evaluate this function for \(C=0.35\) (35\%).
\[
\begin{aligned}
n & =\frac{100(0.35)-20}{0.4-0.35} \\
& =\frac{15}{0.05} \\
& =300
\end{aligned}
\]

We can conclude that 300 mL of the \(40 \%\) solution should be added.

\section*{TRY IT}

Find the inverse of the function \(f(x)=\frac{x+3}{x-2}\). Verify that you are correct by graphing both functions and the function \(f(x)=x\).
Answer
\(f^{-1}(x)=\frac{2 x+3}{x-1}\)
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\section*{SUMMARY: RADICAL FUNCTIONS}

\section*{Key Concepts}
- The inverse of a quadratic function is a square root function.
- If \(f^{-1}\) is the inverse of a function \(f\), then \(f\) is the inverse of the function \(f^{-1}\).
- While it is not possible to find an inverse of most polynomial functions, some basic polynomials are invertible.
- To find the inverse of certain functions, we must restrict the function to a domain on which it will be one-to-one.
- When finding the inverse of a radical function, we need a restriction on the domain of the answer.
- Inverse and radical and functions can be used to solve application problems.

\section*{Glossary}
invertible function any function that has an inverse function

\section*{INTRODUCTION TO VARIATION}

\section*{Learning Outcomes}

By the end of this lesson, you will be able to:
- Solve direct variation problems.
- Solve inverse variation problems.
- Solve problems involving joint variation.

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers \(16 \%\) commission on her sales. Her earnings depend on the amount of her sales. For instance if she sells a vehicle for \(\$ 4,600\), she will earn \(\$ 736\). She wants to evaluate the offer, but she is not sure how. In this section we will look at relationships, such as this one, between earnings, sales, and commission rate.

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\section*{DIRECT VARIATION}

\section*{LEARNING OUTCOMES}
- Solve a direct variation problem
- Use a constant of variation to describe the relationship between two variables

A used-car company has just offered their best candidate, Nicole, a position in sales. The position offers \(16 \%\) commission on her sales. Her earnings depend on the amount of her sales. For instance if she sells a vehicle for \(\$ 4,600\), she will earn \(\$ 736\). She wants to evaluate the offer, but she is not sure how. In this section we will look at relationships, such as this one, between earnings, sales, and commission rate.

In the example above, Nicole's earnings can be found by multiplying her sales by her commission. The formula \(e=0.16 s\) tells us her earnings, \(e\), come from the product of 0.16 , her commission, and the sale price of the vehicle, \(s\). If we create a table, we observe that as the sales price increases, the earnings increase as well, which should be intuitive.
\begin{tabular}{|l|l|l|}
\hline\(s\), sales prices & \(e=0.16 s\) & Interpretation \\
\hline\(\$ 4,600\) & \(e=0.16(4,600)=736\) & A sale of a \(\$ 4,600\) vehicle results in \(\$ 736\) earnings. \\
\hline\(\$ 9,200\) & \(e=0.16(9,200)=1,472\) & A sale of a \(\$ 9,200\) vehicle results in \(\$ 1472\) earnings. \\
\hline\(\$ 18,400\) & \(e=0.16(18,400)=2,944\) & A sale of a \(\$ 18,400\) vehicle results in \(\$ 2944\) earnings. \\
\hline
\end{tabular}

Notice that earnings are a multiple of sales. As sales increase, earnings increase in a predictable way. Double the sales of the vehicle from \(\$ 4,600\) to \(\$ 9,200\), and we double the earnings from \(\$ 736\) to \(\$ 1,472\). As the input increases, the output increases as a multiple of the input. A relationship in which one quantity is a constant multiplied by another quantity is called direct variation. Each variable in this type of relationship varies directly with the other.

The graph below represents the data for Nicole's potential earnings. We say that earnings vary directly with the sales price of the car. The formula \(y=k x^{n}\) is used for direct variation. The value \(k\) is a nonzero constant greater than zero and is called the constant of variation. In this case, \(k=0.16\) and \(n=1\).


\section*{A GENERAL NOTE: DIRECT VARIATION}

If \(x\) and \(y\) are related by an equation of the form
\(y=k x^{n}\)
then we say that the relationship is direct variation and \(y\) varies directly with the \(n\)th power of \(x\). In direct variation relationships, there is a nonzero constant ratio \(k=\frac{y}{x^{n}}\), where \(k\) is called the constant of variation, which help defines the relationship between the variables.

\section*{RECALL ISOLATING A VARIABLE IN A FORMULA}

We've learned to solve certain formulas for one of the variables. For example, in the formula that relates distance, rate, and time, \(d=r t\), we can solve the equation for rate, \(r=\frac{d}{t}\) or for time, \(t=\frac{d}{r}\). We say we are isolating the variable of interest in these cases.
The same idea applies when solving a direct variation problem for the constant of variation, \(k\). Given a direct variation such as \(y=k x^{n}\), we can isolate the constant of variation using the properties of equality to obtain \(\frac{y}{x^{n}}=k\).

\section*{HOW TO: GIVEN A DESCRIPTION OF A DIRECT VARIATION PROBLEM, SOLVE FOR AN UNKNOWN.}
1. Identify the input, \(x\), and the output, \(y\).
2. Determine the constant of variation. You may need to divide \(y\) by the specified power of \(x\) to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Watch this video to see a quick lesson in direct variation. You will see more worked examples.

\section*{EXAMPLE: SOLVING A DIRECT VARIATION PROBLEM}

The quantity \(y\) varies directly with the cube of \(x\). If \(y=25\) when \(x=2\), find \(y\) when \(x\) is 6 .
Answer
The general formula for direct variation with a cube is \(y=k x^{3}\). The constant can be found by dividing \(y\) by the cube of \(x\).
\(k=\frac{y}{x^{3}}\)
\(=\frac{25}{2^{3}}\)
\(=\frac{25}{8}\)
Now use the constant to write an equation that represents this relationship.
\(y=\frac{25}{8} x^{3}\)
Substitute \(x=6\) and solve for \(y\).
\(y=\frac{25}{8}(6)^{3}\)
\(=675\)

Analysis of the Solution
The graph of this equation is a simple cubic, as shown below.


\section*{Q \& A}

Do the graphs of all direct variation equations look like the example above?
No. Direct variation equations are power functions-they may be linear, quadratic, cubic, quartic, radical, etc. But all of the graphs pass through \((0,0)\).

The quantity \(y\) varies directly with the square of \(y\). If \(y=24\) when \(x=3\), find \(y\) when \(x\) is 4 .

\section*{Answer}

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\section*{INVERSE AND JOINT VARIATION}

\section*{LEARNING OUTCOMES}
- Solve an Inverse variation problem.
- Write a formula for an inversely proportional relationship.

Water temperature in an ocean varies inversely to the water's depth. Between the depths of 250 feet and 500 feet, the formula \(T=\frac{14,000}{d}\) gives us the temperature in degrees Fahrenheit at a depth in feet below Earth's surface. Consider the Atlantic Ocean, which covers \(22 \%\) of Earth's surface. At a certain location, at the depth of 500 feet, the temperature may be \(28^{\circ} \mathrm{F}\).

If we create a table we observe that, as the depth increases, the water temperature decreases.
\begin{tabular}{|l|l|l|}
\hline\(d\), depth & \(T=\frac{14,000}{d}\) & Interpretation \\
\hline 500 ft & \(\frac{14,000}{500}=28\) & At a depth of 500 ft, the water temperature is \(28^{\circ} \mathrm{F}\). \\
\hline 350 ft & \(\frac{14,000}{350}=40\) & At a depth of 350 ft, the water temperature is \(40^{\circ} \mathrm{F}\). \\
\hline 250 ft & \(\frac{14,000}{250}=56\) & At a depth of 250 ft, the water temperature is \(56^{\circ} \mathrm{F}\). \\
\hline
\end{tabular}

We notice in the relationship between these variables that, as one quantity increases, the other decreases. The two quantities are said to be inversely proportional and each term varies inversely with the other. Inversely proportional relationships are also called inverse variations.

For our example, the graph depicts the inverse variation. We say the water temperature varies inversely with the depth of the water because, as the depth increases, the temperature decreases. The formula \(y=\frac{k}{x}\) for inverse variation in this case uses \(k=14,000\).


\section*{A GENERAL NOTE: INVERSE VARIATION}

If \(x\) and \(y\) are related by an equation of the form
\(y=\frac{k}{x^{n}}\)
where \(k\) is a nonzero constant, then we say that \(y\) varies inversely with the \(n\)th power of \(x\). In inversely proportional relationships, or inverse variations, there is a constant multiple \(k=x^{n} y\).

\section*{ISOLATING THE CONSTANT OF VARIATION}

To isolate the constant of variation in an inverse variation, use the properties of equality to solve the equation for \(k\).
\(y=\frac{k}{x^{n}}\)
Isolate \(k\) using algebra.
\(y x^{n}=k\)
The following video presents a short lesson on inverse variation and includes more worked examples.

\section*{EXAMPLE: WRITING A FORMULA FOR AN INVERSELY PROPORTIONAL RELATIONSHIP}

A tourist plans to drive 100 miles. Find a formula for the time the trip will take as a function of the speed the tourist drives.

\section*{Answer}

Recall that multiplying speed by time gives distance. If we let \(t\) represent the drive time in hours, and \(v\) represent the velocity (speed or rate) at which the tourist drives, then \(v t=\) distance. Because the distance is fixed at 100 miles, \(v t=100\). Solving this relationship for the time gives us our function.
\(t(v)=\frac{100}{v}\)
\[
=100 v^{-1}
\]

We can see that the constant of variation is 100 and, although we can write the relationship using the negative exponent, it is more common to see it written as a fraction.

\section*{HOW TO: GIVEN A DESCRIPTION OF AN INVERSE VARIATION PROBLEM, SOLVE FOR AN UNKNOWN.}
1. Identify the input, \(x\), and the output, \(y\).
2. Determine the constant of variation. You may need to multiply \(y\) by the specified power of \(x\) to determine the constant of variation.
3. Use the constant of variation to write an equation for the relationship.
4. Substitute known values into the equation to find the unknown.

Many situations are more complicated than a basic direct variation or inverse variation model. One variable often depends on multiple other variables. When a variable is dependent on the product or quotient of two or more variables, this is called joint variation. For example, the cost of busing students for each school trip varies with the number of students attending and the distance from the school. The variable \(c\), cost, varies jointly with the number of students, \(n\), and the distance, \(d\).

The following video provides another worked example of a joint variation problem.

\section*{EXAMPLE: SOLVING AN INVERSE VARIATION PROBLEM}

A quantity \(y\) varies inversely with the cube of \(x\). If \(y=25\) when \(x=2\), find \(y\) when \(x\) is 6 .

\section*{Answer}

The general formula for inverse variation with a cube is \(y=\frac{k}{x^{3}}\). The constant can be found by multiplying \(y\) by the cube of \(x\).
\(k=x^{3} y\)
\(=2^{3} \cdot 25\)
\(=200\)
Now we use the constant to write an equation that represents this relationship.
\[
\begin{aligned}
& y=\frac{k}{x^{3}}, k=200 \\
& y=\frac{200}{x^{3}}
\end{aligned}
\]

Substitute \(x=6\) and solve for \(y\).
\(y=\frac{200}{6^{3}}\)
\[
=\frac{25}{27}
\]

\section*{Analysis of the Solution}

The graph of this equation is a rational function.


\section*{TRY IT}

A quantity \(y\) varies inversely with the square of \(x\). If \(y=8\) when \(x=3\), find \(y\) when \(x\) is 4 .
Answer
\(\frac{9}{2}\)
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\section*{A GENERAL NOTE: JOINT VARIATION}

Joint variation occurs when a variable varies directly or inversely with multiple variables.
For instance, if \(x\) varies directly with both \(y\) and \(z\), we have \(x=k y z\). If \(x\) varies directly with \(y\) and inversely with \(z\), we have \(x=\frac{k y}{z}\). Notice that we only use one constant in a joint variation equation.

\section*{ISOLATING THE CONSTANT OF VARIATION}

To isolate the constant of variation in a joint variation, use the properties of equality to solve the equation for \(k\).
\(x=k y z\)
Isolate \(k\) using algebra.
\(\frac{x}{y z}=k\)

\section*{EXAMPLE: SOLVING PROBLEMS INVOLVING JOINT VARIATION}

A quantity \(x\) varies directly with the square of \(y\) and inversely with the cube root of \(z\). If \(x=6\) when \(y=2\) and \(z=8\), find \(x\) when \(y=1\) and \(z=27\).
Answer
Begin by writing an equation to show the relationship between the variables.
\(x=\frac{k y^{2}}{\sqrt[3]{z}}\)
Substitute \(x=6, y=2\), and \(z=8\) to find the value of the constant \(k\).
\(6=\frac{k 2^{2}}{\sqrt[3]{8}}\)
\(6=\frac{4 k}{2}\)
\(3=k\)
Now we can substitute the value of the constant into the equation for the relationship.
\(x=\frac{3 y^{2}}{\sqrt[3]{z}}\)
To find \(x\) when \(y=1\) and \(z=27\), we will substitute values for \(y\) and \(z\) into our equation.
\(x=\frac{3(1)^{2}}{\sqrt[3]{27}}\)
\(=1\)

\section*{TRY IT}
\(x\) varies directly with the square of \(y\) and inversely with \(z\). If \(x=40\) when \(y=4\) and \(z=2\), find \(x\) when \(y=10\) and \(z=25\).

\section*{Answer}
\(x=20\)
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- Inverse Variation. Authored by: James Sousa (Mathispower4u.com) for Lumen Learning. Located at: https://youtu.be/awp2vxqd-l4. License: CC BY: Attribution
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\section*{SUMMARY: VARIATION}

\section*{Key Equations}
Direct variation \(y=k x^{n}, k\) is a nonzero constant.

Inverse variation \(y=\frac{k}{x^{n}}, k\) is a nonzero constant.

\section*{Key Concepts}
- A relationship where one quantity is a constant multiplied by another quantity is called direct variation.
- Two variables that are directly proportional to one another will have a constant ratio.
- A relationship where one quantity is a constant divided by another quantity is called inverse variation.
- Two variables that are inversely proportional to one another will have a constant multiple.
- In many problems, a variable varies directly or inversely with multiple variables. We call this type of relationship joint variation.

\section*{Glossary}
constant of variation the non-zero value \(k\) that helps define the relationship between variables in direct or inverse variation
direct variation the relationship between two variables that are a constant multiple of each other; as one quantity increases, so does the other
inverse variation the relationship between two variables in which the product of the variables is a constant
inversely proportional a relationship where one quantity is a constant divided by the other quantity; as one quantity increases, the other decreases
joint variation a relationship where a variable varies directly or inversely with multiple variables
varies directly a relationship where one quantity is a constant multiplied by the other quantity
varies inversely a relationship where one quantity is a constant divided by the other quantity

\section*{PUTTING IT TOGETHER: RATIONAL AND RADICAL FUNCTIONS}

Isaac Newton (1642-1726) discovered the Universal Law of Gravitation, which explains how massive objects like planets exert attractive forces on one another. Let \(m_{1}\) and \(m_{2}\) be the masses of two objects which are separated by a distance \(d\). Then the force \(F\) acting on each object varies directly with both \(m_{1}\) and \(m_{2}\) and inversely with the square of \(d\). In other words, there is a certain constant, which we call the universal gravitational constant \(G\), such that:
\(F=\frac{G \cdot m_{1} \cdot m_{2}}{d^{2}}\)
The value of \(G\) has been determined to remarkable precision. In standard units,
\(G \approx 6.67408 \times 10^{-11} \frac{\mathrm{Nm}^{2}}{\mathrm{~kg}^{2}}\)
Don't worry about the exact value of this constant or the strange combination of units following it. What is important is that \(G\) is a constant. Basically, because \(G\) is so incredibly tiny, the force due to gravity is negligible unless the masses of the objects are extremely large and the distances are not too great.

Let's explore the effect on gravitational force when mass or distance between the objects varies. In the table below mass is measured in kilograms (kg), distance in meters ( m ), and force in Newtons ( N ).
\begin{tabular}{|l|l|l|l|}
\hline\(m_{1}\) & \(m_{2}\) & \(d\) & \(F=\frac{\left(6.67 \times 10^{-11}\right) \cdot m_{1} \cdot m_{2}}{d^{2}}\) \\
\hline \(100,000 \mathrm{~kg}\) & \(100,000 \mathrm{~kg}\) & 1 m & 0.667 N \\
\hline \(200,000 \mathrm{~kg}\) & \(100,000 \mathrm{~kg}\) & 1 m & 1.334 N \\
\hline \(300,000 \mathrm{~kg}\) & \(100,000 \mathrm{~kg}\) & 1 m & 2.001 N \\
\hline \(300,000 \mathrm{~kg}\) & \(100,000 \mathrm{~kg}\) & 2 m & 0.500 N \\
\hline \(300,000 \mathrm{~kg}\) & \(100,000 \mathrm{~kg}\) & 3 m & 0.222 N \\
\hline
\end{tabular}

Note that when we double the mass of one object, the force doubles as well. Tripling the mass triples the force. This is because force varies directly with mass. On the other hand, when the distance doubles, the force decreases by a factor of 4, and when the distance is tripled, we find the force is only one-ninth of what it was before! For this reason, the Universal Law of Gravitation is sometimes referred to as an inverse
square law. Other natural phenomena, such as electrostatic force, also work according to an inverse square law. (The force between two charged particles is governed by Coulomb's Law.)

Let's take another look at the Universal Law of Gravitation. Imagine a comet is zooming through space near Earth. Our planet attracts the comet towards it, but as long as the comet is far enough away and travelling at a high enough speed, it should whiz right by without striking the Earth. Because the masses of the Earth and the comet remain relatively constant, the only variable in Newton's Law is distance (d). The Earth has a mass of roughly kg, while a typical comet has a mass on the order of about kg. Therefore, the formula that gives the force of gravity exerted by the Earth on the comet is:
\(F=\frac{\left(6.67 \times 10^{-11}\right)\left(6 \times 10^{24}\right)\left(10^{13}\right)}{d^{2}} \approx \frac{4 \times 10^{27}}{d^{2}}\)

This is a rational function in the variable \(d\), in which the numerator is a constant (degree 0 ) and the denominator has degree 2 . What happens to
 the gravitational force as the comet speeds by and gets further and further away? In other words, what is the end behavior of \(F\) as \(d \rightarrow \infty\) ?

Since the degree of the denominator (2) is greater than the degree of the numerator (0), this function has a horizontal asymptote at \(y=0\). That is, \(f \rightarrow 0\) as \(d \rightarrow \infty\). You can take comfort in the fact that the gravitational force pulling on the comet will become virtually zero if the comet manages to get far enough away.

From gravity to electromagnetism, rational functions and variation seem to be at the heart of how our universe works. If you can analyze rational functions and understand direct and inverse variation, then the sky's the limit!
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\title{
MODULE 11: EXPONENTIAL AND LOGARITHMIC FUNCTIONS
}

\section*{WHY IT MATTERS: EXPONENTIAL AND LOGARITHMIC FUNCTIONS}

\section*{Why It Matters: Exponential and Logarithmic Functions}

Many sources credit Albert Einstein as saying, "Compound interest is the eighth wonder of the world. He who understands it, earns it ... he who doesn't ... pays it." You probably already know this if you have ever invested in an account or taken out a loan. Interest is the amount added to the balance. The beauty, in the case of investing, is that once interest is earned, it also earns interest. This idea of interest earning interest is known as compound interest. (It isn't quite as beautiful on money you owe.)

Interest can be compounded over different time intervals. It might be compounded annually (once per year), or more often, such as semi-annually (twice per year), quarterly (four times per year), monthly ( 12 times per year), weekly ( 52 times per year), or daily ( 365 times per year). There is also one more optioncompounded continuously - which is the theoretical concept of adding interest in infinitesimally small increments. Although not actually possible, it provides the limit of compounding and is therefore a useful quantity in economics.


Suppose you inherit \$10,000. You decide to invest in in an account paying 3\% interest compounded continuously. What will the balance be in 5 years, 10 years, or even 50 years? You'll want to know, especially for retirement planning.

In this module, you will learn about the function you can evaluate to answer these questions. You will also discover how to make changes to the variables involved, such as time or initial investment, to alter your results.
- Use the properties of exponents to rewrite expressions containing exponents.
- Evaluate and solve functions.

\section*{Exponential Functions}
- Identify the base of an exponential function and restrictions for its value.
- Find the equation of an exponential function.
- Use the compound interest formula.
- Evaluate exponential functions with base e.
- Given two data points, write an exponential function.
- Identify initial conditions for an exponential function.
- Find an exponential function given a graph.
- Use a graphing calculator to find an exponential function.
- Find an exponential function that models continuous growth or decay.

\section*{Graphs of Exponential Functions}
- Determine whether an exponential function and its associated graph represents growth or decay.
- Sketch a graph of an exponential function.
- Graph exponential functions shifted horizontally or vertically and write the associated equation.
- Graph a stretched or compressed exponential function.
- Graph a reflected exponential function.
- Write the equation of an exponential function that has been transformed.

\section*{Logarithmic Functions}
- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms with and without a calculator.
- Evaluate logarithms with base 10 and base e.

\section*{Graphs of Logarithmic Functions}
- Determine the domain and range of a logarithmic function.
- Determine the \(x\)-intercept and vertical asymptote of a logarithmic function.
- Identify whether a logarithmic function is increasing or decreasing and give the interval.
- Identify the features of a logarithmic function that make it an inverse of an exponential function.
- Graph horizontal and vertical shifts of logarithmic functions.
- Graph stretches and compressions of logarithmic functions.
- Graph reflections of logarithmic functions.

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\section*{REVIEW TOPICS FOR SUCCESS}

\section*{Learning Outcomes}

By the end of this section, you will be able to:
- Use the properties of exponents to rewrite expressions containing exponents.
- Evaluate and solve functions.

The study of exponential and logarithmic functions requires a thorough understanding of several important concepts that have appeared earlier in this course. In particular, you'll need to be able to use the properties of exponents to rewrite expressions containing them and understand the relationship between corresponding input and output values in a function.

This review section contains concepts from earlier sections in this text. Review them carefully to refresh your understanding before beginning your study of these two, new toolkit functions.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

\section*{THE RULES FOR EXPONENTS}

\section*{LEARNING OUTCOMES}
- Recall the properties of exponents and use them to rewrite expressions containing exponents.

Review the following list of rules for simplifying expressions containing exponents, then try the problems listed below. If you need a refresher, return to the Algebra Essentials module for more explanation and demonstration.

\section*{THE PRODUCT RULE OF EXPONENTS}

For any real number \(a\) and natural numbers \(m\) and \(n\), the product rule of exponents states that
\[
a^{m} \cdot a^{n}=a^{m+n}
\]

\section*{THE QUOTIENT RULE OF EXPONENTS}

For any real number \(a\) and natural numbers \(m\) and \(n\), such that \(m>n\), the quotient rule of exponents states that
\[
\frac{a^{m}}{a^{n}}=a^{m-n}
\]

For any real number \(a\) and positive integers \(m\) and \(n\), the power rule of exponents states that
\[
\left(a^{m}\right)^{n}=a^{m \cdot n}
\]

\section*{THE ZERO EXPONENT RULE OF EXPONENTS}

For any nonzero real number \(a\), the zero exponent rule of exponents states that
\[
a^{0}=1
\]

\section*{THE NEGATIVE RULE OF EXPONENTS}

For any nonzero real number \(a\) and natural number \(n\), the negative rule of exponents states that
\[
a^{-n}=\frac{1}{a^{n}} \text { and } a^{n}=\frac{1}{a^{-n}}
\]

\section*{THE POWER OF A PRODUCT RULE OF EXPONENTS}

For any real numbers \(a\) and \(b\) and any integer \(n\), the power of a product rule of exponents states that
\[
(a b)^{n}=a^{n} b^{n}
\]

\section*{THE POWER OF A QUOTIENT RULE OF EXPONENTS}

For any real numbers \(a\) and \(b\) and any integer \(n\), the power of a quotient rule of exponents states that
\[
\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}
\]

\section*{TRY IT}

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\section*{LEARNING OUTCOMES}
- Evaluate and solve functions.

In this course, we've been using fundamental characteristics of functions to evaluate, solve, identify, transform, and write the equations of each of the toolkit functions. In this module, you'll study two new functions: the exponential function and the logarithmic function. Their graphs and equations may look a little different from the familiar linear, power, and polynomial functions you've seen so far. But they possess the same basic characteristics. That is, they are one-to-one functions, having well-defined corresponding input and output in ordered pairs that represent points in the plane. They are smooth and continuous. And they may be transformed in the same way that other functions can by stretching or compressing, reflecting, and shifting horizontally or vertically.

Recall the form of ordered pairs of input and output for any function: \((x, f(x))\). Using this well-defined relationship along with the fact that any point of form \((x, f(x))\) contained on the graph of a function satisfies the equation of the function (makes a true statement when the values of \(x\) and \(f(x)\) are substituted into the equation). That is, given a value for an input, we can substitute it for \(x\) in the equation of a function to solve for the corresponding output. Likewise, given an value for an output, we can substitute it for \(f(x)\) in the equation to solve for the corresponding input. And, given a point on the graph and an equation containing an unknown such as a leading coefficient or vertical intercept, we can substitute both input and output to solve for the unknown.

Recall the steps to evaluate and solve functions then practice these skills using the problems below. This will help you to prepare for handling the new exponential and logarithmic functions in the same way in this module.

\section*{HOW TO: EVALUATE A FUNCTION GIVEN ITS FORMULA.}
1. Replace the input variable in the formula with the value provided.
2. Calculate the result.

\section*{HOW TO: SOLVE A FUNCTION.}
1. Replace the output in the formula with the value provided.
2. Solve for the input variable that makes the statement true.

\section*{TRY IT}

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\section*{SUMMARY: REVIEW}

\section*{Key Equations}

\section*{Rules of Exponents}

For nonzero real numbers \(a\) and \(b\) and integers \(m\) and \(n\)
\begin{tabular}{|l|l|}
\hline Product rule & \(a^{m} \cdot a^{n}=a^{m+n}\) \\
\hline Quotient rule & \(\frac{a^{m}}{a^{n}}=a^{m-n}\) \\
\hline Power rule & \(\left(a^{m}\right)^{n}=a^{m \cdot n}\) \\
\hline Zero exponent rule & \(a^{0}=1\) \\
\hline Negative rule & \(a^{-n}=\frac{1}{a^{n}}\) \\
\hline Power of a product rule & \((a \cdot b)^{n}=a^{n} \cdot b^{n}\) \\
\hline Power of a quotient rule & \(\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}\) \\
\hline
\end{tabular}

\section*{Key Concepts}
- Products of exponential expressions with the same base can be simplified by adding exponents.
- Quotients of exponential expressions with the same base can be simplified by subtracting exponents.
- Powers of exponential expressions with the same base can be simplified by multiplying exponents.
- An expression with exponent zero is defined as 1 .
- An expression with a negative exponent is defined as a reciprocal.
- The power of a product of factors is the same as the product of the powers of the same factors.
- The power of a quotient of factors is the same as the quotient of the powers of the same factors.
- The rules for exponential expressions can be combined to simplify more complicated expressions.
- Corresponding input and output pairs of a function each represent a point on the graph of the function.
- Each point contained on the graph of a function satisfies the equation of a function.

\section*{INTRODUCTION TO EXPONENTIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Identify the base of an exponential function and restrictions for its value.
- Find the equation of an exponential function.
- Use the compound interest formula.
- Evaluate exponential functions with base e.
- Given two data points, write an exponential function.
- Identify initial conditions for an exponential function.
- Find an exponential function given a graph.
- Use a graphing calculator to find an exponential function.
- Find an exponential function that models continuous growth or decay.

Focus in on a square centimeter of your skin. Look closer. Closer still. If you could look closely enough, you would see hundreds of thousands of microscopic organisms. They are bacteria, and they are not only on your skin, but in your mouth, nose, and even your intestines. In fact, the bacterial cells in your body at any given moment outnumber your own cells. But that is no reason to feel bad about yourself. While some bacteria can cause illness, many are healthy and even essential to the body.


An electron micrograph of E.Coli bacteria. (credit: "Mattosaurus," Wikimedia Commons)
Bacteria commonly reproduce through a process called binary fission during which one bacterial cell splits into two. When conditions are right, bacteria can reproduce very quickly. Unlike humans and other complex organisms, the time required to form a new generation of bacteria is often a matter of minutes or hours as opposed to days or years. (Note: Todar, PhD, Kenneth. Todar's Online Textbook of Bacteriology. http://textbookofbacteriology.net/growth_3.html.)

For simplicity's sake, suppose we begin with a culture of one bacterial cell that can divide every hour. The table below shows the number of bacterial cells at the end of each subsequent hour. We see that the single bacterial cell leads to over one thousand bacterial cells in just ten hours! If we were to extrapolate the table to twenty-four hours, we would have over 16 million!
\begin{tabular}{|l|l|l|l|l|l|l|l|l|l|l|}
\hline Hour & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline Bacteria & 1 & 2 & 4 & 8 & 16 & 32 & 64 & 128 & 256 & 512
\end{tabular}

In this module, we will explore exponential functions which can be used for, among other things, modeling growth patterns such as those found in bacteria. We will also investigate logarithmic functions which are closely related to exponential functions. Both types of functions have numerous real-world applications when it comes to modeling and interpreting data.

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\section*{EVALUATE EXPONENTIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Identify the base of an exponential function and restrictions for its value.
- Evaluate exponential growth functions.
- Use the compound interest formula.
- Evaluate exponential functions with base e.

India is the second most populous country in the world with a population of about 1.25 billion people in 2013. The population is growing at a rate of about \(1.2 \%\) each year. (Note: http://www.worldometers.info/worldpopulation/. Accessed February 24, 2014.) If this rate continues, the population of India will exceed China's population by the year 2031. When populations grow rapidly, we often say that the growth is "exponential." To a mathematician, however, the term exponential growth has a very specific meaning. In this section, we will take a look at exponential functions, which model this kind of rapid growth.

The general form of the exponential formula is
\(f(x)=a b^{x}\)
where \(a\) is any nonzero number and \(b \neq 1\) is a positive real number.
- if \(b>1\), the function grows at a rate proportional to its size.
- if \(0<b<1\), the function decays at a rate proportional to its size.

The base of an exponential function must be a positive real number other than 1 . Why do we limit the base \(b\) to positive values? This is done to ensure that the outputs will be real numbers. Observe what happens if the base is not positive:
- Consider a base of -9 and exponent of \(\frac{1}{2}\). Then \(f(x)=f\left(\frac{1}{2}\right)=(-9)^{\frac{1}{2}}=\sqrt{-9}\), which is not a real number.

Why do we limit the base to positive values other than 1? This is because a base of 1 results in the constant function. Observe what happens if the base is 1 :
- Consider a base of 1 . Then \(f(x)=1^{x}=1\) for any value of \(x\).

To evaluate an exponential function with the form \(f(x)=b^{x}\), we simply substitute \(x\) with the given value, and calculate the resulting power. For example:

Let \(f(x)=2^{x}\). What is \(f(3)\) ?
\[
\begin{aligned}
f(x) & =2^{x} & & \\
f(3) & =2^{3} & & \text { Substitute } x=3 . \\
& =8 & & \text { Evaluate the power. }
\end{aligned}
\]

When evaluating an exponential function, it is important to follow the order of operations. For example:
Let \(f(x)=30(2)^{x}\). What is \(f(3)\) ?
\[
f(x)=30(2)^{x}
\]
\(f(3)=30(2)^{3} \quad\) Substitute \(x=3\).
\(=30(8) \quad\) Simplify the power first.
\(=240\) Multiply .
Note that if the order of operations were not followed, the result would be incorrect:
\(f(3)=30(2)^{3} \neq 60^{3}=216,000\)

\section*{EXAMPLE: EVALUATING EXPONENTIAL FUNCTIONS}

Let \(f(x)=5(3)^{x+1}\). Evaluate \(f(2)\) without using a calculator.
Answer
Follow the order of operations. Be sure to pay attention to the parentheses.
\[
\begin{aligned}
f(x) & =5(3)^{x+1} & & \\
f(2) & =5(3)^{2+1} & & \text { Substitute } x=2 . \\
& =5(3)^{3} & & \text { Add the exponents. } \\
& =5(27) & & \text { Simplify the power. } \\
& =135 & & \text { Multiply. }
\end{aligned}
\]

\section*{TRY IT}

Let \(f(x)=8(1.2)^{x-5}\). Evaluate \(f(3)\) using a calculator. Round to four decimal places.
Answer
5.5556

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Because the output of exponential functions increases very rapidly, the term "exponential growth" is often used in everyday language to describe anything that grows or increases rapidly. However, exponential growth can be defined more precisely in a mathematical sense. If the growth rate is proportional to the amount present, the function models exponential growth.

\section*{A GENERAL NOTE: EXPONENTIAL GROWTH}

A function that models exponential growth grows by a rate proportional to the amount present. For any real number \(x\) and any positive real numbers \(a\) and \(b\) such that \(b \neq 1\), an exponential growth function has the form
\[
f(x)=a b^{x}
\]
where
- \(a\) is the initial or starting value of the function.
- \(b\) is the growth factor or growth multiplier per unit \(x\).

In more general terms, an exponential function consists of a constant base raised to a variable exponent. To

\section*{TIP FOR SUCCESS}

In mathematics, we carefully define terms, objects, and concepts so that there can be no ambiguity about what we mean. But such careful definition can be challenging to decipher. When reading definitions such as the one given above, it is helpful to write them out slowly by hand, perhaps more than once or twice, trying to explain each part to yourself as you go. That's how you'll build intuition for applying the term, object, or concept.
Be sure to read the surrounding text as well, such as the paragraph below. It will help you understand the definition in more general terms.
differentiate between linear and exponential functions, let's consider two companies, A and B. Company A has 100 stores and expands by opening 50 new stores a year, so its growth can be represented by the function \(A(x)=100+50 x\). Company B has 100 stores and expands by increasing the number of stores by \(50 \%\) each year, so its growth can be represented by the function \(B(x)=100(1+0.5)^{x}\).

A few years of growth for these companies are illustrated below.
\begin{tabular}{|l|l|l|}
\hline Year, \(\boldsymbol{x}\) & Stores, Company A & Stores, Company B \\
\hline 0 & \(100+50(0)=100\) & \(100(1+0.5)^{0}=100\) \\
\hline 1 & \(100+50(1)=150\) & \(100(1+0.5)^{1}=150\) \\
\hline 2 & \(100+50(2)=200\) & \(100(1+0.5)^{2}=225\) \\
\hline 3 & \(100+50(3)=250\) & \(100(1+0.5)^{3}=337.5\) \\
\hline\(x\) & \(A(x)=100+50 x\) & \(B(x)=100(1+0.5)^{x}\) \\
\hline
\end{tabular}

The graphs comparing the number of stores for each company over a five-year period are shown below. We can see that, with exponential growth, the number of stores increases much more rapidly than with linear growth.


The graph shows the numbers of stores Companies \(A\) and \(B\) opened over a five-year period.
Notice that the domain for both functions is \([0, \infty)\), and the range for both functions is \([100, \infty)\). After year 1, Company B always has more stores than Company A.

Let's more closely examine the function representing the number of stores for Company B, \(B(x)=100(1+0.5)^{x}\). In this exponential function, 100 represents the initial number of stores, 0.5 represents the growth rate, and \(1+0.5=1.5\) represents the growth factor. Generalizing further, we can write this function as \(B(x)=100(1.5)^{x}\) where 100 is the initial value, 1.5 is called the base, and \(x\) is called the exponent.

\section*{EXAMPLE: EVALUATING A REAL-WORLD EXPONENTIAL MODEL}

At the beginning of this section, we learned that the population of India was about 1.25 billion in the year 2013 with an annual growth rate of about \(1.2 \%\). This situation is represented by the growth function \(P(t)=1.25(1.012)^{t}\) where \(t\) is the number of years since 2013. To the nearest thousandth, what will the population of India be in 2031?

\section*{Answer}

To estimate the population in 2031, we evaluate the models for \(t=18\), because 2031 is 18 years after 2013. Rounding to the nearest thousandth,
\(P(18)=1.25(1.012)^{18} \approx 1.549\)
There will be about 1.549 billion people in India in the year 2031.

\section*{TRY IT}

The population of China was about 1.39 billion in the year 2013 with an annual growth rate of about \(0.6 \%\). This situation is represented by the growth function \(P(t)=1.39(1.006)^{t}\) where \(t\) is the number of years since 2013. To the nearest thousandth, what will the population of China be in the year 2031? How does this compare to the population prediction we made for India in the previous example?

\section*{Answer}

About 1.548 billion people; by the year 2031, India's population will exceed China's by about 0.001 billion, or 1 million people.

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\section*{Using the Compound Interest Formula}

Savings instruments in which earnings are continually reinvested, such as mutual funds and retirement accounts, use compound interest. The term compounding refers to interest earned not only on the original value, but on the accumulated value of the account.

The annual percentage rate (APR) of an account, also called the nominal rate, is the yearly interest rate earned by an investment account. The term nominal is used when the compounding occurs a number of times other than once per year. In fact, when interest is compounded more than once a year, the effective interest rate ends up being greater than the nominal rate! This is a powerful tool for investing.

We can calculate compound interest using the compound interest formula which is an exponential function of the variables time \(t\), principal \(P\), APR \(r\), and number of times compounded in a year \(n\) :
\[
A(t)=P\left(1+\frac{r}{n}\right)^{n t}
\]

For example, observe the table below, which shows the result of investing \(\$ 1,000\) at \(10 \%\) for one year. Notice how the value of the account increases as the compounding frequency increases.
\begin{tabular}{l|l|}
\hline Frequency & Value after 1 year \\
\hline Annually & \(\$ 1100\) \\
\hline Semiannually & \(\$ 1102.50\) \\
\hline Quarterly & \(\$ 1103.81\) \\
\hline Monthly & \(\$ 1104.71\) \\
\hline Daily & \(\$ 1105.16\) \\
\hline
\end{tabular}

\section*{A GENERAL NOTE: THE COMPOUND INTEREST FORMULA}

Compound interest can be calculated using the formula
\(A(t)=P\left(1+\frac{r}{n}\right)^{n t}\)
where
- \(A(t)\) is the accumulated value of the account
- \(t\) is measured in years
- \(P\) is the starting amount of the account, often called the principal, or more generally present value
- \(r\) is the annual percentage rate (APR) expressed as a decimal
- \(n\) is the number of times compounded in a year

\section*{EXAMPLE: CALCULATING COMPOUND INTEREST}

If we invest \(\$ 3,000\) in an investment account paying \(3 \%\) interest compounded quarterly, how much will the account be worth in 10 years?

\section*{Answer}

Because we are starting with \(\$ 3,000, P=3000\). Our interest rate is \(3 \%\), so \(r=0.03\). Because we are compounding quarterly, we are compounding 4 times per year, so \(n=4\). We want to know the value of the account in 10 years, so we are looking for \(A(10)\), the value when \(t=10\).
\(A(t) \quad=P\left(1+\frac{r}{n}\right)^{n t} \quad\) Use the compound interest formula.
\(A(10)=3000\left(1+\frac{0.03}{4}\right)^{4 \cdot 10}\) Substitute using given values.
\(\approx 4045.05 \quad\) Round to two decimal places.
The account will be worth about \$4,045.05 in 10 years.

\section*{TIP FOR SUCCESS}

When calculating the value of an exponential function such as the compound interest formula, be careful when entering your calculation into a calculator. Use as many parentheses as needed to ensure your intent is clear. The calculator will apply order of operations as it is typed, which can cause an incorrect calculation for your equation.
Ex. To find the accumulated investment in the example above, you must calculate
\(A(10)=3000\left(1+\frac{0.03}{4}\right)^{4 \cdot 10}\).
This would be entered in most scientific or graphing calculators as \(3000(1+(0.03 / 4)) \wedge(4 \star 10)\).

\section*{TRY IT}

An initial investment of \(\$ 100,000\) at \(12 \%\) interest is compounded weekly (use 52 weeks in a year). What will the investment be worth in 30 years?
Answer
about \$3,644,675.88
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\section*{Evaluating Exponential Functions with Base e}

As we saw earlier, the amount earned on an account increases as the compounding frequency increases. The table below shows that the increase from annual to semi-annual compounding is larger than the increase from monthly to daily compounding. This might lead us to ask whether this pattern will continue.

Examine the value of \(\$ 1\) invested at \(100 \%\) interest for 1 year compounded at various frequencies.

\section*{EXAMPLE: USING THE COMPOUND INTEREST FORMULA TO SOLVE FOR THE PRINCIPAL}

A 529 Plan is a college-savings plan that allows relatives to invest money to pay for a child's future college tuition; the account grows tax-free. Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to \(\$ 40,000\) over 18 years. She believes the account will earn \(6 \%\) compounded semi-annually (twice a year). To the nearest dollar, how much will Lily need to invest in the account now?
Answer
The nominal interest rate is \(6 \%\), so \(r=0.06\). Interest is compounded twice a year, so \(n=2\).
We want to find the initial investment \(P\) needed so that the value of the account will be worth \(\$ 40,000\) in 18 years. Substitute the given values into the compound interest formula and solve for \(P\).
\(A(t) \quad=P\left(1+\frac{r}{n}\right)^{n t} \quad\) Use the compound interest formula.
\(40,000=P\left(1+\frac{0.06}{2}\right)^{2(18)} \quad\) Substitute using given values \(A, r, n\), and \(t\).
\(40,000=P(1.03)^{36} \quad\) Simplify.
\(\frac{40,000}{(1.03)^{36}}=P \quad\) Isolate \(P\).
\(P \quad \approx 13,801 \quad\) Divide and round to the nearest dollar.
Lily will need to invest \(\$ 13,801\) to have \(\$ 40,000\) in 18 years.

\section*{TRY IT}

Refer to the previous example. To the nearest dollar, how much would Lily need to invest if the account is compounded quarterly?
Answer
\$13,693
\begin{tabular}{|l|l|l|}
\hline Frequency & \(A(t)=\left(1+\frac{1}{n}\right)^{n}\) & Value \\
\hline Annually & \(\left(1+\frac{1}{1}\right)^{1}\) & \(\$ 2\) \\
\hline Semiannually & \(\left(1+\frac{1}{2}\right)^{2}\) & \(\$ 2.25\) \\
\hline Quarterly & \(\left(1+\frac{1}{4}\right)^{4}\) & \(\$ 2.441406\) \\
\hline Monthly & \(\left(1+\frac{1}{12}\right)^{12}\) & \(\$ 2.613035\) \\
\hline Daily & \(\left(1+\frac{1}{365}\right)^{365}\) & \(\$ 2.714567\) \\
\hline Hourly & \(\left(1+\frac{1}{8766}\right)^{8766}\) & \(\$ 2.718127\) \\
\hline Once per minute & \(\left(1+\frac{1}{525960}\right)^{525960}\) & \(\$ 2.718279\) \\
\hline Once per second & \(\left(1+\frac{1}{31557600}\right)^{31557600}\) & \(\$ 2.718282\) \\
\hline
\end{tabular}

These values appear to be approaching a limit as \(n\) increases without bound. In fact, as \(n\) gets larger and larger, the expression \(\left(1+\frac{1}{n}\right)^{n}\) approaches a number used so frequently in mathematics that it has its own
name: the letter \(e\). This value is an irrational number, which means that its decimal expansion goes on forever without repeating. Its approximation to six decimal places is shown below.

\section*{UNDERSTANDING THE NUMBER \(e\)}

The number \(e\) is an irrational constant, like the number \(p i\).
The number \(e\) appears frequently in applications in the real world and is as important as numbers like \(0,1, i\), and \(\pi\), so it is apt that it was named in honor of one of the most prolific and important mathematicians to have lived, Leonhard Euler (pronounced like "oiler").

\section*{A GENERAL NOTE: THE NUMBER \(e\)}

The letter \(e\) represents the irrational number
\(\left(1+\frac{1}{n}\right)^{n}\), as \(n\) increases without bound
The letter \(e\) is used as a base for many real-world exponential models. To work with base e, we use the approximation, \(e \approx 2.718282\). The constant was named by the Swiss mathematician Leonhard Euler (1707-1783) who first investigated and discovered many of its properties.

\section*{EXAMPLE: USING A CALCULATOR TO FIND POWERS OF \(e\)}

Calculate \(e^{3.14}\). Round to five decimal places.

\section*{Answer}

On a calculator, press the button labeled \(\left[e^{x}\right]\). The window shows [ \(e^{\wedge}(]\). Type 3.14 and then close parenthesis, [ \(\mathrm{e}^{\wedge}(3.14)\) ]. Press [ENTER]. Rounding to 5 decimal places, \(e^{3.14} \approx 23.10387\). Caution: Many scientific calculators have an "Exp" button, which is used to enter numbers in scientific notation. It is not used to find powers of \(e\).

\section*{TRY IT}

Use a calculator to find \(e^{-0.5}\). Round to five decimal places.
Answer
\(e^{-0.5} \approx 0.60653\)
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EQUATIONS OF EXPONENTIAL FUNCTIONS

\section*{LEARNING OUTCOMES}
- Given two data points, write an exponential function.
- Identify initial conditions for an exponential function.
- Find an exponential function given a graph.
- Use a graphing calculator to find an exponential function.
- Find an exponential function that models continuous growth or decay.

In the previous examples, we were given an exponential function which we then evaluated for a given input. Sometimes we are given information about an exponential function without knowing the function explicitly. We must use the information to first write the form of the function, determine the constants \(a\) and \(b\), and evaluate the function.

\section*{TIP FOR SUCCESS}

When writing an exponential model from two data points, recall the processes you've learned to write other types of models from data points contained on the graphs of linear, power, polynomial, and rational functions. Each process was different, but each followed a fundamental characteristic of functions: that every point on the graph of a function satisfies the equation of the function. The process of writing an exponential model capitalizes on the same idea.
See the examples below for how to write an exponential model.

\section*{HOW TO: GIVEN TWO DATA POINTS, WRITE AN EXPONENTIAL MODEL}
1. If one of the data points has the form \((0, a)\), then \(a\) is the initial value. Using \(a\), substitute the second point into the equation \(f(x)=a b^{x}\), and solve for \(b\).
2. If neither of the data points have the form \((0, a)\), substitute both points into two equations with the form \(f(x)=a b^{x}\). Solve the resulting system of two equations to find \(a\) and \(b\).
3. Using the \(a\) and \(b\) found in the steps above, write the exponential function in the form \(f(x)=a b^{x}\).

\section*{Investigating Continuous Growth}

So far we have worked with rational bases for exponential functions. For most real-world phenomena, however, \(e\) is used as the base for exponential functions. Exponential models that use \(e\) as the base are called continuous growth or decay models. We see these models in finance, computer science, and most of the sciences such as physics, toxicology, and fluid dynamics.

\section*{EXAMPLE: WRITING AN EXPONENTIAL MODEL WHEN THE INITIAL VALUE IS KNOWN}

In 2006, 80 deer were introduced into a wildlife refuge. By 2012, the population had grown to 180 deer. The population was growing exponentially. Write an algebraic function \(N(t)\) representing the population \(N\) of deer over time \(t\).

\section*{Answer}

We let our independent variable \(t\) be the number of years after 2006. Thus, the information given in the problem can be written as input-output pairs: \((0,80)\) and \((6,180)\). Notice that by choosing our input variable to be measured as years after 2006, we have given ourselves the initial value for the function, \(a=\) 80. We can now substitute the second point into the equation \(N(t)=80 b^{t}\) to find \(b\) :
\[
\begin{array}{lll}
N(t) & =80 b^{t} & \\
180 & =80 b^{6} & \text { Substitute using point }(6,180) . \\
\frac{9}{4} & =b^{6} & \text { Divide and write in lowest terms. } \\
b & =\left(\frac{9}{4}\right)^{\frac{1}{6}} & \text { Isolate } b \text { using properties of exponents. } \\
b & \approx 1.1447 & \text { Round to } 4 \text { decimal places. }
\end{array}
\]

NOTE: Unless otherwise stated, do not round any intermediate calculations. Round the final answer to four places for the remainder of this section.
The exponential model for the population of deer is \(N(t)=80(1.1447)^{t}\). Note that this exponential function models short-term growth. As the inputs get larger, the outputs will get increasingly larger resulting in the model not being useful in the long term due to extremely large output values.
We can graph our model to observe the population growth of deer in the refuge over time. Notice that the graph below passes through the initial points given in the problem, \((0,80)\) and \((6,180)\). We can also see that the domain for the function is \([0, \infty)\) and the range for the function is \([80, \infty)\).


\section*{TRY IT}

A wolf population is growing exponentially. In 2011, 129 wolves were counted. By 2013 the population had reached 236 wolves. What two points can be used to derive an exponential equation modeling this situation? Write the equation representing the population \(N\) of wolves over time \(t\).
Answer
\((0,129)\) and \((2,236) ; N(t)=129(1.3526)^{t}\)

\section*{EXAMPLE: WRITING AN EXPONENTIAL MODEL WHEN THE INITIAL VALUE IS NOT KNOWN}

Find an exponential function that passes through the points \((-2,6)\) and \((2,1)\).
Answer
Because we don't have the initial value, we substitute both points into an equation of the form \(f(x)=a b^{x}\) and then solve the system for \(a\) and \(b\).
- Substituting \((-2,6)\) gives \(6=a b^{-2}\)
- Substituting \((2,1)\) gives \(1=a b^{2}\)

Use the first equation to solve for \(a\) in terms of \(b\) :
\(6=a b^{-2}\)
\(\frac{6}{b^{-2}}=a \quad\) Divide.
\(a=6 b^{2} \quad\) Use properties of exponents to rewrite the denominator.
Substitute \(a\) in the second equation and solve for \(b\) :
\(1=a b^{2}\)
\(1=6 b^{2} b^{2}=6 b^{4} \quad\) Substitute \(a\).
\(b=\left(\frac{1}{6}\right)^{\frac{1}{4}} \quad\) Use properties of exponents to isolate \(b\).
\(b \approx 0.6389 \quad\) Round 4 decimal places.
Use the value of \(b\) in the first equation to solve for the value of \(a\) :
\(a=6 b^{2} \approx 6(0.6389)^{2} \approx 2.4492\)
Thus, the equation is \(f(x)=2.4492(0.6389)^{x}\).
We can graph our model to check our work. Notice that the graph below passes through the initial points given in the problem, \((-2,6)\) and \((2,1)\). The graph is an example of an exponential decay function.


The graph of \(f(x)=2.4492(0.6389)^{x}\) models exponential decay.

\section*{TRY IT}

Given the two points \((1,3)\) and \((2,4.5)\), find the equation of the exponential function that passes through these two points.
Answer
\(f(x)=2(1.5)^{x}\)
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\section*{Q \& A}

Do two points always determine a unique exponential function?

Yes, provided the two points are either both above the \(x\)-axis or both below the \(x\)-axis and have different \(x\) coordinates. But keep in mind that we also need to know that the graph is, in fact, an exponential function. Not every graph that looks exponential really is exponential. We need to know the graph is based on a model that shows the same percent growth with each unit increase in \(x\), which in many real world cases involves time.

\section*{HOW TO: GIVEN THE GRAPH OF AN EXPONENTIAL FUNCTION, WRITE ITS EQUATION}
1. First, identify two points on the graph. Choose the \(y\)-intercept as one of the two points whenever possible. Try to choose points that are as far apart as possible to reduce round-off error.
2. If one of the data points is the \(y\)-intercept \((0, a)\), then \(a\) is the initial value. Using \(a\), substitute the second point into the equation \(f(x)=a b^{x}\) and solve for \(b\).
3. If neither of the data points have the form \((0, a)\), substitute both points into two equations with the form \(f(x)=a b^{x}\). Solve the resulting system of two equations to find \(a\) and \(b\).
4. Write the exponential function, \(f(x)=a b^{x}\).

\section*{EXAMPLE: WRITING AN EXPONENTIAL FUNCTION GIVEN ITS GRAPH}

Find an equation for the exponential function graphed below.


Answer
We can choose the \(y\)-intercept of the graph, \((0,3)\), as our first point. This gives us the initial value \(a=3\). Next, choose a point on the curve some distance away from \((0,3)\) that has integer coordinates. One such point is \((2,12)\).
\(y=a b^{x} \quad\) Write the general form of an exponential equation.
\(y=3 b^{x} \quad\) Substitute the initial value 3 for \(a\).
\(12=3 b^{2} \quad\) Substitute in 12 for \(y\) and 2 for \(x\).
\(4=b^{2} \quad\) Divide by 3.
\(b= \pm 2 \quad\) Take the square root.

Because we restrict ourselves to positive values of \(b\), we will use \(b=2\). Substitute \(a\) and \(b\) into standard form to yield the equation \(f(x)=3(2)^{x}\).

\section*{TRY IT}

Find an equation for the exponential function graphed below.


Answer

\section*{A GENERAL NOTE: THE CONTINUOUS GROWTH/DECAY FORMULA}

For all real numbers \(t\), and all positive numbers \(a\) and \(r\), continuous growth or decay is represented by the formula
\[
A(t)=a e^{r t}
\]
where
- \(a\) is the initial value
- \(r\) is the continuous growth rate per unit of time
- \(t\) is the elapsed time

If \(r>0\), then the formula represents continuous growth. If \(r<0\), then the formula represents continuous decay.
For business applications, the continuous growth formula is called the continuous compounding formula and takes the form
\(A(t)=P e^{r t}\)
where
- \(P\) is the principal or the initial investment
- \(r\) is the growth or interest rate per unit of time
- \(t\) is the period or term of the investment

\section*{HOW TO: GIVEN THE INITIAL VALUE, RATE OF GROWTH OR DECAY, AND TIME T, SOLVE A CONTINUOUS GROWTH OR DECAY FUNCTION}
1. Use the information in the problem to determine \(a\), the initial value of the function.
2. Use the information in the problem to determine the growth rate \(r\).
- If the problem refers to continuous growth, then \(r>0\).
- If the problem refers to continuous decay, then \(r<0\).
3. Use the information in the problem to determine the time \(t\).
4. Substitute the given information into the continuous growth formula and solve for \(A(t)\).

\section*{EXAMPLE: CALCULATING CONTINUOUS GROWTH}

A person invested \(\$ 1,000\) in an account earning a nominal interest rate of \(10 \%\) per year compounded continuously. How much was in the account at the end of one year?

\section*{Answer}

Since the account is growing in value, this is a continuous compounding problem with growth rate \(r=0.10\). The initial investment was \(\$ 1,000\), so \(P=1000\). We use the continuous compounding formula to find the value after \(t=1\) year:
\[
\begin{array}{rlrl}
A(t) & =P e^{r t} & \text { Use the continuous compounding formula. } \\
& =1000(e)^{0.1} & & \text { Substitute known values for } P, r, \text { and } t . \\
& \approx 1105.17 & \text { Use a calculator to approximate. }
\end{array}
\]

The account is worth \(\$ 1,105.17\) after one year.

\section*{TRY IT}

A person invests \(\$ 100,000\) at a nominal \(12 \%\) interest per year compounded continuously. What will be the value of the investment in 30 years?
Answer
\$3,659,823.44
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\section*{EXAMPLE: CALCULATING CONTINUOUS DECAY}

Radon-222 decays at a continuous rate of \(17.3 \%\) per day. How much will 100 mg of Radon- 222 decay to in 3 days?

\section*{Answer}

Since the substance is decaying, the rate, 17.3 , is negative. So, \(r=-0.173\). The initial amount of radon222 was 100 mg , so \(a=100\). We use the continuous decay formula to find the value after \(t=3\) days:
\[
\begin{aligned}
A(t) & =a e^{r t} & & \text { Use the continuous growth formula. } \\
& =100 e^{-0.173(3)} & & \text { Substitute known values for } a, r, \text { and } t . \\
& \approx 59.5115 & & \text { Use a calculator to approximate. }
\end{aligned}
\]

So 59.5115 mg of radon- 222 will remain.

\section*{TRY IT}

Using the data in the previous example, how much radon-222 will remain after one year?
Answer
\(3.77115984 E-26\) (This is calculator notation for the number written as \(3.77 \times 10^{-26}\) in scientific notation. While the output of an exponential function is never zero, this number is so close to zero that for all practical purposes we can accept zero as the answer.)
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\section*{SUMMARY: EXPONENTIAL FUNCTIONS}

\section*{Key Equations}
definition of
the
exponential
\(f(x)=b^{x}\), where \(b>0, b \neq 1\)
function
definition of exponential
\(f(x)=a b^{x}\), where \(a>0, b>0, b \neq 1\) growth
\[
A(t)=P\left(1+\frac{r}{n}\right)^{n t}, \text { where }
\]
\(A(t)\) is the account value at time \(t\)
compound
interest
formula
continuous
growth
formula
\(t\) is the number of years
\(P\) is the initial investment, often called the principal \(r\) is the annual percentage rate (APR), or nominal rate
\(n\) is the number of compounding periods in one year

\section*{Key Concepts}
- An exponential function is defined as a function with a positive constant other than 1 raised to a variable exponent.
- A function is evaluated by solving at a specific input value.
- An exponential model can be found when the growth rate and initial value are known.
- An exponential model can be found when two data points from the model are known.
- An exponential model can be found using two data points from the graph of the model.
- The value of an account at any time \(t\) can be calculated using the compound interest formula when the principal, annual interest rate, and compounding periods are known.
- The initial investment of an account can be found using the compound interest formula when the value of the account, annual interest rate, compounding periods, and life span of the account are known.
- The number \(e\) is a mathematical constant often used as the base of real world exponential growth and decay models. Its decimal approximation is \(e \approx 2.718282\).
- Scientific and graphing calculators have the key \(\left[e^{x}\right]\) or \([\exp (x)]\) for calculating powers of \(e\).
- Continuous growth or decay models are exponential models that use e as the base. Continuous growth and decay models can be found when the initial value and growth or decay rate are known.

\section*{Glossary}
annual percentage rate (APR) the yearly interest rate earned by an investment account, also called nominal rate
compound interest interest earned on the total balance, not just the principal
exponential growth a model that grows by a rate proportional to the amount present
nominal rate the yearly interest rate earned by an investment account, also called annual percentage rate

\section*{INTRODUCTION TO GRAPHS OF EXPONENTIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Determine whether an exponential function and its associated graph represents growth or decay.
- Sketch a graph of an exponential function.
- Graph exponential functions shifted horizontally or vertically and write the associated equation.
- Graph a stretched or compressed exponential function.
- Graph a reflected exponential function.
- Write the equation of an exponential function that has been transformed.

As we discussed in the previous section, exponential functions are used for many real-world applications such as finance, forensics, computer science, and most of the life sciences. Working with an equation that describes a real-world situation gives us a method for making predictions. Most of the time, however, the equation itself is not enough. We learn a lot about things by seeing their visual representations, and that is exactly why graphing exponential equations is a powerful tool. It gives us another layer of insight for predicting future events.

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\section*{CHARACTERISTICS OF GRAPHS OF EXPONENTIAL FUNCTIONS}
- Determine whether an exponential function and its associated graph represents growth or decay.
- Sketch a graph of an exponential function.

Before we begin graphing, it is helpful to review the behavior of exponential growth. Recall the table of values for a function of the form \(f(x)=b^{x}\) whose base is greater than one. We'll use the function \(f(x)=2^{x}\) . Observe how the output values in the table below change as the input increases by 1.
\begin{tabular}{|l|c|c|c|c|l|l|l|}
\hline \(\boldsymbol{x}\) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline\(f(x)=2^{x}\) & \(\frac{1}{8}\) & \(\frac{1}{4}\) & \(\frac{1}{2}\) & 1 & 2 & 4 & 8 \\
\hline
\end{tabular}

Each output value is the product of the previous output and the base, 2 . We call the base 2 the constant ratio. In fact, for any exponential function with the form \(f(x)=a b^{x}, b\) is the constant ratio of the function. This means that as the input increases by 1 , the output value will be the product of the base and the previous output, regardless of the value of \(a\).

Notice from the table that:
- the output values are positive for all values of \(x\)
- as \(x\) increases, the output values increase without bound
- as \(x\) decreases, the output values grow smaller, approaching zero

The graph below shows the exponential growth function \(f(x)=2^{x}\).


Notice that the graph gets close to the \(x\)-axis but never touches it.
The domain of \(f(x)=2^{x}\) is all real numbers, the range is \((0, \infty)\), and the horizontal asymptote is \(y=0\).
To get a sense of the behavior of exponential decay, we can create a table of values for a function of the form \(f(x)=b^{x}\) whose base is between zero and one. We'll use the function \(g(x)=\left(\frac{1}{2}\right)^{x}\). Observe how the output values in the table below change as the input increases by 1.
\begin{tabular}{|l|l|l|l|l|l|l|l|}
\hline \(\boldsymbol{x}\) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline\(g(x)=\left(\frac{1}{2}\right)^{x}\) & 8 & 4 & 2 & 1 & \(\frac{1}{2}\) & \(\frac{1}{4}\) & \(\frac{1}{8}\) \\
\hline
\end{tabular}

Again, because the input is increasing by 1, each output value is the product of the previous output and the base or constant ratio \(\frac{1}{2}\).

Notice from the table that:
- the output values are positive for all values of \(x\)
- as \(x\) increases, the output values grow smaller, approaching zero
- as \(x\) decreases, the output values grow without bound

The graph below shows the exponential decay function, \(g(x)=\left(\frac{1}{2}\right)^{x}\).


The \(x\)-axis is an asymptote
The domain of \(g(x)=\left(\frac{1}{2}\right)^{x}\) is all real numbers, the range is \((0, \infty)\), and the horizontal asymptote is \(y=0\).

An exponential function with the form \(f(x)=b^{x}, b>0, b \neq 1\), has these characteristics:
- one-to-one function
- horizontal asymptote: \(y=0\)
- domain: \((-\infty, \infty)\)
- range: \((0, \infty)\)
- \(x\)-intercept: none
- \(y\)-intercept: \((0,1)\)
- increasing if \(b>1\)
- decreasing if \(b<1\)

Use an online graphing tool to graph \(f(x)=b^{x}\).
Adjust the \(b\) values to see various graphs of exponential growth and decay functions.
Try decimal values between 0 and 1 , and also values greater than 1 . Which one is growth and which one is decay?

\section*{HOW TO: GIVEN AN EXPONENTIAL FUNCTION OF THE FORM \(f(x)=b^{x}\), GRAPH THE FUNCTION}
1. Create a table of points.
2. Plot at least 3 point from the table including the \(y\)-intercept \((0,1)\).
3. Draw a smooth curve through the points.
4. State the domain, \((-\infty, \infty)\), the range, \((0, \infty)\), and the horizontal asymptote, \(y=0\).

\section*{TIP FOR SUCCESS}

When sketching the graph of an exponential function by plotting points, include a few input values left and right of zero as well as zero itself.
With few exceptions, such as functions that would be undefined at zero or negative input like the radical or (as you'll see soon) the logarithmic function, it is good practice to let the input equal
\(-3,-2,-1,0,1,2\), and 3 to get the idea of the shape of the graph.
Watch the following video for another example of graphing an exponential function. The base of the exponential term is between 0 and 1 , so this graph will represent decay.

The next example shows how to plot an exponential growth function where the base is greater than 1 .
The next video example includes graphing an exponential growth function and defining the domain and range of the function.

\section*{EXAMPLE: SKETCHING THE GRAPH OF AN EXPONENTIAL FUNCTION OF THE FORM \(f(x)=b^{x}\)}

Sketch a graph of \(f(x)=0.25^{x}\). State the domain, range, and asymptote.

\section*{Answer}

Before graphing, identify the behavior and create a table of points for the graph.
- Since \(b=0.25\) is between zero and one, we know the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote \(y=0\).
- Create a table of points.
\begin{tabular}{|l|l|l|l|l|l|l|l|}
\hline \(\boldsymbol{x}\) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline\(f(x)=0.25^{x}\) & 64 & 16 & 4 & 1 & 0.25 & 0.0625 & 0.015625 \\
\hline
\end{tabular}
- Plot the \(y\)-intercept, \((0,1)\), along with two other points. We can use \((-1,4)\) and \((1,0.25)\).

Draw a smooth curve connecting the points.


The domain is \((-\infty, \infty)\), the range is \((0, \infty)\), and the horizontal asymptote is \(y=0\).


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\section*{EXAMPLE}

Sketch a graph of \(f(x)=\sqrt{2}(\sqrt{2})^{x}\). State the domain and range.
Answer
Before graphing, identify the behavior and create a table of points for the graph.
- Since \(b=\sqrt{2}\), which is greater than one, we know the function is increasing, and we can verify this by creating a table of values. The left tail of the graph will get really close to the \(x\)-axis and the right tail will increase without bound.
- Create a table of points.
\begin{tabular}{l|ll|l|l|l|ll}
\(\boldsymbol{x}\) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline \begin{tabular}{l}
\(f(x)\) \\
\(=\sqrt{2}(\sqrt{2})^{x}\)
\end{tabular} & 0.5 & 0.71 & 1 & 1.41 & 2 & 2.83 & 4
\end{tabular}
- Plot the \(y\)-intercept, \((0,1.41)\), along with two other points. We can use \((-1,1)\) and \((1,2)\).

Draw a smooth curve connecting the points.


The domain is \((-\infty, \infty)\); the range is \((0, \infty)\).


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\section*{TRY IT}

Sketch the graph of \(f(x)=4^{x}\). State the domain, range, and asymptote.
Answer
The domain is \((-\infty, \infty)\), the range is \((0, \infty)\), and the horizontal asymptote is \(y=0\).


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\section*{HORIZONTAL AND VERTICAL TRANSLATIONS OF EXPONENTIAL FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Graph exponential functions shifted horizontally or vertically and write the associated equation.

Transformations of exponential graphs behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations-shifts, reflections, stretches, and compressionsto the parent function \(f(x)=b^{x}\) without loss of shape. For instance, just as the quadratic function maintains its parabolic shape when shifted, reflected, stretched, or compressed, the exponential function also maintains its general shape regardless of the transformations applied.

\section*{TIP FOR SUCCESS}

Translating exponential functions follows the same ideas you've used to translate other functions. Add or subtract a value inside the function argument (in the exponent) to shift horizontally, and add or subtract a value outside the function argument to shift vertically.

\section*{Graphing a Vertical Shift}

The first transformation occurs when we add a constant \(d\) to the parent function \(f(x)=b^{x}\) giving us a vertical shift \(d\) units in the same direction as the sign. For example, if we begin by graphing a parent function, \(f(x)=2^{x}\), we can then graph two vertical shifts alongside it using \(d=3\) : the upward shift, \(g(x)=2^{x}+3\) and the downward shift, \(h(x)=2^{x}-3\). Both vertical shifts are shown in the figure below.


Observe the results of shifting \(f(x)=2^{x}\) vertically:
- The domain \((-\infty, \infty)\) remains unchanged.
- When the function is shifted up 3 units giving \(g(x)=2^{x}+3\) :
- The \(y\)-intercept shifts up 3 units to \((0,4)\).
- The asymptote shifts up 3 units to \(y=3\).
- The range becomes \((3, \infty)\).
- When the function is shifted down 3 units giving \(h(x)=2^{x}-3\) :
- The \(y\)-intercept shifts down 3 units to \((0,-2)\).
- The asymptote also shifts down 3 units to \(y=-3\).
- The range becomes \((-3, \infty)\).

\section*{TRY IT}
1. Use an online graphing calculator to plot \(f(x)=2^{x}+a\)
2. Adjust the value of \(a\) until the graph has been shifted 4 units up.
3. Add a line that represents the horizontal asymptote for this function. What is the equation for this function? What is the new y-intercept? What is its domain and range?
4. Now create a graph of the function \(f(x)=2^{x}\) that has been shifted down 2 units. Add a line that represents the horizontal asymptote. What is the equation for this function? What is the new \(y\) intercept? What is its domain and range?

\section*{Answer}
1. Equation: \(f(x)=2^{x}+4\), Horizontal Asymptote: \(y=4\), y-intercept: \((0,5)\), Domain: \((-\infty, \infty)\), Range: \((4, \infty)\)
2. Equation: \(f(x)=2^{x}-2\), Horizontal Asymptote: \(y=-2\), y-intercept: \((0,-1)\), Domain: \((-\infty, \infty)\), Range: \((-2, \infty)\)

\section*{Graphing a Horizontal Shift}

The next transformation occurs when we add a constant \(c\) to the input of the parent function \(f(x)=b^{x}\) giving us a horizontal shift \(c\) units in the opposite direction of the sign. For example, if we begin by graphing the parent function \(f(x)=2^{x}\), we can then graph two horizontal shifts alongside it using \(c=3\) : the shift left, \(g(x)=2^{x+3}\), and the shift right, \(h(x)=2^{x-3}\). Both horizontal shifts are shown in the graph below.


Observe the results of shifting \(f(x)=2^{x}\) horizontally:
- The domain, \((-\infty, \infty)\), remains unchanged.
- The asymptote, \(y=0\), remains unchanged.
- The \(y\)-intercept shifts such that:
- When the function is shifted left 3 units to \(g(x)=2^{x+3}\), the \(y\)-intercept becomes \((0,8)\). This is because \(2^{x+3}=\left(2^{3}\right) 2^{x}=(8) 2^{x}\), so the initial value of the function is 8 .
- When the function is shifted right 3 units to \(h(x)=2^{x-3}\), the \(y\)-intercept becomes \(\left(0, \frac{1}{8}\right)\). Again, see that \(2^{x-3}=\left(2^{-3}\right) 2^{x}=\left(\frac{1}{8}\right) 2^{x}\), so the initial value of the function is \(\frac{1}{8}\).

\section*{TRY IT}
1. Using an online graphing calculator, plot \(f(x)=2^{(x+a)}\)
2. Adjust the value of \(a\) until the graph is shifted 4 units to the right. What is the equation for this function? What is the new y-intercept? What are its domain and range?
3. Now adjust the value of \(a\) until the graph has been shifted 3 units to the left. What is the equation for this function? What is the new y-intercept? What are its domain and range?

\section*{Answer}
1. Equation: \(f(x)=2^{x-4}\), y-intercept: \(\left(0, \frac{1}{32}\right)\),Domain: \((-\infty, \infty)\), Range: \((0, \infty)\)
2. Equation: \(f(x)=2^{x+3}\), y-intercept: \((0,8)\), Domain: \((-\infty, \infty)\), Range: \((0, \infty)\)

\section*{A GENERAL NOTE: SHIFTS OF THE PARENT FUNCTION \(f(x)=b^{x}\)}

For any constants \(c\) and \(d\), the function \(f(x)=b^{x+c}+d\) shifts the parent function \(f(x)=b^{x}\)
- shifts the parent function \(f(x)=b^{x}\) vertically \(d\) units, in the same direction as the sign of \(d\).
- shifts the parent function \(f(x)=b^{x}\) horizontally \(c\) units, in the opposite direction as the sign of \(c\).
- has a \(y\)-intercept of \(\left(0, b^{c}+d\right)\).
- has a horizontal asymptote of \(y=d\).
- has a range of \((d, \infty)\).
- has a domain of \((-\infty, \infty)\) which remains unchanged.

\section*{HOW TO: GIVEN AN EXPONENTIAL FUNCTION WITH THE FORM \(f(x)=b^{x+c}+d\), GRAPH THE TRANSLATION}
1. Draw the horizontal asymptote \(y=d\).
2. Shift the graph of \(f(x)=b^{x}\) left \(c\) units if \(c\) is positive and right \(c\) units if \(c\) is negative.
3. Shift the graph of \(f(x)=b^{x}\) up \(d\) units if \(d\) is positive and down \(d\) units if \(d\) is negative.
4. State the domain, \((-\infty, \infty)\), the range, \((d, \infty)\), and the horizontal asymptote \(y=d\).

Watch the following video for more examples of the difference between horizontal and vertical shifts of exponential functions and the resulting graphs and equations.

\section*{Using a Graph to Approximate a Solution to an Exponential Equation}

Graphing can help you confirm or find the solution to an exponential equation. An exponential equation is different from a function because a function is a large collection of points made of inputs and corresponding outputs, whereas equations that you have seen typically have one, two, or no solutions. For example, \(f(x)=2^{x}\) is a function and is comprised of many points \((x, f(x))\), and \(4=2^{x}\) can be solved to find the specific value for x that makes it a true statement. The graph below shows the intersection of the line \(f(x)=4\) and \(f(x)=2^{x}\). You can see they cross at \(y=4\).


In the following example you can try this yourself.

\section*{EXAMPLE: GRAPHING A SHIFT OF AN EXPONENTIAL FUNCTION}

Graph \(f(x)=2^{x+1}-3\). State the domain, range, and asymptote.

\section*{Answer}

We have an exponential equation of the form \(f(x)=b^{x+c}+d\), with \(b=2, c=1\), and \(d=-3\).
Draw the horizontal asymptote \(y=d\), so draw \(y=-3\).
Identify the shift; it is \((-1,-3)\).


The domain is \((-\infty, \infty)\), the range is \((-3, \infty)\), and the horizontal asymptote is \(y=-3\).
Shift the graph of \(f(x)=b^{x}\) left 1 unit and down 3 units.
The domain is \((-\infty, \infty)\); the range is \((-3, \infty)\); the horizontal asymptote is \(y=-3\).

Use an online graphing calculator to plot the function \(f(x)=2^{x-1}+3\). State domain, range, and asymptote.

\section*{Answer}

The domain is \((-\infty, \infty)\), the range is \((3, \infty)\), and the horizontal asymptote is \(y=3\).



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\section*{EXAMPLE : APPROXIMATING THE SOLUTION OF AN EXPONENTIAL EQUATION}

Use an online graphing calculator to solve \(42=1.2(5)^{x}+2.8\) graphically.

\section*{Answer}

First, graph the function \(f(x)=1.2(5)^{x}+2.8\) and graph \(f(x)=42\).
Online graphing calculators automatically calculate points of interest including intersections. Essentially, you are looking for the intersection of two functions. Click on the point of intersection, and you will see the the x and y values for the point.

\section*{TRY IT}

Solve \(4=7.85(1.15)^{x}-2.27\) graphically. Round to the nearest thousandth.

\section*{Answer}
\(x \approx-1.608\)

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\title{
STRETCHING, COMPRESSING, OR REFLECTING AN EXPONENTIAL FUNCTION
}

\section*{LEARNING OUTCOMES}
- Graph a stretched or compressed exponential function.
- Graph a reflected exponential function.
- Write the equation of an exponential function that has been transformed.

While horizontal and vertical shifts involve adding constants to the input or to the function itself, a stretch or compression occurs when we multiply the parent function \(f(x)=b^{x}\) by a constant \(|a|>0\). For example, if we begin by graphing the parent function \(f(x)=2^{x}\), we can then graph the stretch, using \(a=3\), to get \(g(x)=3(2)^{x}\) and the compression, using \(a=\frac{1}{3}\), to get \(h(x)=\frac{1}{3}(2)^{x}\).

(a) \(g(x)=3(2)^{x}\) stretches the graph of \(f(x)=2^{x}\) vertically by a factor of 3 . (b) \(h(x)=\frac{1}{3}(2)^{x}\) compresses the graph of \(f(x)=2^{x}\) vertically by a factor of \(\frac{1}{3}\).

\section*{A GENERAL NOTE: STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION \(f(x)=b^{x}\)}

The function \(f(x)=a b^{x}\)
- is stretched vertically by a factor of \(a\) if \(|a|>1\).
- is compressed vertically by a factor of \(a\) if \(|a|<1\).
- has a \(y\)-intercept is \((0, a)\).
- has a horizontal asymptote of \(y=0\), range of \((0, \infty)\), and domain of \((-\infty, \infty)\) which are all unchanged from the parent function.

\section*{TIP FOR SUCCESS}

Exponential functions are stretched, compressed or reflected in the same manner you've used to transform other functions. Multipliers or negatives inside the function argument (in the exponent) affect horizontal transformations. Multipliers or negatives outside the function argument affect vertical transformations.

\section*{Graphing Reflections}

In addition to shifting, compressing, and stretching a graph, we can also reflect it about the \(x\)-axis or the \(y\) axis. When we multiply the parent function \(f(x)=b^{x}\) by -1 , we get a reflection about the \(x\)-axis. When we multiply the input by -1 , we get a reflection about the \(y\)-axis. For example, if we begin by graphing the parent function \(f(x)=2^{x}\), we can then graph the two reflections alongside it. The reflection about the \(x\)-axis, \(g(x)=-2^{x}\), and the reflection about the \(y\)-axis, \(h(x)=2^{-x}\), are both shown below.

\section*{EXAMPLE: GRAPHING THE STRETCH OF AN EXPONENTIAL FUNCTION}

Sketch a graph of \(f(x)=4\left(\frac{1}{2}\right)^{x}\). State the domain, range, and asymptote.

\section*{Answer}

Before graphing, identify the behavior and key points on the graph.
- Since \(b=\frac{1}{2}\) is between zero and one, the left tail of the graph will increase without bound as \(x\) decreases, and the right tail will approach the \(x\)-axis as \(x\) increases.
- Since \(a=4\), the graph of \(f(x)=\left(\frac{1}{2}\right)^{x}\) will be stretched vertically by a factor of 4 .
- Create a table of points:
\begin{tabular}{|l|l|l|l|l|l|l|l|}
\hline \(\boldsymbol{x}\) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline\(f(x)=4\left(\frac{1}{2}\right)^{x}\) & 32 & 16 & 8 & 4 & 2 & 1 & 0.5 \\
\hline
\end{tabular}
- Plot the \(y\)-intercept, \((0,4)\), along with two other points. We can use \((-1,8)\) and \((1,2)\).
- Draw a smooth curve connecting the points.


The domain is \((-\infty, \infty)\), the range is \((0, \infty)\), the horizontal asymptote is \(y=0\).

\section*{TRY IT}

Use an online graphing tool to sketch the graph of \(f(x)=\frac{1}{2}(4)^{x}\). State the domain, range, and asymptote.
Answer
The domain is \((-\infty, \infty)\); the range is \((0, \infty)\); the horizontal asymptote is \(y=0\).


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Reflection about the \(x\)-axis


Reflection about the \(y\)-axis

(a) \(g(x)=-2^{x}\) reflects the graph of \(f(x)=2^{x}\) about the \(x\)-axis. (b) \(h(x)=2^{-x}\) reflects the graph of \(f(x)=2^{x}\) about the \(y\)-axis.

\section*{A GENERAL NOTE: REFLECTING THE PARENT FUNCTION \(f(x)=b^{x}\)}

The function \(f(x)=-b^{x}\)
- reflects the parent function \(f(x)=b^{x}\) about the \(x\)-axis.
- has a \(y\)-intercept of \((0,-1)\).
- has a range of \((-\infty, 0)\).
- has a horizontal asymptote of \(y=0\) and domain of \((-\infty, \infty)\) which are unchanged from the parent function.

The function \(f(x)=b^{-x}\)
- reflects the parent function \(f(x)=b^{x}\) about the \(y\)-axis.
- has a \(y\)-intercept of \((0,1)\), a horizontal asymptote at \(y=0\), a range of \((0, \infty)\), and a domain of \((-\infty, \infty)\) which are unchanged from the parent function.

\section*{Summarizing Transformations of the Exponential Function}

Now that we have worked with each type of translation for the exponential function, we can summarize them to arrive at the general equation for transforming exponential functions.

1 , and notes the following changes: the reflected function is decreasing as \(x\) moves from 0 to infinity, the asymptote remains \(x=0\), the \(x\)-intercept remains ( 1,0 ), the key point changes to ( \(b^{\wedge}(-1), 1\) ), the domain remains ( 0 , infinity), and the range remains (-infinity, infinity). The second column shows the left shift of the equation \(g(x)=\log \_b(x)\) when \(b>1\), and notes the following changes: the reflected function is decreasing as \(x\) moves from 0 to infinity, the asymptote remains \(x=0\), the \(x\)-intercept changes to ( \(-1,0\) ), the key point changes to (-b, 1), the domain changes to (-infinity, 0), and the range remains (-infinity, infinity).">

\section*{Transformations of the Parent Function \(f(x)=b^{x}\)}

Translation

\section*{Form}

\section*{Shift}
- Horizontally \(c\) units to the left
\[
f(x)=b^{x+c}+d
\]
- Vertically \(d\) units up

\section*{Stretch and Compress}
- Stretch if |al>1
\[
f(x)=a b^{x}
\]
- Compression if \(0<1 a \mid<1\)
\begin{tabular}{|l|l|}
\hline Reflect about the \(x\)-axis & \(f(x)=-b^{x}\) \\
\hline Reflect about the \(y\)-axis & \(f(x)=b^{-x}=\left(\frac{1}{b}\right)^{x}\) \\
\hline General equation for all transformations & \(f(x)=a b^{x+c}+d\) \\
\hline
\end{tabular}

\section*{EXAMPLE: WRITING AND GRAPHING THE REFLECTION OF AN EXPONENTIAL FUNCTION}

Find and graph the equation for a function, \(g(x)\), that reflects \(f(x)=\left(\frac{1}{4}\right)^{x}\) about the \(x\)-axis. State its domain, range, and asymptote.

\section*{Answer}

Since we want to reflect the parent function \(f(x)=\left(\frac{1}{4}\right)^{x}\) about the \(x\)-axis, we multiply \(f(x)\) by -1 to get \(g(x)=-\left(\frac{1}{4}\right)^{x}\). Next we create a table of points.
\begin{tabular}{|l|l|l|l|l|l|l|l|}
\hline\(x\) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline\(g(x)=-\left(\frac{1}{4}\right)^{x}\) & -64 & -16 & -4 & -1 & -0.25 & -0.0625 & -0.0156 \\
\hline
\end{tabular}

Plot the \(y\)-intercept, \((0,-1)\), along with two other points. We can use \((-1,-4)\) and \((1,-0.25)\).
Draw a smooth curve connecting the points:


The domain is \((-\infty, \infty)\), the range is \((-\infty, 0)\), and the horizontal asymptote is \(y=0\).

Use an online graphing calculator to graph the equation for a function, \(g(x)\), that reflects \(f(x)=1.25^{x}\) about the \(y\)-axis. State its domain, range, and asymptote.

\section*{Answer}

The domain is \((-\infty, \infty)\); the range is \((0, \infty)\); the horizontal asymptote is \(y=0\).


\section*{A GENERAL NOTE: TRANSFORMATIONS OF EXPONENTIAL FUNCTIONS}

A transformation of an exponential function has the form
\(f(x)=a b^{x+c}+d\), where the parent function, \(y=b^{x}, b>1\), is
- shifted horizontally \(c\) units to the left.
- stretched vertically by a factor of lal if lal \(>0\).
- compressed vertically by a factor of lal if \(0<|a|<1\).
- shifted vertically \(d\) units.
- reflected about the \(x\)-axis when \(a<0\).

Note the order of the shifts, transformations, and reflections follow the order of operations.

\section*{EXAMPLE: WRITING A FUNCTION FROM A DESCRIPTION}

Write the equation for the function described below. Give the horizontal asymptote, domain, and range.
- \(f(x)=e^{x}\) is vertically stretched by a factor of 2 , reflected across the \(y\)-axis, and then shifted up 4 units.

\section*{Answer}

We want to find an equation of the general form \(f(x)=a b^{x+c}+d\). We use the description provided to find \(a, b, c\), and \(d\).
- We are given the parent function \(f(x)=e^{x}\), so \(b=e\).
- The function is stretched by a factor of 2 , so \(a=2\).
- The function is reflected about the \(y\)-axis. We replace \(x\) with \(-x\) to get: \(e^{-x}\).
- The graph is shifted vertically 4 units, so \(d=4\).

Substituting in the general form, we get:
\[
\begin{aligned}
f(x) & =a b^{x+c}+d \\
& =2 e^{-x+0}+4 \\
& =2 e^{-x}+4
\end{aligned}
\]

The domain is \((-\infty, \infty)\); the range is \((4, \infty)\); the horizontal asymptote is \(y=4\).

\section*{TRY IT}

Write the equation for the function described below. Give the horizontal asymptote, the domain, and the range.
- \(f(x)=e^{x}\) is compressed vertically by a factor of \(\frac{1}{3}\), reflected across the \(x\)-axis, and then shifted down 2 units.

\section*{Answer}
\(f(x)=-\frac{1}{3} e^{x}-2\); the domain is \((-\infty, \infty)\); the range is \((-\infty, 2)\); the horizontal asymptote is \(y=2\).

\section*{SUMMARY: GRAPHS OF EXPONENTIAL FUNCTIONS}

\section*{Key Concepts}
- The graph of the function \(f(x)=b^{x}\) has a \(y\)-intercept at \((0,1)\), domain of \((-\infty, \infty)\), range of \((0, \infty)\), and horizontal asymptote of \(y=0\).
- If \(b>1\), the function is increasing. The left tail of the graph will approach the asymptote \(y=0\), and the right tail will increase without bound.
- If \(0<b<1\), the function is decreasing. The left tail of the graph will increase without bound, and the right tail will approach the asymptote \(y=0\).
- The equation \(f(x)=b^{x}+d\) represents a vertical shift of the parent function \(f(x)=b^{x}\).
- The equation \(f(x)=b^{x+c}\) represents a horizontal shift of the parent function \(f(x)=b^{x}\).
- The equation \(f(x)=a b^{x}\), where \(a>0\), represents a vertical stretch if \(|a|>1\) or compression if \(0<|a|<1\) of the parent function \(f(x)=b^{x}\).
- When the parent function \(f(x)=b^{x}\) is multiplied by -1 , the result, \(f(x)=-b^{x}\), is a reflection about the \(x\)-axis. When the input is multiplied by -1 , the result, \(f(x)=b^{-x}\), is a reflection about the \(y\)-axis.
- All transformations of the exponential function can be summarized by the general equation \(f(x)=a b^{x+c}+d\).
- Using the general equation \(f(x)=a b^{x+c}+d\), we can write the equation of a function given its description.
- Approximate solutions of the equation \(f(x)=b^{x+c}+d\) can be found using a graphing calculator.

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\section*{INTRODUCTION TO LOGARITHMIC FUNCTIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.
- Evaluate logarithms with and without a calculator.
- Evaluate logarithms with base 10 and base e.


Devastation of March 11, 2011 earthquake in Honshu, Japan. (credit: Daniel Pierce)
In 2010, a major earthquake struck Haiti destroying or damaging over 285,000 homes. (Note: http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/\#summary. Accessed 3/4/2013.) One year later, another, stronger earthquake devastated Honshu, Japan destroying or damaging over 332,000 buildings (Note: http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/\#summary. Accessed 3/4/2013.) like those shown in the picture below. Even though both caused substantial damage, the earthquake in 2011 was 100 times stronger than the earthquake in Haiti. How do we know? The magnitudes of earthquakes are measured on a scale known as the Richter Scale. The Haitian earthquake registered a 7.0 on the Richter Scale (Note:
http://earthquake.usgs.gov/earthquakes/eqinthenews/2010/us2010rja6/. Accessed 3/4/2013.) whereas the Japanese earthquake registered a 9.0. (Note: http://earthquake.usgs.gov/earthquakes/eqinthenews/2011/usc0001xgp/\#details. Accessed 3/4/2013.)

The Richter Scale is a base-ten logarithmic scale. In other words, an earthquake of magnitude 8 is not twice as great as an earthquake of magnitude 4. It is \(10^{8-4}=10^{4}=10,000\) times as great! In this lesson, we will investigate the nature of the Richter Scale and the base-ten function upon which it depends.

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}

\section*{CONVERTING BETWEEN LOGARITHMIC AND EXPONENTIAL FORM}

\section*{LEARNING OUTCOMES}
- Convert from logarithmic to exponential form.
- Convert from exponential to logarithmic form.

In order to analyze the magnitude of earthquakes or compare the magnitudes of two different earthquakes, we need to be able to convert between logarithmic and exponential form. For example, suppose the amount of energy released from one earthquake was 500 times greater than the amount of energy released from another. We want to calculate the difference in magnitude. The equation that represents this problem is \(10^{x}=500\) where \(x\) represents the difference in magnitudes on the Richter Scale. How would we solve for \(x\) ?

We have not yet learned a method for solving exponential equations algebraically. None of the algebraic tools discussed so far is sufficient to solve \(10^{x}=500\). We know that \(10^{2}=100\) and \(10^{3}=1000\), so it is clear that \(x\) must be some value between 2 and 3 since \(y=10^{x}\) is increasing. We can examine a graph to better estimate the solution.


Estimating from a graph, however, is imprecise. To find an algebraic solution, we must introduce a new function. Observe that the graph above passes the horizontal line test. The exponential function \(y=b^{x}\) is one-to-one, so its inverse, \(x=b^{y}\) is also a function. As is the case with all inverse functions, we simply interchange \(x\) and \(y\) and solve for \(y\) to find the inverse function. To represent \(y\) as a function of \(x\), we use a logarithmic function of the form \(y=\log _{b}(x)\). The base \(b\) logarithm of a number is the exponent by which we must raise \(b\) to get that number.

\section*{TIP FOR SUCCESS}

Understanding what a logarithm is requires understanding what an exponent is. A logarithm is an exponent.
Read the paragraphs and boxes below carefully, perhaps more than once or twice, to gain the understanding of the inverse relationship between logarithms and exponents. Keep in mind that the inverse of a function effectively "undoes" what the other does.
You can use the definition of the logarithm given below to solve certain equations involving exponents and logarithms.

We read a logarithmic expression as, "The logarithm with base \(b\) of \(x\) is equal to \(y\)," or, simplified, "log base \(b\) of \(x\) is \(y\)." We can also say, " \(b\) raised to the power of \(y\) is \(x\)," because logs are exponents. For example, the base 2 logarithm of 32 is 5 , because 5 is the exponent we must apply to 2 to get 32 . Since \(2^{5}=32\), we can write \(\log _{2} 32=5\). We read this as "log base 2 of 32 is 5 ."

We can express the relationship between logarithmic form and its corresponding exponential form as follows:
\(\log _{b}(x)=y \Leftrightarrow b^{y}=x, b>0, b \neq 1\)
Note that the base \(b\) is always positive.


Think
\(b\) to the \(y=x\)

Because a logarithm is a function, it is most correctly written as \(\log _{b}(x)\) using parentheses to denote function evaluation just as we would with \(f(x)\). However, when the input is a single variable or number, it is common to see the parentheses dropped and the expression written without parentheses as \(\log _{b} x\). Note that many calculators require parentheses around the \(x\).

We can illustrate the notation of logarithms as follows:


Notice that when comparing the logarithm function and the exponential function, the input and the output are switched. This means \(y=\log _{b}(x)\) and \(y=b^{x}\) are inverse functions.

\section*{A GENERAL NOTE: DEFINITION OF THE LOGARITHMIC FUNCTION}

A logarithm base \(b\) of a positive number \(x\) satisfies the following definition:
For \(x>0, b>0, b \neq 1\),
\(y=\log _{b}(x)\) is equal to \(b^{y}=x\), where
- we read \(\log _{b}(x)\) as, "the logarithm with base \(b\) of \(x\) " or the "log base \(b\) of \(x\)."
- the logarithm \(y\) is the exponent to which \(b\) must be raised to get \(x\).
- if no base \(b\) is indicated, the base of the logarithm is assumed to be 10 .

Also, since the logarithmic and exponential functions switch the \(x\) and \(y\) values, the domain and range of the exponential function are interchanged for the logarithmic function. Therefore,
- the domain of the logarithm function with base \(b\) is \((0, \infty)\).
- the range of the logarithm function with base \(b\) is \((-\infty, \infty)\).

Can we take the logarithm of a negative number?

No. Because the base of an exponential function is always positive, no power of that base can ever be negative. We can never take the logarithm of a negative number. Also, we cannot take the logarithm of zero. Calculators may output a log of a negative number when in complex mode, but the log of a negative number is not a real number.

\section*{HOW TO: GIVEN AN EQUATION IN LOGARITHMIC FORM \(\log _{b}(x)=y\), CONVERT IT TO EXPONENTIAL FORM}
1. Examine the equation \(y=\log _{b} x\) and identify \(b, y\), and \(x\).
2. Rewrite \(\log _{b} x=y\) as \(b^{y}=x\).

\section*{EXAMPLE: CONVERTING FROM LOGARITHMIC FORM TO EXPONENTIAL FORM}

Write the following logarithmic equations in exponential form.
1. \(\log _{6}(\sqrt{6})=\frac{1}{2}\)
2. \(\log _{3}(9)=2\)

\section*{Answer}

First, identify the values of \(b, y\), and \(x\). Then, write the equation in the form \(b^{y}=x\).
1. \(\log _{6}(\sqrt{6})=\frac{1}{2}\) Here, \(b=6, y=\frac{1}{2}\), and \(x=\sqrt{6}\). Therefore, the equation \(\log _{6}(\sqrt{6})=\frac{1}{2}\) is equal to \(6^{\frac{1}{2}}=\sqrt{6}\).
2. \(\log _{3}(9)=2\) Here, \(b=3, y=2\), and \(x=9\). Therefore, the equation \(\log _{3}(9)=2\) is equal to \(3^{2}=9\).

\section*{TRY IT}

Write the following logarithmic equations in exponential form.
a. \(\log _{10}(1,000,000)=6\)
b. \(\log _{5}(25)=2\)

\section*{Answer}
a. \(\log _{10}(1,000,000)=6\) is equal to \(10^{6}=1,000,000\)
b. \(\log _{5}(25)=2\) is equal to \(5^{2}=25\)

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\section*{Convert from Exponential to Logarithmic Form}

To convert from exponential to logarithmic form, we follow the same steps in reverse. We identify the base \(b\), exponent \(x\), and output \(y\). Then we write \(x=\log _{b}(y)\).

\section*{EXAMPLE: CONVERTING FROM EXPONENTIAL FORM TO LOGARITHMIC FORM}

Write the following exponential equations in logarithmic form.
1. \(2^{3}=8\)
2. \(5^{2}=25\)
3. \(10^{-4}=\frac{1}{10,000}\)

\section*{Answer}

First, identify the values of \(b, y\), and \(x\). Then, write the equation in the form \(x=\log _{b}(y)\).
1. \(2^{3}=8\) Here, \(b=2, x=3\), and \(y=8\). Therefore, the equation \(2^{3}=8\) is equal to \(\log _{2}(8)=3\).
2. \(5^{2}=25\) Here, \(b=5, x=2\), and \(y=25\). Therefore, the equation \(5^{2}=25\) is equal to \(\log _{5}(25)=2\).
3. \(10^{-4}=\frac{1}{10,000}\) Here, \(b=10, x=-4\), and \(y=\frac{1}{10,000}\). Therefore, the equation \(10^{-4}=\frac{1}{10,000}\) is equal to \(\log _{10}\left(\frac{1}{10,000}\right)=-4\).

\section*{TRY IT}

Write the following exponential equations in logarithmic form.
a. \(3^{2}=9\)
b. \(5^{3}=125\)
c. \(2^{-1}=\frac{1}{2}\)

Answer
a. \(3^{2}=9\) is equal to \(\log _{3}(9)=2\)
b. \(5^{3}=125\) is equal to \(\log _{5}(125)=3\)
c. \(2^{-1}=\frac{1}{2}\) is equal to \(\log _{2}\left(\frac{1}{2}\right)=-1\)

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\section*{EVALUATING LOGARITHMS}

\section*{LEARNING OUTCOMES}
- Evaluate logarithms with and without a calculator.
- Evaluate logarithms with base 10 and base e.

Knowing the squares, cubes, and roots of numbers allows us to evaluate many logarithms mentally. For example, consider \(\log _{2} 8\). We ask, "To what exponent must 2 be raised in order to get 8 ?" Because we already know \(2^{3}=8\), it follows that \(\log _{2} 8=3\).

Now consider solving \(\log _{7} 49\) and \(\log _{3} 27\) mentally.
- We ask, "To what exponent must 7 be raised in order to get 49 ?" We know \(7^{2}=49\). Therefore, \(\log _{7} 49=2\).
- We ask, "To what exponent must 3 be raised in order to get 27 ?" We know \(3^{3}=27\). Therefore, \(\log _{3} 27=3\).

Even some seemingly more complicated logarithms can be evaluated without a calculator. For example, let's evaluate \(\log _{\frac{2}{3}} \frac{4}{9}\) mentally.
- We ask, "To what exponent must \(\frac{2}{3}\) be raised in order to get \(\frac{4}{9}\) ? " We know \(2^{2}=4\) and \(3^{2}=9\), so \(\left(\frac{2}{3}\right)^{2}=\frac{4}{9}\). Therefore, \(\log _{\frac{2}{3}}\left(\frac{4}{9}\right)=2\).

\section*{TIP FOR SUCCESS}

It may be tempting to use your calculator to evaluate these logarithms but try to evaluate them mentally as it will aid your understanding of what a logarithm is, and will help you navigate more complicated situations.

\section*{HOW TO: GIVEN A LOGARITHM OF THE FORM \(y=\log _{b}(x)\), EVALUATE IT MENTALLY}
1. Rewrite the argument \(x\) as a power of \(b: b^{y}=x\).
2. Use previous knowledge of powers of \(b\) to identify \(y\) by asking, "To what exponent should \(b\) be raised in order to get \(x\) ?"

\section*{EXAMPLE: SOLVING LOGARITHMS MENTALLY}

Solve \(y=\log _{4}\) (64) without using a calculator.

\section*{Answer}

First we rewrite the logarithm in exponential form: \(4^{y}=64\). Next, we ask, "To what exponent must 4 be raised in order to get 64?"
We know \(4^{3}=64\)
Therefore,
\(\log _{4}(64)=3\)

\section*{TRY IT}

Solve \(y=\log _{121}\) (11) without using a calculator.

\section*{Answer}
\(\log _{121}(11)=\frac{1}{2}\left(\right.\) recall that \(\left.\sqrt{121}=(121)^{\frac{1}{2}}=11\right)\)

\section*{EXAMPLE: EVALUATING THE LOGARITHM OF A RECIPROCAL}

Evaluate \(y=\log _{3}\left(\frac{1}{27}\right)\) without using a calculator.

\section*{Answer}

First we rewrite the logarithm in exponential form: \(3^{y}=\frac{1}{27}\). Next, we ask, "To what exponent must 3 be raised in order to get \(\frac{1}{27}\) "?
We know \(3^{3}=27\), but what must we do to get the reciprocal, \(\frac{1}{27}\) ? Recall from working with exponents that \(b^{-a}=\frac{1}{b^{a}}\). We use this information to write
\(3^{-3}=\frac{1}{3^{3}}=\frac{1}{27}\)
Therefore, \(\log _{3}\left(\frac{1}{27}\right)=-3\).

\section*{TRY IT}

Evaluate \(y=\log _{2}\left(\frac{1}{32}\right)\) without using a calculator.
Answer
\(\log _{2}\left(\frac{1}{32}\right)=-5\)
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\section*{Using Common Logarithms}

Sometimes we may see a logarithm written without a base. In this case, we assume that the base is 10 . In other words, the expressions \(\log (x)\) means \(\log _{10}(x)\). We call a base-10 logarithm a common
logarithm. Common logarithms are used to measure the Richter Scale mentioned at the beginning of the section. Scales for measuring the brightness of stars and the pH of acids and bases also use common logarithms.

\section*{A GENERAL NOTE: DEFINITION OF THE COMMON LOGARITHM}

A common logarithm is a logarithm with base 10 . We write \(\log _{10}(x)\). simply as \(\log (x)\). The common logarithm of a positive number \(x\) satisfies the following definition:
For \(x>0\),
\(y=\log (x)\) is equivalent to \(10^{y}=x\)

We read \(\log (x)\) as, "the logarithm with base 10 of \(x\) " or "the common logarithm of \(x\)."
The logarithm \(y\) is the exponent to which 10 must be raised to get \(x\).
Since the functions \(y=10^{x}\) and \(y=\log (x)\) are inverse functions, \(\log \left(10^{x}\right)=x\) for all \(x\) and \(10^{\log (x)}=x\) for \(x>0\).

\section*{HOW TO: GIVEN A COMMON LOGARITHM OF THE FORM \(y=\log (x)\), EVALUATE IT USING A CALCULATOR+}
1. Press [LOG].
2. Enter the value given for \(x\), followed by [ ) ].
3. Press [ENTER].

\section*{Using Natural Logarithms}

The most frequently used base for logarithms is \(e\). Base \(e\) logarithms are important in calculus and some scientific applications; they are called natural logarithms. The base \(e\) logarithm, \(\log _{e}(x)\), has its own notation, \(\ln (x)\).

Most values of \(\ln (x)\) can be found only using a calculator. The major exception is that, because the logarithm of 1 is always 0 in any base, \(\ln 1=0\). For other natural logarithms, we can use the \(\ln\) key that can be found on most scientific calculators. We can also find the natural logarithm of any power of \(e\) using the inverse property of logarithms.

\section*{A GENERAL NOTE: DEFINITION OF THE NATURAL LOGARITHM}

A natural logarithm is a logarithm with base \(e\). We write \(\log _{e}(x)\) simply as \(\ln (x)\). The natural logarithm of a positive number \(x\) satisfies the following definition:
For \(x>0\),
\(y=\ln (x)\) is equivalent to \(e^{y}=x\)
We read \(\ln (x)\) as, "the logarithm with base \(e\) of \(x\) " or "the natural logarithm of \(x\)."
The logarithm \(y\) is the exponent to which \(e\) must be raised to get \(x\).
Since the functions \(y=e^{x}\) and \(y=\ln (x)\) are inverse functions, \(\ln \left(e^{x}\right)=x\) for all \(x\) and \(e^{\ln (x)}=x\) for \(x>0\).

\section*{HOW TO: GIVEN A NATURAL LOGARITHM OF THE FORM \(y=\ln (x)\), EVALUATE IT USING A CALCULATOR+}
1. Press [LN].
2. Enter the value given for \(x\), followed by [ ) ].
3. Press [ENTER].

\section*{EXAMPLE: EVALUATING A NATURAL LOGARITHM USING A CALCULATOR}

Evaluate \(y=\ln (500)\) to four decimal places using a calculator.

\section*{Answer}
- Press [LN].
- Enter 500, followed by [ ) ].
- Press [ENTER].

Rounding to four decimal places, \(\ln (500) \approx 6.2146\)

\section*{TRY IT}

Evaluate \(\ln (-500)\).
Answer
It is not possible to take the logarithm of a negative number in the set of real numbers.
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\section*{SUMMARY: LOGARITHMIC FUNCTIONS}

\section*{Key Equations}

Definition of the logarithmic function
Definition of the common logarithm
Definition of the natural logarithm

For \(x>0, b>0, b \neq 1, y=\log _{b}(x)\) if and only if \(b^{y}=x\).
For \(x>0, y=\log (x)\) if and only if \(10^{y}=x\).
For \(x>0, y=\ln (x)\) if and only if \(e^{y}=x\).

\section*{Key Concepts}
- The inverse of an exponential function is a logarithmic function, and the inverse of a logarithmic function is an exponential function.
- Logarithmic equations can be written in an equivalent exponential form using the definition of a logarithm.
- Exponential equations can be written in an equivalent logarithmic form using the definition of a logarithm.
- Logarithmic functions with base \(b\) can be evaluated mentally using previous knowledge of powers of \(b\).
- Common logarithms can be evaluated mentally using previous knowledge of powers of 10.
- When common logarithms cannot be evaluated mentally, a calculator can be used.
- Natural logarithms can be evaluated using a calculator.

\section*{Glossary}
common logarithm the exponent to which 10 must be raised to get \(x ; \log _{10}(x)\) is written simply as \(\log (x)\)
logarithm the exponent to which \(b\) must be raised to get \(x\); written \(y=\log _{b}(x)\)
natural logarithm the exponent to which the number e must be raised to get \(x ; \log _{e}(x)\) is written as \(\ln (x)\)

\title{
INTRODUCTION TO GRAPHS OF LOGARITHMIC FUNCTIONS
}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Determine the domain and range of a logarithmic function.
- Determine the x-intercept and vertical asymptote of a logarithmic function.
- Identify whether a logarithmic function is increasing or decreasing and give the interval.
- Identify the features of a logarithmic function that make it an inverse of an exponential function.
- Graph horizontal and vertical shifts of logarithmic functions.
- Graph stretches and compressions of logarithmic functions.
- Graph reflections of logarithmic functions.

Previously, we saw how creating a graphical representation of an exponential model gives us another layer of insight for predicting future events. How do logarithmic graphs give us insight into situations? Because every logarithmic function is the inverse function of an exponential function, we can think of every output on a logarithmic graph as the input for the corresponding inverse exponential equation. In other words, logarithms give the cause for an effect.

To illustrate, suppose we invest \(\$ 2500\) in an account that offers an annual interest rate of \(5 \%\) compounded continuously. We already know that the balance in our account for any year \(t\) can be found with the equation \(A=2500 e^{0.05 t}\).

What if we wanted to know the year for any balance? We would need to create a corresponding new function by interchanging the input and the output; thus we would need to create a logarithmic model for this situation. By graphing the model, we can see the output (year) for any input (account balance). For instance, what if we wanted to know how many years it would take for our initial investment to double? The graph below shows this point.


In this section we will discuss the values for which a logarithmic function is defined and then turn our attention to graphing the family of logarithmic functions.

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\section*{CHARACTERISTICS OF GRAPHS OF LOGARITHMIC FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Determine the domain and range of a logarithmic function.
- Determine the x-intercept and vertical asymptote of a logarithmic function.
- Identify whether a logarithmic function is increasing or decreasing and give the interval.
- Identify the features of a logarithmic function that make it an inverse of an exponential function.

Before working with graphs, we will take a look at the domain (the set of input values) for which the logarithmic function is defined.

Recall that the exponential function is defined as \(y=b^{x}\) for any real number \(x\) and constant \(b>0, b \neq 1\), where
- The domain of \(y\) is \((-\infty, \infty)\).
- The range of \(y\) is \((0, \infty)\).

In the last section we learned that the logarithmic function \(y=\log _{b}(x)\) is the inverse of the exponential function \(y=b^{x}\). So, as inverse functions:
- The domain of \(y=\log _{b}(x)\) is the range of \(y=b^{x}:(0, \infty)\).
- The range of \(y=\log _{b}(x)\) is the domain of \(y=b^{x}:(-\infty, \infty)\).

Transformations of the parent function \(y=\log _{b}(x)\) behave similarly to those of other functions. Just as with other parent functions, we can apply the four types of transformations-shifts, stretches, compressions, and reflections-to the parent function without loss of shape.

Previously we saw that certain transformations can change the range of \(y=b^{x}\). Similarly, applying transformations to the parent function \(y=\log _{b}(x)\) can change the domain. Therefore, when finding the domain of a logarithmic function, it is important to remember that the domain consists only of positive real numbers. That is, the value you are applying the logarithmic function to, also known as the argument of the logarithmic function, must be greater than zero.

For example, consider \(f(x)=\log _{4}(2 x-3)\). This function is defined for any values of \(x\) such that the argument, in this case \(2 x-3\), is greater than zero. To find the domain, we set up an inequality and solve for \(x\) :
\(2 x-3>0\) Show the argument greater than zero.
\(2 x>3 \quad\) Add 3 .
\(x>\frac{3}{2} \quad\) Divide by 2.
In interval notation, the domain of \(f(x)=\log _{4}(2 x-3)\) is \(\left(\frac{3}{2}, \infty\right)\).

\section*{TIP FOR SUCCESS}

Finding the domain of the logarithm function is similar to finding the domain of the square root function.
Recall that for \(y=\sqrt{u}, u \geq 0\), so the domain of \(y=\sqrt{2 x-3}\) is all \(x\) for which \(2 x-3 \geq 0\), or all \(x \geq \frac{3}{2}\).
Since the logarithm function takes input only strictly greater than zero, we see in the example above that the domain of \(f(x)=\log _{4}(2 x-3)\) is \(\left(\frac{3}{2}, \infty\right)\).

\section*{HOW TO: GIVEN A LOGARITHMIC FUNCTION, IDENTIFY THE DOMAIN}
1. Set up an inequality showing the argument greater than zero.
2. Solve for \(x\).
3. Write the domain in interval notation.

\section*{Graphing a Logarithmic Function Using a Table of Values}

Now that we have a feel for the set of values for which a logarithmic function is defined, we move on to graphing logarithmic functions. The family of logarithmic functions includes the parent function \(y=\log _{b}(x)\) along with all of its transformations: shifts, stretches, compressions, and reflections.

We begin with the parent function \(y=\log _{b}(x)\). Because every logarithmic function of this form is the inverse of an exponential function of the form \(y=b^{x}\), their graphs will be reflections of each other across the line \(y=x\). To illustrate this, we can observe the relationship between the input and output values of \(y=2^{x}\) and

\section*{EXAMPLE: IDENTIFYING THE DOMAIN RESULTING FROM A LOGARITHMIC SHIFT}

What is the domain of \(f(x)=\log _{2}(x+3)\) ?
Answer
\(x+3>0\) The argument must be positive.
\(x>-3 \quad\) Subtract 3 .

The logarithmic function is defined only when the argument is positive, so this function is defined when \(x+3>0\).
The domain of \(f(x)=\log _{2}(x+3)\) is \((-3, \infty)\).

\section*{TRY IT}

What is the domain of \(f(x)=\log _{5}(x-2)+1\) ?
Answer
\((2, \infty)\)

\section*{EXAMPLE: IDENTIFYING THE DOMAIN RESULTING FROM A LOGARITHMIC SHIFT AND REFLECTION}

What is the domain of \(f(x)=\log (5-2 x)\) ?
Answer
The logarithmic function is defined only when the argument is positive, so this function is defined when \(5-2 x>0\). Solving this inequality,
\(5-2 x>0\) The argument must be positive.
\(-2 x>-5\) Subtract 5 .
\(x<\frac{5}{2} \quad\) Divide by -2 and switch the inequality.
The domain of \(f(x)=\log (5-2 x)\) is \(\left(-\infty, \frac{5}{2}\right)\).

\section*{TRY IT}

What is the domain of \(f(x)=\log (x-5)+2\) ?
Answer
\((5, \infty)\)
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its equivalent logarithmic form \(x=\log _{2}(y)\) in the table below.

\section*{RECALL INVERSE RELATIONSHIPS}

Recall that if an invertible function \(f(x)\) contains a point, \((a, b)\), then the inverse function \(f^{-1}(x)\) must contain the point \((b, a)\).
\begin{tabular}{|l|l|l|l|l|l|l|l|}
\hline \(\boldsymbol{x}\) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline \(2^{x}=y\) & \(\frac{1}{8}\) & \(\frac{1}{4}\) & \(\frac{1}{2}\) & 1 & 2 & 4 & 8 \\
\hline \(\log _{2}(y)=x\) & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
\hline
\end{tabular}

Using the inputs and outputs from the table above, we can build another table to observe the relationship between points on the graphs of the inverse functions \(f(x)=2^{x}\) and \(g(x)=\log _{2}(x)\).
\begin{tabular}{l|l|l|l|l|l|l|}
\(f(x)=2^{x}\) & \(\left(-3, \frac{1}{8}\right)\) & \(\left(-2, \frac{1}{4}\right)\) & \(\left(-1, \frac{1}{2}\right)\) & \((0,1)\) & \((1,2)\) & \((2,4)\) \\
\((3,8)\) \\
\hline\(g(x)=\log _{2}(x)\) & \(\left(\frac{1}{8},-3\right)\) & \(\left(\frac{1}{4},-2\right)\) & \(\left(\frac{1}{2},-1\right)\) & \((1,0)\) & \((2,1)\) & \((4,2)\) \\
\hline
\end{tabular}

As we would expect, the \(x\) and \(y\)-coordinates are reversed for the inverse functions. The figure below shows the graphs of \(f\) and \(g\).


Notice that the graphs of \(f(x)=2^{x}\) and \(g(x)=\log _{2}(x)\) are reflections about the line \(y=x\) since they are inverses of each other.
Observe the following from the graph:
- \(f(x)=2^{x}\) has a \(y\)-intercept at \((0,1)\) and \(g(x)=\log _{2}(x)\) has an \(x\)-intercept at \((1,0)\).
- The domain of \(f(x)=2^{x},(-\infty, \infty)\), is the same as the range of \(g(x)=\log _{2}(x)\).
- The range of \(f(x)=2^{x},(0, \infty)\), is the same as the domain of \(g(x)=\log _{2}(x)\).

\section*{A GENERAL NOTE: CHARACTERISTICS OF THE GRAPH OF THE PARENT FUNCTION \(f(x)=\log _{b}(x)\)}

For any real number \(x\) and constant \(b>0, b \neq 1\), we can see the following characteristics in the graph of \(f(x)=\log _{b}(x)\) :
- one-to-one function
- vertical asymptote: \(x=0\)
- domain: \((0, \infty)\)
- range: \((-\infty, \infty)\)
- \(x\)-intercept: \((1,0)\) and key point \((b, 1)\)
- \(y\)-intercept: none
- increasing if \(b>1\)
- decreasing if \(0<b<1\)

\section*{TRY IT}

Use an online graphing tool to graph the function \(f(x)=\log _{b}(x)\)
Adjust the \(b\) value to investigate how changes in the base of the logarithmic function affect the graph of that function. Which function is an increasing function? Which is decreasing? Does the x-intercept change when you change the base? Make sure to use values between 0 and 1 as well as values greater than 1 .

The graphs above show how changing the base \(b\) in \(f(x)=\log _{b}(x)\) can affect the graphs. Observe that the graphs compress vertically as the value of the base increases. (Note: recall that the function \(\ln (x)\) has base \(e \approx 2.718\).)


The graphs of three logarithmic functions with different bases all greater than 1.

\section*{HOW TO: GIVEN A LOGARITHMIC FUNCTION OF THE FORM \(f(x)=\log _{b}(x)\), GRAPH THE FUNCTION}
1. Draw and label the vertical asymptote, \(x=0\).
2. Plot the \(x\)-intercept, \((1,0)\).
3. Plot the key point \((b, 1)\).
4. Draw a smooth curve through the points.
5. State the domain, \((0, \infty)\), the range, \((-\infty, \infty)\), and the vertical asymptote, \(x=0\).

\section*{EXAMPLE: GRAPHING A LOGARITHMIC FUNCTION OF THE FORM}
\[
f(x)=\log _{b}(x)
\]

Graph \(f(x)=\log _{5}(x)\). State the domain, range, and asymptote.

\section*{Answer}

Before graphing, identify the behavior and key points for the graph.
- Since \(b=5\) is greater than one, we know the function is increasing. The left tail of the graph will approach the vertical asymptote \(x=0\), and the right tail will increase slowly without bound.
- The \(x\)-intercept is \((1,0)\).
- The key point \((5,1)\) is on the graph.
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.


The domain is \((0, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).

\section*{TRY IT}

Graph \(f(x)=\log _{\frac{1}{5}}(x)\). State the domain, range, and asymptote.

Answer
The domain is \((0, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).


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\section*{HORIZONTAL AND VERTICAL SHIFTS OF LOGARITHMIC FUNCTIONS}

\section*{LEARNING OUTCOMES}
- Graph horizontal and vertical shifts of logarithmic functions.

As we mentioned in the beginning of the section, transformations of logarithmic functions behave similar to those of other parent functions. We can shift, stretch, compress, and reflect the parent function \(y=\log _{b}(x)\) without loss of shape.

\section*{Graphing a Horizontal Shift of \(f(x)=\log _{b}(x)\)}

When a constant \(c\) is added to the input of the parent function \(f(x)=\log _{b}(x)\), the result is a horizontal shift \(c\) units in the opposite direction of the sign on \(c\). To visualize horizontal shifts, we can observe the general graph of the parent function \(f(x)=\log _{b}(x)\) alongside the shift left, \(g(x)=\log _{b}(x+c)\), and the shift right, \(h(x)=\log _{b}(x-c)\) where \(c>0\).

\section*{TRY IT}

Using an online graphing calculator, plot the functions \(g(x)=\log _{b}(x+c)\) and \(h(x)=\log _{b}(x-c)\) Investigate the following questions:
- Adjust the \(c\) value to 4 .
- Which direction does the graph of \(g(x)\) shift? What is the vertical asymptote, x -intercept, and equation for this new function? How do the domain and range change?
- Which direction does the graph of \(h(x)\) shift? What is the vertical asymptote, x-intercept, and equation for this new function? How do the domain and range change?

The graphs below summarize the changes in the x-intercepts, vertical asymptotes, and equations of a logarithmic function that has been shifted either right or left.
\begin{tabular}{|c|c|}
\hline Shift left
\[
g(x)=\log _{b}(x+c)
\] & \[
\begin{gathered}
\text { Shift right } \\
h(x)=\log _{b}(x-c)
\end{gathered}
\] \\
\hline \begin{tabular}{l}
 \\
-The asymptote changes to \(x=-c\). \\
-The domain changes to \((-c, \infty)\). \\
-The range remains \((-\infty, \infty)\).
\end{tabular} & \begin{tabular}{l}
 \\
-The asymptote changes to \(x=c\). \\
-The domain changes to ( \(c, \infty\) ). \\
-The range remains \((-\infty, \infty)\).
\end{tabular} \\
\hline
\end{tabular}

\section*{A GENERAL NOTE: HORIZONTAL SHIFTS OF THE PARENT FUNCTION \(y=\log _{b}(x)\)}

For any constant \(c\), the function \(f(x)=\log _{b}(x+c)\)
- shifts the parent function \(y=\log _{b}(x)\) left \(c\) units if \(c>0\).
- shifts the parent function \(y=\log _{b}(x)\) right \(c\) units if \(c<0\).
- has the vertical asymptote \(x=-c\).
- has domain \((-c, \infty)\).
- has range \((-\infty, \infty)\).

\section*{HOW TO: GIVEN A LOGARITHMIC FUNCTION OF THE FORM \(f(x)=\log _{b}(x+c)\), GRAPH THE HORIZONTAL SHIFT}
1. Identify the horizontal shift:
- If \(c>0\), shift the graph of \(f(x)=\log _{b}(x)\) left \(c\) units.
- If \(c<0\), shift the graph of \(f(x)=\log _{b}(x)\) right \(c\) units.
2. Draw the vertical asymptote \(x=-c\).
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by subtracting \(c\) from the \(x\) coordinate in each point.
4. Label the three points.
5. The domain is \((-c, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=-\mathrm{c}\).

\section*{TIP FOR SUCCESS}

Logarithm functions follow the same principles as the other toolkit functions with regard to translations.

\section*{EXAMPLE: GRAPHING A HORIZONTAL SHIFT OF THE PARENT FUNCTION \\ \[
y=\log _{b}(x)
\]}

Sketch the horizontal shift \(f(x)=\log _{3}(x-2)\) alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.
Answer
Since the function is \(f(x)=\log _{3}(x-2)\), we notice \(x+(-2)=x-2\).
Thus \(c=-2\), so \(c<0\). This means we will shift the function \(f(x)=\log _{3}(x)\) right 2 units.
The vertical asymptote is \(x=-(-2)\) or \(x=2\).
Consider the three key points from the parent function: \(\left(\frac{1}{3},-1\right),(1,0)\), and \((3,1)\).
The new coordinates are found by adding 2 to the \(x\) coordinates of each point.
Plot and label the points \(\left(\frac{7}{3},-1\right),(3,0)\), and \((5,1)\).
The domain is \((2, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=2\).


\section*{TRY IT}

Sketch a graph of \(f(x)=\log _{3}(x+4)\) alongside its parent function. Include the key points and asymptotes on the graph. State the domain, range, and asymptote.

\section*{Answer}

The domain is \((-4, \infty)\), the range \((-\infty, \infty)\), and the asymptote \(x=-4\).


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\section*{Graphing a Vertical Shift of \(y=\log _{b}(x)\)}

When a constant \(d\) is added to the parent function \(f(x)=\log _{b}(x)\), the result is a vertical shift \(d\) units in the direction of the sign of \(d\). To visualize vertical shifts, we can observe the general graph of the parent function \(f(x)=\log _{b}(x)\) alongside the shift up, \(g(x)=\log _{b}(x)+d\), and the shift down,
\(h(x)=\log _{b}(x)-d\).
\begin{tabular}{|c|c|}
\hline Shift up
\[
g(x)=\log _{b}(x)+d
\] & Shift down
\[
h(x)=\log _{b}(x)-d
\] \\
\hline  &  \\
\hline \begin{tabular}{l}
-The asymptote remains \(x=0\). \\
-The domain remains to \((0, \infty)\). \\
-The range remains \((-\infty, \infty)\).
\end{tabular} & \begin{tabular}{l}
-The asymptote remains \(x=0\). \\
-The domain remains to \((0, \infty)\). \\
-The range remains \((-\infty, \infty)\).
\end{tabular} \\
\hline
\end{tabular}

\section*{A GENERAL NOTE: VERTICAL SHIFTS OF THE PARENT FUNCTION \(y=\log _{b}(x)\)}

For any constant \(d\), the function \(f(x)=\log _{b}(x)+d\)
- shifts the parent function \(y=\log _{b}(x)\) up \(d\) units if \(d>0\).
- shifts the parent function \(y=\log _{b}(x)\) down \(d\) units if \(d<0\).
- has the vertical asymptote \(x=0\).
- has domain \((0, \infty)\).
- has range \((-\infty, \infty)\).

\section*{HOW TO: GIVEN A LOGARITHMIC FUNCTION OF THE FORM \(f(x)=\log _{b}(x)+d\), GRAPH THE VERTICAL SHIFT}
1. Identify the vertical shift:
- If \(d>0\), shift the graph of \(f(x)=\log _{b}(x)\) up \(d\) units.
- If \(d<0\), shift the graph of \(f(x)=\log _{b}(x)\) down \(d\) units.
2. Draw the vertical asymptote \(x=0\).
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by adding \(d\) to the \(y\) coordinate of each point.
4. Label the three points.
5. The domain is \((0, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).

\section*{EXAMPLE: GRAPHING A VERTICAL SHIFT OF THE PARENT FUNCTION \\ \(y=\log _{b}(x)\)}

Sketch a graph of \(f(x)=\log _{3}(x)-2\) alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

\section*{Answer}

Since the function is \(f(x)=\log _{3}(x)-2\), we notice \(d=-2\). Thus \(d<0\).
This means we will shift the function \(f(x)=\log _{3}(x)\) down 2 units.
The vertical asymptote is \(x=0\).
Consider the three key points from the parent function, \(\left(\frac{1}{3},-1\right),(1,0)\), and \((3,1)\).
The new coordinates are found by subtracting 2 from the \(y\) coordinates of each point. Label the points \(\left(\frac{1}{3},-3\right),(1,-2)\), and \((3,-1)\).


The domain is \((0, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).

\section*{TRY IT}

Sketch a graph of \(f(x)=\log _{2}(x)+2\) alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.
Answer
The domain is \((0, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).


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\title{
STRETCHING, COMPRESSING, OR REFLECTING A LOGARITHMIC FUNCTION
}

\section*{LEARNING OUTCOMES}
- Graph stretches and compressions of logarithmic functions.
- Graph reflections of logarithmic functions.

\section*{Graphing Stretches and Compressions of \(y=\log _{b}(x)\)}

When the parent function \(f(x)=\log _{b}(x)\) is multiplied by a constant \(a>0\), the result is a vertical stretch or compression of the original graph. To visualize stretches and compressions, we set \(a>1\) and observe the general graph of the parent function \(f(x)=\log _{b}(x)\) alongside the vertical stretch, \(g(x)=a \log _{b}(x)\), and the vertical compression, \(h(x)=\frac{1}{a} \log _{b}(x)\).

\section*{TRY IT}

Using an online graphing calculator plot the functions \(g(x)=a \log _{b} x\) and \(h(x)=\frac{1}{a} \log _{b} x\). One represents a vertical compression of the other. You may select any value for \(b\), though one between 2 and 5 will be easier to see. Experiment with various \(a\) values between 1 and 10 . As you investigate, consider the following questions:
- Both the vertical stretch and compression produce graphs that are increasing. Which transformation produces a function that increases faster?
- One of the key points that is commonly defined for transformations of a logarithmic function comes from finding the input that gives an output of \(y=1\). This point can help you determine whether a graph is the result of a vertical compression or stretch. Explain why.

The graphs below summarize the key features of the resulting graphs of vertical stretches and compressions of logarithmic functions.
\begin{tabular}{|c|c|}
\hline Vertical Stretch
\[
g(x)=\operatorname{alog}_{b}(x), a>1
\] & Vertical Compression
\[
h(x)=\frac{1}{a} \log _{b}(x), a>1
\] \\
\hline \begin{tabular}{l}
 \\
-The asymptote remains \(x=0\). \\
-The \(x\)-intercept remains \((1,0)\). \\
-The domain remains \((0, \infty)\). \\
-The range remains \((-\infty, \infty)\).
\end{tabular} & \begin{tabular}{l}
 \\
-The asymptote remains \(x=0\). \\
-The \(x\)-intercept remains \((1,0)\). \\
-The domain remains \((0, \infty)\). \\
-The range remains \((-\infty, \infty)\).
\end{tabular} \\
\hline
\end{tabular}

\section*{A GENERAL NOTE: VERTICAL STRETCHES AND COMPRESSIONS OF THE PARENT FUNCTION \(y=\log _{b}(x)\)}

For any constant \(a>1\), the function \(f(x)=a \log _{b}(x)\)
- stretches the parent function \(y=\log _{b}(x)\) vertically by a factor of \(a\) if \(a>1\).
- compresses the parent function \(y=\log _{b}(x)\) vertically by a factor of \(a\) if \(0<a<1\).
- has the vertical asymptote \(x=0\).
- has the \(x\)-intercept \((1,0)\).
- has domain \((0, \infty)\).
- has range \((-\infty, \infty)\).

\section*{HOW TO: GIVEN A LOGARITHMIC FUNCTION OF THE FORM \(f(x)=a \log _{b}(x), a>0\), GRAPH THE STRETCH OR COMPRESSION}
1. Identify the vertical stretch or compression:
- If \(|a|>1\), the graph of \(f(x)=\log _{b}(x)\) is stretched by a factor of \(a\) units.
- If \(|a|<1\), the graph of \(f(x)=\log _{b}(x)\) is compressed by a factor of \(a\) units.
2. Draw the vertical asymptote \(x=0\).
3. Identify three key points from the parent function. Find new coordinates for the shifted functions by multiplying the \(y\) coordinates in each point by \(a\).
4. Label the three points.
5. The domain is \((0, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).

\section*{TIP FOR SUCCESS}

Logarithm functions follow the same principles as the other toolkit functions with regard to stretches, compressions, and reflections.

\section*{Graphing Reflections of \(f(x)=\log _{b}(x)\)}

When the parent function \(f(x)=\log _{b}(x)\) is multiplied by -1 , the result is a reflection about the \(x\)-axis. When the input is multiplied by -1 , the result is a reflection about the \(y\)-axis. To visualize reflections, we restrict \(b>1\) and observe the general graph of the parent function \(f(x)=\log _{b}(x)\) alongside the reflection about the \(x\)-axis, \(g(x)=-\log _{b}(x)\), and the reflection about the \(y\)-axis, \(h(x)=\log _{b}(-x)\).

The graphs below summarize the key characteristics of reflecting \(f(x)=\log _{b} x\) horizontally and vertically.

\section*{EXAMPLE: GRAPHING A STRETCH OR COMPRESSION OF THE PARENT FUNCTION \(y=\log _{b}(x)\)}

Sketch the graph of \(f(x)=2 \log _{4}(x)\) alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.

\section*{Answer}

Since the function is \(f(x)=2 \log _{4}(x)\), we will note that \(=2\).
This means we will stretch the function \(f(x)=\log _{4}(x)\) by a factor of 2 .
The vertical asymptote is \(x=0\).
Consider the three key points from the parent function, \(\left(\frac{1}{4},-1\right),(1,0)\), and \((4,1)\).
The new coordinates are found by multiplying the \(y\) coordinates of each point by 2 .
Label the points \(\left(\frac{1}{4},-2\right),(1,0)\), and \((4,2)\).


The domain is \((0, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).

\section*{TRY IT}

Sketch a graph of \(f(x)=\frac{1}{2} \log _{4}(x)\) alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.
Answer
The domain is \((0, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).

\begin{tabular}{|c|c|}
\hline Reflection about the \(x\)-axis
\[
g(x)=\log _{b}(x), b>1
\] & Reflection about the \(\boldsymbol{y}\)-axis
\[
h(x)=\log _{b}(-x), b>1
\] \\
\hline \begin{tabular}{l}
 \\
-The reflected function is decreasing as \(x\) moves from zero to infinity. \\
-The asymptote remains \(x=0\). \\
-The \(x\)-intercept remains \((1,0)\). \\
-The key point changes to \(\left(b^{-1}, 1\right)\) \\
-The domain remains \((0, \infty)\). \\
-The range remains \((-\infty, \infty)\).
\end{tabular} & \begin{tabular}{l}
 \\
-The reflected function is decreasing as \(x\) moves from negative infinity to zero. \\
-The asymptote remains \(x=0\). \\
-The \(x\)-intercept changes to \((-1,0)\). \\
-The key point changes to \((-b, 1)\) \\
-The domain changes to \((-\infty, 0)\). \\
-The range remains \((-\infty, \infty)\).
\end{tabular} \\
\hline
\end{tabular}

\section*{Summarizing Transformations of Logarithmic Functions}

Now that we have worked with each type of transformation for the logarithmic function, we can summarize each in the table below to arrive at the general equation for transforming exponential functions.

\section*{EXAMPLE: COMBINING A SHIFT AND A STRETCH}

Sketch the graph of \(f(x)=5 \log (x+2)\). State the domain, range, and asymptote.
Answer
Remember, what happens inside parentheses happens first. First, we move the graph left 2 units and then stretch the function vertically by a factor of 5 . The vertical asymptote will be shifted to \(x=-2\). The \(x\) intercept will be \((-1,0)\). The domain will be \((-2, \infty)\). Two points will help give the shape of the graph: \((-1,0)\) and \((8,5)\). We chose \(x=8\) as the \(x\)-coordinate of one point to graph because when \(x=8, x+2=\) 10, the base of the common logarithm.


The domain is \((-2, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=-2\).

\section*{TRY IT}

Sketch a graph of the function \(f(x)=3 \log (x-2)+1\). State the domain, range, and asymptote.
Answer
The domain is \((2, \infty)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=2\).


\section*{TRY IT}

Using an online graphing calculator, plot the functions
\(f(x)=\log _{b} x, g(x)=-\log _{b} x\), and \(h(x)=\log _{b}(-x)\). You may select any value for \(b\), though one between 2 and 5 will be easier to see. Also plot the point \((b, 1)\). Consider the following questions:
- Which graph, \(g(x)=-\log _{b} x\) or \(h(x)=\log _{b}(-x)\) represents a vertical reflection? Which one represents a horizontal reflection?
- You already added the point \((b, 1)\) as a point of interest for the function \(f(x)\). Using the variable \(b\) as your \(x\) value, add the corresponding points of interest for \(g(x)\) and \(h(x)\).
- Does the vertical asymptote change when you reflect the graph of \(f(x)\) either vertically or horizontally?

\section*{A GENERAL NOTE: REFLECTIONS OF THE PARENT FUNCTION \(y=\log _{b}(x)\)}

The function \(f(x)=-\log _{b}(x)\)
- reflects the parent function \(y=\log _{b}(x)\) about the \(x\)-axis.
- has domain \((0, \infty)\), range \((-\infty, \infty)\), and vertical asymptote \(x=0\) which are unchanged from the parent function.

The function \(f(x)=\log _{b}(-x)\)
- reflects the parent function \(y=\log _{b}(x)\) about the \(y\)-axis.
- has domain \((-\infty, 0)\).
- has range \((-\infty, \infty)\) and vertical asymptote \(x=0\) which are unchanged from the parent function.

\section*{HOW TO: GIVEN A LOGARITHMIC FUNCTION WITH THE PARENT FUNCTION \(f(x)=\log _{b}(x)\), GRAPH A REFLECTION}

If \(f(x)=-\log _{b}(x) \quad\) If \(f(x)=\log _{b}(-x)\)
1. Draw the vertical asymptote, \(x=0\).
2. Plot the \(x\)-intercept, \((1,0)\).
3. Reflect the graph of the parent function \(f(x)=\log _{b}(x)\) about the \(x\)-axis.
4. Draw a smooth curve through the points.
5. State the domain \((0, \infty)\), the range \((-\infty, \infty)\), and the vertical asymptote \(x=0\).
1. Draw the vertical asymptote, \(x=0\).
2. Plot the \(x\)-intercept, \((1,0)\).
3. Reflect the graph of the parent function \(f(x)=\log _{b}(x)\) about the \(y\)-axis.
4. Draw a smooth curve through the points.
5. State the domain, \((-\infty, 0)\), the range, \((-\infty, \infty)\), and the vertical asymptote \(x=0\).

Transformations of the Parent Function \(y=\log _{b}(x)\)
\begin{tabular}{l|l} 
Translation & Form \\
\hline Shift & \\
\hline - Horizontally \(c\) units to the left & \(y=\log _{b}(x+c)+d\) \\
- Vertically \(d\) units up &
\end{tabular}

\section*{Stretch and Compression}
- Stretch if \(|a|>1\)
- Compression if \(|a|<1\)

Reflection about the \(x\)-axis
Reflection about the \(y\)-axis
\[
y=a \log _{b}(x)
\]

General equation for all transformations
\[
\begin{aligned}
& y=-\log _{b}(x) \\
& y=\log _{b}(-x) \\
& y=a \log _{b}(x+c)+d
\end{aligned}
\]

\section*{EXAMPLE: GRAPHING A REFLECTION OF A LOGARITHMIC FUNCTION}

Sketch a graph of \(f(x)=\log (-x)\) alongside its parent function. Include the key points and asymptote on the graph. State the domain, range, and asymptote.
Answer
Before graphing \(f(x)=\log (-x)\), identify the behavior and key points for the graph.
- Since \(b=10\) is greater than one, we know that the parent function is increasing. Since the input value is multiplied by -1 , \(f\) is a reflection of the parent graph about the \(y\)-axis. Thus, \(f(x)=\log (-x)\) will be decreasing as \(x\) moves from negative infinity to zero, and the right tail of the graph will approach the vertical asymptote \(x=0\).
- The \(x\)-intercept is \((-1,0)\).
- We draw and label the asymptote, plot and label the points, and draw a smooth curve through the points.


The domain is \((-\infty, 0)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).

\section*{TRY IT}

Graph \(f(x)=-\log (-x)\). State the domain, range, and asymptote.
Answer
The domain is \((-\infty, 0)\), the range is \((-\infty, \infty)\), and the vertical asymptote is \(x=0\).


\section*{HOW TO: GIVEN A LOGARITHMIC EQUATION, USE A GRAPHING CALCULATOR TO APPROXIMATE SOLUTIONS}
1. Press \([\mathbf{Y}=]\). Enter the given logarithmic equation or equations as \(\mathbf{Y}_{1}=\) and, if needed, \(\mathbf{Y}_{2}=\).
2. Press [GRAPH] to observe the graphs of the curves and use [WINDOW] to find an appropriate view of the graphs, including their point(s) of intersection.
3. To find the value of \(x\), we compute the point of intersection. Press [2ND] then [CALC]. Select "intersect" and press [ENTER] three times. The point of intersection gives the value of \(x\) for the point(s) of intersection.

\section*{EXAMPLE: APPROXIMATING THE SOLUTION OF A LOGARITHMIC EQUATION}

Solve \(4 \ln (x)+1=-2 \ln (x-1)\) graphically. Round to the nearest thousandth.

\section*{Answer}

Press [ \(\mathbf{Y}=\) ] and enter \(4 \ln (x)+1\) next to \(\mathbf{Y}_{\mathbf{1}}=\). Then enter \(-2 \ln (x-1)\) next to \(\mathbf{Y}_{\mathbf{2}}=\). For a window, use the values 0 to 5 for \(x\) and -10 to 10 for \(y\). Press [GRAPH]. The graphs should intersect somewhere a little to the right of \(x=1\).
For a better approximation, press [2ND] then [CALC]. Select [5: intersect] and press [ENTER] three times. The \(x\)-coordinate of the point of intersection is displayed as 1.3385297. (Your answer may be different if you use a different window or use a different value for Guess?) So, to the nearest thousandth, \(x \approx 1.339\).

\section*{TRY IT}

Solve \(5 \log (x+2)=4-\log (x)\) graphically. Round to the nearest thousandth.

\section*{Answer}
\(x \approx 3.049\)

\section*{A GENERAL NOTE: TRANSFORMATIONS OF LOGARITHMIC FUNCTIONS}

All transformations of the parent logarithmic function \(y=\log _{b}(x)\) have the form \(f(x)=a \log _{b}(x+c)+d\)
where the parent function, \(y=\log _{b}(x), b>1\), is
- shifted vertically up \(d\) units.
- shifted horizontally to the left \(c\) units.
- stretched vertically by a factor of lal if lal \(>0\).
- compressed vertically by a factor of lal if \(0<l a l<1\).
- reflected about the \(x\)-axis when \(a<0\).

For \(f(x)=\log (-x)\), the graph of the parent function is reflected about the \(y\)-axis.

\section*{EXAMPLE: FINDING THE VERTICAL ASYMPTOTE OF A LOGARITHMIC FUNCTION}

What is the vertical asymptote of \(f(x)=-2 \log _{3}(x+4)+5\) ?
Answer
The vertical asymptote is at \(x=-4\).

\section*{Analysis of the Solution}

The coefficient, the base, and the upward translation do not affect the asymptote. The shift of the curve 4 units to the left shifts the vertical asymptote to \(x=-4\).

\section*{TRY IT}

What is the vertical asymptote of \(f(x)=3+\ln (x-1) ?\)
Answer
\(x=1\)

\section*{TIP FOR SUCCESS}

In the example below, you'll write a common logarithmic function for the graph shown. Remember that all the functions studied in this course possess the characteristic that every point contained on the graph of a function satisfies the equation of the function.
As you have done before, begin with the form of a transformed logarithm function, \(f(x)=a \log (x+c)+d\) , then fill in the parts you can discern from the graph.
- Find the horizontal shift by locating the vertical asymptote.
- Examine the shape of the graph to see if it has been reflected.
- Once you have filled in what you know, substitute one or more points in integer coordinates if possible to solve for any remaining unknowns.
- Remember that if there are more than one unknown, you'll need more than one point and more than one equation to solve for all the unknowns.

Work through the example step-by-step with a pencil on paper, perhaps more than once or twice, to gain understanding.

\section*{EXAMPLE: FINDING THE EQUATION FROM A GRAPH}

Find a possible equation for the common logarithmic function graphed below.


\section*{Answer}

This graph has a vertical asymptote at \(x=-2\) and has been vertically reflected. We do not know yet the vertical shift or the vertical stretch. We know so far that the equation will have the form:
\(f(x)=-a \log (x+2)+k\)
It appears the graph passes through the points \((-1,1)\) and \((2,-1)\). Substituting \((-1,1)\),
\(1=-a \log (-1+2)+k\) Substitute \((-1,1)\).
\(1=-a \log (1)+k \quad\) Arithmetic.
\(1=k\)
\(\log (1)=0\).
Next, substituting \((2,-1)\),
\(-1=-a \log (2+2)+1\)
Plug in \((2,-1)\).
\(-2=-a \log (4)\)
Arithmetic.
\(a=\frac{2}{\log (4)}\)
Solve for \(a\).

This gives us the equation \(f(x)=-\frac{2}{\log (4)} \log (x+2)+1\).

\section*{Analysis of the Solution}

We can verify this answer by comparing the function values in the table below with the points on the graph in this example.
\begin{tabular}{|l|l|l|l|l|l|}
\hline \(\boldsymbol{x}\) & -1 & 0 & 1 & 2 & 3 \\
\hline \(\boldsymbol{f}(\boldsymbol{x})\) & 1 & 0 & -0.58496 & -1 & -1.3219 \\
\hline \(\boldsymbol{x}\) & 4 & 5 & 6 & 7 & 8 \\
\hline \(\boldsymbol{f}(\boldsymbol{x})\) & -1.5850 & -1.8074 & -2 & -2.1699 & -2.3219 \\
\hline
\end{tabular}

\section*{TRY IT}

Give the equation of the natural logarithm graphed below.


Answer
\(f(x)=2 \ln (x+3)-1\)

\section*{Q \& A}

Is it possible to tell the domain and range and describe the end behavior of a function just by looking at the graph?
Yes if we know the function is a general logarithmic function. For example, look at the graph in the previous example. The graph approaches \(x=-3\) (or thereabouts) more and more closely, so \(x=-3\) is, or is very close to, the vertical asymptote. It approaches from the right, so the domain is all points to the right, \(\{x \mid x>-3\}\). The range, as with all general logarithmic functions, is all real numbers. And we can see the end behavior because the graph goes down as it goes left and up as it goes right. The end behavior is as \(x \rightarrow-3^{+}, f(x) \rightarrow-\infty\) and as \(x \rightarrow \infty, f(x) \rightarrow \infty\).

\section*{SUMMARY: GRAPHS OF LOGARITHMIC FUNCTIONS}

\section*{Key Equations}

General Form for the Transformation of the Parent Logarithmic Function \(f(x)=\log _{b}(x)\)
\[
f(x)=a \log _{b}(x+c)+d
\]

\section*{Key Concepts}
- To find the domain of a logarithmic function, set up an inequality showing the argument greater than zero and solve for \(x\).
- The graph of the parent function \(f(x)=\log _{b}(x)\) has an \(x\)-intercept at \((1,0)\), domain \((0, \infty)\), range \((-\infty, \infty)\), vertical asymptote \(x=0\), and
- if \(b>1\), the function is increasing.
\(\circ\) if \(0<b<1\), the function is decreasing.
- The equation \(f(x)=\log _{b}(x+c)\) shifts the parent function \(y=\log _{b}(x)\) horizontally
- left \(c\) units if \(c>0\).
- right \(c\) units if \(c<0\).
- The equation \(f(x)=\log _{b}(x)+d\) shifts the parent function \(y=\log _{b}(x)\) vertically
- up \(d\) units if \(d>0\).
- down \(d\) units if \(d<0\).
- For any constant \(a>0\), the equation \(f(x)=a \log _{b}(x)\)
- stretches the parent function \(y=\log _{b}(x)\) vertically by a factor of \(a\) if lal \(>1\).
- compresses the parent function \(y=\log _{b}(x)\) vertically by a factor of \(a\) if \(|a|<1\).
- When the parent function \(y=\log _{b}(x)\) is multiplied by -1 , the result is a reflection about the \(x\)-axis. When the input is multiplied by -1 , the result is a reflection about the \(y\)-axis.
- The equation \(f(x)=-\log _{b}(x)\) represents a reflection of the parent function about the \(x\)-axis.
- The equation \(f(x)=\log _{b}(-x)\) represents a reflection of the parent function about the \(y\)-axis.
- A graphing calculator may be used to approximate solutions to some logarithmic equations.
- All transformations of the logarithmic function can be summarized by the general equation \(f(x)=a \log _{b}(x+c)+d\).
- Given an equation with the general form \(f(x)=a \log _{b}(x+c)+d\), we can identify the vertical asymptote \(x=-c\) for the transformation.
- Using the general equation \(f(x)=a \log _{b}(x+c)+d\), we can write the equation of a logarithmic function given its graph.

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\section*{PUTTING IT TOGETHER: EXPONENTIAL AND LOGARITHMIC FUNCTIONS}

At the start of this module, you were considering investing your inheritance possibly to save for retirement. Now you can use what you've learned to figure it out. The final value of your investment can be represented by the equation
\(f(t)=P e^{r t}\)
where
\(P=\) the initial investment
\(t=\) number of years invested
\(r=\) interest rate, expressed as a decimal
Now remember that you had \(\$ 10,000\) to invest, so \(P=10,000\). Also recall that the interest rate was \(3 \%\), so \(r=0.03\).

Let's start with 5 years, so \(t=5\).
\begin{tabular}{l|l|}
\hline Start with the function: & \(f(t)=P e^{r t}\) \\
\hline Substitute P, r, and t: & \(f(5)=10,000 e^{0.03(5)}\) \\
\hline Evaluate: & \(f(5)=11,618.34\) \\
\hline
\end{tabular}

Now let's look at 10 years, so \(t=10\).
\begin{tabular}{l|l|}
\hline Start with the function: & \(f(t)=P e^{r t}\) \\
\hline Substitute P, r, and t: & \(f(10)=10,000 e^{0.03(10)}\) \\
\hline Evaluate: & \(f(10)=13,498.59\) \\
\hline
\end{tabular}

Now let's look at 50 years, so \(t=50\).
\begin{tabular}{|l|l|}
\hline Start with the function: & \(f(t)=P e^{t r}\) \\
\hline Substitute P, r, and t: & \(f(50)=10,000 e^{0.03(50)}\) \\
\hline Evaluate: & \(f(10)=44,816.89\) \\
\hline
\end{tabular}

Using the function for continuously compounded interest, you can see how your initial investment will grow over time.
\begin{tabular}{|l|l|l|}
\hline\(t\) & Interest rate & \(f(t)\) \\
\hline 5 & 0.03 & \(\$ 11,618.34\) \\
\hline 10 & 0.03 & \(\$ 13.498 .59\) \\
\hline 50 & 0.03 & \(\$ 44,816.89\) \\
\hline
\end{tabular}

Now you know that your \(\$ 10,000\) can grow to over \(\$ 44,000\) in 50 years! With that knowledge under your belt, you can decide if you want to add to your investment or find an account with a greater interest rate. Either way, thanks to your knowledge of exponential functions, you can make sound financial decisions.

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\section*{MODULE 12: EXPONENTIAL AND LOGARITHMIC EQUATIONS AND MODELS}

\section*{WHY IT MATTERS: EXPONENTIAL AND LOGARITHMIC EQUATIONS AND MODELS}

\section*{Why It Matters: Exponential and Logarithmic Equations}


You have a job assisting an archeologist who has just discovered a fossil that appears to be an animal bone. She assigns you the task of determining how old the bone is. Where do you start?

Fortunately, you know that living things contain a radioactive form of carbon called carbon-14. Like all radioactive elements, carbon-14 decays at a predictable rate known as its half-life. The half-life of a radioactive element is the amount of time required for half of a sample to decay. The half-life of carbon-14 is 5,730 years. Given an original sample of carbon-14 of 100 g , the table shows the mass remaining after each half-life.
\begin{tabular}{|l|l|l|l|l|l|}
\hline Amount of sample (g) & 100 & 50 & 25 & 12.5 & 6.25 \\
\hline Time (years) & 0 & 5730 & 11,460 & 17,190 & 22,920 \\
\hline
\end{tabular}

But, what if the bone started with a different mass of carbon-14 or a different number of years has passed? To better study the bone, you need to know that the rate of decay of a radioactive element can be modeled with an exponential function. Given a couple of data points, you can build a model that represents the decay of carbon over time for your specimen.

As you complete this module, keep the following questions in mind. Then at the end of the module, we will return to develop a model for the decay of carbon-14.
- How do you develop a model for the decay of carbon-14?
- How can you use the model to determine the amount of carbon-14 that remains after any number of years?
- What would a graph of the decay of carbon-14 look like?

\section*{LEARNING OUTCOMES}

\section*{Review for Success}
- Define the identity and zero properties of exponents and logarithms.
- Define the inverse property of exponents and logarithms.
- Define the one-to-one properties of exponents and logarithms.
- Define the product and quotient properties of exponents and logarithms.

\section*{Properties of Logarithms}
- Rewrite a logarithmic expression using the power rule, product rule, or quotient rule.
- Expand logarithmic expressions using a combination of logarithm rules.
- Condense logarithmic expressions using logarithm rules.
- Expand a logarithm using a combination of logarithm rules.
- Condense a logarithmic expression into one logarithm.
- Rewrite logarithms with a different base using the change of base formula.

\section*{Exponential and Logarithmic Equations}
- Solve an exponential equation with a common base.
- Rewrite an exponential equation so all terms have a common base then solve.
- Recognize when an exponential equation does not have a solution.
- Use logarithms to solve exponential equations.
- Solve a logarithmic equation algebraically.
- Solve a logarithmic equation graphically.
- Use the one-to-one property of logarithms to solve a logarithmic equation.
- Solve a radioactive decay problem.

Exponential and Logarithmic Models
- Graph exponential growth and decay functions.
- Solve problems involving radioactive decay, carbon dating, and half life.
- Use Newton's Law of Cooling.
- Use a logistic growth model.
- Choose an appropriate model for data.
- Use a graphing utility to create an exponential regression from a set of data.

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\section*{REVIEW TOPICS FOR SUCCESS}

By the end of this section, you will be able to:
- Define the identity and zero properties of exponents and logarithms.
- Define the inverse property of exponents and logarithms.
- Define the one-to-one properties of exponents and logarithms.
- Define the product and quotient properties of exponents and logarithms.

This module will introduce techniques for solving exponential and logarithmic equations. In order to solve these types of equations, you'll need to obtain the properties that allow you to rewrite logarithms, which will be defined and demonstrated in the first part of the module. It will be helpful for you to first see how the logarithmic properties compare to the familiar exponential properties, though, which is the focus of this review section.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

\section*{DEFINE AND IDENTIFY EXPONENTIAL AND LOGARITHMIC PROPERTIES}

\section*{LEARNING OUTCOMES}

Define the identity and zero properties of exponents and logarithms.
Define the inverse property of exponents and logarithms.
Define the one-to-one properties of exponents and logarithms.
- Define the product and quotient properties of exponents and logarithms.

When you learned how to solve linear equations, you were likely introduced to the properties of real numbers. These properties help us know what the rules are for isolating and combining numbers and variables. For example, it is advantageous to know that multiplication and division "undo" each other when you want to solve an equation for a variable that is multiplied by a number. You may have also been introduced to properties and rules for writing and using exponents. In this section, we will show you how the properties and rules for logarithms compare to those for exponents.

Recall that we can express the relationship between logarithmic form and its corresponding exponential form as follows:
\(\log _{b}(x)=y \Leftrightarrow b^{y}=x, b>0, b \neq 1\)
That is, to say that the logarithm to base b of \(x\) is \(y\) is equivalent to saying that \(y\) is the exponent on the base \(b\) that produces \(x\).

Note that the base \(b\) is always a positive number other than 1 and that the logarithmic and exponential forms "undo" each other. This means that logarithms have properties similar to exponents. Let's compare several such properties side by side.

\section*{Identity and Zero Properties of Exponents and Logarithms}

\section*{ZERO EXPONENT RULE FOR LOGARITHMS AND EXPONENTIALS}

Recall that for exponents,
\(b^{0}=1\).
Using the definition of a logarithm, we can rewrite that statement in logarithm form:
\(\log _{b} 1=0\)

\section*{IDENTITY EXPONENT RULE FOR LOGARITHMS AND EXPONENTIALS}

Recall that for exponents,
\(b^{1}=b\)
Using the definition of a logarithm, we can rewrite that statement in logarithm form:
\(\log _{b} b=1\)

\section*{EXAMPLE}

Use the the fact that exponentials and logarithms are inverses to prove the zero and identity exponent rule for the following:
1. \(\log _{5} 1=0\)
2. \(\log _{5} 5=1\)

Answer
1. \(\log _{5} 1=0\) since \(5^{0}=1\)
\(2 \cdot \log _{5} 5=1\) since \(5^{1}=5\)

\section*{Inverse Property of Logarithms and Exponents}

Exponential and logarithmic functions are inverses of each other, and we can take advantage of this to evaluate and solve expressions and equations involving logarithms and exponentials. The inverse property of logarithms and exponentials gives us an explicit way to rewrite an exponential as a logarithm or a logarithm as an exponential.

\section*{INVERSE PROPERTY OF LOGARITHMS AND EXPONENTIALS}

Recall that we can use the definition of a logarithm to rewrite it as an exponent. That is \(\log _{b} x=y \Leftrightarrow b^{y}=x\).
By extension,
\(\log _{b} b^{x}=x \quad\) and \(\quad b^{\log _{b} x}=x\).

\section*{EXAMPLE}

Evaluate:
1. \(\log\) (100)
2. \(e^{\ln (7)}\)

Answer
1. Rewrite the logarithm as \(\log _{10}\left(10^{2}\right)\), and then apply the inverse property \(\log _{b}\left(b^{x}\right)=x\) to get \(\log _{10}\left(10^{2}\right)=2\).
2. Rewrite the logarithm as \(e^{\log _{e} 7}\), and then apply the inverse property \(b^{\log _{b} x}=x\) to get \(e^{\log _{e} 7}=7\)

\section*{One-to-One Property of Logarithms and Exponents}

Consider the mathematical statement \(2^{x}=2^{y}\). The only possible way this could be a true statement is if the variables \(x\) and \(y\) are equivalent. This statement is an example of the one-to-one property of exponents. More formally, it states
\(a^{m}=a^{n} \Rightarrow m=n\).
In more general terms, if like bases raised to perhaps different exponents are given to be equivalent, then the exponents themselves must also be equivalent.

The same idea applies to logarithms. If two logarithms of like base but with perhaps different arguments are given to be equivalent, then the arguments themselves must also be equivalent. More formally, the one-toone property of logarithms states
\(\log _{b}(M)=\log _{b}(N) \Rightarrow M=N\).

\section*{EXAMPLE}

Solve the equation \(\log _{3}(3 x)=\log _{3}(2 x+5)\) for \(x\).

\section*{Answer}

In order for this equation to be true, we must find a value for x such that \(3 x=2 x+5\)
\(3 x=2 x+5 \quad\) Set the arguments equal to each other.
\(x=5 \quad\) Subtract \(2 x\).
Check your answer by substituting 5 for \(x\).
\[
\begin{aligned}
& \log _{3}(3 \cdot 5)=\log _{3}(2 \cdot 5+5) \\
& \log _{3}(15)=\log _{3}(15)
\end{aligned}
\]

This is a true statement, so we must have found the correct value for \(x\).

\section*{The Product and Quotient Rules for Logarithms}

Recall that we use the product rule of exponents to combine the product of like bases raised to exponents by adding the exponents:
\(x^{a} x^{b}=x^{a+b}\).

We have a similar property for logarithms, called the product rule for logarithms, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents, and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

\section*{THE PRODUCT RULE FOR LOGARITHMS}

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.
\(\log _{b}(M N)=\log _{b}(M)+\log _{b}(N)\)

\section*{EXAMPLE}

Using the product rule for logarithms, rewrite the logarithm of a product as the sum of logarithms of its factors.
\(\log _{b}(w x y z)\)
Answer
\(\log _{b}(w x y z)=\log _{b} w+\log _{b} x+\log _{b} y+\log _{b} z\)
For quotients, we have a similar rule for logarithms. Recall that we use the quotient rule of exponents to simplify division of like bases raised to powers by subtracting the exponents:
\(\frac{x^{a}}{x^{b}}=x^{a-b}\).
The quotient rule for logarithms says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

\section*{THE QUOTIENT RULE FOR LOGARITHMS}

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.
\[
\log _{b}\left(\frac{M}{N}\right)=\log _{b} M-\log _{b} N
\]

Later in the module, you'll extend these properties and rules to more complicated examples and use them to write compact logarithms in expanded form and contract those already expanded. You'll also obtain the power rule for logarithms, which will be particularly helpful when writing and solving exponential models of real-world situations.

\section*{EXAMPLE}

Expand the following expression using the quotient rule for logarithms.
\(\log \left(\frac{2 x^{2}+6 x}{3 x+9}\right)\)

\section*{Answer}

Factoring and canceling, we get:
\[
\begin{array}{rlrl}
\log \left(\frac{2 x^{2}+6 x}{3 x+9}\right) & =\log \left(\frac{2 x(x+3)}{3(x+3)}\right) & \text { Factor the numerator and denominator. } \\
& =\log \left(\frac{2 x}{3}\right) & & \text { Cancel the common factors. }
\end{array}
\]

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.
\[
\begin{aligned}
\log \left(\frac{2 x}{3}\right) & =\log (2 x)-\log (3) \\
& =\log (2)+\log (x)-\log (3)
\end{aligned}
\] Free at : http://cnx.org/contents/fd53eae1-fa23-47c7-bb1b-972349835c3c@5.175.

\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
- Logarithms have properties similar to exponential properties because logarithms and exponents are of inverse forms to one another.
- Use the inverse nature and the definition of the logarithm to prove the properties of logarithms.
- Recall that exponents are added when like bases are multiplied as a reminder that the logarithm of a product is equivalent to a sum of logarithms to the same base.
- Recall that exponents are subtracted when like bases are divided as a reminder that the logarithm of a quotient is equivalent to a difference of logarithms to the same base.

\section*{Key Equations}
- \(\log _{b} x=y \Leftrightarrow b^{y}=x\)
- \(\log _{b}(M N)=\log _{b}(M)+\log _{b}(N)\)
- \(\log _{b}\left(\frac{M}{N}\right)=\log _{b}(M)-\log _{b}(N)\)

\section*{Glossary}
one-to-one property of exponents states that \(a^{m}=a^{n} \Leftrightarrow m=n\)


\section*{INTRODUCTION TO PROPERTIES OF LOGARITHMS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Rewrite a logarithmic expression using the power rule, product rule, or quotient rule.
- Expand logarithmic expressions using a combination of logarithm rules.
- Condense logarithmic expressions using logarithm rules.
- Expand a logarithm using a combination of logarithm rules.
- Condense a logarithmic expression into one logarithm.
- Rewrite logarithms with a different base using the change of base formula.

In chemistry, pH is used as a measure of the acidity or alkalinity of a substance. The pH scale runs from 0 to 14 . Substances with a pH less than 7 are considered acidic and substances with a pH greater than 7 are said to be alkaline. Our bodies, for instance, must maintain a pH close to 7.35 in order for enzymes to work properly. To get a feel for what is acidic and what is alkaline, consider the following pH levels of some common substances:
- Battery acid: 0.8
- Stomach acid: 2.7
- Orange juice: 3.3
- Pure water: 7 (at \(25^{\circ} \mathrm{C}\) )
- Human blood: 7.35
- Fresh coconut: 7.8
- Sodium hydroxide (lye): 14

To determine whether a solution is acidic or alkaline, we find its pH which is a measure of the number of active positive hydrogen ions in the solution. The pH is defined by the following formula where \(a\) is the


The pH of hydrochloric acid is tested with litmus paper. (credit: David Berardan) concentration of hydrogen ion in the solution.
\[
\begin{aligned}
\mathrm{pH} & =-\log \left(\left[H^{+}\right]\right) \\
& =\log \left(\frac{1}{\left[H^{+}\right]}\right)
\end{aligned}
\]
\(-\log \left(\left[H^{+}\right]\right)\)is equal to \(\log \left(\frac{1}{\left[H^{+}\right]}\right)\)due to one of the logarithm properties we will examine in this section.

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\section*{PROPERTIES OF LOGARITHMS}

\section*{LEARNING OUTCOMES}
- Rewrite a logarithmic expression using the power rule, product rule, or quotient rule.
- Expand logarithmic expressions using a combination of logarithm rules.
- Condense logarithmic expressions using logarithm rules.

\section*{Properties of Logarithms}

Recall that the logarithmic and exponential functions "undo" each other. This means that logarithms have similar properties to exponents. Some important properties of logarithms are given here. First, the following properties are easy to prove.
\[
\begin{aligned}
& \log _{b} 1=0 \\
& \log _{b} b=1
\end{aligned}
\]

For example, \(\log _{5} 1=0\) since \(5^{0}=1\) and \(\log _{5} 5=1\) since \(5^{1}=5\).
Next, we have the inverse property.
\[
\begin{aligned}
& \log _{b}\left(b^{x}\right)=x \\
& b^{\log _{b} x}=x, x>0
\end{aligned}
\]

For example, to evaluate \(\log (100)\), we can rewrite the logarithm as \(\log _{10}\left(10^{2}\right)\) and then apply the inverse property \(\log _{b}\left(b^{x}\right)=x\) to get \(\log _{10}\left(10^{2}\right)=2\).

To evaluate \(e^{\ln (7)}\), we can rewrite the logarithm as \(e^{\log _{e} 7}\) and then apply the inverse property \(b^{\log _{b} x}=x\) to get \(e^{\log _{e} 7}=7\).

Finally, we have the one-to-one property.
\(\log _{b} M=\log _{b} N\) if and only if \(M=N\)
We can use the one-to-one property to solve the equation \(\log _{3}(3 x)=\log _{3}(2 x+5)\) for \(x\). Since the bases are the same, we can apply the one-to-one property by setting the arguments equal and solving for \(x\) :
\(3 x=2 x+5\) Set the arguments equal.
\(x=5 \quad\) Subtract \(2 x\).
But what about the equation \(\log _{3}(3 x)+\log _{3}(2 x+5)=2\) ? The one-to-one property does not help us in this instance. Before we can solve an equation like this, we need a method for combining logarithms on the left side of the equation.

\section*{Using the Product Rule for Logarithms}

Recall that we use the product rule of exponents to combine the product of exponents by adding: \(x^{a} x^{b}=x^{a+b}\). We have a similar property for logarithms, called the product rule for logarithms, which says that the logarithm of a product is equal to a sum of logarithms. Because logs are exponents and we multiply like bases, we can add the exponents. We will use the inverse property to derive the product rule below.

Given any real number \(x\) and positive real numbers \(M, N\), and \(b\), where \(b \neq 1\), we will show
\(\log _{b}(M N)=\log _{b}(M)+\log _{b}(N)\).
Let \(m=\log _{b} M\) and \(n=\log _{b} N\). In exponential form, these equations are \(b^{m}=M\) and \(b^{n}=N\). It follows that
\[
\begin{array}{rlrl}
\log _{b}(M N) & =\log _{b}\left(b^{m} b^{n}\right) & & \text { Substitute for } M \text { and } N . \\
& =\log _{b}\left(b^{m+n}\right) & & \text { Apply the product rule for exponents. } \\
& =m+n & & \text { Apply the inverse property of logs. } \\
& =\log _{b}(M)+\log _{b}(N) & \text { Substitute for } m \text { and } n .
\end{array}
\]

\section*{A GENERAL NOTE: THE PRODUCT RULE FOR LOGARITHMS}

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.
\(\log _{b}(M N)=\log _{b}(M)+\log _{b}(N)\) for \(b>0\)

\section*{EXAMPLE: USING THE PRODUCT RULE FOR LOGARITHMS}

Expand \(\log _{3}(30 x(3 x+4))\).
Answer
We begin by writing an equal equation by summing the logarithms of each factor. \(\log _{3}(30 x(3 x+4))=\log _{3}(30 x)+\log _{3}(3 x+4)=\log _{3}(30)+\log _{3}(x)+\log _{3}(3 x+4)\)
The final expansion looks like this. Note how the factor \(30 x\) can be expanded into the sum of two logarithms:
\(\log _{3}(30)+\log _{3}(x)+\log _{3}(3 x+4)\)

\section*{TRY IT}

Expand \(\log _{b}(8 k)\).
Answer
\(\log _{b} 8+\log _{b} k\)
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=354

\section*{Using the Quotient Rule for Logarithms}

For quotients, we have a similar rule for logarithms. Recall that we use the quotient rule of exponents to combine the quotient of exponents by subtracting: \(x^{\frac{a}{b}}=x^{a-b}\). The quotient rule for logarithms says that the logarithm of a quotient is equal to a difference of logarithms. Just as with the product rule, we can use the inverse property to derive the quotient rule.

Given any real number \(x\) and positive real numbers \(M, N\), and \(b\), where \(b \neq 1\), we will show
\(\log _{b}\left(\frac{M}{N}\right)=\log _{b}(M)-\log _{b}(N)\).
Let \(m=\log _{b} M\) and \(n=\log _{b} N\). In exponential form, these equations are \(b^{m}=M\) and \(b^{n}=N\). It follows that
\[
\begin{aligned}
\log _{b}\left(\frac{M}{N}\right) & =\log _{b}\left(\frac{b^{m}}{b^{n}}\right) & & \text { Substitute for } M \text { and } N . \\
& =\log _{b}\left(b^{m-n}\right) & & \text { Apply the quotient rule for exponents. } \\
& =m-n & & \text { Apply the inverse property of logs. } \\
& =\log _{b}(M)-\log _{b}(N) & & \text { Substitute for } m \text { and } n .
\end{aligned}
\]

For example, to expand \(\log \left(\frac{2 x^{2}+6 x}{3 x+9}\right)\), we must first express the quotient in lowest terms. Factoring and canceling, we get
\[
\begin{array}{rlrl}
\log \left(\frac{2 x^{2}+6 x}{3 x+9}\right) & =\log \left(\frac{2 x(x+3)}{3(x+3)}\right) & \text { Factor the numerator and denominator. } \\
& =\log \left(\frac{2 x}{3}\right) \quad \text { Cancel the common factors. }
\end{array}
\]

Next we apply the quotient rule by subtracting the logarithm of the denominator from the logarithm of the numerator. Then we apply the product rule.
\[
\begin{aligned}
\log \left(\frac{2 x}{3}\right) & =\log (2 x)-\log (3) \\
& =\log (2)+\log (x)-\log (3)
\end{aligned}
\]

\section*{A GENERAL NOTE: THE QUOTIENT RULE FOR LOGARITHMS}

The quotient rule for logarithms can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

\section*{HOW TO: GIVEN THE LOGARITHM OF A QUOTIENT, USE THE QUOTIENT RULE OF LOGARITHMS TO WRITE AN EQUIVALENT DIFFERENCE OF LOGARITHMS}
1. Express the argument in lowest terms by factoring the numerator and denominator and canceling common terms.
2. Write the equivalent expression by subtracting the logarithm of the denominator from the logarithm of the numerator.
3. Check to see that each term is fully expanded. If not, apply the product rule for logarithms to expand completely.

\section*{EXAMPLE: USING THE QUOTIENT RULE FOR LOGARITHMS}

Expand \(\log _{2}\left(\frac{15 x(x-1)}{(3 x+4)(2-x)}\right)\).

\section*{Answer}

First we note that the quotient is factored and in lowest terms, so we apply the quotient rule.
\(\log _{2}\left(\frac{15 x(x-1)}{(3 x+4)(2-x)}\right)=\log _{2}(15 x(x-1))-\log _{2}((3 x+4)(2-x))\)
Notice that the resulting terms are logarithms of products. To expand completely, we apply the product rule.
\(\log _{2}(15 x(x-1))-\log _{2}((3 x+4)(2-x))\)
\(=\left[\log _{2}(15)+\log _{2}(x)+\log _{2}(x-1)\right]-\left[\log _{2}(3 x+4)+\log _{2}(2-x)\right]\)
\(=\log _{2}(15)+\log _{2}(x)+\log _{2}(x-1)-\log _{2}(3 x+4)-\log _{2}(2-x)\)

\section*{Analysis of the Solution}

There are exceptions to consider in this and later examples. First, because denominators must never be zero, this expression is not defined for \(x=-\frac{4}{3}\) and \(x=2\). Also, since the argument of a logarithm must be positive, we note as we observe the expanded logarithm that \(x>0, x>1, x>-\frac{4}{3}\), and \(x<2\). Combining these conditions is beyond the scope of this section, and we will not consider them here or in subsequent exercises.

\section*{TRY IT}

Expand \(\log _{3}\left(\frac{7 x^{2}+21 x}{7 x(x-1)(x-2)}\right)\).
Answer
\(\log _{3}(x+3)-\log _{3}(x-1)-\log _{3}(x-2)\)
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\section*{RECALL RULES FOR REWRITING EXPONENTS}

The rules and properties of exponents will appear frequently when manipulating logarithms.
Product Rule \(a^{m} \cdot a^{n}=a^{m+n}\)
Quotient Rule \(\frac{a^{m}}{a^{n}}=a^{m-n}\)
Power Rule \(\left(a^{m}\right)^{n}=a^{m \cdot n}\)
Zero Exponent \(a^{0}=1\)
Negative Exponent \(a^{-n}=\frac{1}{a^{n}}\) and \(a^{n}=\frac{1}{a^{-n}}\)
Power of a Product \((a b)^{n}=a^{n} b^{n}\)
Power of a Quotient \(\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}\)
We have explored the product rule and the quotient rule, but how can we take the logarithm of a power, such as \(x^{2}\) ? One method is as follows:
\[
\begin{aligned}
\log _{b}\left(x^{2}\right) & =\log _{b}(x \cdot x) \\
& =\log _{b} x+\log _{b} x \\
& =2 \log _{b} x
\end{aligned}
\]

Notice that we used the product rule for logarithms to find a solution for the example above. By doing so, we have derived the power rule for logarithms, which says that the log of a power is equal to the exponent times the log of the base. Keep in mind that although the input to a logarithm may not be written as a power, we may be able to change it to a power. For example,
\[
100=10^{2}, \quad \sqrt{3}=3^{\frac{1}{2}}, \quad \frac{1}{e}=e^{-1}
\]

\section*{A GENERAL NOTE: THE POWER RULE FOR LOGARITHMS}

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.
\(\log _{b}\left(M^{n}\right)=n \log _{b} M\)

\section*{HOW TO: GIVEN THE LOGARITHM OF A POWER, USE THE POWER RULE OF LOGARITHMS TO WRITE AN EQUIVALENT PRODUCT OF A FACTOR AND A LOGARITHM}
1. Express the argument as a power, if needed.
2. Write the equivalent expression by multiplying the exponent times the logarithm of the base.

\section*{EXAMPLE: EXPANDING A LOGARITHM WITH POWERS}

Rewrite \(\log _{2} x^{5}\).
Answer
The argument is already written as a power, so we identify the exponent, 5 , and the base, \(x\), and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.
\(\log _{2}\left(x^{5}\right)=5 \log _{2} x\)

\section*{TRY IT}

Rewrite \(\ln x^{2}\).

\section*{Answer}
\(2 \ln x\)
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=354

\section*{EXAMPLE: REWRITING AN EXPRESSION AS A POWER BEFORE USING THE POWER RULE}

Rewrite \(\log _{3}(25)\) using the power rule for logs.

\section*{Answer}

Expressing the argument as a power, we get \(\log _{3}(25)=\log _{3}\left(5^{2}\right)\).
Next we identify the exponent, 2 , and the base, 5 , and rewrite the equivalent expression by multiplying the exponent times the logarithm of the base.
\(\log _{3}\left(5^{2}\right)=2 \log _{3}(5)\)

\section*{TRY IT}

Rewrite \(\ln \left(\frac{1}{x^{2}}\right)\).

\section*{Answer}
\(-2 \ln (x)\)
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=354

\section*{EXPANDING AND CONDENSING LOGARITHMS}

\section*{LEARNING OUTCOMES}
- Expand a logarithm using a combination of logarithm rules.
- Condense a logarithmic expression into one logarithm.

\section*{Expanding Logarithms}

Taken together, the product rule, quotient rule, and power rule are often called "properties of logs." Sometimes we apply more than one rule in order to expand an expression. For example:
\[
\begin{aligned}
\log _{b}\left(\frac{6 x}{y}\right) & =\log _{b}(6 x)-\log _{b} y \\
& =\log _{b} 6+\log _{b} x-\log _{b} y
\end{aligned}
\]

We can use the power rule to expand logarithmic expressions involving negative and fractional exponents. Here is an alternate proof of the quotient rule for logarithms using the fact that a reciprocal is a negative power:
\[
\begin{aligned}
\log _{b}\left(\frac{A}{C}\right) & =\log _{b}\left(A C^{-1}\right) \\
& =\log _{b}(A)+\log _{b}\left(C^{-1}\right) \\
& =\log _{b} A+(-1) \log _{b} C \\
& =\log _{b} A-\log _{b} C
\end{aligned}
\]

We can also apply the product rule to express a sum or difference of logarithms as the logarithm of a product.

With practice, we can look at a logarithmic expression and expand it mentally and then just writing the final answer. Remember, however, that we can only do this with products, quotients, powers, and roots-never with addition or subtraction inside the argument of the logarithm.

In the next example we will recall that we can write roots as exponents, and use this quality to simplify logarithmic expressions.

Now we will provide some examples that will require careful attention.

\section*{Condensing Logarithms}

We can use the rules of logarithms we just learned to condense sums, differences, and products with the same base as a single logarithm. It is important to remember that the logarithms must have the same base to be combined. We will learn later how to change the base of any logarithm before condensing.

In our next few examples we will use a combination of logarithm rules to condense logarithms.

\section*{EXAMPLE: USING A COMBINATION OF THE RULES FOR LOGARITHMS TO EXPAND A LOGARITHM}

Rewrite \(\ln \left(\frac{x^{4} y}{7}\right)\) as a sum or difference of logs.

\section*{Answer}

First, because we have a quotient of two expressions, we can use the quotient rule:
\(\ln \left(\frac{x^{4} y}{7}\right)=\ln \left(x^{4} y\right)-\ln (7)\)
Then seeing the product in the first term, we use the product rule:
\(\ln \left(x^{4} y\right)-\ln (7)=\ln \left(x^{4}\right)+\ln (y)-\ln (7)\)
Finally, we use the power rule on the first term:
\(\ln \left(x^{4}\right)+\ln (y)-\ln (7)=4 \ln (x)+\ln (y)-\ln (7)\)

\section*{TIP FOR SUCCESS}

When expanding and condensing logarithms, keep in mind that there are often more than one or two good ways to reach a good conclusion. The rules for manipulating exponents and logarithms can be combined creatively. You should try a few different ideas for using the rules on complicated expressions to get practice for finding the most efficient path to take in different situations.

\section*{TRY IT}

Expand \(\log \left(\frac{x^{2} y^{3}}{z^{4}}\right)\).

\section*{Answer}
\(2 \log x+3 \log y-4 \log z\)
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\section*{EXAMPLE: USING THE POWER RULE FOR LOGARITHMS TO SIMPLIFY THE LOGARITHM OF A RADICAL EXPRESSION}

Expand \(\log (\sqrt{x})\).

\section*{Answer}
\[
\begin{aligned}
\log (\sqrt{x}) & =\log x^{\left(\frac{1}{2}\right)} \\
& =\frac{1}{2} \log x
\end{aligned}
\]

\section*{TRY IT}

Expand \(\ln \left(\sqrt[3]{x^{2}}\right)\).

An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=355

\section*{Q \& A}

Can we expand \(\ln \left(x^{2}+y^{2}\right)\) ?
No. There is no way to expand the logarithm of a sum or difference inside the argument of the logarithm.

\section*{TIP FOR SUCCESS}

When working with more complicated examples, write your work down step by step to avoid making incorrect assumptions. It's okay to try different rules as you practice creativity as long as you use each rule correctly.

\section*{EXAMPLE: EXPANDING COMPLEX LOGARITHMIC EXPRESSIONS}

Expand \(\log _{6}\left(\frac{64 x^{3}(4 x+1)}{(2 x-1)}\right)\).
Answer
We can expand by applying the product and quotient rules.
\[
\begin{aligned}
& \log _{6}\left(\frac{64 x^{3}(4 x+1)}{(2 x-1)}\right)=\log _{6} 64+\log _{6} x^{3}+\log _{6}(4 x+1) \\
&-\log _{6}(2 x-1) \\
&=\log _{6} 2^{6}+\log _{6} x^{3}+\log _{6}(4 x+1) \\
&-\log _{6}(2 x-1) \\
&=6 \log _{6} 2+3 \log _{6} x+\log _{6}(4 x+1) \\
& \text { Simplify by writing } 64 \text { as } 2^{6} . \\
& \text { Apply the power rule. }
\end{aligned}
\]

\section*{TRY IT}

Expand \(\ln \left(\frac{\sqrt{(x-1)(2 x+1)^{2}}}{\left(x^{2}-9\right)}\right)\).
Answer
\(\frac{1}{2} \ln (x-1)+\ln (2 x+1)-\ln (x+3)-\ln (x-3)\)
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\section*{Applications of Properties of Logarithms}

\title{
HOW TO: GIVEN A SUM, DIFFERENCE, OR PRODUCT OF LOGARITHMS WITH THE SAME BASE, WRITE AN EQUIVALENT EXPRESSION AS A SINGLE LOGARITHM
}
1. Apply the power property first. Identify terms that are products of factors and a logarithm and rewrite each as the logarithm of a power.
2. From left to right, apply the product and quotient properties. Rewrite sums of logarithms as the logarithm of a product and differences of logarithms as the logarithm of a quotient.

\section*{EXAMPLE: USING THE POWER RULE IN REVERSE}

Use the power rule for logs to rewrite \(4 \ln (x)\) as a single logarithm with a leading coefficient of 1 .

\section*{Answer}

Because the logarithm of a power is the product of the exponent times the logarithm of the base, it follows that the product of a number and a logarithm can be written as a power. For the expression \(4 \ln (x)\), we identify the factor, 4 , as the exponent and the argument, \(x\), as the base and rewrite the product as a logarithm of a power:
\(4 \ln (x)=\ln \left(x^{4}\right)\).

\section*{TRY IT}

Use the power rule for logs to rewrite \(2 \log _{3} 4\) as a single logarithm with a leading coefficient of 1 .

\section*{Answer}
\(\log _{3} 16\)

\section*{EXAMPLE: USING THE PRODUCT AND QUOTIENT RULES TO COMBINE LOGARITHMS}

Write \(\log _{3}(5)+\log _{3}(8)-\log _{3}(2)\) as a single logarithm.

\section*{Answer}

From left to right, since we have the addition of two logs, we first use the product rule:
\(\log _{3}(5)+\log _{3}(8)=\log _{3}(5 \cdot 8)=\log _{3}(40)\)
This simplifies our original expression to:
\(\log _{3}(40)-\log _{3}(2)\)
Using the quotient rule:
\(\log _{3}(40)-\log _{3}(2)=\log _{3}\left(\frac{40}{2}\right)=\log _{3}(20)\)

\section*{TRY IT}

Condense \(\log 3-\log 4+\log 5-\log 6\).

\section*{Answer}
\(\log \left(\frac{3.5}{4 \cdot 6}\right)\); can also be written \(\log \left(\frac{5}{8}\right)\) by simplifying the fraction to lowest terms.

\section*{EXAMPLE: CONDENSING COMPLEX LOGARITHMIC EXPRESSIONS}

Condense \(\log _{2}\left(x^{2}\right)+\frac{1}{2} \log _{2}(x-1)-3 \log _{2}\left((x+3)^{2}\right)\).

\section*{Answer}

We apply the power rule first:
\[
\log _{2}\left(x^{2}\right)+\frac{1}{2} \log _{2}(x-1)-3 \log _{2}\left((x+3)^{2}\right)=\log _{2}\left(x^{2}\right)+\log _{2}(\sqrt{x-1})-\log _{2}\left((x+3)^{6}\right)
\]

From left to right, since we have the addition of two logs, we apply the product rule to the sum:
\[
\log _{2}\left(x^{2}\right)+\log _{2}(\sqrt{x-1})-\log _{2}\left((x+3)^{6}\right)=\log _{2}\left(x^{2} \sqrt{x-1}\right)-\log _{2}\left((x+3)^{6}\right)
\]

Finally we apply the quotient rule to the difference:
\[
\log _{2}\left(x^{2} \sqrt{x-1}\right)-\log _{2}\left((x+3)^{6}\right)=\log _{2} \frac{x^{2} \sqrt{x-1}}{(x+3)^{6}}
\]

\section*{EXAMPLE: REWRITING AS A SINGLE LOGARITHM}

Rewrite \(2 \log x-4 \log (x+5)+\frac{1}{x} \log (3 x+5)\) as a single logarithm.

\section*{Answer}

We apply the power rule first:
\(2 \log x-4 \log (x+5)+\frac{1}{x} \log (3 x+5)=\log \left(x^{2}\right)-\log \left((x+5)^{4}\right)+\log \left((3 x+5)^{x^{-1}}\right)\)
From left to right, since we have the difference of two logs, we apply the quotient rule to the difference:
\(\log \left(x^{2}\right)-\log \left((x+5)^{4}\right)+\log \left((3 x+5)^{x^{-1}}\right)=\log \left(\frac{x^{2}}{(x+5)^{4}}\right)+\log \left((3 x+5)^{x^{-1}}\right)\)
Finally we apply the product rule to the sum:
\(\log \left(\frac{x^{2}}{(x+5)^{4}}\right)+\log \left((3 x+5)^{x^{-1}}\right)=\log \left(\frac{x^{2}(3 x+5)^{x^{-1}}}{(x+5)^{4}}\right)\)

\section*{TRY IT}

Rewrite \(\log (5)+0.5 \log (x)-\log (7 x-1)+3 \log (x-1)\) as a single logarithm.

\section*{Answer}
\(\log \left(\frac{5(x-1)^{3} \sqrt{x}}{(7 x-1)}\right)\)
Condense \(4(3 \log (x)+\log (x+5)-\log (2 x+3))\).

\section*{Answer}
\(\log \frac{x^{12}(x+5)^{4}}{(2 x+3)^{4}}\); this answer could also be written as \(\log \left(\frac{x^{3}(x+5)}{(2 x+3)}\right)^{4}\).
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In chemistry, pH is a measure of how acidic or basic a liquid is. It is essentially a measure of the concentration of hydrogen ions in a solution. The scale for measuring pH is standardized across the world, the scientific community having agreed upon its values and methods for acquiring them.

Measurements of pH can help scientists, farmers, doctors, and engineers solve problems and identify sources of problems.
pH is defined as the decimal logarithm of the reciprocal of the hydrogen ion activity, \(a_{H}+\), in a solution.
\(\mathrm{pH}=-\log _{10}\left(a_{\mathrm{H}^{+}}\right)=\log _{10}\left(\frac{1}{a_{\mathrm{H}^{+}}}\right)\)
For example, a solution with a hydrogen ion activity of \(2.5 \times 10^{-6}\) (at that level essentially the number of moles of hydrogen ions per liter of solution) has a pH of \(\log _{10}\left(\frac{1}{2.5 \times 10^{-6}}\right)=5.6\)

In the next examples, we will solve some problems involving pH .

\section*{EXAMPLE: APPLYING PROPERTIES OF LOGS}

Recall that, in chemistry, \(\mathrm{pH}=-\log \left[H^{+}\right]\). If the concentration of hydrogen ions in a liquid is doubled, what is the effect on pH ?

\section*{Answer}

Suppose \(C\) is the original concentration of hydrogen ions and \(P\) is the original pH of the liquid. Then \(\mathrm{P}=-\log (C)\). If the concentration is doubled, the new concentration is \(2 C\). Then the pH of the new liquid is \(\mathrm{pH}=-\log (2 C)\)
Using the product rule of logs
\(\mathrm{pH}=-\log (2 C)=-(\log (2)+\log (C))=-\log (2)-\log (C)\)
Since \(P=-\log (C)\), the new pH is
\(\mathrm{pH}=P-\log (2) \approx P-0.301\)
When the concentration of hydrogen ions is doubled, the pH decreases by about 0.301 .

\section*{TRY IT}

How does the pH change when the concentration of positive hydrogen ions is decreased by half?

\section*{Answer}

The pH increases by about 0.301.
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\section*{CHANGE OF BASE FORMULA}

\section*{LEARNING OUTCOMES}
- Rewrite logarithms with a different base using the change of base formula.

\section*{Using the Change-of-Base Formula for Logarithms}

Most calculators can only evaluate common and natural logs. In order to evaluate logarithms with a base other than 10 or \(e\), we use the change-of-base formula to rewrite the logarithm as the quotient of logarithms of any other base; when using a calculator, we would change them to common or natural logs.

To derive the change-of-base formula, we use the one-to-one property and power rule for logarithms.
Given any positive real numbers \(M, b\), and \(n\), where \(n \neq 1\) and \(b \neq 1\), we show
\(\log _{b} M=\frac{\log _{n} M}{\log _{n} b}\)
Let \(y=\log _{b} M\). Converting to exponential form, we obtain \(b^{y}=M\). It follows that:
\(\log _{n}\left(b^{y}\right)=\log _{n} M\) Apply the one-to-one property.
\(y \log _{n} b \quad=\log _{n} M \quad\) Apply the power rule for logarithms.
\(y \quad=\frac{\log _{n} M}{\log _{n} b} \quad\) Isolate \(y\).
\(\log _{b} M \quad=\frac{\log _{n} M}{\log _{n} b} \quad\) Substitute for \(y\).

\section*{APPLYING THE ONE-TO-ONE PROPERTY OF LOGARITHMS AND EXPONENTS}

In the demonstration above, deriving the change-of-base formula from the definition of the logarithm, we applied the one-to-one property of logarithms to
\(b^{y}=M\)
to obtain
\(\log _{n} b^{y}=\log _{n} M\).
The application of the property is sometimes referred to as a property of equality with regard to taking the \(\log\) base \(n\) on both sides, where \(n\) is any real number.
Recall that the one-to-one property states that \(\log _{b} M=\log _{b} N \Leftrightarrow M=N\). We take the double-headed arrow to mean if and only if and use it when the equality can be implied in either direction. Therefore, it is just as appropriate to state that
\(M=N \Leftrightarrow \log _{b} M=\log _{b} N\), which is what we did in the derivation above. That is,
\(b^{y}=M \Leftrightarrow \log _{n} b^{y}=\log _{n} M\).
The same idea applies to the one-to-one property of exponents. Since \(a^{m}=a^{n} \Leftrightarrow m=n\), it is also true that given \(m=n\), we can write \(q^{m}=q^{n}\) for \(q\), any real number.
This idea leads to important techniques for solving logarithmic and exponential equations. Keep it in mind as you work through the rest of the module.

For example, to evaluate \(\log _{5} 36\) using a calculator, we must first rewrite the expression as a quotient of common or natural logs. We will use the common log.
\[
\begin{aligned}
\log _{5} 36 & =\frac{\log (36)}{\log (5)} \quad \text { Apply the change of base formula using base } 10 . \\
& \approx 2.2266 \text { Use a calculator to evaluate to } 4 \text { decimal places. }
\end{aligned}
\]

\section*{A GENERAL NOTE: THE CHANGE-OF-BASE FORMULA}

The change-of-base formula can be used to evaluate a logarithm with any base.
For any positive real numbers \(M, b\), and \(n\), where \(n \neq 1\) and \(b \neq 1\),
\(\log _{b} M=\frac{\log _{n} M}{\log _{n} b}\).
It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.
\(\log _{b} M=\frac{\ln M}{\ln b}\)
and
\(\log _{b} M=\frac{\log M}{\log b}\)

\section*{HOW TO: GIVEN A LOGARITHM OF THE FORM \(\log _{b} M\), USE THE CHANGE-OF-BASE FORMULA TO REWRITE IT AS A QUOTIENT OF LOGS WITH ANY POSITIVE BASE \(n\), WHERE \(n \neq 1\)}
1. Determine the new base \(n\), remembering that the common \(\log , \log (x)\), has base 10 and the natural \(\log , \ln (x)\), has base \(e\).
2. Rewrite the log as a quotient using the change-of-base formula:
- The numerator of the quotient will be a logarithm with base \(n\) and argument \(M\).
- The denominator of the quotient will be a logarithm with base \(n\) and argument \(b\).

\section*{EXAMPLE: CHANGING LOGARITHMIC EXPRESSIONS TO EXPRESSIONS INVOLVING ONLY NATURAL LOGS}

Change \(\log _{5} 3\) to a quotient of natural logarithms.

\section*{Answer}

Because we will be expressing \(\log _{5} 3\) as a quotient of natural logarithms, the new base \(n=e\).
We rewrite the log as a quotient using the change-of-base formula. The numerator of the quotient will be the natural log with argument 3 . The denominator of the quotient will be the natural log with argument 5 .
\[
\begin{aligned}
\log _{b} M & =\frac{\ln M}{\ln b} \\
\log _{5} 3 & =\frac{\ln 3}{\ln 5}
\end{aligned}
\]

Even if your calculator has a logarithm function for bases other than 10 or \(e\), you should become familiar with the change-of-base formula. Being able to manipulate formulas by hand is a useful skill in any quantitative or STEM-related field.

\section*{TRY IT}

Change \(\log _{0.5} 8\) to a quotient of natural logarithms.
Answer
\(\frac{\ln 8}{\ln 0.5}\)
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\section*{Q \& A}

Can we change common logarithms to natural logarithms?
Yes. Remember that \(\log 9\) means \(\log _{10} 9\). So, \(\log 9=\frac{\ln 9}{\ln 10}\).

\section*{EXAMPLE: USING THE CHANGE-OF-BASE FORMULA WITH A CALCULATOR}

Evaluate \(\log _{2}(10)\) using the change-of-base formula with a calculator.

\section*{Answer}

According to the change-of-base formula, we can rewrite the log base 2 as a logarithm of any other base. Since our calculators can evaluate the natural log, we might choose to use the natural logarithm which is the \(\log\) base \(e\).
\(\log _{2} 10=\frac{\ln 10}{\ln 2} \quad\) Apply the change of base formula using base \(e\).
\(\approx 3.3219\) Use a calculator to evaluate to 4 decimal places.

\section*{TRY IT}

Evaluate \(\log _{5}\) (100) using the change-of-base formula.

\section*{Answer}
\[
\frac{\ln 100}{\ln 5} \approx \frac{4.6051}{1.6094}=2.861
\]

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\section*{TRY IT}

The first graphing calculators were programmed to only handle logarithms with base 10. One clever way to create the graph of a logarithm with a different base was to change the base of the logarithm using the
principles from this section.
Use an online graphing tool to plot \(f(x)=\frac{\log _{10} x}{\log _{10} 2}\).
Follow these steps to see a clever way to graph a logarithmic function with base other than 10 on a graphing tool that only knows base 10.
- Enter the function \(g(x)=\log _{2} x\)
- Can you tell the difference between the graph of this function and the graph of \(f(x)\) ? Explain what you think is happening.
- Your challenge is to write two new functions \(h(x)\), and \(k(x)\) that include a slider so you can change the base of the functions. Remember that there are restrictions on what values the base of a logarithm can take. You can click on the endpoints of the slider to change the input values.

\section*{SUMMARY: PROPERTIES OF LOGARITHMS}

\section*{Key Equations}
\begin{tabular}{|l|l|}
\hline The Product Rule for Logarithms & \(\log _{b}(M N)=\log _{b}(M)+\log _{b}(N)\) \\
\hline The Quotient Rule for Logarithms & \(\log _{b}\left(\frac{M}{N}\right)=\log _{b} M-\log _{b} N\) \\
\hline The Power Rule for Logarithms & \(\log _{b}\left(M^{n}\right)=n \log _{b} M\) \\
\hline The Change-of-Base Formula & \(\log _{b} M=\frac{\log _{n} M}{\log _{n} b} n>0, n \neq 1, b \neq 1\) \\
\hline
\end{tabular}

\section*{Key Concepts}
- We can use the product rule of logarithms to rewrite the log of a product as a sum of logarithms.
- We can use the quotient rule of logarithms to rewrite the log of a quotient as a difference of logarithms.
- We can use the power rule for logarithms to rewrite the log of a power as the product of the exponent and the log of its base.
- We can use the product rule, quotient rule, and power rule together to combine or expand a logarithm with a complex input.
- The rules of logarithms can also be used to condense sums, differences, and products with the same base as a single logarithm.
- We can convert a logarithm with any base to a quotient of logarithms with any other base using the change-of-base formula.
- The change-of-base formula is often used to rewrite a logarithm with a base other than 10 or \(e\) as the quotient of natural or common logs. A calculator can then be used to evaluate it.

\section*{Glossary}
change-of-base formula a formula for converting a logarithm with any base to a quotient of logarithms with any other base
power rule for logarithms a rule of logarithms that states that the log of a power is equal to the product of the exponent and the log of its base
product rule for logarithms a rule of logarithms that states that the log of a product is equal to a sum of logarithms
quotient rule for logarithms a rule of logarithms that states that the log of a quotient is equal to a difference of logarithms

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\section*{INTRODUCTION TO EXPONENTIAL AND LOGARITHMIC EQUATIONS}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Solve an exponential equation with a common base.
- Rewrite an exponential equation so all terms have a common base then solve.
- Recognize when an exponential equation does not have a solution.
- Use logarithms to solve exponential equations.
- Solve a logarithmic equation algebraically.
- Solve a logarithmic equation graphically.
- Use the one-to-one property of logarithms to solve a logarithmic equation.
- Solve a radioactive decay problem.

In 1859, an Australian landowner named Thomas Austin released 24 rabbits into the wild for hunting. Because Australia had few predators and ample food, the rabbit population exploded. In fewer than ten years, the rabbit population numbered in the millions.

Uncontrolled population growth, as in the wild rabbits in Australia, can be modeled with exponential functions. Equations resulting from those exponential functions can be solved to analyze and make predictions about exponential growth. In this section we will learn techniques for solving exponential and logarithmic equations.

\section*{EXPONENTIAL EQUATIONS}

\section*{LEARNING OUTCOMES}
- Solve an exponential equation with a common base.
- Rewrite an exponential equation so all terms have a common base then solve.
- Recognize when an exponential equation does not have a solution.
- Use logarithms to solve exponential equations.

\section*{Exponential Equations}

The first technique we will introduce for solving exponential equations involves two functions with like bases. The one-to-one property of exponential functions tells us that, for any real numbers \(b, S\), and \(T\), where \(b>0, b \neq 1, b^{S}=b^{T}\) if and only if \(S=T\).

In other words, when an exponential equation has the same base on each side, the exponents must be equal. This also applies when the exponents are algebraic expressions. Therefore, we can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.

For example, consider the equation \(3^{4 x-7}=\frac{3^{2 x}}{3}\). To solve for \(x\), we use the division property of exponents to rewrite the right side so that both sides have the common base 3 . Then we apply the one-to-one property of exponents by setting the exponents equal to one another and solving for \(x\) :
\[
\begin{array}{lll}
3^{4 x-7} & =\frac{3^{2 x}}{3} & \\
3^{4 x-7} & =\frac{3^{2 x}}{3^{1}} & \\
\text { Rewrite } 3 \text { as } 3^{1} . \\
3^{4 x-7} & =3^{2 x-1} & \text { Use the division property of exponents. } \\
4 x-7 & =2 x-1 & \text { Apply the one-to-one property of exponents. } \\
2 x & =6 & \\
\text { Subtract } 2 x \text { and add } 7 \text { to both sides. } \\
x & =3 & \text { Divide by } 3 .
\end{array}
\]

\section*{A GENERAL NOTE: USING THE ONE-TO-ONE PROPERTY OF EXPONENTIAL FUNCTIONS TO SOLVE EXPONENTIAL EQUATIONS}

For any algebraic expressions \(S\) and \(T\), and any positive real number \(b \neq 1\), \(b^{S}=b^{T}\) if and only if \(S=T\)

\section*{HOW TO: GIVEN AN EXPONENTIAL EQUATION OF THE FORM \(b^{S}=b^{T}\), WHERE S AND TARE ALGEBRAIC EXPRESSIONS WITH AN UNKNOWN, SOLVE FOR THE UNKNOWN}
1. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form \(b^{S}=b^{T}\).
2. Use the one-to-one property to set the exponents equal to each other.

3 . Solve the resulting equation, \(S=T\), for the unknown.

\section*{EXAMPLE: SOLVING AN EXPONENTIAL EQUATION WITH A COMMON BASE}

Solve \(2^{x-1}=2^{2 x-4}\).

\section*{Answer}
\(2^{x-1}=2^{2 x-4} \quad\) The common base is 2.
\(x-1=2 x-4\) By the one-to-one property the exponents must be equal.
\(x=3 \quad\) Solve for \(x\).

\section*{TRY IT}

Solve \(5^{2 x}=5^{3 x+2}\).
Answer
\(x=-2\)
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\section*{Rewriting Equations So All Powers Have the Same Base}

Sometimes the common base for an exponential equation is not explicitly shown. In these cases we simply rewrite the terms in the equation as powers with a common base and solve using the one-to-one property.

For example, consider the equation \(256=4^{x-5}\). We can rewrite both sides of this equation as a power of 2 . Then we apply the rules of exponents along with the one-to-one property to solve for \(x\) :
\[
\begin{array}{ll}
256=4^{x-5} & \\
2^{8}=\left(2^{2}\right)^{x-5} & \text { Rewrite each side as a power with base } 2 . \\
2^{8}=2^{2 x-10} & \text { To take a power of a power, multiply the exponents } \\
8=2 x-10 & \text { Apply the one-to-one property of exponents. } \\
18=2 x & \text { Add } 10 \text { to both sides } . \\
x=9 & \text { Divide by } 2 .
\end{array}
\]

\section*{HOW TO: GIVEN AN EXPONENTIAL EQUATION WITH UNLIKE BASES, USE THE ONE-TO-ONE PROPERTY TO SOLVE IT}
1. Rewrite each side in the equation as a power with a common base.
2. Use the rules of exponents to simplify, if necessary, so that the resulting equation has the form \(b^{S}=b^{T}\).
3. Use the one-to-one property to set the exponents equal to each other.
4. Solve the resulting equation, \(S=T\), for the unknown.

\section*{EXAMPLE: SOLVING EQUATIONS BY REWRITING THEM TO HAVE A COMMON BASE}

Solve \(8^{x+2}=16^{x+1}\).

\section*{Answer}
\[
8^{x+2}=16^{x+1}
\]
\(\left(2^{3}\right)^{x+2}=\left(2^{4}\right)^{x+1} \quad\) Write 8 and 16 as powers of 2.
\(2^{3 x+6}=2^{4 x+4} \quad\) To take a power of a power, multiply the exponents.
\(3 x+6=4 x+4 \quad\) Use the one-to-one property to set the exponents equal to each other.
\(x=2 \quad\) Solve for \(x\).

\section*{TRY IT}

Solve \(5^{2 x}=25^{3 x+2}\).
Answer
\(x=-1\)
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\section*{EXAMPLE: SOLVING EQUATIONS BY REWRITING ROOTS WITH FRACTIONAL EXPONENTS TO HAVE A COMMON BASE}

Solve \(2^{5 x}=\sqrt{2}\).

\section*{Answer}
\(2^{5 x}=2^{\frac{1}{2}} \quad\) Write the square root of 2 as a power of 2.
\(5 x=\frac{1}{2} \quad\) Use the one-to-one property.
\(x=\frac{1}{10} \quad\) Solve for \(x\).

\section*{TRY IT}

Solve \(5^{x}=\sqrt{5}\).
Answer
\(x=\frac{1}{2}\)
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=359

\section*{Q \& A}

Do all exponential equations have a solution? If not, how can we tell if there is a solution during the problem-solving process?
No. Recall that the range of an exponential function is always positive. While solving the equation we may obtain an expression that is undefined.

\section*{Using Logarithms to Solve Exponential Equations}

Sometimes the terms of an exponential equation cannot be rewritten with a common base. In these cases, we solve by taking the logarithm of each side. Recall that since \(\log (a)=\log (b)\) is equivalent to \(a=b\), we may apply logarithms with the same base to both sides of an exponential equation.

\section*{Equations Containing \(e\)}

One common type of exponential equations are those with base \(e\). This constant occurs again and again in nature, mathematics, science, engineering, and finance. When we have an equation with a base e on either side, we can use the natural logarithm to solve it.

\section*{Extraneous Solutions}

Sometimes the methods used to solve an equation introduce an extraneous solution, which is a solution that is correct algebraically but does not satisfy the conditions of the original equation. One such situation arises in solving when taking the logarithm of both sides of the equation. In such cases, remember that the argument of the logarithm must be positive. If the number we are evaluating in a logarithm function is

\section*{EXAMPLE: DETERMINE WHEN AN EQUATION HAS NO SOLUTION}

Solve \(3^{x+1}=-2\).

\section*{Answer}

This equation has no solution. There is no real value of \(x\) that will make the equation a true statement because any power of a positive number is positive.

\section*{Analysis of the Solution}

The figure below shows that the two graphs do not cross so the left side of the equation is never equal to the right side of the equation. Thus the equation has no solution.


\section*{TRY IT}

Solve \(2^{x}=-100\).

\section*{Answer}

The equation has no solution.
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\section*{APPLYING THE ONE-TO-ONE PROPERTY OF LOGARITHMS}

Note in the paragraph above the reiteration of the one-to-one property of logarithms. When applying logarithms with the same base to both sides of an exponential equation, we often use the common logarithm, log or the natural logarithm, \(\ln\). The choice is yours which to use in most situations, but if either base in a given exponential equation is 10 , use \(\log\) or if the base is \(e\), use \(\ln\) to take advantage of the identity property of logarithms.
negative, there is no output.

\section*{HOW TO: GIVEN AN EXPONENTIAL EQUATION WHERE A COMMON BASE CANNOT BE FOUND, SOLVE FOR THE UNKNOWN}
1. Apply the logarithm to both sides of the equation.
- If one of the terms in the equation has base 10 , use the common logarithm.
- If none of the terms in the equation has base 10 , use the natural logarithm.
2. Use the rules of logarithms to solve for the unknown.

\section*{EXAMPLE: SOLVING AN EQUATION CONTAINING POWERS OF DIFFERENT BASES}

Solve \(5^{x+2}=4^{x}\).

\section*{Answer}
\(5^{x+2}=4^{x} \quad\) There is no easy way to get the powers to have the same base.
\(\ln 5^{x+2}=\ln 4^{x} \quad\) Take \(\ln\) of both sides.
\((x+2) \ln 5=x \ln 4 \quad\) Use the power rule for logs.
\(x \ln 5+2 \ln 5=x \ln 4 \quad\) Use the distributive property.
\(x \ln 5-x \ln 4=-2 \ln 5 \quad\) Get terms containing \(x\) on one side, terms without \(x\) on the other.
\(x(\ln 5-\ln 4)=-2 \ln 5 \quad\) On the left hand side, factor out \(x\).
\(x \ln \left(\frac{5}{4}\right)=\ln \left(\frac{1}{25}\right) \quad\) Use the properties of logs.
\(x=\frac{\ln \left(\frac{1}{25}\right)}{\ln \left(\frac{5}{4}\right)} \quad\) Divide by the coefficient of \(x\).

\section*{TRY IT}

Solve \(2^{x}=3^{x+1}\).
Answer
\(x=\frac{\ln 3}{\ln \left(\frac{2}{3}\right)}\)
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\section*{Q \& A}

Is there any way to solve \(2^{x}=3^{x}\) ?
Yes. The solution is \(x=0\).

\section*{HOW TO: GIVEN AN EQUATION OF THE FORM \(y=A e^{k t}\), SOLVE FOR \(t\)}
1. Divide both sides of the equation by \(A\).
2. Apply the natural logarithm to both sides of the equation.
3. Divide both sides of the equation by \(k\).

\section*{EXAMPLE: SOLVE AN EQUATION OF THE FORM \(y=A e^{k t}\)}

Solve \(100=20 e^{2 t}\).

\section*{Answer}
\(100=20 e^{2 t}\)
\(5=e^{2 t} \quad\) Divide by the coefficient 20.
\(\ln 5=\ln e^{2 t} \quad\) Take \(\ln\) of both sides.
\(\ln 5=2 t\)
Use the fact that \(\ln\)
\((x)\) and \(e^{x}\) are inverse functions.
\(t \quad=\frac{\ln 5}{2} \quad\) Divide by the coefficient of \(t\).

\section*{Analysis of the Solution}

Using laws of logs, we can also write this answer in the form \(t=\ln \sqrt{5}\). If we want a decimal approximation of the answer, then we use a calculator.

\section*{TIP FOR SUCCESS}

Just as you have done when solving various types of equations, isolate the term containing the variable for which you are solving before applying any properties of equality or inverse operations. That's why, in the example above, you must divide away the \(A\) first.

\section*{TRY IT}

Solve \(3 e^{0.5 t}=11\).
Answer
\(t=2 \ln \left(\frac{11}{3}\right)\) or \(\ln \left(\frac{11}{3}\right)^{2}\)
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\section*{Q \& A}

Does every equation of the form \(y=A e^{k t}\) have a solution?

No. There is a solution when \(k \neq 0\), and when \(y\) and [latex]A[/latex] are either both 0 or neither 0 and they have the same sign. An example of an equation with this form that has no solution is \(2=-3 e^{t}\).

\section*{EXAMPLE: SOLVING AN EQUATION THAT CAN BE SIMPLIFIED TO THE FORM \(y=A e^{k t}\)}

Solve \(4 e^{2 x}+5=12\).
Answer
\(4 e^{2 x}+5=12\)
\(4 e^{2 x}=7 \quad\) Subtract 5 from both sides.
\(e^{2 x}=\frac{7}{4} \quad\) Divide both sides by 4.
\(2 x=\ln \left(\frac{7}{4}\right) \quad\) Take \(\ln\) of both sides.
\(x=\frac{1}{2} \ln \left(\frac{7}{4}\right) \quad\) Solve for \(x\).

\section*{TRY IT}

Solve \(3+e^{2 t}=7 e^{2 t}\).
Answer
\(t=\ln \left(\frac{1}{\sqrt{2}}\right)=-\frac{1}{2} \ln (2)\)
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=359

\section*{TIP FOR SUCCESS}

Always check solutions to logarithmic equations to see if they satisfy the domain restriction on the logarithm.

\section*{EXAMPLE: SOLVING EXPONENTIAL FUNCTIONS IN QUADRATIC FORM}

Solve \(e^{2 x}-e^{x}=56\).
Answer
\(e^{2 x}-e^{x}=56\)
\(e^{2 x}-e^{x}-56=0 \quad\) Get one side of the equation equal to zero.
\(\left(e^{x}+7\right)\left(e^{x}-8\right)=0 \quad\) Factor by the FOIL method.
\(e^{x}+7=0\) or \(e^{x}-8=0\) If a product is zero, then one factor must be zero.
\(e^{x}=-7\) or \(\mathrm{e}^{x}=8 \quad\) Isolate the exponentials.
\(e^{x}=8 \quad\) Reject the equation in which the power equals a negative number.
\(x=\ln 8 \quad\) Solve the equation in which the power equals a positive number.

\section*{Analysis of the Solution}

When we plan to use factoring to solve a problem, we always get zero on one side of the equation because zero has the unique property that when a product is zero, one or both of the factors must be zero. We reject the equation \(e^{x}=-7\) because a positive number never equals a negative number. The solution \(x=\ln (-7)\) is not a real number and in the real number system, this solution is rejected as an extraneous solution.

\section*{TRY IT}

Solve \(e^{2 x}=e^{x}+2\).
Answer
\(x=\ln 2\)
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\section*{Q \& A}

\section*{Does every logarithmic equation have a solution?}

No. Keep in mind that we can only apply the logarithm to a positive number. Always check for extraneous solutions.
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\section*{LOGARITHMIC EQUATIONS}

\section*{LEARNING OUTCOMES}
- Solve a logarithmic equation algebraically.
- Solve a logarithmic equation graphically.
- Use the one-to-one property of logarithms to solve a logarithmic equation.
- Solve a radioactive decay problem.

\section*{Logarithmic Equations}

We have already seen that every logarithmic equation \(\log _{b}(x)=y\) is equal to the exponential equation \(b^{y}=x\). We can use this fact, along with the rules of logarithms, to solve logarithmic equations where the argument is an algebraic expression.

For example, consider the equation \(\log _{2}(2)+\log _{2}(3 x-5)=3\). To solve this equation, we can use rules of logarithms to rewrite the left side as a single log and then apply the definition of logs to solve for \(x\) :
\begin{tabular}{ll}
\(\log _{2}(2)+\log _{2}(3 x-5)=3\) & \\
\(\log _{2}(2(3 x-5))=3\) & Apply the product rule of logarithms. \\
\(\log _{2}(6 x-10)=3\) & Distribute. \\
\(2^{3}=6 x-10\) & Convert to exponential form. \\
\(8=6 x-10\) & Calculate \(2^{3}\). \\
\(18=6 x\) & Add 10 to both sides. \\
\(x=3\) & Divide both sides by 6.
\end{tabular}

\section*{A GENERAL NOTE: USING THE DEFINITION OF A LOGARITHM TO SOLVE LOGARITHMIC EQUATIONS}

For any algebraic expression \(S\) and real numbers \(b\) and \(c\), where \(b>0, b \neq 1\), \(\log _{b}(S)=c\) if and only if \(b^{c}=S\)

\section*{Using the One-to-One Property of Logarithms to Solve Logarithmic Equations}

As with exponential equations, we can use the one-to-one property to solve logarithmic equations. The one-to-one property of logarithmic functions tells us that, for any real numbers \(x>0, S>0, T>0\) and any positive real number \(b\), where \(b \neq 1\),
\(\log _{b} S=\log _{b} T\) if and only if \(S=T\)

\section*{EXAMPLE: USING ALGEBRA TO SOLVE A LOGARITHMIC EQUATION}

Solve \(2 \ln x+3=7\).

\section*{Answer}
\(2 \ln x+3=7\)
\(2 \ln x=4 \quad\) Subtract 3 from both sides.
\(\ln x=2 \quad\) Divide both sides by 2 .
\(x=e^{2} \quad\) Rewrite in exponential form.

\section*{TIP FOR SUCCESS}

As was the case when using the properties and rules of exponents and logarithms to rewrite expressions containing them, there can be more than one good way to solve a logarithmic equation. It is good practice to follow the examples given for each of the situations in this section, but you should think about alternative ways to creatively and correctly apply the properties and rules.

\section*{TRY IT}

Solve \(6+\ln x=10\).

\section*{Answer}
\(x=e^{4}\)
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\section*{EXAMPLE: USING ALGEBRA BEFORE AND AFTER USING THE DEFINITION OF THE NATURAL LOGARITHM}

Solve \(2 \ln (6 x)=7\).

\section*{Answer}
\(2 \ln (6 x)=7\)
\(\ln (6 x)=\frac{7}{2} \quad\) Divide both sides by 2.
\(6 x=e^{\left(\frac{7}{2}\right)} \quad\) Use the definition of \(\ln\).
\(x=\frac{1}{6} e^{\left(\frac{7}{2}\right)} \quad\) Divide both sides by 6.

Solve \(2 \ln (x+1)=10\).

Answer
\(x=e^{5}-1\)
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=360

\section*{EXAMPLE: USING A GRAPH TO UNDERSTAND THE SOLUTION TO A LOGARITHMIC EQUATION}

Solve \(\ln x=3\).

\section*{Answer}
\(\ln x=3\)
\(x=e^{3} \quad\) Use the definition of \(\ln\).

Below is a graph of the equation. On the graph the \(x\)-coordinate of the point where the two graphs intersect is close to 20 . In other words \(e^{3} \approx 20\). A calculator gives a better approximation: \(e^{3} \approx 20.0855\).


The graphs of \(y=\ln x\) and \(y=3\) cross at the point \(\left(e^{3}, 3\right)\) which is approximately \((20.0855,3)\).

\section*{TRY IT}

Use a graphing calculator to estimate the approximate solution to the logarithmic equation \(2^{x}=1000\) to 2 decimal places.
Answer
\(x \approx 9.97\)
For example,

If \(\log _{2}(x-1)=\log _{2}(8)\), then \(x-1=8\)
So if \(x-1=8\), then we can solve for \(x\) and we get \(x=9\). To check, we can substitute \(x=9\) into the original equation: \(\log _{2}(9-1)=\log _{2}(8)=3\). In other words, when a logarithmic equation has the same base on each side, the arguments must be equal. This also applies when the arguments are algebraic expressions. Therefore, when given an equation with logs of the same base on each side, we can use rules of logarithms to rewrite each side as a single logarithm. Then we use the fact that logarithmic functions are one-to-one to set the arguments equal to one another and solve for the unknown.

For example, consider the equation \(\log (3 x-2)-\log (2)=\log (x+4)\). To solve this equation, we can use the rules of logarithms to rewrite the left side as a single logarithm and then apply the one-to-one property to solve for \(x\) :
\[
\begin{array}{ll}
\log (3 x-2)-\log (2)=\log (x+4) & \\
\log \left(\frac{3 x-2}{2}\right)=\log (x+4) & \text { Apply the quotient rule of logarithms. } \\
\frac{3 x-2}{2}=x+4 & \text { Apply the one-to-one property. } \\
3 x-2=2 x+8 & \text { Multiply both sides of the equation by } 2 . \\
x=10 & \text { Subtract } 2 x \text { and add } 2 .
\end{array}
\]

To check the result, substitute \(x=10\) into \(\log (3 x-2)-\log (2)=\log (x+4)\).
\(\log (3(10)-2)-\log (2)=\log ((10)+4)\)
\(\log (28)-\log (2)=\log (14)\)
\(\log \left(\frac{28}{2}\right)=\log (14)\)
The solution checks.

\section*{A GENERAL NOTE: USING THE ONE-TO-ONE PROPERTY OF LOGARITHMS TO SOLVE LOGARITHMIC EQUATIONS}

For any algebraic expressions \(S\) and \(T\) and any positive real number \(b\), where \(b \neq 1\), \(\log _{b} S=\log _{b} T\) if and only if \(S=T\)
Note, when solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

\section*{HOW TO: GIVEN AN EQUATION CONTAINING LOGARITHMS, SOLVE IT USING THE ONE-TO-ONE PROPERTY}
1. Use the rules of logarithms to combine like terms, if necessary, so that the resulting equation is of the form \(\log _{b} S=\log _{b} T\).
2. Use the one-to-one property to set the arguments equal to each other.
3. Solve the resulting equation, \(S=T\), for the unknown.

\section*{EXAMPLE: SOLVING AN EQUATION USING THE ONE-TO-ONE PROPERTY OF LOGARITHMS}

Solve \(\ln \left(x^{2}\right)=\ln (2 x+3)\).
Answer
\(\ln \left(x^{2}\right)=\ln (2 x+3)\)
\(x^{2}=2 x+3 \quad\) Use the one-to-one property of the logarithm.
\(x^{2}-2 x-3=0 \quad\) Get zero on one side before factoring.
\((x-3)(x+1)=0 \quad\) Factor using FOIL.
\(x-3=0\) or \(x+1=0\) If a product is zero, one of the factors must be zero.
\(x=3\) or \(x=-1 \quad\) Solve for \(x\).

\section*{Analysis of the Solution}

There are two solutions: \(x=3\) or \(x=-1\). The solution \(x=-1\) is negative, but it checks when substituted into the original equation because the argument of the logarithm function is still positive.

\section*{TRY IT}

Solve \(\ln \left(x^{2}\right)=\ln 1\).
Answer
\(x=1\) or \(x=-1\)
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\section*{SUMMARY: EXPONENTIAL AND LOGARITHMIC EQUATIONS}

One-to-one property for exponential functions

Definition of a logarithm

One-to-one property for logarithmic functions

For any algebraic expressions \(S\) and \(T\) and any positive real number \(b\), where \(b>0, b \neq 1, b^{S}=b^{T}\) if and only if \(S=T\).

For any algebraic expression \(S\) and positive real numbers \(b\) and \(c\), where \(b \neq 1, \log _{b}(S)=c\) if and only if \(b^{c}=S\).

For any algebraic expressions \(S\) and \(T\) and any positive real number \(b\), where \(b \neq 1\), \(\log _{b} S=\log _{b} T\) if and only if \(S=T\).

\section*{Key Concepts}
- We can solve many exponential equations by using the rules of exponents to rewrite each side as a power with the same base. Then we use the fact that exponential functions are one-to-one to set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are explicitly shown as being equal, set the exponents equal to one another and solve for the unknown.
- When we are given an exponential equation where the bases are not explicitly shown as being equal, rewrite each side of the equation as powers of the same base, then set the exponents equal to one another and solve for the unknown.
- When an exponential equation cannot be rewritten with a common base, solve by taking the logarithm of each side.
- We can solve exponential equations with base \(e\) by applying the natural logarithm to both sides because exponential and logarithmic functions are inverses of each other.
- After solving an exponential equation, check each solution in the original equation to find and eliminate any extraneous solutions.
- When given an equation of the form \(\log _{b}(S)=c\), where \(S\) is an algebraic expression, we can use the definition of a logarithm to rewrite the equation as the equivalent exponential equation \(b^{c}=S\) and solve for the unknown.
- We can also use graphing to solve equations of the form \(\log _{b}(S)=c\). We graph both equations \(y=\log _{b}(S)\) and \(y=c\) on the same coordinate plane and identify the solution as the \(x\)-value of the point of intersecting.
- When given an equation of the form \(\log _{b} S=\log _{b} T\), where \(S\) and \(T\) are algebraic expressions, we can use the one-to-one property of logarithms to solve the equation \(S=T\) for the unknown.
- Combining the skills learned in this and previous sections, we can solve equations that model real world situations whether the unknown is in an exponent or in the argument of a logarithm.

\section*{Glossary}

\section*{extraneous solution}
a solution introduced while solving an equation that does not satisfy the conditions of the original equation

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\title{
INTRODUCTION TO EXPONENTIAL AND LOGARITHMIC MODELS
}

\section*{LEARNING OUTCOMES}

By the end of this lesson, you will be able to:
- Graph exponential growth and decay functions.
- Solve problems involving radioactive decay, carbon dating, and half life.
- Use Newton's Law of Cooling.
- Use a logistic growth model.
- Choose an appropriate model for data.
- Use a graphing utility to create an exponential regression from a set of data.


A nuclear research reactor inside the Neely Nuclear Research Center on the Georgia Institute of Technology campus. (credit: Georgia Tech Research Institute)

We have already explored some basic applications of exponential and logarithmic functions. In this section, we explore some important applications in more depth including radioactive isotopes and Newton's Law of Cooling.

\section*{EXPONENTIAL GROWTH AND DECAY}

\section*{LEARNING OUTCOMES}
- Graph exponential growth and decay functions.
- Solve problems involving radioactive decay, carbon dating, and half life.

\section*{Exponential Growth and Decay}

\section*{TIP FOR SUCCESS}

As you learn about modelling exponential growth and decay, recall familiar techniques that have helped you to model situations using other types of functions. The exponential growth and decay models possess input and corresponding output like other functions graphed in the real plane. They also adhere to the characteristic feature of the graphs of all functions; each point (corresponding input and output pairs) contained on the graph of the function satisfies the equation of the function.
You can rely on these features as you work with the form of the function to fill in known quantities from a given situation to solve for unknowns.

In real-world applications, we need to model the behavior of a function. In mathematical modeling, we choose a familiar general function with properties that suggest that it will model the real-world phenomenon we wish to analyze. In the case of rapid growth, we may choose the exponential growth function:
\(y=A_{0} e^{k t}\)
where \(A_{0}\) is equal to the value at time zero, \(e\) is Euler's constant, and \(k\) is a positive constant that determines the rate (percentage) of growth. We may use the exponential growth function in applications involving doubling time, the time it takes for a quantity to double. Such phenomena as wildlife populations, financial investments, biological samples, and natural resources may exhibit growth based on a doubling time. In some applications, however, as we will see when we discuss the logistic equation, the logistic model sometimes fits the data better than the exponential model.

On the other hand, if a quantity is falling rapidly toward zero, without ever reaching zero, then we should probably choose the exponential decay model. Again, we have the form \(y=A_{0} e^{-k t}\) where \(A_{0}\) is the starting value, and \(e\) is Euler's constant. Now \(k\) is a negative constant that determines the rate of decay. We may use the exponential decay model when we are calculating half-life, or the time it takes for a substance to exponentially decay to half of its original quantity. We use half-life in applications involving radioactive isotopes.

In our choice of a function to serve as a mathematical model, we often use data points gathered by careful observation and measurement to construct points on a graph and hope we can recognize the shape of the graph. Exponential growth and decay graphs have a distinctive shape, as we can see in the graphs below. It is important to remember that, although parts of each of the two graphs seem to lie on the \(x\)-axis, they are really a tiny distance above the \(x\)-axis.


A graph showing exponential growth. The equation is \(y=2 e^{3 x}\).


A graph showing exponential decay. The equation is \(y=3 e^{-2 x}\).
Exponential growth and decay often involve very large or very small numbers. To describe these numbers, we often use orders of magnitude. The order of magnitude is the power of ten when the number is expressed in scientific notation with one digit to the left of the decimal. For example, the distance to the nearest star, Proxima Centauri, measured in kilometers, is 40,113,497,200,000 kilometers. Expressed in scientific notation, this is \(4.01134972 \times 10^{13}\). We could describe this number as having order of magnitude \(10^{13}\).

\section*{A GENERAL NOTE: CHARACTERISTICS OF THE EXPONENTIAL FUNCTION \\ \[
y=A_{0} e^{k t}
\]}

An exponential function of the form \(y=A_{0} e^{k t}\) has the following characteristics:
- one-to-one function
- horizontal asymptote: \(y=0\)
- domain: \((-\infty, \infty)\)
- range: \((0, \infty)\)
- x intercept: none
- y-intercept: \(\left(0, A_{0}\right)\)
- increasing if \(k>0\)
- decreasing if \(k<0\)

An exponential function models exponential growth when \(\mathrm{k}>0\) and exponential decay when \(\mathrm{k}<0\).

\section*{TRY IT}

The graphs below include important characteristics of an exponential function that can illustrate whether the function represents growth or decay and the speed at which the function grows or decays.



0 and with the labeled
points ( \(\left.1 / k,\left(A \_0\right) e\right),\left(0, A \_0\right)\), and \(\left(-1 / k,\left(A \_0\right) / e\right)\). The second graph is of when \(k<0\) and the with the labeled points ( \(\left.-1 / k,\left(A \_0\right) e\right)\) ), ( \(0, A \_0\) ), and ( \(1 / k,\left(A \_0\right) / e\) )" width=" 731 " height="337">
Your task is to add the key features from the graphs above to graph made using an online graphing calculator and adjusting the \(a\) and \(k\) values. The function \(f(x)=a \cdot e^{k x}\) will be what you use as your starting point.
Include the following key features:
- The horizontal asymptote
- y-intercept
- The key points labeled in the still graphs above

\section*{Answer}

Below is what the graph would look like in Desmos, but you can use any online graphing tool.


\section*{TIP FOR SUCCESS}

Work through the examples in this section on paper, perhaps more than once or twice, to obtain a thorough understanding of the processes involved.

\section*{Calculating Doubling Time}

For growing quantities, we might want to find out how long it takes for a quantity to double. As we mentioned above, the time it takes for a quantity to double is called the doubling time.

Given the basic exponential growth equation \(A=A_{0} e^{k t}\), doubling time can be found by solving for when the original quantity has doubled, that is, by solving \(2 A_{0}=A_{0} e^{k t}\).

The formula is derived as follows:
\(2 A_{0}=A_{0} e^{k t}\)
\(2=e^{k t} \quad\) Divide both sides by \(A_{0}\).
\(\ln 2=k t \quad\) Take the natural logarithm of both sides.
\(t=\frac{\ln 2}{k} \quad\) Divide by the coefficient of \(t\).
Thus the doubling time is
\(t=\frac{\ln 2}{k}\)

\section*{Half-Life}

\section*{EXAMPLE: GRAPHING EXPONENTIAL GROWTH}

A population of bacteria doubles every hour. If the culture started with 10 bacteria, graph the population as a function of time.
Answer
When an amount grows at a fixed percent per unit time, the growth is exponential. To find \(A_{0}\) we use the fact that \(A_{0}\) is the amount at time zero, so \(A_{0}=10\). To find \(k\), use the fact that after one hour \((t=1)\) the population doubles from 10 to 20 . The formula is derived as follows:
\[
20=10 e^{k \cdot 1}
\]
\(2=e^{k} \quad\) Divide both sides by 10
\(\ln 2=k \quad\) Take the natural logarithm of both sides
so \(k=\ln (2)\). Thus the equation we want to graph is \(y=10 e^{(\ln 2) t}=10\left(e^{\ln 2}\right)^{t}=10 \cdot 2^{t}\). The graph is shown below.


The graph of \(y=10 e^{(\ln 2) t}\).

\section*{Analysis of the Solution}

The population of bacteria after ten hours is 10,240 . We could describe this amount as being of the order of magnitude \(10^{4}\). The population of bacteria after twenty hours is \(10,485,760\) which is of the order of magnitude \(10^{7}\), so we could say that the population has increased by three orders of magnitude in ten hours.

\section*{EXAMPLE: FINDING A FUNCTION THAT DESCRIBES EXPONENTIAL GROWTH}

According to Moore's Law, the doubling time for the number of transistors that can be put on a computer chip is approximately two years. Give a function that describes this behavior.

\section*{Answer}

The formula is derived as follows:
\(t=\frac{\ln 2}{k} \quad\) The doubling time formula.
\(2=\frac{\ln 2}{k} \quad\) Use a doubling time of two years.
\(k=\frac{\ln 2}{2} \quad\) Multiply by \(k\) and divide by 2.
\(A=A_{0} e^{\frac{\ln 2}{2} t} \quad\) Substitute \(k\) into the continuous growth formula.

The function is \(A=A_{0} e^{\frac{\ln 2}{2} t}\).

\section*{TRY IT}

Recent data suggests that, as of 2013, the rate of growth predicted by Moore's Law no longer holds. Growth has slowed to a doubling time of approximately three years. Find the new function that takes that longer doubling time into account.
Answer
\(f(t)=A_{0} e^{\frac{\ln 2}{3} t}\)

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We now turn to exponential decay. One of the common terms associated with exponential decay, as stated above, is half-life, the length of time it takes an exponentially decaying quantity to decrease to half its original amount. Every radioactive isotope has a half-life, and the process describing the exponential decay of an isotope is called radioactive decay.

To find the half-life of a function describing exponential decay, solve the following equation:
\(\frac{1}{2} A_{0}=A_{o} e^{k t}\)
We find that the half-life depends only on the constant \(k\) and not on the starting quantity \(A_{0}\).
The formula is derived as follows
\(\frac{1}{2} A_{0}=A_{o} e^{k t}\)
Divide both sides by \(A_{0}\).
\(\frac{1}{2}=e^{k t}\)
\(\ln \left(\frac{1}{2}\right)=k t \quad\) Take the natural \(\log\) of both sides.
\(-\ln (2)=k t \quad\) Apply properties of logarithms.
\(-\frac{\ln (2)}{k}=t \quad\) Divide by \(k\).
Since \(t\), the time, is positive, \(k\) must, as expected, be negative. This gives us the half-life formula
\(t=-\frac{\ln (2)}{k}\)
In previous sections, we learned the properties and rules for both exponential and logarithmic functions. We have seen that any exponential function can be written as a logarithmic function and vice versa. We have used exponents to solve logarithmic equations and logarithms to solve exponential equations. We are now ready to combine our skills to solve equations that model real-world situations, whether the unknown is in an exponent or in the argument of a logarithm.

\section*{TIP FOR SUCCESS}

As you learn to model real-world applications using exponential and logarithmic functions, it will be helpful to have created a reference list of the properties and rules for exponents and logarithms as well as the techniques for solving exponential and logarithmic equations. Keep this reference list nearby as you practice working through the examples. Eventually, you'll be able to work through new applications without referring to your notes.
The applications below use exponential and logarithmic functions to model behavior in the real world.
One such application is in science, in calculating the time it takes for half of the unstable material in a sample of a radioactive substance to decay, called its half-life. The table below lists the half-life for several of the more common radioactive substances.
\begin{tabular}{|l|l|l|}
\hline \multicolumn{1}{|c|}{ Substance } & \multicolumn{1}{c|}{ Use } & \multicolumn{1}{c|}{ Half-life } \\
\hline gallium-67 & nuclear medicine & 80 hours \\
\hline cobalt-60 & manufacturing & 5.3 years \\
\hline technetium-99m & nuclear medicine & 6 hours \\
\hline americium-241 & construction & 432 years \\
\hline carbon-14 & archeological dating & 5,715 years \\
\hline uranium-235 & atomic power & \(703,800,000\) years \\
\hline
\end{tabular}

We can see how widely the half-lives for these substances vary. Knowing the half-life of a substance allows us to calculate the amount remaining after a specified time. We can use the formula for radioactive decay:
\(A(t)=A_{0} e^{\frac{\ln (0.5)}{T} t}\)
\(A(t)=A_{0} e^{\ln (0.5) \frac{t}{T}}\)
\(A(t)=A_{0}\left(e^{\ln (0.5)}\right)^{\frac{t}{T}}\)
\(A(t)=A_{0}\left(\frac{1}{2}\right)^{\frac{t}{T}}\)
where
- \(A_{0}\) is the amount initially present
- \(T\) is the half-life of the substance
- \(t\) is the time period over which the substance is studied
- \(y\) is the amount of the substance present after time \(t\)

\section*{HOW TO: GIVEN THE HALF-LIFE, FIND THE DECAY RATE}
1. Write \(A=A_{o} e^{k t}\).
2. Replace \(A\) by \(\frac{1}{2} A_{0}\) and replace \(t\) by the given half-life.
3. Solve to find \(k\). Express \(k\) as an exact value (do not round).

Note: It is also possible to find the decay rate using \(k=-\frac{\ln (2)}{t}\).

\section*{EXAMPLE: USING THE FORMULA FOR RADIOACTIVE DECAY TO FIND THE QUANTITY OF A SUBSTANCE}

How long will it take for \(10 \%\) of a 1000-gram sample of uranium- 235 to decay?
Answer
\(y=1000 e \frac{\ln (0.5)}{703,800,000} t\)
\(900=1000 e^{\frac{\ln (0.5)}{703,80,000} t} \quad \quad\) After \(10 \%\) decays, 900 grams are left.
\(0.9=e^{\frac{\ln (0.5)}{703,80,000} t} \quad\) Divide by 1000.
\(\ln (0.9)=\ln \left(e^{\frac{\ln (0.5)}{703,800,000} t}\right) \quad\) Take \(\ln\) of both sides.
\(\ln (0.9)=\frac{\ln (0.5)}{703,800,000} t \quad \ln \left(e^{M}\right)=M\)
\(t=703,800,000 \times \frac{\ln (0.9)}{\ln (0.5)}\) years Solve for \(t\).
\(t \approx 106,979,777\) years

Analysis of the Solution
Ten percent of 1000 grams is 100 grams. If 100 grams decay, the amount of uranium- 235 remaining is 900 grams.

\section*{TRY IT}

How long will it take before twenty percent of our 1000-gram sample of uranium-235 has decayed?
Answer
\(t=703,800,000 \times \frac{\ln (0.8)}{\ln (0.5)}\) years \(\approx 226,572,993\) years.

\section*{Radiocarbon Dating}

The formula for radioactive decay is important in radiocarbon dating which is used to calculate the

\section*{EXAMPLE: FINDING THE FUNCTION THAT DESCRIBES RADIOACTIVE DECAY}

The half-life of carbon-14 is 5,730 years. Express the amount of carbon-14 remaining as a function of time, \(t\).

\section*{Answer}

This formula is derived as follows.
\(A=A_{0} e^{k t} \quad\) The continuous growth formula.
\(0.5 A_{0}=A_{0} e^{k .5730} \quad\) Substitute the half-life for \(t\) and \(0.5 A_{0}\) for \(f(t)\).
\(0.5=e^{5730 k} \quad\) Divide both sides by \(A_{0}\).
\(\ln (0.5)=5730 k \quad\) Take the natural log of both sides.
\(k=\frac{\ln (0.5)}{5730} \quad\) Divide by the coefficient of \(k\).
\(A=A_{0} e^{\left(\frac{\ln (0.5)}{5730}\right) t} \quad\) Substitute for \(r\) in the continuous growth formula.
The function that describes this continuous decay is \(f(t)=A_{0} e^{\left(\frac{\ln (0.5)}{5730}\right) t}\). We observe that the coefficient of \(t, \frac{\ln (0.5)}{5730} \approx-1.2097\) is negative, as expected in the case of exponential decay.

\section*{TRY IT}

The half-life of plutonium-244 is \(80,000,000\) years. Find a function that gives the amount of carbon-14 remaining as a function of time measured in years.

\section*{Answer}
\(f(t)=A_{0} e^{-0.0000000087 t}\)
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approximate date a plant or animal died. Radiocarbon dating was discovered in 1949 by Willard Libby who won a Nobel Prize for his discovery. It compares the difference between the ratio of two isotopes of carbon in an organic artifact or fossil to the ratio of those two isotopes in the air. It is believed to be accurate to within about \(1 \%\) error for plants or animals that died within the last 60,000 years.

Carbon-14 is a radioactive isotope of carbon that has a half-life of 5,730 years. It occurs in small quantities in the carbon dioxide in the air we breathe. Most of the carbon on Earth is carbon-12 which has an atomic weight of 12 and is not radioactive. Scientists have determined the ratio of carbon-14 to carbon-12 in the air for the last 60,000 years using tree rings and other organic samples of known dates-although the ratio has changed slightly over the centuries.

As long as a plant or animal is alive, the ratio of the two isotopes of carbon in its body is close to the ratio in the atmosphere. When it dies, the carbon-14 in its body decays and is not replaced. By comparing the ratio of carbon-14 to carbon-12 in a decaying sample to the known ratio in the atmosphere, the date the plant or animal died can be approximated.

Since the half-life of carbon-14 is 5,730 years, the formula for the amount of carbon-14 remaining after \(t\) years is
\(A \approx A_{0} e^{\left(\frac{\ln (0.5)}{5730}\right) t}\)
where
- \(A\) is the amount of carbon-14 remaining
- \(A_{0}\) is the amount of carbon-14 when the plant or animal began decaying.

This formula is derived as follows:
\(A=A_{0} e^{k t} \quad\) The continuous growth formula.
\(0.5 A_{0}=A_{0} e^{k .5730}\) Substitute the half-life for \(t\) and \(0.5 A_{0}\) for \(f(t)\).
\(0.5=e^{5730 k} \quad\) Divide both sides by \(A_{0}\).
\(\ln (0.5)=5730 k \quad\) Take the natural log of both sides.
\(k=\frac{\ln (0.5)}{5730} \quad\) Divide both sides by the coefficient of \(k\).
\(A=A_{0} e^{\left(\frac{\ln (0.5)}{5730}\right) t} \quad\) Substitute for \(r\) in the continuous growth formula.
To find the age of an object we solve this equation for \(t\) :
\(t=\frac{\ln \left(\frac{A}{A_{0}}\right)}{-0.000121}\)
Out of necessity, we neglect here the many details that a scientist takes into consideration when doing carbon- 14 dating, and we only look at the basic formula. The ratio of carbon-14 to carbon-12 in the atmosphere is approximately \(0.0000000001 \%\). Let \(r\) be the ratio of carbon- 14 to carbon-12 in the organic artifact or fossil to be dated determined by a method called liquid scintillation. From the equation \(A \approx A_{0} e^{-0.000121 t}\) we know the ratio of the percentage of carbon-14 in the object we are dating to the percentage of carbon-14 in the atmosphere is \(r=\frac{A}{A_{0}} \approx e^{-0.000121 t}\). We solve this equation for \(t\), to get
\(t=\frac{\ln (r)}{-0.000121}\)

\section*{HOW TO: GIVEN THE PERCENTAGE OF CARBON-14 IN AN OBJECT, DETERMINE ITS AGE}
1. Express the given percentage of carbon-14 as an equivalent decimal \(r\).
2. Substitute for \(r\) in the equation \(t=\frac{\ln (r)}{-0.000121}\) and solve for the age, \(t\).

\section*{EXAMPLE: FINDING THE AGE OF A BONE}

A bone fragment is found that contains \(20 \%\) of its original carbon-14. To the nearest year, how old is the bone?
Answer
We substitute \(20 \%=0.20\) for \(r\) in the equation and solve for \(t\) :
\(t=\frac{\ln (r)}{-0.000121}\) Use the general form of the equation.
\(=\frac{\ln (0.20)}{-0.000121} \quad\) Substitute for \(r\).
\(\approx 13301\) Round to the nearest year.
The bone fragment is about 13,301 years old.

\section*{Analysis of the Solution}

The instruments that measure the percentage of carbon-14 are extremely sensitive and, as we mention above, a scientist will need to do much more work than we did in order to be satisfied. Even so, carbon dating is only accurate to about \(1 \%\), so this age should be given as 13,301 years \(\pm 1 \%\) or 13,301 years \(\pm 133\) years.

\section*{TRY IT}

Cesium-137 has a half-life of about 30 years. If we begin with 200 mg of cesium-137, will it take more or less than 230 years until only 1 milligram remains?
Answer
less than 230 years; 229.3157 to be exact
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\section*{BOUNDED GROWTH AND DECAY}

\section*{LEARNING OUTCOMES}
- Use Newton's Law of Cooling.
- Use a logistic growth model.
- Choose an appropriate model for data.

\section*{TIP FOR SUCCESS}

The applications in this section rely upon all the knowledge we've built up from our first study of the language and notation of functions to the most recent work you've done in exponential and logarithmic functions.
If you need a refresher, return to Algebra Basics, the module on function basics,or any section you've covered with regard to modeling using functions.

\section*{Using Newton's Law of Cooling}

Exponential decay can also be applied to temperature. When a hot object is left in surrounding air that is at a lower temperature, the object's temperature will decrease exponentially, leveling off as it approaches the surrounding air temperature. On a graph of the temperature function, the leveling off will correspond to a horizontal asymptote at the temperature of the surrounding air. Unless the room temperature is zero, this will correspond to a vertical shift of the generic exponential decay function. This translation leads to Newton's Law of Cooling, the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature.

The formula is derived as follows:
\(T(t)=A b^{c t}+T_{s}\)
\(T(t)=A e^{\ln \left(b^{c t}\right)}+T_{s} \quad\) Properties of logarithms.
\(T(t)=A e^{c t \ln b}+T_{s} \quad\) Properties of logarithms.
\(T(t)=A e^{k t}+T_{s} \quad\) Rename the constant \(c \ln b\), calling it \(k\).

\section*{A GENERAL NOTE: NEWTON'S LAW OF COOLING}

The temperature of an object, \(T\), in surrounding air with temperature \(T_{s}\) will behave according to the formula
\(T(t)=A e^{k t}+T_{s}\)
where
- \(t\) is time
- \(A\) is the difference between the initial temperature of the object and the surroundings
- \(k\) is a constant, the continuous rate of cooling of the object

\section*{HOW TO: GIVEN A SET OF CONDITIONS, APPLY NEWTON’S LAW OF COOLING}
1. Set \(T_{s}\) equal to the \(y\)-coordinate of the horizontal asymptote (usually the ambient temperature).
2. Substitute the given values into the continuous growth formula \(T(t)=A e^{k t}+T_{s}\) to find the parameters \(A\) and \(k\).
3. Substitute in the desired time to find the temperature or the desired temperature to find the time.

\section*{EXAMPLE: USING NEWTON'S LAW OF COOLING}

A cheesecake is taken out of the oven with an ideal internal temperature of \(165^{\circ} \mathrm{F}\) and is placed into a \(35^{\circ} \mathrm{F}\) refrigerator. After 10 minutes, the cheesecake has cooled to \(150^{\circ} \mathrm{F}\). If we must wait until the cheesecake has cooled to \(70^{\circ} \mathrm{F}\) before we eat it, how long will we have to wait?
Answer
Because the surrounding air temperature in the refrigerator is 35 degrees, the cheesecake's temperature will decay exponentially toward 35 , following the equation
\(T(t)=A e^{k t}+35\)
We know the initial temperature was 165 , so \(T(0)=165\).
\(165=A e^{k 0}+35\) Substitute \((0,165)\).
\(A=130 \quad\) Solve for \(A\).
We were given another data point, \(T(10)=150\), which we can use to solve for \(k\).
\(150=130 e^{k 10}+35 \quad\) Substitute \((10,150)\).
\(115=130 e^{k 10} \quad\) Subtract 35 from both sides.
\(\frac{115}{130}=e^{10 k} \quad\) Divide both sides by 130.
\(\ln \left(\frac{115}{130}\right)=10 k \quad\) Take the natural \(\log\) of both sides.
\(k=\frac{\ln \left(\frac{115}{130}\right)}{10}=-0.0123\) Divide both sides by the coefficient of \(k\).
This gives us the equation for the cooling of the cheesecake: \(T(t)=130 e^{-0.0123 t}+35\).
Now we can solve for the time it will take for the temperature to cool to 70 degrees.
\(70=130 e^{-0.0123 t}+35\) Substitute in 70 for \(T(t)\).
\(35=130 e^{-0.0123 t} \quad\) Subtract 35 from both sides.
\(\frac{35}{130}=e^{-0.0123 t} \quad\) Divide both sides by 130.
\(\ln \left(\frac{35}{130}\right)=-0.0123 t \quad\) Take the natural \(\log\) of both sides.
\(t=\frac{\ln \left(\frac{35}{130}\right)}{-0.0123} \approx 106.68 \quad\) Divide both sides by the coefficient of \(t\).
It will take about 107 minutes, or one hour and 47 minutes, for the cheesecake to cool to \(70^{\circ} \mathrm{F}\).

\section*{TRY IT}

A pitcher of water at 40 degrees Fahrenheit is placed into a 70 degree room. One hour later, the temperature has risen to 45 degrees. How long will it take for the temperature to rise to 60 degrees?

\section*{Answer}
6.026 hours

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Exponential growth cannot continue forever. Exponential models, while they may be useful in the short term, tend to fall apart the longer they continue. Consider an aspiring writer who writes a single line on day one and plans to double the number of lines she writes each day for a month. By the end of the month, she must write over 17 billion lines or one-half-billion pages. It is impractical, if not impossible, for anyone to write that much in such a short period of time. Eventually an exponential model must begin to approach some limiting value and then the growth is forced to slow. For this reason, it is often better to use a model with an upper bound instead of an exponential growth model although the exponential growth model is still useful over a short term before approaching the limiting value.

The logistic growth model is approximately exponential at first, but it has a reduced rate of growth as the output approaches the model's upper bound called the carrying capacity. For constants \(a, b\), and \(c\), the logistic growth of a population over time \(x\) is represented by the model
\(f(x)=\frac{c}{1+a e^{-b x}}\)
The graph below shows how the growth rate changes over time. The graph increases from left to right, but the growth rate only increases until it reaches its point of maximum growth rate at which the rate of increase decreases.


\section*{A GENERAL NOTE: LOGISTIC GROWTH}

The logistic growth model is
\(f(x)=\frac{c}{1+a e^{-b x}}\)
where
- \(\frac{c}{1+a}\) is the initial value
- \(c\) is the carrying capacity or limiting value
- \(b\) is a constant determined by the rate of growth.

\section*{Choosing an Appropriate Model}

Now that we have discussed various mathematical models, we need to learn how to choose the appropriate model for the raw data we have. Many factors influence the choice of a mathematical model among which are experience, scientific laws, and patterns in the data itself. Not all data can be described by elementary functions. Sometimes a function is chosen that approximates the data over a given interval. For instance, suppose data were gathered on the number of homes bought in the United States from the years 1960 to 2013. After plotting these data in a scatter plot, we notice that the shape of the data from the years 2000 to 2013 follow a logarithmic curve. We could restrict the interval from 2000 to 2010, apply regression analysis using a logarithmic model, and use it to predict the number of home buyers for the year 2015.

\section*{EXAMPLE: USING THE LOGISTIC-GROWTH MODEL}

An influenza epidemic spreads through a population rapidly at a rate that depends on two factors. The more people who have the flu, the more rapidly it spreads, and also the more uninfected people there are, the more rapidly it spreads. These two factors make the logistic model good for studying the spread of communicable diseases. And, clearly, there is a maximum value for the number of people infected: the entire population.
For example, at time \(t=0\) there is one person in a community of 1,000 people who has the flu. So, in that community, at most 1,000 people can have the flu. Researchers find that for this particular strain of the flu, the logistic growth constant is \(b=0.6030\). Estimate the number of people in this community who will have had this flu after ten days. Predict how many people in this community will have had this flu after a long period of time has passed.

\section*{Answer}

We substitute the given data into the logistic growth model
\(f(x)=\frac{c}{1+a e^{-b x}}\)
Because at most 1,000 people, the entire population of the community, can get the flu, we know the limiting value is \(c=1000\). To find \(a\), we use the formula that the number of cases at time \(t=0\) is \(\frac{c}{1+a}=1\), from which it follows that \(a=999\). This model predicts that, after ten days, the number of people who have had the flu is \(f(x)=\frac{1000}{1+999 e^{-0.0630 x}} \approx 293.8\). Because the actual number must be a whole number (a person has either had the flu or not) we round to 294 . In the long term, the number of people who will contract the flu is the limiting value, \(c=1000\).

\section*{Analysis of the Solution}

Remember that because we are dealing with a virus, we cannot predict with certainty the number of people infected. The model only approximates the number of people infected and will not give us exact or actual values.
The graph below gives a good picture of how this model fits the data.


The graph of \(f(x)=\frac{1000}{1+999 e^{-0.6030 x}}\).

\section*{TRY IT}

Using the model in the previous example, estimate the number of cases of flu on day 15.

\section*{Answer}

895 cases on day 15
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Three kinds of functions that are often useful in mathematical models are linear functions, exponential functions, and logarithmic functions. If the data lies on a straight line or seems to lie approximately along a straight line, a linear model may be best. If the data is non-linear, we often consider an exponential or logarithmic model although other models, such as quadratic models, may also be considered.

In choosing between an exponential model and a logarithmic model, we look at the way the data curves. This is called the concavity. If we draw a line between two data points, and all (or most) of the data between those two points lies above that line, we say the curve is concave down. We can think of it as a bowl that bends downward and therefore cannot hold water. If all (or most) of the data between those two points lies below the line, we say the curve is concave up. In this case, we can think of a bowl that bends upward and can therefore hold water. An exponential curve, whether rising or falling, whether representing growth or decay, is always concave up away from its horizontal asymptote. A logarithmic curve is always concave down away from its vertical asymptote. In the case of positive data, which is the most common case, an exponential curve is always concave up and a logarithmic curve always concave down.

A logistic curve changes concavity. It starts out concave up and then changes to concave down beyond a certain point, called a point of inflection.

After using the graph to help us choose a type of function to use as a model, we substitute points, and solve to find the parameters. We reduce round-off error by choosing points as far apart as possible.

\section*{Expressing an Exponential Model in Base e}

While powers and logarithms of any base can be used in modeling, the two most common bases are 10 and \(e\). In science and mathematics, the base \(e\) is often preferred. We can use properties of exponents and properties of logarithms to change any base to base \(e\).

\section*{EXAMPLE: CHOOSING A MATHEMATICAL MODEL}

Does a linear, exponential, logarithmic, or logistic model best fit the values listed below? Find the model, and use a graph to check your choice.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|l|}
\hline \(\boldsymbol{x}\) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline \(\boldsymbol{y}\) & 0 & 1.386 & 2.197 & 2.773 & 3.219 & 3.584 & 3.892 & 4.159 & 4.394 \\
\hline
\end{tabular}

First, plot the data on a graph as in the graph below. For the purpose of graphing, round the data to two significant digits.


Clearly, the points do not lie on a straight line, so we reject a linear model. If we draw a line between any two of the points, most or all of the points between those two points lie above the line, so the graph is concave down, suggesting a logarithmic model. We can try \(y=a \ln (b x)\). Plugging in the first point, \((1,0)\), gives \(0=a \ln b\). We reject the case that \(a=0\) (if it were, all outputs would be 0 ), so we know \(\ln (b)=0\). Thus \(b=1\) and \(y=a \ln (\mathrm{x})\). Next we can use the point \((9,4.394)\) to solve for \(a\) :
\(y=a \ln (x)\)
\(4.394=a \ln (9)\)
\(a=\frac{4.394}{\ln (9)}\)
Because \(a=\frac{4.394}{\ln (9)} \approx 2\), an appropriate model for the data is \(y=2 \ln (x)\).
To check the accuracy of the model, we graph the function together with the given points.


The graph of \(y=2 \ln x\).
We can conclude that the model is a good fit to the data. Compare the figure above to the graph of \(y=\ln \left(x^{2}\right)\) shown below.


The graphs appear to be identical when \(x>0\). A quick check confirms this conclusion:
\(y=\ln \left(x^{2}\right)=2 \ln (x)\) for \(x>0\). However, if \(x<0\), the graph of \(y=\ln \left(x^{2}\right)\) includes an "extra" branch as shown below. This occurs because while \(y=2 \ln (x)\) cannot have negative values in the domain (as such values would force the argument to be negative), the function \(y=\ln \left(x^{2}\right)\) can have negative domain
values.


\section*{TRY IT}

Does a linear, exponential, or logarithmic model best fit the data in the table below? Find the model.
\begin{tabular}{|l|l|l|l|l|l|l|l|l|l|}
\hline \(\boldsymbol{x}\) & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline \(\boldsymbol{y}\) & 3.297 & 5.437 & 8.963 & 14.778 & 24.365 & 40.172 & 66.231 & 109.196 & 180.034 \\
\hline
\end{tabular}

\section*{Answer}

Exponential. \(y=2 e^{0.5 x}\).

\section*{HOW TO: GIVEN A MODEL WITH THE FORM \(y=a b^{x}\), CHANGE IT TO THE FORM \(y=A_{0} e^{k x}\)}
1. Rewrite \(y=a b^{x}\) as \(y=a e^{\ln \left(b^{x}\right)}\).
2. Use the power rule of logarithms to rewrite as \(y=a e^{x \ln (b)}=a e^{\ln (b) x}\).
3. Note that \(a=A_{0}\) and \(k=\ln (b)\) in the equation \(y=A_{0} e^{k x}\).

\section*{EXAMPLE: CHANGING TO BASE \(e\)}

Change the function \(y=2.5(3.1)^{x}\) so that this same function is written in the form \(y=A_{0} e^{k x}\).
Answer
The formula is derived as follows
\[
\begin{array}{rlrl}
y & =2.5(3.1)^{x} & \\
& =2.5 e^{\ln \left(3.1^{x}\right)} & & \text { Insert exponential and its inverse. } \\
& =2.5 e^{x \ln 3.1} & & \text { Properties of logs. } \\
& =2.5 e^{(\ln 3.1) x} & & \text { Commutative law of multiplication }
\end{array}
\]

\section*{TRY IT}

Change the function \(y=3(0.5)^{x}\) to one having \(e\) as the base.
Answer
\(y=3 e^{(\ln 0.5) x}\)

\section*{LEARNING OUTCOMES}
- Use a graphing utility to create an exponential regression from a set of data.

\section*{Exponential Regression}

As we have learned, there are a multitude of situations that can be modeled by exponential functions, such as investment growth, radioactive decay, atmospheric pressure changes, and temperatures of a cooling object. What do these phenomena have in common? For one thing, all the models either increase or decrease as time moves forward. But that's not the whole story. It's the way data increase or decrease that helps us determine whether it is best modeled by an exponential function. Knowing the behavior of exponential functions in general allows us to recognize when to use exponential regression, so let's review exponential growth and decay.

Recall that exponential functions have the form \(y=a b^{x}\) or \(y=A_{0} e^{k x}\). When performing regression analysis, we use the form most commonly used on graphing utilities, \(y=a b^{x}\). Take a moment to reflect on the characteristics we've already learned about the exponential function \(y=a b^{x}\) (assume \(a>0\) ):
- \(b\) must be greater than zero and not equal to one.
- The initial value of the model is a.
- If \(b>1\), the function models exponential growth. As \(x\) increases, the outputs of the model increase slowly at first, but then increase more and more rapidly, without bound.
- If \(0<b<1\), the function models exponential decay. As \(x\) increases, the outputs for the model decrease rapidly at first and then level off to become asymptotic to the \(x\)-axis. In other words, the outputs never become equal to or less than zero.

As part of the results, your calculator will display a number known as the correlation coefficient, labeled by the variable ror \(r^{2}\). (You may have to change the calculator's settings for these to be shown.) The values are an indication of the "goodness of fit" of the regression equation to the data. We more commonly use the value of \(r^{2}\) instead of \(r\), but the closer either value is to 1 , the better the regression equation approximates the data.

\section*{A GENERAL NOTE: EXPONENTIAL REGRESSION}

Exponential regression is used to model situations in which growth begins slowly and then accelerates rapidly without bound, or where decay begins rapidly and then slows down to get closer and closer to zero. We use the command "ExpReg" on a graphing utility to fit an exponential function to a set of data points. This returns an equation of the form \(y=a b^{x}\).
Note that:
- \(b\) must be non-negative.
- When \(b>1\), we have an exponential growth model.
- When \(0<b<1\), we have an exponential decay model.

\section*{HOW TO: GIVEN A SET OF DATA, PERFORM EXPONENTIAL REGRESSION USING AN ONLINE GRAPHING TOOL}
1. Create a table by clicking on the + in the upper left and selecting the table icon.
2. Enter your data into the table.
3. Enter \(y_{1} \sim a b^{x_{1}}\) in the next line.
4. You can check the quality of the fit by looking at the \(R^{2}\) value provided by the calculator. The closer it is to 1 , the better the model fits the data.
\(\square\)

\section*{EXAMPLE: USING EXPONENTIAL REGRESSION TO FIT A MODEL TO DATA}

In 2007, a university study was published investigating the crash risk of alcohol impaired driving. Data from 2,871 crashes were used to measure the association of a person's blood alcohol level (BAC) with the risk of being in an accident. The table below shows results from the study. (Note: Source: Indiana University Center for Studies of Law in Action, 2007) The relative risk is a measure of how many times more likely a person is to crash. So, for example, a person with a BAC of 0.09 is 3.54 times as likely to crash as a person who has not been drinking alcohol.
\begin{tabular}{|l|l|l|l|l|l|l|}
\hline BAC & 0 & 0.01 & 0.03 & 0.05 & 0.07 & 0.09 \\
\hline Relative Risk of Crashing & 1 & 1.03 & 1.06 & 1.38 & 2.09 & 3.54 \\
\hline BAC & 0.11 & 0.13 & 0.15 & 0.17 & 0.19 & 0.21 \\
\hline Relative Risk of Crashing & 6.41 & 12.6 & 22.1 & 39.05 & 65.32 & 99.78 \\
\hline
\end{tabular}
1. Let \(x\) represent the BAC level and let \(y\) represent the corresponding relative risk. Use exponential regression to fit a model to these data.
2. After 6 drinks, a person weighing 160 pounds will have a BAC of about 0.16 . How many times more likely is a person with this weight to crash if they drive after having a 6-pack of beer? Round to the nearest hundredth.

\section*{Answer}
1. Using an online graphing tool, create a table by clicking on the + in the upper left and selecting the table icon. Enter the data.


Below your table enter \(y_{1} \sim a b^{x_{1}}\).
Notice that \(r^{2}\) is very close to 1 which indicates the model is a good fit to the data.
If using an online graphing tool, the model obtained is \(y=0.55877(57,700,000,000)^{x}\)
If using a graphing calculator, the model obtained is \(y=0.58304829(22,072,021,300)^{x}\)

2. Use the model to estimate the risk associated with a BAC of 0.16 . Substitute 0.16 for \(x\) in the model and solve for \(y\).
Using the online graphing tool model:
\[
\begin{aligned}
y & =0.55877(57,700,000,000)^{x} & & \text { Use the regression model found in part (a). } \\
& =0.55877(57,700,000,000)^{0.16} & & \text { Substitute 0.16 for } x . \\
& \approx 29.44 & & \text { Round to the nearest hundredth. }
\end{aligned}
\]

If a 160 -pound person drives after having 6 drinks, he or she is about 29.44 times more likely to crash than if driving while sober.
Using the graphing calculator model:
\(y=0.58304829(22,072,021,300)^{x} \quad\) Use the regression model found in part (a).
\(=0.58304829(22,072,021,300)^{0.16}\) Substitute 0.16 for \(x\).
\(\approx 26.35\) Round to the nearest hundredth.
If a 160-pound person drives after having 6 drinks, he or she is about 26.35 times more likely to crash than if driving while sober.

\section*{TRY IT}

The table below shows a recent graduate's credit card balance each month after graduation.
\begin{tabular}{|l|l|l|l|l|l|l|l|}
\hline Month & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline Debt (\$) & 620.00 & 761.88 & 899.80 & 1039.93 & 1270.63 & 1589.04 & 1851.31 \\
\hline
\end{tabular}
a. Use exponential regression to fit a model to these data.
b. If spending continues at this rate, what will the graduate's credit card debt be one year after graduating?

Answer
a.

Using an online graphing tool, the exponential regression model that fits these data is
\(y=528.25(1.1943)^{x}\).
Using a graphing calculator, the exponential regression model that fits these data is \(y=522.88585984(1.19645256)^{x}\).
b.

Using an online graphing tool, if spending continues at this rate, the graduate's credit card debt will be \(\$ 4,448.37\) after one year.
Using a graphing calculator, if spending continues at this rate, the graduate's credit card debt will be \$4,499.38 after one year.

\section*{Q \& A}

Is it reasonable to assume that an exponential regression model will represent a situation indefinitely?
No. Remember that models are formed by real-world data gathered for regression. It is usually reasonable to make estimates within the interval of original observation (interpolation). However, when a model is used to make predictions, it is important to use reasoning skills to determine whether the model makes sense for inputs far beyond the original observation interval (extrapolation).

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\section*{SUMMARY: EXPONENTIAL AND LOGARITHMIC MODELS}

\section*{Key Equations}
\begin{tabular}{l|l} 
Half-life formula & If \(A=A_{0} e^{k t}, k<0\), the half-life is \(t=-\frac{\ln (2)}{k}\). \\
Carbon-14 dating & \(t=\frac{\ln \left(\frac{A}{A_{0}}\right)}{-0.000121} \cdot A_{0}\) is the amount of carbon-14 when the plant or animal died \\
\(A\) is the amount of carbon-14 remaining today. \\
\(t\) is that age of the fossil.
\end{tabular}

\section*{Key Concepts}
- The basic exponential function is \(f(x)=a b^{x}\). If \(b>1\), we have exponential growth; if \(0<b<1\), we have exponential decay.
- We can also write \(f(x)=a b^{x}\) in terms of continuous growth as \(A=A_{0} e^{k x}\), where \(A_{0}\) is the starting value. If \(A_{0}\) is positive, then we have exponential growth when \(k>0\) and exponential decay when \(k<0\).
- In general, we solve problems involving exponential growth or decay in two steps. First, we set up a model and use the model to find the parameters. Then we use the formula with these parameters to predict growth and decay.
- We can find the age, \(t\), of an organic artifact by measuring the amount, \(k\), of carbon-14 remaining in the artifact and using the formula \(t=\frac{\ln (k)}{-0.000121}\) to solve for \(t\).
- Given a substance's doubling time or half-life, we can find a function that represents its exponential growth or decay.
- We can use Newton's Law of Cooling to find how long it will take for a cooling object to reach a desired temperature or to find what temperature an object will be after a given time.
- We can use logistic growth functions to model real-world situations where the rate of growth changes over time, such as population growth, spread of disease, and spread of rumors.
- We can use real-world data gathered over time to observe trends. Knowledge of linear, exponential, logarithmic, and logistic graphs help us to develop models that best fit our data.
- Any exponential function of the form \(y=a b^{x}\) can be rewritten as an equivalent exponential function of the form \(y=A_{0} e^{k x}\) where \(k=\ln b\).
- Exponential regression is used to model situations where growth begins slowly and then accelerates rapidly without bound or where decay begins rapidly and then slows down to get closer and closer to zero.

\section*{Glossary}
carrying capacity in a logistic model, the limiting value of the output
doubling time the time it takes for a quantity to double
half-life the length of time it takes for a substance to exponentially decay to half of its original quantity
logistic growth model a function of the form \(f(x)=\frac{c}{1+a e^{-b x}}\) where \(\frac{c}{1+a}\) is the initial value, \(c\) is the carrying capacity, or limiting value, and \(b\) is a constant determined by the rate of growth

Newton's Law of Cooling the scientific formula for temperature as a function of time as an object's temperature is equalized with the ambient temperature
order of magnitude the power of ten when a number is expressed in scientific notation with one non-zero digit to the left of the decimal
```

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\section*{PUTTING IT TOGETHER: EXPONENTIAL AND LOGARITHMIC EQUATIONS AND MODELS}

At the start of this module, you were assigned the task of analyzing a fossilized bone to determine its age. To make that estimate, you need to model for the decay rate of carbon-14.

The decay of a radioactive element is an exponential function of the form:
\[
\begin{aligned}
& A(t)= A_{0} e^{-k t} \\
& \text { where } \\
& A(t)=\text { mass of element remaining after } \mathrm{t} \text { years } \\
& A_{0}=\text { original mass of element } \\
& k=\text { rate of decay } \\
& t=\text { time in years }
\end{aligned}
\]

So to create a model for the decay function of carbon-14, assume for simplicity that the sample you started with had a mass of 1 g . We know that half the starting mass of the sample will remain after one half-life which is 5,730 years. We can substitute these values for \(A(t)\) and \(A_{0}\) as follows:
\[
\begin{aligned}
& A(t)=A_{0} e^{-k t} \\
& \frac{1}{2}=(1) e^{-k(5730)} \\
& 1 n\left(\frac{1}{2}\right)=1 n\left(e^{-k(5730)}\right) \\
& 1 n\left(2^{-1}\right)=(-5730 k) \ln (e) \\
& -1 n(2)=-5730 k(1)
\end{aligned}
\]
\[
k \approx 1.21 \times 10^{-4}
\]

Now you know the decay rate so you can write the equation for the exponential decay of carbon-14 and you can represent it as a graph.


The next step is to evaluate the function for a given mass. Assume a starting mass of 100 grams and that there are 20 grams remaining. Substitute these values into the model in the following way:
\begin{tabular}{|l|l|}
\hline Write the equation & \(A(t)=100 e^{-(0.000121) t}\) \\
\hline Substitute 20 grams for \(\mathrm{A}(\mathrm{t})\) & \(20=100 e^{-(0.000121) t}\) \\
\hline Divide both sides by 100 & \(0.20=e^{-(0.000121) t}\) \\
\hline Change to logarithmic form & \(1 n(0.20)=-(0.000121) t\) \\
\hline Divide both sides by -0.000121 & \(t=\frac{1 n(0.20)}{-0.000121}\) \\
\hline Solve & \(t \approx 13,301\) years \\
\hline
\end{tabular}

Now you know that it would take 13,301 years for a 100-gram sample of carbon-14 to decay to the point that only 20 grams are left. Confirm that this number makes sense by looking at the graph.

You can also determine the amount of a 100-gram sample that would remain after a given number of years such as 8,000 . To do this, substitute the number of years into the function and evaluate.
\(A(t)=100 e^{-(0.000121) t}\)
\(A(8000)=100 e^{-(0.000121)(8000)} \approx 38\) grams
About 38 grams would remain after 8,000 years.
Understanding exponential functions helps scientists better understand radioactive decay and provides insights into past civilizations and species.
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\title{
MODULE 13: SYSTEMS OF EQUATIONS AND INEQUALITIES
}

\section*{WHY IT MATTERS: SYSTEMS OF EQUATIONS AND INEQUALITIES}

After years of saving, you have finally done it-you rented a space to open your very own coffee shop. You've already painted the walls and set up the furniture. The next step is to plan your menu. In considering both taste and cost, you have developed a coffee flavor that is sure to bring customers coming back for more.


This soon-to-be signature flavor is made from a mixture of coffee beans. The first type of coffee bean sells for \(\$ 8\) per pound and the other sells for \(\$ 14\) per pound.

You will combine the coffee beans in batches of 100 pounds, and sell them for \(\$ 9.80\) per pound. Now all you need to figure out is how much of each type of coffee bean should you mix together. You can use an equation to relate the amount of each type of coffee bean. If you set the number of pounds of the first coffee bean equal to \(x\) and the number of pounds of the first coffee bean equal to \(y\), you know that the total is 100 .
\(x+y=100\)
You can also relate the cost of each type of bean to the cost of the mixture. The cost of the first type of coffee bean is the number of pounds times the cost per pound, \(8 x\). Similarly, cost of the second type of coffee bean is the number of pounds times its cost per pound, \(14 x\). And the total cost is \(\$ 980\) for 100 pounds.
\(8 x+14 y=980\)
Now you have two equations, but what can you do to solve for the values of \(x\) and \(y\) ? And what information do those values give you?

To find the answers to these and other questions, read on in this module. There you will learn about combinations of equations, called systems of equations, along with different methods of solving them.

\section*{LEARNING OUTCOMES}

\section*{Review Topics for Success}
- Identify and graph a linear inequality in two variables.
- Identify whether an ordered pair is in the solution set of a linear inequality.
- Identify solutions to systems of linear inequalities.
- Graph systems of linear inequalities.
- Interpret graphs and solutions to systems of linear inequalities.
- Write and graph a system of inequalities that models a given situation.

\section*{Systems of Linear Equations: Two Variables}
- Solve systems of equations by graphing, substitution, and addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables using standard notations.

Systems of Nonlinear Equations and Inequalities
- Solve a system of nonlinear equations using substitution or elimination.
- Graph a nonlinear inequality.
- Graph a system of nonlinear inequalities.

Systems of Linear Equations: Three Variables
- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables using standard notations.

\section*{Partial Fractions: an Application of Systems}
- Decompose \(\frac{P(x)}{Q(x)}\), where \(Q(x)\) has only nonrepeated linear factors.
- Decompose \(\frac{P(x)}{Q(x)}\), where \(Q(x)\) has repeated linear factors.
- Decompose \(\frac{P(x)}{Q(x)}\), where \(Q(x)\) has a nonrepeated irreducible quadratic factor.
- Decompose \(\frac{P(x)}{Q(x)}\), where \(Q(x)\) has a repeated irreducible quadratic factor.

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\section*{REVIEW TOPICS FOR SUCCESS}

\section*{Learning Outcomes}

By the end of this section, you will be able to:
- Identify and graph a linear inequality in two variables
- Identify whether an ordered pair is in the solution set of a linear inequality
- Identify solutions to systems of linear inequalities
- Graph systems of linear inequalities
- Interpret graphs and solutions to systems of linear inequalities
- Write and graph a system of inequalities that models a given situation

You'll explore techniques in this module that will help you set up and interpret situations involving systems of two or three linear equations and systems involving non-linear inequalities. You've already learned about identifying and graphing linear equations, but a good understanding of linear inequalities and systems of linear inequalities will be necessary before beginning your study of non-linear inequalities.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

\section*{GRAPH A LINEAR INEQUALITY IN TWO VARIABLES}

\section*{Learning Outcome}
- Identify and graph a linear inequality in two variables

So how do you get from the algebraic form of an inequality, like \(y>3 x+1\), to a graph of that inequality? Plotting inequalities is fairly straightforward if you follow a couple steps.

To graph an inequality:
- Graph the related boundary line. Replace the given inequality symbol, \(<,>, \leq\) or \(\geq\), in the inequality with the equality symbol, \(=\), to find the equation of the boundary line.
- Identify at least one ordered pair on either side of the boundary line and substitute those \((x, y)\) values into the inequality. Shade the region that contains the ordered pairs that make the inequality a true statement.
- If points on the boundary line are solutions, then use a solid line for drawing the boundary line. This will be the case for inequality with equality, \(\leq\) or \(\geq\).
- If points on the boundary line are not solutions, then use a dotted line for the boundary line. This will be the case for strict inequality, <or >.

Graph the inequality \(x+4 y \leq 4\).
To graph the boundary line, find at least two values that lie on the line \(x+4 y=4\). You can use the \(x\) and \(y\) intercepts for this equation by substituting 0 in for \(x\) first and finding the value of \(y\); then substitute 0 in for \(y\) and find \(x\).
\begin{tabular}{|l|l|}
\hline \(\boldsymbol{x}\) & \(\boldsymbol{y}\) \\
\hline 0 & 1 \\
\hline 4 & 0 \\
\hline
\end{tabular}

Plot the points \((0,1)\) and \((4,0)\), and draw a line through these two points for the boundary line. The line is solid because \(\leq\) means "less than or equal to," so all ordered pairs along the line are included in the solution set.


The next step is to find the region that contains the solutions. Is it above or below the boundary line? To identify the region where the inequality holds true, you can test a couple of ordered pairs, one on each side of the boundary line.

If you substitute \((-1,3)\) into \(x+4 y \leq 4\) :
\(-1+4(3) \leq 4\)
\(-1+12 \leq 4\)
\(11 \leq 4\)
This is a false statement since 11 is not less than or equal to 4 .
On the other hand, if you substitute \((2,0)\) into \(x+4 y \leq 4\) :
\(2+4(0) \leq 4\)
\(2+0 \leq 4\)
\(2 \leq 4\)
This is true! The region that includes \((2,0)\) should be shaded, as this is the region of solutions for the inequality.


And there you have it, the graph of the set of solutions for \(x+4 y \leq 4\).
Below is a video about how to graph inequalities with two variables.


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=371

\section*{Example}

Graph the inequality \(2 y>4 x-6\).
Answer
Solve for \(y\).
\[
\begin{aligned}
& 2 y>4 x-6 \\
& \frac{2 y}{2}>\frac{4 x}{2}-\frac{6}{2} \\
& y>2 x-3
\end{aligned}
\]

Create a table of values to find two points on the line \(y=2 x-3\).
\begin{tabular}{|c|c|}
\hline \(\mathbf{x}\) & \(\mathbf{y}\) \\
\hline 0 & -3 \\
\hline 2 & 1 \\
\hline
\end{tabular}

Plot the points and graph the line. The line is dotted because the sign in the inequality is \(>\), not \(\geq\) and therefore points on the line are not solutions to the inequality.


The inequality is \(2 y>4 x-6\). Find an ordered pair on either side of the boundary line. Insert the \(x\) and \(y\) values into the inequality. If the simplified result is true, then shade on the side of the line the point is located.

Test \(1:(-3,1)\)
\(2(1)>4(-3)-6\)
\[
2>-12-6
\]
\(2>-18\)

\section*{TRUE}

Test \(2:(4,1)\)
\[
\begin{gathered}
2(1)>4(4)-6 \\
2>16-6
\end{gathered}
\]
\(2>10\)
FALSE
Since \((-3,1)\) results in a true statement, the region that includes \((-3,1)\) should be shaded.
The graph of the inequality \(2 y>4 x-6\) is:


A quick note about the problem above: notice that you can use the points \((0,-3)\) and \((2,1)\) to graph the boundary line, but these points are not included in the region of solutions since the region does not include the boundary line.

Below is a video about how to graph inequalities with two variables when the equation is in slope-intercept form.


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\section*{Summary}

When inequalities are graphed on a coordinate plane, the solutions are located in a region of the coordinate plane which is represented as a shaded area on the plane. The boundary line for the inequality is drawn as a solid line if the points on the line itself do satisfy the inequality, as in the cases of inequality with equality, \(\leq\) and \(\geq\). It is drawn as a dashed line if the points on the line do not satisfy the inequality, as in the cases of
strict inequality, < and >. You can tell which region to shade by testing some points in the inequality. Using a coordinate plane is especially helpful for visualizing the region of solutions for inequalities with two variables.

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\section*{SOLUTION SETS OF INEQUALITIES}

\section*{LEARNING OUTCOME}
- Identify whether an ordered pair is in the solution set of a linear inequality

The graph below shows the region of values that makes the inequality \(3 x+2 y \leq 6\) true (shaded red), the boundary line \(3 x+2 y=6\), as well as a handful of ordered pairs. The boundary line is solid because points on the boundary line \(3 x+2 y=6\) will make the inequality \(3 x+2 y \leq 6\) true.


You can substitute the \(x\) and \(y\)-values of each of the \((x, y)\) ordered pairs into the inequality to find solutions. Sometimes making a table of values makes sense for more complicated inequalities.
\begin{tabular}{|c|c|c|}
\hline Ordered Pair & Makes the inequality \(3 x+2 y \leq 6\) a true statement & Makes the inequality \(3 x+2 y \leq 6\) a false statement \\
\hline \((-5,5)\) & \[
\begin{aligned}
& 3(-5)+2(5) \leq 6 \\
& -15+10 \leq 6 \\
& -5 \leq 6
\end{aligned}
\] & \\
\hline \((-2,-2)\) & \[
\begin{aligned}
& 3(-2)+2(-2) \leq 6 \\
& -6+(-4) \leq 6 \\
& -10 \leq 6
\end{aligned}
\] & \\
\hline \((2,3)\) & & \[
\begin{aligned}
& 3(2)+2(3) \leq 6 \\
& 6+6 \leq 6 \\
& 12 \leq 6
\end{aligned}
\] \\
\hline \((2,0)\) & \[
\begin{aligned}
& 3(2)+2(0) \leq 6 \\
& 6+0 \leq 6 \\
& 6 \leq 6
\end{aligned}
\] & \\
\hline \((4,-1)\) & & \[
\begin{aligned}
& 3(4)+2(-1) \leq 6 \\
& 12+(-2) \leq 6 \\
& 10 \leq 6
\end{aligned}
\] \\
\hline
\end{tabular}

If substituting \((x, y)\) into the inequality yields a true statement, then the ordered pair is a solution to the inequality, and the point will be plotted within the shaded region or the point will be part of a solid boundary line. A false statement means that the ordered pair is not a solution, and the point will graph outside the shaded region, or the point will be part of a dotted boundary line.

\section*{Example}

Use the graph to determine which ordered pairs plotted below are solutions of the inequality \(x-y<3\).


\section*{Answer}

Solutions will be located in the shaded region. Since this is a "less than" problem, ordered pairs on the boundary line are not included in the solution set.

The values below are located in the shaded region so they are solutions. When substituted into the inequality \(x-y<3\), they produce true statements.
\((-1,1)\)
\((-2,-2)\)
These values are not located in the shaded region, so are not solutions. When substituted into the inequality \(x-y<3\), they produce false statements.
\((1,-2)\)
\((3,-2)\)
\((4,0)\)
\((-1,1)\) and \((-2,-2)\) are the ordered pairs indicated in the graph that are solutions to the inequality.
The following video shows an example of determining whether an ordered pair is a solution to an inequality.


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\section*{Example}

Is \((2,-3)\) a solution of the inequality \(y<-3 x+1\) ?
Answer
If \((2,-3)\) is a solution, then it will yield a true statement when substituted into the inequality \(y<-3 x+1\).
\(y<-3 x+1\)
Substitute \(x=2\) and \(y=-3\) into inequality.
\(-3<-3(2)+1\)
Evaluate.
\(-3<-6+1\)
\(-3<-5\)

This statement is not true, so the ordered pair \((2,-3)\) is not a solution.

The following video shows another example of determining whether an ordered pair is a solution to an inequality.


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\section*{LEARNING OUTCOMES}
- Graph a system of linear inequalities and define the solutions region
- Verify whether a point is a solution to a system of inequalities
- Identify when a system of inequalities has no solution

\section*{Graph a System of Two Inequalities}

Recall that the graph of a single linear inequality splits the coordinate plane into two regions. On one side lie all the solutions to the inequality. On the other side, there are no solutions. Consider the graph of the inequality \(y<2 x+5\).


The dashed line is \(y=2 x+5\). Every ordered pair in the shaded area below the line is a solution to \(y<2 x+5\), as all of the points below the line will make the inequality true. If you doubt that, try substituting the \(x\) and \(y\) coordinates of Points A and B into the inequality; you will see that they work. So, the shaded area shows all of the solutions for this inequality.

The boundary line divides the coordinate plane in half. In this case, it is shown as a dashed line as the points on the line do not satisfy the inequality. If the inequality had been \(y \leq 2 x+5\), then the boundary line would have been solid.

Now graph another inequality: \(y>-x\). You can check a couple of points to determine which side of the boundary line to shade. Checking points M and N yield true statements. So, we shade the area above the line. The line is dashed as points on the line are not true.


To create a system of inequalities, you need to graph two or more inequalities together. Let us use \(y<2 x+5\) and \(y>-x\) since we have already graphed each of them.


The purple area shows where the solutions of the two inequalities overlap. This area is the solution to the system of inequalities. Any point within this purple region will be true for both \(y>-x\) and \(y<2 x+5\).

Watch the following video examples to see how to graph a system of linear inequalities and define the solution region.


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In the next section, we will see that points can be solutions to systems of equations and inequalities. We will verify algebraically whether a point is a solution to a linear equation or inequality.

\section*{Determine Whether an Ordered Pair is a Solution to a System of Linear Inequalities}


On the graph above, you can see that the points B and N are solutions for the system because their coordinates will make both inequalities true statements.

In contrast, points M and A both lie outside the solution region (purple). While point M is a solution for the inequality \(y>-x\) and point A is a solution for the inequality \(y<2 x+5\), neither point is a solution for the system. The following example shows how to test a point to see whether it is a solution to a system of inequalities.

\section*{Example}

Is the point \((2,1)\) a solution of the system \(x+y>1\) and \(2 x+y<8\) ?

\section*{Answer}

Check the point with each of the inequalities. Substitute 2 for \(x\) and 1 for \(y\). Is the point a solution of both inequalities?
\(x+y>1\)
\(2+1>1\)
\(3>1\)
TRUE
\((2,1)\) is a solution for \(x+y>1\).
\(2 x+y<8\)
\(2(2)+1<8\)
\(4+1<8\)
\(5<8\)
TRUE
\((2,1)\) is a solution for \(2 x+y<8\).
Since \((2,1)\) is a solution of each inequality, it is also a solution of the system.
Here is a graph of the system in the example above. Notice that \((2,1)\) lies in the purple area which is the overlapping area for the two inequalities.


\section*{Example}

Is the point \((2,1)\) a solution of the system \(x+y>1\) and \(3 x+y<4\) ?
Answer
Check the point with each of the inequalities. Substitute 2 for \(x\) and 1 for \(y\). Is the point a solution of both inequalities?
\(x+y>1\)
\(2+1>1\)
\(3>1\)
TRUE
\((2,1)\) is a solution for \(x+y>1\).
\(3 x+y<4\)
\(3(2)+1<4\)
\(6+1<4\)
\(7<4\)

\section*{FALSE}
\((2,1)\) is not a solution for \(3 x+y<4\).
Since \((2,1)\) is not a solution of one of the inequalities, it is not a solution of the system.
Here is a graph of the system above. Notice that \((2,1)\) is not in the purple area which is the overlapping area; it is a solution for one inequality (the red region), but it is not a solution for the second inequality (the blue region).


The following video offers another example of determining whether a point is in the solution of a system of linear inequalities.


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As shown above, finding the solutions of a system of inequalities can be done by graphing each inequality and identifying the region they share. Below, you are given more examples that show the entire process of defining the region of solutions on a graph for a system of two linear inequalities. The general steps are outlined below:
- Graph each inequality as a line and determine whether it will be solid or dashed.
- Determine which side of each boundary line represents solutions to the inequality by testing a point on each side.
- Shade the region that represents solutions for both inequalities.

\section*{Systems with No Solutions}

In the next example, you'll see the solution to a system of two inequalities whose boundary lines are parallel to each other. When the graphs of a system of two linear equations are parallel to each other there is no solution to the system. That is, there is no point where the lines intersect. We will get a similar result for the following system of linear inequalities.

\section*{Summary}

\section*{EXAMPLES}

Graph the system
\(y \geq 2 x+1\)
\(y<2 x-3\)

\section*{Answer}

The boundary lines for this system are parallel to each other. Note how they have the same slopes.
\[
\begin{aligned}
& y=2 x+1 \\
& y=2 x-3
\end{aligned}
\]

Plotting the boundary lines will give the graph below. Note that the inequality \(y<2 x-3\) requires that we draw a dashed line, while the inequality \(y \geq 2 x+1\) requires a solid line.


Now we need to shade the regions that represent the inequalities. For the inequality \(y \geq 2 x+1\), we can test a point on either side of the line to see which region to shade. Test \((0,0)\) to make the task less challenging.
Substitute ( 0,0 ) into \(y \geq 2 x+1\)
\(y \geq 2 x+1\)
\(0 \geq 2(0)+1\)
\(0 \geq 1\)
This is not true, so we know that we need to shade the other side of the boundary line for the inequality \(y \geq 2 x+1\). The graph will now look like this:


Now shade the region that shows the solutions to the inequality \(y<2 x-3\). Again, we can pick \((0,0)\) to test to make our work less challenging.
Substitute ( 0,0 ) into \(y<2 x-3\)
\(y<2 x-3\)
\(0<2(0) x-\),
\(0<-3\)
This is not true, so we know that we need to shade the other side of the boundary line for the inequality \(y<2 x-3\). The graph will now look like this:


This system of inequalities has no points in common so has no solution.
- Solutions to systems of linear inequalities are entire regions of points.
- You can verify whether a point is a solution to a system of linear inequalities in the same way you verify whether a point is a solution to a system of equations.
- Systems of inequalities can have no solutions when boundary lines are parallel.
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\section*{GRAPH SOLUTIONS TO SYSTEMS OF LINEAR INEQUALITIES}

\section*{LEARNING OUTCOMES}
- Solve systems of linear inequalities by graphing the solution region
- Graph solutions to a system that contains a compound inequality

We will continue to practice graphing the solution region for systems of linear inequalities. We will also graph the solutions to a system that includes a compound inequality.

The videos that follow show more examples of graphing the solution set of a system of linear inequalities.
The system in our last example includes a compound inequality. We will see that you can treat a compound inequality like two lines when you are graphing them.

In the video that follows, we show how to solve another system of inequalities that contains a compound inequality.

\section*{EXAMPLE}

Shade the region of the graph that represents solutions for both inequalities. \(x+y \geq 1\) and \(y-x \geq 5\).

\section*{Answer}

Graph one inequality. First graph the boundary line using a table of values, intercepts, or any other method you prefer. The boundary line for \(x+y \geq 1\) is \(x+y=1\), or \(y=-x+1\). Since the equal sign is included with the greater than sign, the boundary line is solid.


Find an ordered pair on either side of the boundary line. Insert the \(x\) and \(y\)-values into the inequality \(x+y \geq 1\) and see which ordered pair results in a true statement.
Test \(1:(-3,0)\)
\(x+y \geq 1\)
\(-3+0 \geq 1\)
\(-3 \geq 1\)

\section*{FALSE}

Test \(2:(4,1)\)
\(x+y \geq 1\)
\(4+1 \geq 1\)
\(5 \geq 1\)
TRUE
Since \((4,1)\) results in a true statement, the region that includes \((4,1)\) should be shaded.


Do the same with the second inequality. Graph the boundary line, then test points to find which region is the solution to the inequality. In this case, the boundary line is \(y-x=5\) (or \(y=x+5\) ) and is solid. Test point \((-3,0)\) is not a solution of \(y-x \geq 5\) and test point \((0,6)\) is a solution.


The purple region in this graph shows the set of all solutions of the system.



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\section*{EXAMPLE}

Find the solution to the system \(3 x+2 y<12\) and \(-1 \leq y \leq 5\).
Answer
Graph one inequality. First graph the boundary line, then test points.
Remember, because the inequality \(3 x+2 y<12\) does not include the equal sign, draw a dashed border line.
Testing a point like \((0,0)\) will show that the area below the line is the solution to this inequality.


The inequality \(-1 \leq y \leq 5\) is actually two inequalities: \(-1 \leq y\), and \(y \leq 5\). Another way to think of this is \(y\) must be between -1 and 5 . The border lines for both are horizontal. The region between those two lines contains the solutions of \(-1 \leq y \leq 5\). We make the lines solid because we also want to include \(y=-1\) and \(y=5\).
Graph this region on the same axes as the other inequality.


The purple region shows the set of all solutions of the system.



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\section*{APPLICATIONS OF SYSTEMS OF LINEAR INEQUALITIES}

\section*{Learning Outcomes}
- Write and graph a system that models the quantity that must be sold to achieve a given amount of sales

We can use what we now know about inequalities to write and graph a system of linear inequalities that models the amount of sales needed to obtain a specific amount of money.

Watch the video below for another example of solving an application using a system of linear inequalities.
We have seen that systems of linear equations and inequalities can help to define market behaviors that are very helpful to businesses. You'll explore this idea more deeply later in this module.

\section*{EXAMPLE}

Cathy is selling ice cream cones at a school fundraiser. She is selling two sizes: small (which has 1 scoop) and large (which has 2 scoops). She knows that she can get a maximum of 70 scoops of ice cream out of her supply. She charges \(\$ 3\) for a small cone and \(\$ 5\) for a large cone.
Cathy wants to earn at least \(\$ 120\) to give back to the school. Write and graph a system of inequalities that models this situation.
Answer
First, identify the variables. There are two variables: the number of small cones sold and the number of large cones sold.
\(s=\) the number of small cones sold
\(I=\) the number of large cones sold
Write the first equation: the maximum number of scoops she can give out. The scoops she has available (70) must be greater than or equal to the number of scoops used to make the small cones and the large cones she sells. That is, 1 scoop each times the number of small cones sold together with 2 scoops each times the number of large cones sold cannot exceed the total supply of 70 scoops.
\(s+2 l \leq 70\)
Write the second equation: the amount of money she raises. She wants the total amount of money earned from small cones, \(\$ 3\) times the number sold, and large cones \(\$ 5\) times the number sold to be at least \(\$ 120\).
\(3 s+5 l \geq 120\)
Write the system.
\(\left\{\begin{array}{l}s+2 l \leq 70 \\ 3 s+5 l \geq 120 \\ s>=0 \\ l>=0\end{array}\right.\)
Now graph the system. The variables \(x\) and \(y\) have been replaced by \(s\) and \(l\); graph \(s\) along the \(x\)-axis and I along the \(y\)-axis.
First graph the region \(s+2 l \leq 70\). Graph the boundary line and then test individual points to see which region to shade. We only shade the regions that also satisfy \(x \geq 0, y \geq 0\). The graph is shown below.
(s)

Now graph the region \(3 s+5 l \geq 120\) Graph the boundary line and then test individual points to see which region to shade. The graph is shown below.


Graphing the regions together, you find the following:


Looking at just the overlapping region, you have:


The region in purple is the solution. As long as the combination of the number of small cones and large cones that Cathy sells can be mapped in the purple region, she will have earned at least \(\$ 120\) and not used more than 70 scoops of ice cream.


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\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
- To graph an inequality, first graph the related boundary line by replacing the given inequality symbol with the equality symbol to find its equation. Use a dashed line for a strict inequality. Then, test ordered pairs on either side of the boundary line and shade the side of the graph containing the point that satisfies the inequality equation.
- The solution to a system of linear inequalities is the region containing all the ordered pairs satisfying the system.
- You can verify whether a point is a solution to a system of linear inequalities in the same way you verify whether a point is a solution to a system of equations.
- Systems of inequalities have no solutions when boundary lines are parallel.
- Systems of inequalities can be used to model certain business situations.

\section*{Glossary}
boundary line replacing the inequality symbol of an inequality with an equality symbol and graphing the resulting line will yield the boundary line of the inequality
solution region all possible points \((x, y)\) on a graph that satisfy an inequality or system of inequalities
system of inequalities two or more inequalities with the requirement that solutions for the system must be solutions of all inequalities in the system

\section*{INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS: TWO VARIABLES}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Solve systems of equations by graphing.
- Solve systems of equations by substitution.
- Solve systems of equations by addition.
- Identify inconsistent systems of equations containing two variables.
- Express the solution of a system of dependent equations containing two variables using standard notations.

(credit: Thomas Sørenes)
A skateboard manufacturer introduces a new line of boards. The manufacturer tracks its costs, which is the amount it spends to produce the boards, and its revenue, which is the amount it earns through sales of its boards. How can the company determine if it is making a profit with its new line? How many skateboards must be produced and sold before a profit is possible? In this section we will consider linear equations with two variables to answer these and similar questions.

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\section*{INTRODUCTION TO SOLUTIONS OF SYSTEMS}

\section*{LEARNING OUTCOMES}
- Identify the three types of solutions possible from a system of two linear equations.
- Use a graph to find solution(s) to a system of two linear equations.

In order to investigate situations such as that of the skateboard manufacturer, we need to recognize that we are dealing with more than one variable and likely more than one equation. A system of linear equations consists of two or more linear equations made up of two or more variables such that all equations in the system are considered simultaneously. To find the unique solution to a system of linear equations, we must find a numerical value for each variable in the system that will satisfy all equations in the system at the same
time. Some linear systems may not have a solution and others may have an infinite number of solutions. In order for a linear system to have a unique solution, there must be at least as many equations as there are variables. Even so, this does not guarantee a unique solution.

In this section, we will look at systems of linear equations in two variables, which consist of two equations that contain two different variables. For example, consider the following system of linear equations in two variables.
\(2 x+y=15\)
\(3 x-y=5\)
The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently. In this example, the ordered pair \((4,7)\) is the solution to the system of linear equations. We can verify the solution by substituting the values into each equation to see if the ordered pair satisfies both equations. Shortly we will investigate methods of finding such a solution if it exists.
\(2(4)+(7)=15 \quad\) True
\(3(4)-(7)=5 \quad\) True
In addition to considering the number of equations and variables, we can categorize systems of linear equations by the number of solutions. A consistent system of equations has at least one solution. A consistent system is considered to be an independent system if it has a single solution, such as the example we just explored. The two lines have different slopes and intersect at one point in the plane. A consistent system is considered to be a dependent system if the equations have the same slope and the same \(y\)-intercepts. In other words, the lines coincide so the equations represent the same line. Every point on the line represents a coordinate pair that satisfies the system. Thus, there are an infinite number of solutions.

Another type of system of linear equations is an inconsistent system, which is one in which the equations represent two parallel lines. The lines have the same slope and different \(y\)-intercepts. There are no points common to both lines; hence, there is no solution to the system.

\section*{TIP FOR SUCCESS}

Flashcards are a good method for memorizing specific terminology used in mathematics. Your instructor will be using the terms introduced in the text on quizzes and tests, so it is important to know what they mean when you encounter them in context.

\section*{A GENERAL NOTE: TYPES OF LINEAR SYSTEMS}

There are three types of systems of linear equations in two variables, and three types of solutions.
- An independent system has exactly one solution pair \((x, y)\). The point where the two lines intersect is the only solution.
- An inconsistent system has no solution. Notice that the two lines are parallel and will never intersect.
- A dependent system has infinitely many solutions. The lines are coincident. They are the same line, so every coordinate pair on the line is a solution to both equations.

Below is a comparison of graphical representations of each type of system.


Independent System


Inconsistent System


Dependent System

\section*{HOW TO: GIVEN A SYSTEM OF LINEAR EQUATIONS AND AN ORDERED PAIR, DETERMINE WHETHER THE ORDERED PAIR IS A SOLUTION.}
1. Substitute the ordered pair into each equation in the system.
2. Determine whether true statements result from the substitution in both equations; if so, the ordered pair is a solution.

Watch the following video for another example of how to verify whether an ordered pair is a solution to a system of equations.

\section*{Solving Systems of Equations by Graphing}

There are multiple methods of solving systems of linear equations. For a system of linear equations in two variables, we can determine both the type of system and the solution by graphing the system of equations on the same set of axes.

Watch the following video for an example of how to identify what type of system and solution are represented in the graph of a system.

Watch the next video to see how to solve a system of equations by first graphing the lines and then identifying the type of solution.

\section*{EXAMPLE: DETERMINING WHETHER AN ORDERED PAIR IS A SOLUTION TO A SYSTEM OF EQUATIONS}

Determine whether the ordered pair \((5,1)\) is a solution to the given system of equations.
\(x+3 y=8\)
\(2 x-9=y\)
Answer
Substitute the ordered pair \((5,1)\) into both equations.
\[
\begin{aligned}
(5)+3(1) & =8 & & \\
8 & =8 & & \text { True } \\
2(5)-9 & =(1) & & \\
1 & =1 & & \text { True }
\end{aligned}
\]

The ordered pair \((5,1)\) satisfies both equations, so it is the solution to the system.

\section*{Analysis of the Solution}

We can see the solution clearly by plotting the graph of each equation. Since the solution is an ordered pair that satisfies both equations, it is a point on both of the lines and thus the point of intersection of the two lines.



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\section*{TRY IT}

Determine whether the ordered pair \((8,5)\) is a solution to the following system.
\(5 x-4 y=20\)
\(2 x+1=3 y\)
Answer
Not a solution.

\section*{TIP FOR SUCCESS}

Making a quick sketch of any mathematical situation is often a good idea to help you visualize it. Recall the techniques for graphing linear equations include using the \(y\)-intercept and slope to plot two points as well as using the intercepts. With practice, you'll get a feel for which technique to use in a given situation.

\section*{EXAMPLE: SOLVING A SYSTEM OF EQUATIONS IN TWO VARIABLES BY GRAPHING}

Solve the following system of equations by graphing. Identify the type of system.
\(2 x+y=-8\)
\(x-y=-1\)
Answer
Solve the first equation for \(y\).
\(2 x+y=-8\)
\(y=-2 x-8\)
Solve the second equation for \(y\).
\(x-y=-1\)
\[
y=x+1
\]

Graph both equations on the same set of axes, use function notation so you can check your solution more easily later. For example, \(f(x)=-2 x-8\), and \(g(x)=x+1\)
The lines appear to intersect at the point \((-3,-2)\). You can check to make sure that this is the solution to the system by substituting the ordered pair into both equations.
Enter \(f(-3)\) and \(g(-3)\) on the next two lines. Both functions have a value of -2 when \(x=-3\) therefore the point \((-3,-2)\) is a solution to the system.
This system is independent.

\section*{TRY IT}

Solve the following system of equations by graphing.
\(2 x-5 y=-25\)
\(-4 x+5 y=35\)
Answer
The solution to the system is the ordered pair \((-5,3)\).

\section*{Q\& A}

Can graphing be used if the system is inconsistent or dependent?

Yes, in both cases we can still graph the system to determine the type of system and solution. If the two lines are parallel, the system has no solution and is inconsistent. If the two lines are identical, the system has infinite solutions and is a dependent system.


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\section*{TRY IT}

Plot the three different systems with an online graphing calculator. Categorize each solution as either consistent or inconsistent. If the system is consistent determine whether it is dependent or independent. You may find it easier to plot each system individually, then clear out your entries before you plot the next.
1)
\(5 x-3 y=-19\)
\(x=2 y-1\)
2)
\(4 x+y=11\)
\(-2 y=-25+8 x\)
3)
\(y=-3 x+6\)
\(-\frac{1}{3} y+2=x\)
Answer
1. One solution - consistent, independent
2. No solutions, inconsistent, neither dependent nor independent
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\section*{THE SUBSTITUTION AND ADDITION METHODS}

\section*{LEARNING OUTCOMES}
- Use the substitution method to find solution(s) to a system of two linear equations.
- Use the addition method to find solution(s) to a system of linear equations.

\section*{Solving Systems of Equations by Substitution}

Solving a linear system in two variables by graphing works well when the solution consists of integer values, but if our solution contains decimals or fractions, it is not the most precise method. We will consider two more methods of solving a system of linear equations that are more precise than graphing. One such method is solving a system of equations by the substitution method, in which we solve one of the equations for one variable and then substitute the result into the second equation to solve for the second variable. Recall that we can solve for only one variable at a time, which is the reason the substitution method is both valuable and practical.

\section*{HOW TO: GIVEN A SYSTEM OF TWO EQUATIONS IN TWO VARIABLES, SOLVE USING THE SUBSTITUTION METHOD.}
1. Solve one of the two equations for one of the variables in terms of the other.
2. Substitute the expression for this variable into the second equation, then solve for the remaining variable.
3. Substitute that solution into either of the original equations to find the value of the first variable. If possible, write the solution as an ordered pair.
4. Check the solution in both equations.

In the following video, you will be given an example of solving a system of two equations using the substitution method.

If you had chosen the other equation to start with in the previous example, you would still be able to find the same solution. It is really a matter of preference, because sometimes solving for a variable will result in having to work with fractions. As you become more experienced with algebra, you will be able to anticipate

\section*{EXAMPLE: SOLVING A SYSTEM OF EQUATIONS IN TWO VARIABLES BY SUBSTITUTION}

Solve the following system of equations by substitution.
\[
\begin{aligned}
-x+y & =-5 \\
2 x-5 y & =1
\end{aligned}
\]

\section*{Answer}

First, we will solve the first equation for \(y\).
\[
\begin{aligned}
-x+y & =-5 \\
y & =x-5
\end{aligned}
\]

Now we can substitute the expression \(x-5\) for \(y\) in the second equation.
\[
\begin{aligned}
2 x-5 y & =1 \\
2 x-5(x-5) & =1 \\
2 x-5 x+25 & =1 \\
-3 x & =-24 \\
x & =8
\end{aligned}
\]

Now, we substitute \(x=8\) into the first equation and solve for \(y\).
\(-(8)+y=-5\)
\[
y=3
\]

Our solution is \((8,3)\).
Check the solution by substituting \((8,3)\) into both equations.
\[
\begin{aligned}
-x+y & =-5 & & \text { True } \\
-(8)+(3) & =-5 & & \\
2 x-5 y & =1 & & \\
2(8)-5(3) & =1 & & \text { True }
\end{aligned}
\]


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what choices will lead to more desirable outcomes.

Since the concepts in this module rely heavily upon those from earlier modules, feel free to go back to review information you may not have fully obtained the first time. Math is best digested over time with repetition that leads to deeper appreciation of the ideas involved.
Additional practice will also help you build skills using learned concepts in creative ways to make good choices of technique in different situations.

\section*{TRY IT}

You can use an online graphing calculator to help you solve a system of equations by substitution. We will use the following system to show you how:
\(x=y+3\)
\(4=3 x-2 y\)
First, solve both equations for y :
\(y=x-3\)
\(y=\frac{3}{2} x-2\)
Now enter \(x-3=\frac{3}{2} x-2\) into the calculator. You will see that calculator has provided you with \(x=-2\).
You now can substitute \(x=-2\) into both equations. If you get the same result for both, you have found an ordered pair solution. Give it a try.

\section*{Answer}
\((-2,-5)\)
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\section*{Q \& A}

Can the substitution method be used to solve any linear system in two variables?
Yes, but the method works best if one of the equations contains a coefficient of 1 or -1 so that we do not have to deal with fractions.

The following video is \(\sim 10\) minutes long and provides a mini-lesson on using the substitution method to solve a system of linear equations. We present three different examples, and also use a graphing tool to help summarize the solution for each example.


\section*{Solving Systems of Equations in Two Variables by the Addition Method}

A third method of solving systems of linear equations is the addition method, this method is also called the elimination method. In this method, we add two terms with the same variable, but opposite coefficients, so that the sum is zero. Of course, not all systems are set up with the two terms of one variable having opposite coefficients. Often we must adjust one or both of the equations by multiplication so that one variable will be eliminated by addition.

\section*{HOW TO: GIVEN A SYSTEM OF EQUATIONS, SOLVE USING THE ADDITION METHOD.}
1. Write both equations with \(x\) - and \(y\)-variables on the left side of the equal sign and constants on the right.
2. Write one equation above the other, lining up corresponding variables. If one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, add the equations together, eliminating one variable. If not, use multiplication by a nonzero number so that one of the variables in the top equation has the opposite coefficient of the same variable in the bottom equation, then add the equations to eliminate the variable.
3. Solve the resulting equation for the remaining variable.
4. Substitute that value into one of the original equations and solve for the second variable.
5. Check the solution by substituting the values into the other equation.

In the following video, you will see another example of how to use the method of elimination to solve a system of linear equations.

Below is another video example of using the elimination method to solve a system of linear equations when multiplication of one equation is required.

In the following video, you will find one more example of using the elimination method to solve a system; this one has coefficients that are fractions.
in the following video we present more examples of how to use the addition (elimination) method to solve a system of two linear equations.

\section*{EXAMPLE: SOLVING A SYSTEM BY THE ADDITION METHOD}

Solve the given system of equations by addition.
\[
\begin{aligned}
x+2 y & =-1 \\
-x+y & =3
\end{aligned}
\]

\section*{Answer}

Both equations are already set equal to a constant. Notice that the coefficient of \(x\) in the second equation, -1 , is the opposite of the coefficient of \(x\) in the first equation, 1 . We can add the two equations to eliminate \(x\) without needing to multiply by a constant.
\[
\begin{aligned}
x+2 y & =-1 \\
-x+y & =3 \\
\hline 3 y & =2
\end{aligned}
\]

Now that we have eliminated \(x\), we can solve the resulting equation for \(y\).
\[
\begin{aligned}
3 y & =2 \\
y & =\frac{2}{3}
\end{aligned}
\]

Then, we substitute this value for \(y\) into one of the original equations and solve for \(x\).
\[
\begin{aligned}
-x+y & =3 \\
-x+\frac{2}{3} & =3 \\
-x & =3-\frac{2}{3} \\
-x & =\frac{7}{3} \\
x & =-\frac{7}{3}
\end{aligned}
\]

The solution to this system is \(\left(-\frac{7}{3}, \frac{2}{3}\right)\).
Check the solution in the first equation.
\[
\begin{aligned}
x+2 y & =-1 \\
\left(-\frac{7}{3}\right)+2\left(\frac{2}{3}\right) & = \\
-\frac{7}{3}+\frac{4}{3} & = \\
-\frac{3}{3} & = \\
-1 & =-1 \quad \text { True }
\end{aligned}
\]

\section*{Analysis of the Solution}

We gain an important perspective on systems of equations by looking at the graphical representation. See the graph below to find that the equations intersect at the solution. We do not need to ask whether there may be a second solution because observing the graph confirms that the system has exactly one solution.



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\section*{TRY IT}

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\section*{EXAMPLE: USING THE ADDITION METHOD WHEN MULTIPLICATION OF ONE EQUATION IS REQUIRED}

Solve the given system of equations by the addition method.
\[
\begin{aligned}
3 x+5 y & =-11 \\
x-2 y & =11
\end{aligned}
\]

\section*{Answer}

Adding these equations as presented will not eliminate a variable. However, we see that the first equation has \(3 x\) in it and the second equation has \(x\). So if we multiply the second equation by -3 , the \(x\)-terms will add to zero.
\[
\begin{aligned}
x-2 y & =11 & & \\
-3(x-2 y) & =-3(11) & & \text { Multiply both sides by }-3 \\
-3 x+6 y & =-33 & & \text { Use the distributive property. }
\end{aligned}
\]

Now, let's add them.
\[
\begin{aligned}
3 x+5 y & =-11 \\
-3 x+6 y & =-33 \\
\hline 11 y & =-44 \\
y & =-4
\end{aligned}
\]

For the last step, we substitute \(y=-4\) into one of the original equations and solve for \(x\).
\[
\begin{aligned}
3 x+5 y & =-11 \\
3 x+5(-4) & =-11 \\
3 x-20 & =-11 \\
3 x & =9 \\
x & =3
\end{aligned}
\]

Our solution is the ordered pair \((3,-4)\). Check the solution in the original second equation.
\[
x-2 y=11
\]
(3) \(-2(-4)=3+8\)
\[
=11 \quad \text { True }
\]



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\section*{TIP FOR SUCCESS}

Remember to work through the given examples and practice problems on paper and more than once to deepen your understanding.

\section*{TRY IT}

Solve the system of equations by addition.
\(2 x-7 y=2\)
\(3 x+y=-20\)
Answer
\((-6,-2)\)
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\section*{EXAMPLE: USING THE ADDITION METHOD WHEN MULTIPLICATION OF BOTH EQUATIONS IS REQUIRED}

Solve the given system of equations in two variables by addition.
\[
2 x+3 y=-16
\]
\(5 x-10 y=30\)

\section*{Answer}

One equation has \(2 x\) and the other has \(5 x\). The least common multiple is \(10 x\) so we will have to multiply both equations by a constant in order to eliminate one variable. Let's eliminate \(x\) by multiplying the first equation by -5 and the second equation by 2 .
\[
\begin{aligned}
-5(2 x+3 y) & =-5(-16) \\
-10 x-15 y & =80 \\
2(5 x-10 y) & =2(30) \\
10 x-20 y & =60
\end{aligned}
\]

Then, we add the two equations together.
\[
\begin{aligned}
-10 x-15 y & =80 \\
10 x-20 y & =60 \\
\hline-35 y & =140 \\
y & =-4
\end{aligned}
\]

Substitute \(y=-4\) into the original first equation.
\[
\begin{aligned}
2 x+3(-4) & =-16 \\
2 x-12 & =-16 \\
2 x & =-4 \\
x & =-2
\end{aligned}
\]

The solution is \((-2,-4)\). Check it in the other equation.
\[
\begin{aligned}
5 x-10 y & =30 \\
5(-2)-10(-4) & =30 \\
-10+40 & =30 \\
30 & =30
\end{aligned}
\]


\section*{EXAMPLE: USING THE ADDITION METHOD IN SYSTEMS OF EQUATIONS CONTAINING FRACTIONS}

Solve the given system of equations in two variables by addition.
\(\frac{x}{3}+\frac{y}{6}=3\)
\(\frac{x}{2}-\frac{y}{4}=1\)
Answer
First clear each equation of fractions by multiplying both sides of the equation by the least common denominator.
\(\begin{aligned} 6\left(\frac{x}{3}+\frac{y}{6}\right) & =6(3) \\ 2 x+y & =18\end{aligned}\)
\(4\left(\frac{x}{2}-\frac{y}{4}\right)=4(1)\)
\(2 x-y=4\)
Now multiply the second equation by -1 so that we can eliminate \(x\).
\[
\begin{aligned}
-1(2 x-y) & =-1(4) \\
-2 x+y & =-4
\end{aligned}
\]

Add the two equations to eliminate \(x\) and solve the resulting equation for \(y\).
\[
\begin{aligned}
2 x+y & =18 \\
-2 x+y & =-4 \\
\hline 2 y & =14 \\
y & =7
\end{aligned}
\]

Substitute \(y=7\) into the first equation.
\[
\begin{aligned}
2 x+(7) & =18 \\
2 x & =11 \\
x & =\frac{11}{2} \\
& =7.5
\end{aligned}
\]

The solution is \(\left(\frac{11}{2}, 7\right)\). Check it in the other equation.
\[
\begin{aligned}
\frac{x}{2}-\frac{y}{4} & =1 \\
\frac{11}{2}-\frac{7}{4} & =1 \\
\frac{11}{4}-\frac{7}{4} & =1 \\
\frac{4}{4} & =1
\end{aligned}
\]


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\section*{TRY IT}

Solve the system of equations by addition.
\(2 x+3 y=8\)
\(3 x+5 y=10\)

\section*{Answer}
(10, -4)
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\section*{CLASSIFY SOLUTIONS TO SYSTEMS}

\section*{LEARNING OUTCOMES}
- Determine whether a solution reveals that a system has one, many, or no solutions
- Express the general solution of a dependent system
- Interpret the solution to a system of equations that represents profits for a business
- Write a system of equations that models a given situation

Now that we have several methods for solving systems of equations, we can use the methods to identify inconsistent systems. Recall that an inconsistent system consists of parallel lines that have the same slope but different \(y\)-intercepts. They will never intersect. When searching for a solution to an inconsistent system, we will come up with a false statement, such as \(12=0\).

In the next video, see another example of using substitution to solve a system that has no solution.

\section*{Expressing the Solution of a System of Dependent Equations Containing Two Variables}

Recall that a dependent system of equations in two variables is a system in which the two equations represent the same line. Dependent systems have an infinite number of solutions because all of the points on one line are also on the other line. After using substitution or addition, the resulting equation will be an identity, such as \(0=0\).

Watch the next video for an example of a dependent system.

\section*{Using Systems of Equations to Investigate Profits}

Using what we have learned about systems of equations, we can return to the skateboard manufacturing problem at the beginning of the section. The skateboard manufacturer's revenue function is the function used to calculate the amount of money that comes into the business. It can be represented by the equation \(R=x p\), where \(x=\) quantity and \(p=\) price. The revenue function is shown in orange in the graph below.

The cost function is the function used to calculate the costs of doing business. It includes fixed costs, such as rent and salaries, and variable costs, such as utilities. The cost function is shown in blue in the graph below. The \(x\)-axis represents quantity in hundreds of units. The \(y\)-axis represents either cost or revenue in hundreds of dollars.

\section*{EXAMPLE: SOLVING AN INCONSISTENT SYSTEM OF EQUATIONS}

Solve the following system of equations.
\[
\begin{gathered}
x=9-2 y \\
x+2 y=13
\end{gathered}
\]

\section*{Answer}

We can approach this problem in two ways. Because one equation is already solved for \(x\), the most obvious step is to use substitution.
\[
\begin{aligned}
x+2 y & =13 \\
(9-2 y)+2 y & =13 \\
9+0 y & =13 \\
9 & =13
\end{aligned}
\]

Clearly, this statement is a contradiction because \(9 \neq 13\). Therefore, the system has no solution.
The second approach would be to first manipulate the equations so that they are both in slope-intercept form. We manipulate the first equation as follows.
\[
\begin{aligned}
x & =9-2 y \\
2 y & =-x+9 \\
y & =-\frac{1}{2} x+\frac{9}{2}
\end{aligned}
\]

We then convert the second equation expressed to slope-intercept form.
\[
\begin{gathered}
x+2 y=13 \\
2 y=-x+13 \\
y=-\frac{1}{2} x+\frac{13}{2}
\end{gathered}
\]

Comparing the equations, we see that they have the same slope but different \(y\)-intercepts. Therefore, the lines are parallel and do not intersect.
\(y=-\frac{1}{2} x+\frac{9}{2}\)
\(y=-\frac{1}{2} x+\frac{13}{2}\)

\section*{Analysis of the Solution}

Writing the equations in slope-intercept form confirms that the system is inconsistent because all lines will intersect eventually unless they are parallel. Parallel lines will never intersect; thus, the two lines have no points in common. The graphs of the equations in this example are shown below.



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\section*{TRY IT}

Solve the following system of equations in two variables.
\(2 y-2 x=2\)
\(2 y-2 x=6\)
Answer
No solution. It is an inconsistent system.
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\section*{TIP FOR SUCCESS}

Writing a general solution for a dependent system takes practice to feel natural. Work through the following example on paper, then apply the information in Writing the General Solution below.


The point at which the two lines intersect is called the break-even point. We can see from the graph that if 700 units are produced, the cost is \(\$ 3,300\) and the revenue is also \(\$ 3,300\). In other words, the company breaks even if they produce and sell 700 units. They neither make money nor lose money.

\section*{EXAMPLE: FINDING A SOLUTION TO A DEPENDENT SYSTEM OF LINEAR EQUATIONS}

Find a solution to the system of equations using the addition method.
\[
\begin{gathered}
x+3 y=2 \\
3 x+9 y=6
\end{gathered}
\]

\section*{Answer}

With the addition method, we want to eliminate one of the variables by adding the equations. In this case, let's focus on eliminating \(x\). If we multiply both sides of the first equation by -3 , then we will be able to eliminate the \(x\)-variable.
\[
\begin{aligned}
x+3 y & =2 \\
(-3)(x+3 y) & =(-3)(2) \\
-3 x-9 y & =-6
\end{aligned}
\]

Now add the equations.
\[
\begin{aligned}
-3 x-9 y & =-6 \\
+3 x+9 y & =6 \\
\hline 0 & =0
\end{aligned}
\]

We can see that there will be an infinite number of solutions that satisfy both equations.

\section*{Analysis of the Solution}

If we rewrote both equations in the slope-intercept form, we might know what the solution would look like before adding. Let's look at what happens when we convert the system to slope-intercept form.
\[
\begin{array}{cc}
x+3 y=2 & 9 x+9 y=6 \\
3 y=-x+2 & 9 y=-3 x+6 \\
y=-\frac{1}{3} x+\frac{2}{3} & y=-\frac{3}{9} x+\frac{6}{9} \\
& y=-\frac{1}{3} x+\frac{2}{3}
\end{array}
\]

Look at the graph below. Notice the results are the same. The general solution to the system is \(\left(x,-\frac{1}{3} x+\frac{2}{3}\right)\).


The shaded region to the right of the break-even point represents quantities for which the company makes a

\section*{WRITING THE GENERAL SOLUTION}

In the previous example, we presented an analysis of the solution to the following system of equations:
\(x+3 y=2\)
\(3 x+9 y=6\)
After a little algebra, we found that these two equations were exactly the same. We then wrote the general solution as \(\left(x,-\frac{1}{3} x+\frac{2}{3}\right)\). Why would we write the solution this way? In some ways, this representation tells us a lot. It tells us that \(x\) can be anything, \(x\) is \(x\). It also tells us that \(y\) is going to depend on \(x\), just like when we write a function rule. In this case, depending on what you put in for \(x, y\) will be defined in terms of \(x\) as \(-\frac{1}{3} x+\frac{2}{3}\).
In other words, there are infinitely many \((x, y)\) pairs that will satisfy this system of equations, and they all fall on the line \(f(x)-\frac{1}{3} x+\frac{2}{3}\).


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\section*{TRY IT}

Solve the following system of equations in two variables.
\[
\begin{aligned}
y-2 x & =5 \\
-3 y+6 x & =-15
\end{aligned}
\]

\section*{Answer}

The system is dependent so there are infinitely many solutions of the form \((x, 2 x+5)\).
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\section*{TRY IT}

Use an online graphing calculator to write three different systems:
- A system of equations with one solution
- A system of equations with no solutions
- A system of equations with infinitely many solutions
\(\qquad\)
profit. The shaded region to the left represents quantities for which the company suffers a loss. The profit function is the revenue function minus the cost function, written as \(P(x)=R(x)-C(x)\). Clearly, knowing the quantity for which the cost equals the revenue is of great importance to businesses.

In the following video, you will see an example of how to find the break-even point for a small sno-cone business.

\section*{Writing a System of Linear Equations Given a Situation}

It is rare to be given equations that neatly model behaviors that you encounter in business, rather, you will probably be faced with a situation for which you know key information as in the example above. Below, we summarize three key factors that will help guide you in translating a situation into a system.

Now let's practice putting these key factors to work. In the next example, we determine how many different types of tickets are sold given information about the total revenue and amount of tickets sold to an event.

See the video below for another example of translating a given situation into a system of two linear equations.

Sometimes, a system of equations can inform a decision. In our next example, we help answer the question, "Which truck rental company will give the best value?"

The applications for systems seems almost endless, but we will just show one more. The next few problems involve mixtures.

\section*{EXAMPLE: FINDING THE BREAK-EVEN POINT AND THE PROFIT FUNCTION USING SUBSTITUTION}

Given the cost function \(C(x)=0.85 x+35,000\) and the revenue function \(R(x)=1.55 x\), find the breakeven point and the profit function.
Answer
Write the system of equations using \(y\) to replace function notation.
\(y=0.85 x+35,000\)
\(y=1.55 x\)
Substitute the expression \(0.85 x+35,000\) from the first equation into the second equation and solve for \(x\).
\(0.85 x+35,000=1.55 x\)
\[
35,000=0.7 x
\]
\[
50,000=x
\]

Then, we substitute \(x=50,000\) into either the cost function or the revenue function.
\(1.55(50,000)=77,500\)
The break-even point is \((50,000,77,500)\).
The profit function is found using the formula \(P(x)=R(x)-C(x)\).
\[
P(x)=1.55 x-(0.85 x+35,000)
\]
\[
=0.7 x-35,000
\]

The profit function is \(P(x)=0.7 x-35,000\).

\section*{Analysis of the Solution}

The cost to produce 50,000 units is \(\$ 77,500\), and the revenue from the sales of 50,000 units is also \(\$ 77,500\). To make a profit, the business must produce and sell more than 50,000 units.


We see from the graph below that the profit function has a negative value until \(x=50,000\), when the graph crosses the \(x\)-axis. Then, the graph emerges into positive \(y\)-values and continues on this path as the profit function is a straight line. This illustrates that the break-even point for businesses occurs when the profit function is 0 . The area to the left of the break-even point represents operating at a loss.



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\section*{HOW TO: GIVEN A SITUATION THAT REPRESENTS A SYSTEM OF LINEAR EQUATIONS, WRITE THE SYSTEM OF EQUATIONS AND IDENTIFY THE SOLUTION.}
1. Identify the input and output of each linear model.
2. Identify the slope and \(y\)-intercept of each linear model.
3. Find the solution by setting the two linear functions equal to another and solving for \(x\), or find the point of intersection on a graph.

Work through the following examples on paper perhaps more than once to be sure you understand how the situations can be translated into a mathematical model. They use techniques you used earlier in the course to write models for real-world situations.

\section*{EXAMPLE: WRITING AND SOLVING A SYSTEM OF EQUATIONS IN TWO VARIABLES}

The cost of a ticket to the circus is \(\$ 25.00\) for children and \(\$ 50.00\) for adults. On a certain day, attendance at the circus is 2,000 and the total gate revenue is \(\$ 70,000\). How many children and how many adults bought tickets?

\section*{Answer}

Let \(c=\) the number of children and \(a=\) the number of adults in attendance.
The total number of people is 2,000 . We can use this to write an equation for the number of people at the circus that day.
\(c+a=2,000\)
The revenue from all children can be found by multiplying \(\$ 25.00\) by the number of children, \(25 c\). The revenue from all adults can be found by multiplying \(\$ 50.00\) by the number of adults, \(50 a\). The total revenue is \(\$ 70,000\). We can use this to write an equation for the revenue.
\(25 c+50 a=70,000\)
We now have a system of linear equations in two variables.
\[
c+a=2,000
\]
\[
25 c+50 a=70,000
\]

In the first equation, the coefficient of both variables is 1 . We can quickly solve the first equation for either \(c\) or \(a\). We will solve for \(a\).
\(c+a=2,000\)
\(a=2,000-c\)
Substitute the expression \(2,000-c\) in the second equation for \(a\) and solve for \(c\).
\[
\begin{aligned}
25 c+50(2,000-c) & =70,000 \\
25 c+100,000-50 c & =70,000 \\
-25 c & =-30,000 \\
c & =1,200
\end{aligned}
\]

Substitute \(c=1,200\) into the first equation to solve for \(a\).
\[
\begin{aligned}
1,200+a & =2,000 \\
a & =800
\end{aligned}
\]

We find that 1,200 children and 800 adults bought tickets to the circus that day.


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\section*{TRY IT}

Meal tickets at the circus cost \(\$ 4.00\) for children and \(\$ 12.00\) for adults. If 1,650 meal tickets were bought for a total of \(\$ 14,200\), how many children and how many adults bought meal tickets?
Answer
700 children, 950 adults

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\section*{EXAMPLE: BUILDING A SYSTEM OF LINEAR MODELS TO CHOOSE A TRUCK RENTAL COMPANY}

Jamal is choosing between two truck-rental companies. The first, Keep on Trucking, Inc., charges an upfront fee of \(\$ 20\), then 59 cents a mile. The second, Move It Your Way, charges an up-front fee of \(\$ 16\), then 63 cents a mile. (Note: Rates retrieved Aug 2, 2010 from http://www.budgettruck.com and http://www.uhaul.com/) When will Keep on Trucking, Inc. be the better choice for Jamal?

\section*{Answer}

The two important quantities in this problem are the cost and the number of miles driven. Because we have two companies to consider, we will define two functions.

Input d, distance driven in miles
Outputs \(\quad K(d)\) : cost, in dollars, for renting from Keep on Trucking \(M(d)\) cost, in dollars, for renting from Move It Your Way

Initial Value Up-front fee: \(K(0)=20\) and \(M(0)=16\)
\begin{tabular}{l|l}
\(\begin{array}{l}\text { Rate of } \\
\text { Change }\end{array}\) & \(K(d)=\$ 0.59 /\) mile and \(P(d)=\$ 0.63 /\) mile
\end{tabular}
A linear function is of the form \(f(x)=m x+b\). Using the rates of change and initial charges, we can write the equations
\(K(d)=0.59 d+20\)
\(M(d)=0.63 d+16\)
Using these equations, we can determine when Keep on Trucking, Inc., will be the better choice. Because all we have to make that decision from is the costs, we are looking for when Move It Your Way, will cost less, or when \(K(d)<M(d)\). The solution pathway will lead us to find the equations for the two functions, find the intersection, and then see where the \(K(d)\) function is smaller.


These graphs are sketched above, with \(K(d)\) in blue.
To find the intersection, we set the equations equal and solve:
\[
\begin{aligned}
K(d) & =M(d) \\
0.59 d+20 & =0.63 d+16 \\
4 & =0.04 d \\
100 & =d \\
d & =100
\end{aligned}
\]

This tells us that the cost from the two companies will be the same if 100 miles are driven. Either by looking at the graph, or noting that \(K(d)\) is growing at a slower rate, we can conclude that Keep on

\section*{TIP FOR SUCCESS}

These examples are challenging. Work through them on paper perhaps more than once or twice to get a feel for how the situations are being translated into mathematical models.

\section*{EXAMPLE: SOLVE A CHEMICAL MIXTURE PROBLEM}

A chemist has 70 mL of a \(50 \%\) methane solution. How much of a \(80 \%\) solution must she add so the final solution is \(60 \%\) methane?
Answer
We will use the following table to help us solve this mixture problem:
\begin{tabular}{|l|l|l|l|}
\hline & Amount & Part & Total \\
\hline Start & & & \\
\hline Add & & & \\
\hline Final & & & \\
\hline
\end{tabular}

We start with 70 mL of solution, and the unknown amount can be \(x\). The part is the percentages, or concentration of solution 0.5 for start, 0.8 for add.
\begin{tabular}{|l|l|l|l|}
\hline & Amount & Part & Total \\
\hline Start & 70 mL & 0.5 & \\
\hline Add & \(x\) & 0.8 & \\
\hline Final & \(70+x\) & 0.6 & \\
\hline
\end{tabular}

Add amount column to get final amount. The part for this amount is 0.6 because we want the final solution to be \(60 \%\) methane.
\begin{tabular}{|l|l|l|l|}
\hline & Amount & Part & Total \\
\hline Start & 70 mL & 0.5 & 35 \\
\hline Add & \(x\) & 0.8 & \(0.8 x\) \\
\hline Final & \(70+x\) & 0.6 & \(42+0.6 x\) \\
\hline
\end{tabular}

Multiply amount by part to get total. be sure to distribute on the last row: \((70+x) 0.6\).
If we add the start and add entries in the Total column, we get the final equation that represents the total amount and it's concentration.
\(35+0.8 x=42+0.6 x\)
\[
\begin{aligned}
0.2 x & =7 \\
\frac{0.2}{0.2} x & =\frac{7}{0.2} \\
x & =35
\end{aligned}
\]

35 mL of \(80 \%\) solution must be added to 70 mL of \(50 \%\) solution to get a \(60 \%\) solution of Methane.
The same process can be used if the starting and final amount have a price attached to them, rather than a percentage.

\section*{TRY IT}

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\section*{SUMMARY: SYSTEMS OF LINEAR EQUATIONS: TWO VARIABLES}

\section*{Key Concepts}
- A system of linear equations consists of two or more equations made up of two or more variables such that all equations in the system are considered simultaneously.
- The solution to a system of linear equations in two variables is any ordered pair that satisfies each equation independently.
- Systems of equations are classified as independent with one solution, dependent with an infinite number of solutions, or inconsistent with no solution.
- One method of solving a system of linear equations in two variables is by graphing. In this method, we graph the equations on the same set of axes.
- Another method of solving a system of linear equations is by substitution. In this method, we solve for one variable in one equation and substitute the result into the second equation.
- A third method of solving a system of linear equations is by addition, in which we can eliminate a variable by adding opposite coefficients of corresponding variables.
- It is often necessary to multiply one or both equations by a constant to facilitate elimination of a variable when adding the two equations together.
- Either method of solving a system of equations results in a false statement for inconsistent systems because they are made up of parallel lines that never intersect.
- The solution to a system of dependent equations will always be true because both equations describe the same line.
- Systems of equations can be used to solve real-world problems that involve more than one variable, such as those relating to revenue, cost, and profit.

\section*{Glossary}
addition method an algebraic technique used to solve systems of linear equations in which the equations are added in a way that eliminates one variable, allowing the resulting equation to be solved for the remaining variable; substitution is then used to solve for the first variable
break-even point the point at which a cost function intersects a revenue function; where profit is zero
consistent system a system for which there is a single solution to all equations in the system and it is an independent system, or if there are an infinite number of solutions and it is a dependent system
cost function the function used to calculate the costs of doing business; it usually has two parts, fixed costs and variable costs
dependent system a system of linear equations in which the two equations represent the same line; there are an infinite number of solutions to a dependent system
inconsistent system a system of linear equations with no common solution because they represent parallel lines, which have no point or line in common
independent system a system of linear equations with exactly one solution pair \((x, y)\)
profit function the profit function is written as \(P(x)=R(x)-C(x)\), revenue minus cost
revenue function the function that is used to calculate revenue, simply written as \(R=x p\), where \(x=\) quantity and \(p=\) price
substitution method an algebraic technique used to solve systems of linear equations in which one of the two equations is solved for one variable and then substituted into the second equation to solve for the second variable
system of linear equations a set of two or more equations in two or more variables that must be considered simultaneously.

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\section*{INTRODUCTION TO SYSTEMS OF NONLINEAR EQUATIONS AND INEQUALITIES}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Solve a system of nonlinear equations using substitution.
- Solve a system of nonlinear equations using elimination.
- Graph a nonlinear inequality.
- Graph a system of nonlinear inequalities.

Halley's Comet orbits the sun about once every 75 years. Its path can be considered to be a very elongated ellipse. Other comets follow similar paths in space. These orbital paths can be studied using systems of equations. These systems, however, are different from the ones we considered in the previous section because the equations are not linear.


Halley's Comet (credit: "NASA Blueshift"/Flickr)
In this section we will consider the intersection of a parabola and a line, a circle and a line, and a circle and an ellipse. The methods for solving systems of nonlinear equations are similar to those for linear equations.

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}

\title{
GRAPH NONLINEAR INEQUALITIES AND SYSTEMS OF NONLINEAR INEQUALITIES
}

\section*{LEARNING OUTCOMES}
- Graph a non linear inequality.
- Graph a system of nonlinear inequalities.

All of the equations in the systems that we have encountered so far have involved equalities, but we may also encounter systems that involve inequalities. We have already learned to graph linear inequalities by graphing the corresponding equation, and then shading the region represented by the inequality symbol. As a reminder, here is one to try. Choose the type of line first, then graph it, and click on the side that satisfies the inequality.

\section*{TRY IT}

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Now we will follow similar steps to graph a nonlinear inequality so that we can learn to solve systems of nonlinear inequalities. A nonlinear inequality is an inequality containing a nonlinear expression. Graphing a nonlinear inequality is much like graphing a linear inequality.

\section*{RECALL GRAPHING NONLINEAR EQUATIONS}

Nonlinear equations may be any type of equation you have studied so far than is not the equation of a line. They may be parabolas, as in the examples below, or any of the curves you've studied.
Feel free to return to earlier sections on functions to recall how to graph them by plotting points or by transforming the parent function.

Recall that when the inequality is strictly greater than, \(y>a\), or less than, \(y<a\), the graph is drawn with a dashed line. When the inequality is greater than or equal to, \(y \geq a\), or less than or equal to, \(y \leq a\), the graph is drawn with a solid line. The graphs will create regions in the plane, and we will test each region for a solution. If one point in the region works, the whole region works. That is the region we shade.

(a) an example of \(y>a\); (b) an example of \(y \geq a\); (c) an example of \(y<a\); (d) an example of \(y \leq a\)

\section*{HOW TO: GIVEN AN INEQUALITY BOUNDED BY A PARABOLA, SKETCH A GRAPH.}
1. Graph the parabola as if it were an equation. This is the boundary for the region that is the solution set.
2. If the boundary is included in the region (the operator is \(\leq\) or \(\geq\) ), the parabola is graphed as a solid line.
3. If the boundary is not included in the region (the operator is \(<\) or \(>\) ), the parabola is graphed as a dashed line.
4. Test a point in one of the regions to determine whether it satisfies the inequality statement. If the statement is true, the solution set is the region including the point. If the statement is false, the solution set is the region on the other side of the boundary line.
5 . Shade the region representing the solution set.

\section*{Graphing a System of Nonlinear Inequalities}

Now that we have learned to graph nonlinear inequalities, we can learn how to graph systems of nonlinear inequalities. A system of nonlinear inequalities is a system of two or more inequalities in two or more variables containing at least one inequality that is not linear. Graphing a system of nonlinear inequalities is similar to graphing a system of linear inequalities. The difference is that our graph may result in more shaded regions that represent a solution than we find in a system of linear inequalities. The solution to a nonlinear system of inequalities is the region of the graph where the shaded regions of the graph of each inequality overlap, or where the regions intersect, called the feasible region.

\section*{EXAMPLE: GRAPHING AN INEQUALITY FOR A PARABOLA}

Graph the inequality \(y>x^{2}+1\).

\section*{Answer}

First, graph the corresponding equation \(y=x^{2}+1\).
Since \(y>x^{2}+1\) has a greater than symbol, we draw the graph with a dashed line. Then we choose points to test both inside and outside the parabola. Let's test the points \((0,2)\) and \((2,0)\). One point is clearly inside the parabola and the other point is clearly outside.
\(y>x^{2}+1\)
\(2>(0)^{2}+1\)
\(2>1 \quad\) True
\(0>(2)^{2}+1\)
\(0>5\)
False
The graph is shown below. We can see that the solution set consists of all points inside the parabola, but not on the graph itself.


\section*{TRY IT}

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\section*{HOW TO: GIVEN A SYSTEM OF NONLINEAR INEQUALITIES, SKETCH A GRAPH.}
1. Find the intersection points by solving the corresponding system of nonlinear equations.
2. Graph the nonlinear equations.
3. Find the shaded regions of each inequality.
4. Identify the feasible region as the intersection of the shaded regions of each inequality or the set of points common to each inequality.

\section*{EXAMPLE: GRAPHING A SYSTEM OF INEQUALITIES}

Graph the given system of inequalities.
\[
\begin{aligned}
x^{2}-y & \leq 0 \\
2 x^{2}+y & \leq 12
\end{aligned}
\]

\section*{Answer}

These two equations are clearly parabolas. We can find the points of intersection by the elimination process: Add both equations and the variable \(y\) will be eliminated. Then we solve for \(x\).
\[
\begin{aligned}
x^{2}-y & =0 \\
2 x^{2}+y & =12 \\
\hline 3 x^{2} & =12 \\
x^{2} & =4 \\
x & = \pm 2
\end{aligned}
\]

Substitute the \(x\)-values into one of the equations and solve for \(y\).
\[
x^{2}-y=0
\]
\[
(2)^{2}-y=0
\]
\[
4-y=0
\]
\[
y=4
\]
\[
\begin{aligned}
(-2)^{2}-y & =0 \\
4-y & =0 \\
y & =4
\end{aligned}
\]

The two points of intersection are \((2,4)\) and \((-2,4)\). Notice that the equations can be rewritten as follows.
\(x^{2}-y \leq 0\)
\(x^{2} \leq y\)
\(y \geq x^{2}\)
\(2 x^{2}+y \leq 12\)
\(y \leq-2 x^{2}+12\)
Graph each inequality. The feasible region is the region between the two equations bounded by \(2 x^{2}+y \leq 12\) on the top and \(x^{2}-y \leq 0\) on the bottom.


\section*{TRY IT}

Graph the given system of inequalities.
\(y \geq x^{2}-1\)
\(x-y \geq-1\)

\section*{Answer}

Shade the area bounded by the two curves, above the quadratic and below the line.


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\section*{METHODS FOR SOLVING A SYSTEM OF NONLINEAR EQUATIONS}

\section*{LEARNING OUTCOMES}
- Solve a system that represents the intersection of a parabola and a line using substitution.
- Solve a system that represents the intersection of a circle and a line using substitution.
- Solve a system that represents the intersection of a circle and an ellipse using elimination.

A system of nonlinear equations is a system of two or more equations in two or more variables containing at least one equation that is not linear. Recall that a linear equation can take the form \(A x+B y+C=0\). Any equation that cannot be written in this form in nonlinear. The substitution method we used for linear systems is the same method we will use for nonlinear systems. We solve one equation for one variable and then substitute the result into the second equation to solve for another variable, and so on. There is, however, a variation in the possible outcomes.

\section*{TIP FOR SUCCESS}

A quick sketch of the equations in a system can help you anticipate the possible number of solutions to the system, and can help you avoid missing solutions.

\section*{Intersection of a Parabola and a Line}

There are three possible types of solutions for a system of nonlinear equations involving a parabola and a line.

\section*{A GENERAL NOTE: POSSIBLE TYPES OF SOLUTIONS FOR POINTS OF INTERSECTION OF A PARABOLA AND A LINE}

The graphs below illustrate possible solution sets for a system of equations involving a parabola and a line.
- No solution. The line will never intersect the parabola.
- One solution. The line is tangent to the parabola and intersects the parabola at exactly one point.
- Two solutions. The line crosses on the inside of the parabola and intersects the parabola at two points.

(a)

(b)

(c)

\section*{HOW TO: GIVEN A SYSTEM OF EQUATIONS CONTAINING A LINE AND A PARABOLA, FIND THE SOLUTION.}
1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the parabola equation.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

\section*{EXAMPLE: SOLVING A SYSTEM OF NONLINEAR EQUATIONS REPRESENTING A PARABOLA AND A LINE}

Solve the system of equations.
\(x-y=-1\)
\(y=x^{2}+1\)

\section*{Answer}

Solve the first equation for \(x\) and then substitute the resulting expression into the second equation.
\[
\begin{array}{ll}
x-y=-1 & \\
x=y-1 & \text { Solve for } x . \\
y=x^{2}+1 & \\
y=(y-1)^{2}+1 & \text { Substitute expression for } x . \\
y=\left(y^{2}-2 y+1\right)+1 & \text { Expand } \\
y=y^{2}-2 y+2 & \\
0=y^{2}-3 y+2 & \text { Set equal to } 0 \text { and solve. } \\
0=(y-2)(y-1) &
\end{array}
\]

Solving for \(y\) gives \(y=2\) and \(y=1\). Next, substitute each value for \(y\) into the first equation to solve for \(x\). Always substitute the value into the linear equation to check for extraneous solutions.
\[
\begin{gathered}
x-y=-1 \\
x-(2)=-1 \\
x=1 \\
x-(1)=-1 \\
x=0
\end{gathered}
\]

The solutions are \((1,2)\) and \((0,1)\), which can be verified by substituting these \((x, y)\) values into both of the original equations.


\section*{Q \& A \\ COULD WE HAVE SUBSTITUTED VALUES FOR \(y\) INTO THE SECOND EQUATION TO SOLVE FOR \(x\) IN EXAMPLE 1?}

Yes, but because \(x\) is squared in the second equation this could give us extraneous solutions for \(x\).
For \(y=1\)
\[
\begin{aligned}
& y=x^{2}+1 \\
& y=x^{2}+1 \\
& x^{2}=0 \\
& x= \pm \sqrt{0}=0
\end{aligned}
\]

This gives us the same value as in the solution.
For \(y=2\)
\[
\begin{aligned}
& y=x^{2}+1 \\
& 2=x^{2}+1 \\
& x^{2}=1 \\
& x= \pm \sqrt{1}= \pm 1
\end{aligned}
\]

Notice that -1 is an extraneous solution.

\section*{TRY IT}

Solve the given system of equations by substitution.
\[
\begin{gathered}
3 x-y=-2 \\
2 x^{2}-y=0
\end{gathered}
\]

Answer
\(\left(-\frac{1}{2}, \frac{1}{2}\right)\) and \((2,8)\)
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\section*{TRY IT}

Use an online graphing calculator to graph the parabola \(y=x^{2}+2 x-3\). Now graph the line \(a x+b y+c=0\) and adjust the values of \(a, b\), and \(c\) to find equations for lines that produce systems with the following types of solutions:
- One solution
- Two solutions
- No solutions

\section*{Intersection of a Circle and a Line}

\section*{A GENERAL NOTE: POSSIBLE TYPES OF SOLUTIONS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND A LINE}

The graph below illustrates possible solution sets for a system of equations involving a circle and a line.
- No solution. The line does not intersect the circle.
- One solution. The line is tangent to the circle and intersects the circle at exactly one point.
- Two solutions. The line crosses the circle and intersects it at two points.


\section*{HOW TO: GIVEN A SYSTEM OF EQUATIONS CONTAINING A LINE AND A CIRCLE, FIND THE SOLUTION.}
1. Solve the linear equation for one of the variables.
2. Substitute the expression obtained in step one into the equation for the circle.
3. Solve for the remaining variable.
4. Check your solutions in both equations.

\section*{Solving a System of Nonlinear Equations Using Elimination}

We have seen that substitution is often the preferred method when a system of equations includes a linear equation and a nonlinear equation. However, when both equations in the system have like variables of the second degree, solving them using elimination by addition is often easier than substitution. Generally, elimination is a far simpler method when the system involves only two equations in two variables (a two-bytwo system), rather than a three-by-three system, as there are fewer steps. As an example, we will investigate the possible types of solutions when solving a system of equations representing a circle and an ellipse.

In the following video, we present an example of how to solve a system of non-linear equations that represent the intersection of an ellipse and a hyperbola.

\section*{EXAMPLE: FINDING THE INTERSECTION OF A CIRCLE AND A LINE BY SUBSTITUTION}

Find the intersection of the given circle and the given line by substitution.
\[
\begin{gathered}
x^{2}+y^{2}=5 \\
y=3 x-5
\end{gathered}
\]

Answer
One of the equations has already been solved for \(y\). We will substitute \(y=3 x-5\) into the equation for the circle.
\[
\begin{gathered}
x^{2}+(3 x-5)^{2}=5 \\
x^{2}+9 x^{2}-30 x+25=5 \\
10 x^{2}-30 x+20=0
\end{gathered}
\]

Now, we factor and solve for \(x\).
\[
\begin{gathered}
10\left(x^{2}-3 x+2\right)=0 \\
10(x-2)(x-1)=0 \\
x=2 \quad x=1
\end{gathered}
\]

Substitute the two \(x\)-values into the original linear equation to solve for \(y\).
\[
\begin{aligned}
y & =3(2)-5 \\
& =1 \\
y & =3(1)-5 \\
& =-2
\end{aligned}
\]

The line intersects the circle at \((2,1)\) and \((1,-2)\), which can be verified by substituting these \((x, y)\) values into both of the original equations.


\section*{TRY IT}

Solve the system of nonlinear equations.
\(x^{2}+y^{2}=10\)
\(x-3 y=-10\)

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\section*{TIP FOR SUCCESS}

It is not necessary that you know the form of an equation given in a system in order to solve the system (such as with an ellipse or hyperbola in the examples below). If you are unable to sketch the graph of an equation given in the system, you must be extra diligent with your algebra though to avoid missing or extraneous solutions.

\section*{A GENERAL NOTE: POSSIBLE TYPES OF SOLUTIONS FOR THE POINTS OF INTERSECTION OF A CIRCLE AND AN ELLIPSE}

The figure below illustrates possible solution sets for a system of equations involving a circle and an ellipse.
- No solution. The circle and ellipse do not intersect. One shape is inside the other or the circle and the ellipse are a distance away from the other.
- One solution. The circle and ellipse are tangent to each other, and intersect at exactly one point.
- Two solutions. The circle and the ellipse intersect at two points.
- Three solutions. The circle and the ellipse intersect at three points.
- Four solutions. The circle and the ellipse intersect at four points.


No solution


One solution


Two solutions


Three solutions


Four solutions

\section*{EXAMPLE: SOLVING A SYSTEM OF NONLINEAR EQUATIONS REPRESENTING A CIRCLE AND AN ELLIPSE}

Solve the system of nonlinear equations.
\[
\begin{array}{r}
x^{2}+y^{2}=26 \\
3 x^{2}+25 y^{2}=100 \tag{2}
\end{array}
\]

\section*{Answer}

Let's begin by multiplying equation (1) by -3 , and adding it to equation (2).
\[
\begin{aligned}
(-3)\left(x^{2}+y^{2}\right) & =(-3)(26) \\
-3 x^{2}-3 y^{2} & =-78 \\
3 x^{2}+25 y^{2} & =100 \\
\hline 22 y^{2} & =22
\end{aligned}
\]

After we add the two equations together, we solve for \(y\).
\[
\begin{aligned}
& y^{2}=1 \\
& y= \pm \sqrt{1}= \pm 1
\end{aligned}
\]

Substitute \(y= \pm 1\) into one of the equations and solve for \(x\).
\[
\begin{aligned}
& x^{2}+(1)^{2}=26 \\
& x^{2}+1=26 \\
& x^{2}=25 \\
& x= \pm \sqrt{25}= \pm 5 \\
& x^{2}+(-1)^{2}=26 \\
& x^{2}+1=26 \\
& x^{2}=25= \pm 5
\end{aligned}
\]

There are four solutions:
\((5,1),(-5,1),(5,-1)\), and \((-5,-1)\).


\section*{TRY IT}

Use an online graphing calculator to find the solution to the system of equations.
\[
\begin{gathered}
4 x^{2}+y^{2}=13 \\
x^{2}+y^{2}=10
\end{gathered}
\]

Answer
\(\{(1,3),(1,-3),(-1,3),(-1,-3)\}\)


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\section*{SUMMARY: SYSTEMS OF NONLINEAR EQUATIONS AND INEQUALITIES}

\section*{Key Concepts}
- There are three possible types of solutions to a system of equations representing a line and a parabola: (1) no solution, the line does not intersect the parabola; (2) one solution, the line is tangent to the parabola; and (3) two solutions, the line intersects the parabola in two points.
- There are three possible types of solutions to a system of equations representing a circle and a line: (1) no solution, the line does not intersect the circle; (2) one solution, the line is tangent to the parabola; (3) two solutions, the line intersects the circle in two points.
- There are five possible types of solutions to the system of nonlinear equations representing an ellipse and a circle:
(1) no solution, the circle and the ellipse do not intersect; (2) one solution, the circle and the ellipse are tangent to each other; (3) two solutions, the circle and the ellipse intersect in two points; (4) three solutions, the circle and ellipse intersect in three places; (5) four solutions, the circle and the ellipse intersect in four points.
- An inequality is graphed in much the same way as an equation, except for \(>\) or <, we draw a dashed line and shade the region containing the solution set.
- Inequalities are solved the same way as equalities, but solutions to systems of inequalities must satisfy both inequalities.

\section*{Glossary}
feasible region the solution to a system of nonlinear inequalities that is the region of the graph where the shaded regions of each inequality intersect
nonlinear inequality an inequality containing a nonlinear expression
system of nonlinear equations a system of equations containing at least one equation that is of degree larger than one
system of nonlinear inequalities a system of two or more inequalities in two or more variables containing at least one inequality that is not linear

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\section*{INTRODUCTION TO SYSTEMS OF LINEAR EQUATIONS: THREE VARIABLES}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Solve systems of three equations in three variables.
- Identify inconsistent systems of equations containing three variables.
- Express the solution of a system of dependent equations containing three variables using standard notations.

(credit: "Elembis," Wikimedia Commons)
John received an inheritance of \(\$ 12,000\) that he divided into three parts and invested in three ways: in a money-market fund paying \(3 \%\) annual interest; in municipal bonds paying \(4 \%\) annual interest; and in mutual funds paying \(7 \%\) annual interest. John invested \(\$ 4,000\) more in municipal funds than in municipal bonds. He earned \(\$ 670\) in interest the first year. How much did John invest in each type of fund?

Understanding the correct approach to setting up problems such as this one makes finding a solution a matter of following a pattern. We will solve this and similar problems involving three equations and three variables in this section. Doing so uses similar techniques as those used to solve systems of two equations in two variables. However, finding solutions to systems of three equations requires a bit more organization and a touch of visual gymnastics.

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\section*{SOLVE SYSTEMS OF THREE EQUATIONS IN THREE VARIABLES}

\section*{LEARNING OUTCOMES}
- Determine whether an ordered triple is a solution to a system of three equations.
- Use back substitution to find a solution to a system of three equations.
- Write the equations for a system given a scenario, and solve.

In order to solve systems of equations in three variables, known as three-by-three systems, the primary goal is to eliminate one variable at a time to achieve back-substitution. A solution to a system of three equations in three variables \((x, y, z)\), is called an ordered triple.

To find a solution, we can perform the following operations:
1. Interchange the order of any two equations.
2. Multiply both sides of an equation by a nonzero constant.
3. Add a nonzero multiple of one equation to another equation.

Graphically, the ordered triple defines the point that is the intersection of three planes in space. You can visualize such an intersection by imagining any corner in a rectangular room. A corner is defined by three planes: two adjoining walls and the floor (or ceiling). Any point where two walls and the floor meet represents the intersection of three planes.

\section*{A GENERAL NOTE: NUMBER OF POSSIBLE SOLUTIONS}

The planes illustrate possible solution scenarios for three-by-three systems.
- Systems that have a single solution are those which, after elimination, result in a solution set consisting of an ordered triple \(\{(x, y, z)\}\). Graphically, the ordered triple defines a point that is the intersection of three planes in space.
- Systems that have an infinite number of solutions are those which, after elimination, result in an expression that is always true, such as \(0=0\). Graphically, an infinite number of solutions represents a line or coincident plane that serves as the intersection of three planes in space.
- Systems that have no solution are those that, after elimination, result in a statement that is a contradiction, such as \(3=0\). Graphically, a system with no solution is represented by three planes with no point in common.

(a)
(a)Three planes intersect at a single point, representing a three-by-three system with a single solution. (b) Three planes intersect in a line, representing a three-by-three system with infinite solutions.


\section*{EXAMPLE: DETERMINING WHETHER AN ORDERED TRIPLE IS A SOLUTION TO A SYSTEM}

Determine whether the ordered triple \((3,-2,1)\) is a solution to the system.
\(x+y+z=2\)
\(6 x-4 y+5 z=31\)
\(5 x+2 y+2 z=13\)

\section*{Answer}

We will check each equation by substituting in the values of the ordered triple for \(x, y\), and \(z\).
\(x+y+z=2\)
\((3)+(-2)+(1)=2\)
True
\(6(3)-4(-2)+5(1)=31\)
\(18+8+5=31\)
True
\[
\begin{aligned}
& 5(3)+2(-2)+2(1)=13 \\
& 15-4+2=13 \\
& \text { True }
\end{aligned}
\]

The ordered triple \((3,-2,1)\) is indeed a solution to the system.

\section*{HOW TO: GIVEN A LINEAR SYSTEM OF THREE EQUATIONS, SOLVE FOR THREE UNKNOWNS.}
1. Pick any pair of equations and solve for one variable.
2. Pick another pair of equations and solve for the same variable.
3. You have created a system of two equations in two unknowns. Solve the resulting two-by-two system.
4. Back-substitute known variables into any one of the original equations and solve for the missing variable.

\section*{TIP FOR SUCCESS}

Work through the following examples on paper in the order given. They slowly build up the technique of solving a three-by-three system in stages. Then you will have an opportunity to practice the 4 -step process given in the How To box above.

Solving a system with three variables is very similar to solving one with two variables. It is important to keep track of your work as the addition of one more equation creates more steps in the solution process.

We'll take the steps slowly in the following few examples. First, we'll look just at the last step in the process: back-substitution. Then, we'll look at an example that requires the addition (elimination) method to reach the first solution. Then we'll see some video examples that illustrate some of the different kinds of situations you may encounter when solving three-by-three systems. Finally, you'll have the opportunity to practice applying the complete process.

In the example that follows, we will solve the system by using back-substitution.
In the following video, we show another example of using back-substitution to solve a system in three variables.

In the next example we'll need to use the addition method (elimination) to find our first solution.
Watch the following videos for more examples of the algebra you may encounter when solving systems with three variables.

Now, try the example and problems that follow to see if the process is becoming familiar to you. Solving three-by-three systems involves both creativity and careful, well-organized work. It will take some practice before it begins to feel natural.

In the following video, you will see a visual representation of the three possible outcomes for solutions to a system of equations in three variables. There is also a worked example of solving a system using elimination.

\section*{Applications of Systems of Three Equations in Three Variables}

Now we are ready to handle the problem we encountered as we began this section by using what we know about linear equations to translate the situation into a system of three equations. Then, we'll use our new understanding of three-by-three systems to find the solution.

Systems of three equations in three variables apply to other types of real-world situations as well.
In this example, we will write three equations that model sales at an art fair to learn how many prints should be sold to break even for the cost of the booth rental.

In the following video example, we show how to define a system of three equations in three variables that represents a mixture needed by a chemist.

Our last example shows you how to write a system of three equations that represents ticket sales for a theater that has three different prices for tickets.

\section*{EXAMPLE}

Solve the given system.
\(\left\{\begin{aligned} x-\frac{1}{3} y+\frac{1}{2} z & =1 \\ y-\frac{1}{2} z & =4 \\ z & =-1\end{aligned}\right.\)

\section*{Answer}

The third equation states that \(z=-1\), so we substitute this into the second equation to obtain a solution for \(y\).
\[
\begin{aligned}
& y-\frac{1}{2}(-1)=4 \\
& y+\frac{1}{2}=4 \\
& y=4-\frac{1}{2} \\
& y=\frac{8}{2}-\frac{1}{2} \\
& y=\frac{7}{2}
\end{aligned}
\]

Now we have two of our solutions, and we can substitute them both into the first equation to solve for \(x\).
\[
\begin{aligned}
& x-\frac{1}{3}\left(\frac{7}{2}\right)+\frac{1}{2}(-1)=1 \\
& x-\frac{7}{6}-\frac{1}{2}=1 \\
& x-\frac{7}{6}-\frac{3}{6}=1 \\
& x-\frac{10}{6}=1 \\
& x=1+\frac{10}{6} \\
& x=\frac{6}{6}+\frac{10}{6} \\
& x=\frac{16}{6}=\frac{8}{3}
\end{aligned}
\]

Now we have our ordered triple; remember to place each variable solution in order.
\((x, y, z)=\left(\frac{8}{3}, \frac{7}{2},-1\right)\)
Analysis of the Solution:
Each of the lines in the system above represents a plane (think about a sheet of paper). If you imagine three sheets of notebook paper each representing a portion of these planes, you will start to see the complexities involved in how three such planes can intersect. Below is a sketch of the three planes. It turns out that any two of these planes intersect in a line, so our intersection point is where all three of these lines meet.


Three planes intersecting


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\section*{EXAMPLE}

Find a solution to the following system:
\[
\begin{align*}
& x-y+z=5 \\
& -2 y+z=6  \tag{2}\\
& 2 y-2 z=-12
\end{align*}
\]

\section*{Answer}

We labeled the equations this time to be able to keep track of things a little more easily. The most obvious first step here is to eliminate \(y\) by adding equations (2) and (3).
\[
\begin{array}{r}
-2 y+z=6 \\
2 y-2 z=-12  \tag{3}\\
\hline-z=-6 \\
z=6
\end{array}
\]

Now we can substitute the value for \(z\) that we obtained into equation (2).
\[
\begin{aligned}
& -2 y+(6)=6 \\
& -2 y=6-6 \\
& -2 y=0 \\
& y=0
\end{aligned}
\]

Be careful here not to get confused with a solution of \(y=0\) and an inconsistent solution. It is ok for variables to equal 0 .
Now we can substitute \(z=6\) and \(y=0\) back into the first equation.
\[
\begin{aligned}
& x-y+z=5 \\
& x-0+6=5 \\
& x+6=5 \\
& x=5-6 \\
& x=-1 \\
& (x, y, z)=(-1,0,6)
\end{aligned}
\]


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\section*{EXAMPLE: SOLVING A SYSTEM OF THREE EQUATIONS IN THREE VARIABLES BY ELIMINATION}

Find a solution to the following system:
\[
\begin{array}{r}
x-2 y+3 z=9 \\
-x+3 y-z=-6 \\
2 x-5 y+5 z=17 \tag{3}
\end{array}
\]

\section*{Answer}

There will always be several choices as to where to begin, but the most obvious first step here is to eliminate \(x\) by adding equations (1) and (2).
\[
\begin{align*}
x-2 y+3 z & =9  \tag{1}\\
-x+3 y-z & =-6  \tag{2}\\
\hline y+2 z & =3
\end{align*}
\]

The second step is multiplying equation (1) by -2 and adding the result to equation (3). These two steps will eliminate the variable \(x\).
\(-2 x+4 y-6 z=-18\)
(2) multiplied by -2
\(2 x-5 y+5 z=17\)

In equations (4) and (5), we have created a new two-by-two system. We can solve for \(z\) by adding the two equations.
\[
\begin{align*}
y+2 z & =3  \tag{4}\\
-y-z & =-1  \tag{5}\\
\hline z & =2
\end{align*}
\]

Choosing one equation from each new system, we obtain the upper triangular form:
\[
\begin{array}{r}
x-2 y+3 z=9 \\
y+2 z=3 \\
z=2 \tag{6}
\end{array}
\]

Next, we back-substitute \(z=2\) into equation (4) and solve for \(y\).
\[
\begin{aligned}
y+2(2) & =3 \\
y+4 & =3 \\
y & =-1
\end{aligned}
\]

Finally, we can back-substitute \(z=2\) and \(y=-1\) into equation (1). This will yield the solution for \(x\).
\[
\begin{aligned}
x-2(-1)+3(2) & =9 \\
x+2+6 & =9 \\
x & =1
\end{aligned}
\]

The solution is the ordered triple \((1,-1,2)\).


\section*{TRY IT}

Solve the system of equations in three variables.
\(2 x+y-2 z=-1\)
\(3 x-3 y-z=5\)
\(x-2 y+3 z=6\)

\section*{Answer}
\((1,-1,1)\)
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\section*{TIP FOR SUCCESS}

Applications of three-by-three systems are complicated. Work through each of the examples below perhaps more than once or twice. Don't be discouraged if you don't understand the process right away. It

\section*{EXAMPLE: SOLVING A REAL-WORLD PROBLEM USING A SYSTEM OF THREE EQUATIONS IN THREE VARIABLES}

In the problem posed at the beginning of the section, John invested his inheritance of \$12,000 in three different funds: part in a money-market fund paying \(3 \%\) interest annually; part in municipal bonds paying \(4 \%\) annually; and the rest in mutual funds paying \(7 \%\) annually. John invested \(\$ 4,000\) more in mutual funds than he invested in municipal bonds. The total interest earned in one year was \(\$ 670\). How much did he invest in each type of fund?

\section*{Answer}

To solve this problem, we use all of the information given and set up three equations. First, we assign a variable to each of the three investment amounts:
\(x=\) amount invested in money-market fund
\(y=\) amount invested in municipal bonds
\(z=\) amount invested in mutual funds
The first equation indicates that the sum of the three principal amounts is \(\$ 12,000\).
\(x+y+z=12,000\)
We form the second equation according to the information that John invested \(\$ 4,000\) more in mutual funds than he invested in municipal bonds.
\(z=y+4,000\)
The third equation shows that the total amount of interest earned from each fund equals \(\$ 670\).
\(0.03 x+0.04 y+0.07 z=670\)
Then, we write the three equations as a system.
\[
\begin{array}{r}
x+y+z=12,000 \\
-y+z=4,000 \\
0.03 x+0.04 y+0.07 z=670
\end{array}
\]

To make the calculations simpler, we can multiply the third equation by 100. Thus,
\[
\begin{array}{r}
x+y+z=12,000 \\
-y+z=4,000 \\
3 x+4 y+7 z=67,000 \tag{3}
\end{array}
\]

Step 1. Interchange equation (2) and equation (3) so that the two equations with three variables will line up.
\(x+y+z=12,000\)
\(3 x+4 y+7 z=67,000\)
\(-y+z=4,000\)
Step 2. Multiply equation (1) by -3 and add to equation (2). Write the result as row 2.
\(x+y+z=12,000\)
\(y+4 z=31,000\)
\(-y+z=4,000\)
Step 3. Add equation (2) to equation (3) and write the result as equation (3).
\(x+y+z=12,000\)
\[
y+4 z=31,000
\]
\[
5 z=35,000
\]

Step 4. Solve for \(z\) in equation (3). Back-substitute that value in equation (2) and solve for \(y\). Then, backsubstitute the values for \(z\) and \(y\) into equation (1) and solve for \(x\).
\(5 z=35,000\)
\(z=7,000\)
\(y+4(7,000)=31,000\)
\(y=3,000\)
\(x+3,000+7,000=12,000\)
\(x=2,000\)
John invested \(\$ 2,000\) in a money-market fund, \(\$ 3,000\) in municipal bonds, and \(\$ 7,000\) in mutual funds.

TRY IT

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\section*{EXAMPLE}

Andrea sells photographs at art fairs. She prices the photos according to size: small photos cost \(\$ 10\), medium photos cost \(\$ 15\), and large photos cost \(\$ 40\). She usually sells as many small photos as medium and large photos combined. She also sells twice as many medium photos as large. A booth at the art fair costs \(\$ 300\).
If her sales go as usual, how many of each size photo must she sell to pay for the booth?

\section*{Answer}

To set up the system, first choose the variables. In this case the unknown values are the number of small, medium, and large photos.
\(S\) = number of small photos sold
\(M\) = number of medium photos sold
\(L=\) number of large photos sold
The total of her sales must be \(\$ 300\) to pay for the booth. We can find the total by multiplying the cost for each size by the number of that size sold.
\(10 S=\) money received for small photos
\(15 M=\) money received for medium photos
\(40 L=\) money received for large photos
Total Sales: \(10 S+15 M+40 L=300\)
The number of small photos is the same as the total of medium and large photos combined.
S = M + L
She sells twice as many medium photos as large photos.
\(\mathrm{M}=2 \mathrm{~L}\)
To make things easier, rewrite the equations to be in the same format, with all variables on the left side of the equal sign and only a constant number on the right.
\(\left\{\begin{array}{r}10 S+15 M+40 L=300 \\ S-M-L=0 \\ M-2 L=0\end{array}\right.\)
Now solve the system.
Step 1: First choose two equations and eliminate a variable. Since one equation has no \(S\) variable, it may be helpful to use the other two equations and eliminate the \(S\) variable from them. Multiply both sides of the second equation by -10 .
\(-10(S-M-L)=-10(0)\)
\(-10 s+10 M+10 L=0\)
Now add this modified equation with the first equation in the original list of equation.
\(10 S+15 M+40 L=300\)
\(+(-10 s+10 M+10 L=0)\)
\(25 M+50 L=300\)
Step 2: The other equation for our two-variable system will be the remaining equation (that has no \(S\) variable). Eliminate a second variable using the equation from step 1 . While you could multiply the third of the original equations by 25 to eliminate \(L\), the numbers will stay nicer if you divide the resulting equation from step 1 by 25 . Do not forget to be careful of the signs!
Divide first:
\[
\frac{25}{25} M+\frac{50}{25} L=\frac{300}{25}
\]
\(M+2 L=12\)
Now eliminate L by adding \(\mathrm{M}-2 \mathrm{~L}=0\) to this new equation.
\(M+2 L=12\)
\(\underline{M-2 L=0}\)
\(2 M=12\)
\(M=\frac{12}{2}=6\)
Step 3: Use \(\mathrm{M}=6\) and one of the equations containing just two variables to solve for the second variable. It is best to use one of the original equation in case an error was made in multiplication.
\(M-2 L=0\)
\(6-2 L=0\)
\(-2 L=-6\)
\(L=3\)
Step 4: Use the two found values and one of the original equations to solve for the third variable.
\(S-M-L=0\)
\(S-6-3=0\)
\(S-9=0\)
\(S=9\)
Step 5: Check your answer. With application problems, it is sometimes easier (and better) to use the original wording of the problem rather than the equations you write.
She usually sells as many small photos as medium and large photos combined.
- Medium and large photos combined: \(6+3=9\), which is the number of small photos.

She also sells twice as many medium photos as large.
- Medium photos is 6 , which is twice the number of large photos (3).

A booth at the art fair costs \(\$ 300\).
- Andrea receives \(\$ 10(9)\) or \(\$ 90\) for the 9 small photos, \(\$ 15(6)\) or \(\$ 90\) for the 6 medium photos, and \(\$ 40(3)\) or \(\$ 120\) for the large photos. \(\$ 90+\$ 90+\$ 120=\$ 300\).

If Andrea sells 9 small photos, 6 medium photos, and 3 large photos, she will receive exactly the amount of money needed to pay for the booth.

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\section*{CLASSIFY SOLUTIONS TO SYSTEMS IN THREE VARIABLES}

\section*{LEARNING OUTCOMES}
- Recognize whether a system has one, none, or an infinite number of solutions based on its solution.
- Use correct notation to express solutions to systems of three equations.

Just as with systems of equations in two variables, we may come across an inconsistent system of equations in three variables, which means that it does not have a solution that satisfies all three equations. The equations could represent three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location. The process of elimination will result in a false statement, such as \(3=7\) or some other contradiction.

\section*{TIP FOR SUCCESS}

As you discovered when solving systems that have one solution, well-organized work is essential to being certain about the result you obtain. It can take several steps for the contradiction in a system with no solution to appear, as in the example below. Patient effort is essential.

\section*{EXAMPLE: SOLVING AN INCONSISTENT SYSTEM OF THREE EQUATIONS IN THREE VARIABLES}

Solve the following system.
\[
\begin{array}{r}
x-3 y+z=4 \\
-x+2 y-5 z=3 \\
5 x-13 y+13 z=8 \tag{3}
\end{array}
\]

Answer
Looking at the coefficients of \(x\), we can see that we can eliminate \(x\) by adding equation (1) to equation (2).
\[
\begin{array}{r}
x-3 y+z=4 \\
-x+2 y-5 z=3  \tag{2}\\
\hline-y-4 z=7
\end{array}
\]

Next, we multiply equation (1) by -5 and add it to equation (3).
\(-5 x+15 y-5 z=-20\)
(1) multiplied by -5
\begin{tabular}{rl}
\(5 x-13 y+13 z\) & \(=8\) \\
\hline \(2 y+8 z\) & \(=-12\)
\end{tabular}

Then, we multiply equation (4) by 2 and add it to equation (5).
\[
\begin{align*}
-2 y-8 z & =14 \quad(4) \text { multiplied by } 2 \\
2 y+8 z & =-12  \tag{5}\\
\hline 0 & =2
\end{align*}
\]

The final equation \(0=2\) is a contradiction, so we conclude that the system of equations in inconsistent and, therefore, has no solution.

\section*{Analysis of the Solution}

In this system, each plane intersects the other two, but not at the same location. Therefore, the system is inconsistent.

Watch the video below for another example of using elimination in a system that has no solution.


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\section*{TRY IT}

Solve the system of three equations in three variables.
\(x+y+z=2\)
\(y-3 z=1\)
\(2 x+y+5 z=0\)
Answer
No solution.

\section*{Expressing the Solution of a System of Dependent Equations Containing Three Variables}

We know from working with systems of equations in two variables that a dependent system of equations has an infinite number of solutions. The same is true for dependent systems of equations in three variables. An infinite number of solutions can result from several situations. The three planes could be the same, so that a solution to one equation will be the solution to the other two equations. All three equations could be different but they intersect on a line, which has infinite solutions. Or two of the equations could be the same and intersect the third on a line.

\section*{TIP FOR SUCCESS}

As you did with dependent two-by-two systems, you can write the general solution to a three-by-three system. See the example below for a demonstration, the try the following problem. It will take practice before it feels natural to write the general solution to a dependent system.

See the following video for another example of a dependent three-by-three system.

\section*{EXAMPLE: FINDING THE SOLUTION TO A DEPENDENT SYSTEM OF EQUATIONS}

Find the solution to the given system of three equations in three variables.
\[
\begin{array}{r}
2 x+y-3 z=0 \\
4 x+2 y-6 z=0 \\
x-y+z=0 \tag{3}
\end{array}
\]

Answer
First, we can multiply equation (1) by -2 and add it to equation (2).
\(-4 x-2 y+6 z=0\)
(1) multiplied by -2
\(4 x+2 y-6 z=0\)

We do not need to proceed any further. The result we get is an identity, \(0=0\), which tells us that this system has an infinite number of solutions. There are other ways to begin to solve this system, such as multiplying equation (3) by -2 , and adding it to equation (1). We then perform the same steps as above and find the same result, \(0=0\).
When a system is dependent, we can find general expressions for the solutions. Adding equations (1) and (3), we have
\[
\begin{array}{r}
2 x+y-3 z=0 \\
x-y+z=0 \\
\hline 3 x-2 z=0
\end{array}
\]

We then solve the resulting equation for \(z\).
\[
\begin{array}{r}
3 x-2 z=0 \\
z=\frac{3}{2} x
\end{array}
\]

We back-substitute the expression for \(z\) into one of the equations and solve for \(y\).
\(2 x+y-3\left(\frac{3}{2} x\right)=0\)
\(2 x+y-\frac{9}{2} x=0\)
\(y=\frac{9}{2} x-2 x\)
\(y=\frac{5}{2} x\)
So the general solution is \(\left(x, \frac{5}{2} x, \frac{3}{2} x\right)\). In this solution, \(x\) can be any real number. The values of \(y\) and \(z\) are dependent on the value selected for \(x\).

\section*{Analysis of the Solution}

As shown below, two of the planes are the same and they intersect the third plane on a line. The solution set is infinite, as all points along the intersection line will satisfy all three equations.


\section*{Q \& A}

Does the generic solution to a dependent system always have to be written in terms of \(x\) ?

No, you can write the generic solution in terms of any of the variables, but it is common to write it in terms of \(x\) and if needed \(x\) and \(y\).


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\section*{TRY IT}

Solve the following system.
\(x+y+z=7\)
\(3 x-2 y-z=4\)
\(x+6 y+5 z=24\)
Answer
Infinitely many number of solutions of the form \((x, 4 x-11,-5 x+18)\).
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\section*{SUMMARY: SYSTEMS OF LINEAR EQUATIONS: THREE VARIABLES}

\section*{Key Concepts}
- A solution set is an ordered triple \(\{(x, y, z)\}\) that represents the intersection of three planes in space.
- A system of three equations in three variables can be solved by using a series of steps that forces a variable to be eliminated. The steps include interchanging the order of equations, multiplying both sides of an equation by a nonzero constant, and adding a nonzero multiple of one equation to another equation.
- Systems of three equations in three variables are useful for solving many different types of real-world problems.
- A system of equations in three variables is inconsistent if no solution exists. After performing elimination operations, the result is a contradiction.
- Systems of equations in three variables that are inconsistent could result from three parallel planes, two parallel planes and one intersecting plane, or three planes that intersect the other two but not at the same location.
- A system of equations in three variables is dependent if it has an infinite number of solutions. After performing elimination operations, the result is an identity.
- Systems of equations in three variables that are dependent could result from three identical planes, three planes intersecting at a line, or two identical planes that intersect the third on a line.

\section*{Glossary}
solution set the set of all ordered pairs or triples that satisfy all equations in a system of equations
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\section*{INTRODUCTION TO PARTIAL FRACTIONS: AN APPLICATION OF SYSTEMS}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Decompose \(\frac{P(x)}{Q(x)}\), where \(\mathrm{Q}(\mathrm{x})\) has only nonrepeated linear factors.
- Decompose \(\frac{P(x)}{Q(x)}\), where \(\mathrm{Q}(\mathrm{x})\) has repeated linear factors.
- Decompose \(\frac{P(x)}{Q(x)}\), where \(\mathrm{Q}(\mathrm{x})\) has a nonrepeated irreducible quadratic factor.
- Decompose \(\frac{P(x)}{Q(x)}\), where \(\mathrm{Q}(\mathrm{x})\) has a repeated irreducible quadratic factor.

Earlier in this chapter, we studied systems of two equations in two variables, systems of three equations in three variables, and nonlinear systems. Here we introduce another way that systems of equations can be utilized-the decomposition of rational expressions.

Fractions can be complicated; adding a variable in the denominator makes them even more so. The methods studied in this section will help simplify the concept of a rational expression.

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\section*{LINEAR FACTORS}

\section*{LEARNING OUTCOMES}
- Decompose a rational expression with distinct linear factors.
- Decompose a rational expression with repeated linear factors.

Recall the algebra regarding adding and subtracting rational expressions.
1. Factor the numerator and denominator.
2. Find the LCD of the expressions.
3. Multiply the expressions by a form of 1 that changes the denominators to the LCD.
4. Add or subtract the numerators over the common denominator.
5. Simplify.

These operations depend on finding a common denominator so that we can write the sum or difference as a single, simplified rational expression. In this section, we will look at partial fraction decomposition, which is the undoing of the procedure to add or subtract rational expressions. In other words, it is a return from the single simplified rational expression to the original expressions, called the partial fractions. Some types of rational expressions require solving a system of equations in order to decompose them.

For example, suppose we add the following fractions:
\[
\frac{2}{x-3}+\frac{-1}{x+2}
\]

We would first need to find a common denominator,
\((x+2)(x-3)\).
Next, we would write each expression with this common denominator and find the sum of the terms.
\[
\begin{aligned}
& \frac{2}{x-3}\left(\frac{x+2}{x+2}\right)+\frac{-1}{x+2}\left(\frac{x-3}{x-3}\right) \\
& =\frac{2 x+4-x+3}{(x+2)(x-3)} \\
& =\frac{x+7}{x^{2}-x-6}
\end{aligned}
\]

Partial fraction decomposition is the reverse of this procedure. We would start with the solution and rewrite (decompose) it as the sum of two fractions.
\[
\frac{x+7}{x^{2}-x-6}=\frac{2}{x-3}+\frac{-1}{x+2}
\]

Simplified sum Partial fraction decomposition
We will investigate rational expressions with linear factors and quadratic factors in the denominator where the degree of the numerator is less than the degree of the denominator. Regardless of the type of expression we are decomposing, the first and most important thing to do is factor the denominator.

When the denominator of the simplified expression contains distinct linear factors, it is likely that each of the original rational expressions, which were added or subtracted, had one of the linear factors as the denominator. In other words, using the example above, the factors of \(x^{2}-x-6\) are \((x-3)(x+2)\), the denominators of the decomposed rational expression. So we will rewrite the simplified form as the sum of individual fractions and use a variable for each numerator. Then, we will solve for each numerator using one of several methods available for partial fraction decomposition.

\section*{TIP FOR SUCCESS}

The techniques to decompose fractions are challenging and will require some practice to become familiar to you. The explanations of how to perform the decomposition may be confusing the first time (or more) that you read through them. If so, work through the given examples on paper first, slowly, step by step. Then return to the General Note and How To boxes to gain an understanding. It will take time and effort, so don't be discouraged if it takes multiple attempts for each example.

\title{
A GENERAL NOTE: PARTIAL FRACTION DECOMPOSITION OF \(\frac{P(x)}{Q(x)}: Q(x)\) HAS NONREPEATED LINEAR FACTORS
}

The partial fraction decomposition of \(\frac{P(x)}{Q(x)}\) when \(Q(x)\) has nonrepeated linear factors and the degree of \(P(x)\) is less than the degree of \(Q(x)\) is
\[
\frac{P(x)}{Q(x)}=\frac{A_{1}}{\left(a_{1} x+b_{1}\right)}+\frac{A_{2}}{\left(a_{2} x+b_{2}\right)}+\frac{A_{3}}{\left(a_{3} x+b_{3}\right)}+\cdots+\frac{A_{n}}{\left(a_{n} x+b_{n}\right)}
\]

\section*{HOW TO: GIVEN A RATIONAL EXPRESSION WITH DISTINCT LINEAR FACTORS IN THE DENOMINATOR, DECOMPOSE IT.}
1. Use a variable for the original numerators, usually \(A, B\), or \(C\), depending on the number of factors, placing each variable over a single factor. For the purpose of this definition, we use \(A_{n}\) for each numerator
\[
\frac{P(x)}{Q(x)}=\frac{A_{1}}{\left(a_{1} x+b_{1}\right)}+\frac{A_{2}}{\left(a_{2} x+b_{2}\right)}+\cdots+\frac{A_{n}}{\left(a_{n} x+b_{n}\right)}
\]
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

\section*{RECALL HOW TO ELIMINATE DENOMINATORS FROM AN EQUATION USING THE LCD}

The following example relies upon the technique of multiplying both sides of a rational equation by the lowest common denominator (LCD) to eliminate the denominators from the equation.
1. Factor all denominators in the equation
2. Find the LCD
3. Multiply the whole equation by the LCD (each term). As you do, cancel out the denominator in each term. If the LCD has been chose correctly, there will be no denominators remaining at the end of this step.

In this video, you will see another example of how to find a partial fraction decomposition when you have distinct linear factors.

\section*{Decomposing \(\mathrm{P}(\mathrm{x}) / \mathrm{Q}(\mathrm{x})\), Where \(\mathrm{Q}(\mathrm{x})\) Has Repeated Linear Factors}

Some fractions we may come across are special cases that we can decompose into partial fractions with repeated linear factors. We must remember that we account for repeated factors by writing each factor in increasing powers.

In this video, you will see an example of how to find the partial fraction decomposition of a rational expression with repeated linear factors.

\section*{EXAMPLE: DECOMPOSING A RATIONAL EXPRESSION WITH DISTINCT LINEAR FACTORS}

Decompose the given rational expression with distinct linear factors.
\(\frac{3 x}{(x+2)(x-1)}\)
Answer
We will separate the denominator factors and give each numerator a symbolic label, like \(A, B\), or \(C\).
\(\frac{3 x}{(x+2)(x-1)}=\frac{A}{(x+2)}+\frac{B}{(x-1)}\)
Multiply both sides of the equation by the common denominator to eliminate the fractions:


The resulting equation is
\(3 x=A(x-1)+B(x+2)\)
Expand the right side of the equation and collect like terms.
\(3 x=A x-A+B x+2 B\)
\(3 x=(A+B) x-A+2 B\)
Set up a system of equations associating corresponding coefficients.
\[
\begin{gathered}
3=A+B \\
0=-A+2 B
\end{gathered}
\]

\section*{TIP FOR SUCCESS}

Associating corresponding coefficients is a well-used mathematical technique. In this example, if
(3) \(x+0=(A+B) x+(-A+2 B)\)
then it must be true that
\(3=A+B\) and \(0=-A+2 B\).
Add the two equations and solve for \(B\).
\[
\begin{aligned}
& 3=A+B \\
& 0=-A+2 B \\
& \hline 3=0+3 B \\
& B=1
\end{aligned}
\]

Substitute \(B=1\) into one of the original equations in the system.
\[
3=A+1
\]
\[
2=A
\]

Thus, the partial fraction decomposition is
\[
\frac{3 x}{(x+2)(x-1)}=\frac{2}{(x+2)}+\frac{1}{(x-1)}
\]

Another method to use to solve for \(A\) or \(B\) is by considering the equation that resulted from eliminating the fractions and substituting a value for \(x\) that will make either the \(A\) - or \(B\)-term equal 0 . If we let \(x=1\), the \(A\) - term becomes 0 and we can simply solve for \(B\).
\[
\begin{aligned}
3 x & =A(x-1)+B(x+2) \\
3(1) & =A[(1)-1]+B[(1)+2] \\
3 & =0+3 B \\
B & =1
\end{aligned}
\]

Next, either substitute \(B=1\) into the equation and solve for \(A\), or make the \(B\)-term 0 by substituting \(x=-2\) into the equation.
\[
\begin{aligned}
3 x & =A(x-1)+B(x+2) \\
3(-2) & =A[(-2)-1]+B[(-2)+2] \\
-6 & =-3 A+0 \\
\frac{-6}{-3} & =A \\
A & =2
\end{aligned}
\]

We obtain the same values for \(A\) and \(B\) using either method, so the decompositions are the same using either method.
\[
\frac{3 x}{(x+2)(x-1)}=\frac{2}{(x+2)}+\frac{1}{(x-1)}
\]

Although this method is not seen very often in textbooks, we present it here as an alternative that may make some partial fraction decompositions easier. It is known as the Heaviside method, named after Charles Heaviside, a pioneer in the study of electronics.

\section*{TRY IT}

Find the partial fraction decomposition of the following expression.
\(\frac{x}{(x-3)(x-2)}\)
Answer
\(\frac{3}{x-3}-\frac{2}{x-2}\)
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\title{
A GENERAL NOTE: PARTIAL FRACTION DECOMPOSITION OF \(\frac{P(x)}{Q(x)}: Q(x)\) HAS REPEATED LINEAR FACTORS
}

The partial fraction decomposition of \(\frac{P(x)}{Q(x)}\), when \(Q(x)\) has a repeated linear factor occurring \(n\) times and the degree of \(P(x)\) is less than the degree of \(Q(x)\), is
\(\frac{P(x)}{Q(x)}=\frac{A_{1}}{(a x+b)}+\frac{A_{2}}{(a x+b)^{2}}+\frac{A_{3}}{(a x+b)^{3}}+\cdots+\frac{A_{n}}{(a x+b)^{n}}\)
Write the denominator powers in increasing order.

\section*{HOW TO: GIVEN A RATIONAL EXPRESSION WITH REPEATED LINEAR FACTORS, DECOMPOSE IT.}
1. Use a variable like \(A, B\), or \(C\) for the numerators and account for increasing powers of the denominators.
\[
\frac{P(x)}{Q(x)}=\frac{A_{1}}{(a x+b)}+\frac{A_{2}}{(a x+b)^{2}}+\ldots+\frac{A_{n}}{(a x+b)^{n}}
\]
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

\section*{EXAMPLE: DECOMPOSING WITH REPEATED LINEAR FACTORS}

Decompose the given rational expression with repeated linear factors.
\(\frac{-x^{2}+2 x+4}{x^{3}-4 x^{2}+4 x}\)

\section*{Answer}

The denominator factors are \(x(x-2)^{2}\). To allow for the repeated factor of \((x-2)\), the decomposition will include three denominators: \(x,(x-2)\), and \((x-2)^{2}\). Thus,
\(\frac{-x^{2}+2 x+4}{x^{3}-4 x^{2}+4 x}=\frac{A}{x}+\frac{B}{(x-2)}+\frac{C}{(x-2)^{2}}\)
Next, we multiply both sides by the common denominator.
\[
\begin{gathered}
x(x-2)^{2}\left[\frac{-x^{2}+2 x+4}{x(x-2)^{2}}\right]=\left[\frac{A}{x}+\frac{B}{(x-2)}+\frac{C}{(x-2)^{2}}\right] x(x-2)^{2} \\
-x^{2}+2 x+4=A(x-2)^{2}+B x(x-2)+C x
\end{gathered}
\]

On the right side of the equation, we expand and collect like terms.
\[
\begin{aligned}
& -x^{2}+2 x+4=A\left(x^{2}-4 x+4\right)+B\left(x^{2}-2 x\right)+C x \\
& =A x^{2}-4 A x+4 A+B x^{2}-2 B x+C x \\
& =(A+B) x^{2}+(-4 A-2 B+C) x+4 A
\end{aligned}
\]

Next, we compare the coefficients of both sides. This will give the system of equations in three variables:
\(-x^{2}+2 x+4=(A+B) x^{2}+(-4 A-2 B+C) x+4 A\)
\(A+B=-1\)
\(-4 A-2 B+C=2\)
\(4 A=4\)
Solving for \(A\), we have
\[
\begin{aligned}
4 A & =4 \\
A & =1
\end{aligned}
\]

Substitute \(A=1\) into equation (1).
\[
A+B=-1
\]
(1) \(+B=-1\)
\[
B=-2
\]

Then, to solve for \(C\), substitute the values for \(A\) and \(B\) into equation (2).
\[
\begin{aligned}
-4 A-2 B+C & =2 \\
-4(1)-2(-2)+C & =2 \\
-4+4+C & =2 \\
C & =2
\end{aligned}
\]

Thus,
\(\frac{-x^{2}+2 x+4}{x^{3}-4 x^{2}+4 x}=\frac{1}{x}-\frac{2}{(x-2)}+\frac{2}{(x-2)^{2}}\)

\section*{TRY IT}

Find the partial fraction decomposition of the expression with repeated linear factors.
\(\frac{6 x-11}{(x-1)^{2}}\)

\section*{Answer}
\(\frac{6}{x-1}-\frac{5}{(x-1)^{2}}\)


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\section*{QUADRATIC FACTORS}

\section*{LEARNING OUTCOMES}
- Decompose a rational expression with non-repeated irreducible quadratic factors.
- Decompose a rational expression that has repeated irreducible quadratic factors.

So far we have performed partial fraction decomposition with expressions that have had linear factors in the denominator, and we applied numerators \(A, B\), or \(C\) representing constants. Now we will look at an example where one of the factors in the denominator is a quadratic expression that does not factor. This is referred to as an irreducible quadratic factor. In cases like this, we use a linear numerator such as \(A x+B, B x+C\), etc.

In the following video, you will see another example of how to find the partial fraction decomposition for a

\section*{TIP FOR SUCCESS}

As with decomposing fractions with linear factors in the previous section, the techniques to decompose fractions with quadratic factors are challenging and will require some practice to become familiar to you. Do work through the given examples on paper first, slowly, step by step before trying to make sense of the General Note and How To boxes.
It will take time and effort, so don't be discouraged if it takes multiple attempts for each example.

\section*{A GENERAL NOTE: DECOMPOSITION OF \(\frac{P(x)}{Q(x)}: Q(x)\) HAS A NONREPEATED IRREDUCIBLE QUADRATIC FACTOR}

The partial fraction decomposition of \(\frac{P(x)}{Q(x)}\) such that \(Q(x)\) has a nonrepeated irreducible quadratic factor and the degree of \(P(x)\) is less than the degree of \(Q(x)\) is written as
\[
\frac{P(x)}{Q(x)}=\frac{A_{1} x+B_{1}}{\left(a_{1} x^{2}+b_{1} x+c_{1}\right)}+\frac{A_{2} x+B_{2}}{\left(a_{2} x^{2}+b_{2} x+c_{2}\right)}+\cdots+\frac{A_{n} x+B_{n}}{\left(a_{n} x^{2}+b_{n} x+c_{n}\right)}
\]

The decomposition may contain more rational expressions if there are linear factors. Each linear factor will have a different constant numerator: \(A, B, C\), and so on.

\section*{HOW TO: GIVEN A RATIONAL EXPRESSION WHERE THE FACTORS OF THE DENOMINATOR ARE DISTINCT, IRREDUCIBLE QUADRATIC FACTORS, DECOMPOSE IT.}
1. Use variables such as \(A, B\), or \(C\) for the constant numerators over linear factors, and linear expressions such as \(A_{1} x+B_{1}, A_{2} x+B_{2}\), etc., for the numerators of each quadratic factor in the denominator.
\[
\frac{P(x)}{Q(x)}=\frac{A}{a x+b}+\frac{A_{1} x+B_{1}}{\left(a_{1} x^{2}+b_{1} x+c_{1}\right)}+\frac{A_{2} x+B_{2}}{\left(a_{2} x^{2}+b_{2} x+c_{2}\right)}+\cdots+\frac{A_{n} x+B_{n}}{\left(a_{n} x^{2}+b_{n} x+c_{n}\right)}
\]
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.
rational expression that has quadratic factors.

\section*{Decomposing \(\mathrm{P}(\mathrm{x})\) / \(\mathrm{Q}(\mathrm{x})\), When \(\mathrm{Q}(\mathrm{x})\) Has a Repeated Irreducible Quadratic Factor}

Now that we can decompose a simplified rational expression with an irreducible quadratic factor, we will learn how to do partial fraction decomposition when the simplified rational expression has repeated irreducible quadratic factors. The decomposition will consist of partial fractions with linear numerators over each irreducible quadratic factor represented in increasing powers.

This video provides you with another worked example of how to find the partial fraction decomposition for a rational expression that has repeating quadratic factors.

\title{
EXAMPLE: DECOMPOSING \(\frac{P(x)}{Q(x)}\) WHEN \(Q(X)\) CONTAINS A NONREPEATED IRREDUCIBLE QUADRATIC FACTOR
}

Find a partial fraction decomposition of the given expression.
\[
\frac{8 x^{2}+12 x-20}{(x+3)\left(x^{2}+x+2\right)}
\]

\section*{Answer}

We have one linear factor and one irreducible quadratic factor in the denominator, so one numerator will be a constant and the other numerator will be a linear expression. Thus,
\(\frac{8 x^{2}+12 x-20}{(x+3)\left(x^{2}+x+2\right)}=\frac{A}{(x+3)}+\frac{B x+C}{\left(x^{2}+x+2\right)}\)
We follow the same steps as in previous problems. First, clear the fractions by multiplying both sides of the equation by the common denominator.
\[
\begin{aligned}
(x+3)\left(x^{2}+x+2\right)\left[\frac{8 x^{2}+12 x-20}{(x+3)\left(x^{2}+x+2\right)}\right] & =\left[\frac{A}{(x+3)}+\frac{B x+C}{\left(x^{2}+x+2\right)}\right](x+3)\left(x^{2}+x+2\right) \\
8 x^{2}+12 x-20 & =A\left(x^{2}+x+2\right)+(B x+C)(x+3)
\end{aligned}
\]

Notice we could easily solve for \(A\) by choosing a value for \(x\) that will make the \(B x+C\) term equal 0 . Let \(x=-3\) and substitute it into the equation.
\[
\begin{aligned}
8 x^{2}+12 x-20 & =A\left(x^{2}+x+2\right)+(B x+C)(x+3) \\
8(-3)^{2}+12(-3)-20 & =A\left((-3)^{2}+(-3)+2\right)+(B(-3)+C)((-3)+3) \\
16 & =8 A \\
A & =2
\end{aligned}
\]

Now that we know the value of \(A\), substitute it back into the equation. Then expand the right side and collect like terms.
\(8 x^{2}+12 x-20=2\left(x^{2}+x+2\right)+(B x+C)(x+3)\)
\(8 x^{2}+12 x-20=2 x^{2}+2 x+4+B x^{2}+3 B+C x+3 C\)
\(8 x^{2}+12 x-20=(2+B) x^{2}+(2+3 B+C) x+(4+3 C)\)
Setting the coefficients of terms on the right side equal to the coefficients of terms on the left side gives the system of equations.
\[
\begin{array}{r}
2+B=8 \\
2+3 B+C=12 \\
4+3 C=-20 \tag{3}
\end{array}
\]

Solve for \(B\) using equation (1) and solve for \(C\) using equation (3).
\[
\begin{align*}
2+B & =8  \tag{1}\\
B & =6
\end{align*}
\]
\[
\begin{array}{r}
4+3 C=-20  \tag{3}\\
3 C=-24 \\
C=-8
\end{array}
\]

Thus, the partial fraction decomposition of the expression is
\(\frac{8 x^{2}+12 x-20}{(x+3)\left(x^{2}+x+2\right)}=\frac{2}{(x+3)}+\frac{6 x-8}{\left(x^{2}+x+2\right)}\)

\section*{Q \& A}

Could we have just set up a system of equations to solve Example 3?

Yes, we could have solved it by setting up a system of equations without solving for \(A\) first. The expansion on the right would be:
\(8 x^{2}+12 x-20=A x^{2}+A x+2 A+B x^{2}+3 B+C x+3 C\)
\(8 x^{2}+12 x-20=(A+B) x^{2}+(A+3 B+C) x+(2 A+3 C)\)
So the system of equations would be:
\[
\begin{array}{r}
A+B=8 \\
A+3 B+C=12 \\
2 A+3 C=-20
\end{array}
\]

\section*{TIP FOR SUCCESS}

Sometimes, a combination of methods is helpful to obtain the decomposition. In the example above, you could have set the system up first, then used \(x=-3\) to obtain the value for \(A\), and it would be quick work to obtain \(B\) and \(C\) from the system.
Remember that creativity is a key component of doing mathematics and that there is often more than one good way to reach a conclusion.

\section*{TRY IT}

Find the partial fraction decomposition of the expression with a nonrepeating irreducible quadratic factor.
\[
\frac{5 x^{2}-6 x+7}{(x-1)\left(x^{2}+1\right)}
\]

Answer
\(\frac{3}{x-1}+\frac{2 x-4}{x^{2}+1}\)
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\section*{TIP FOR SUCCESS}

Do work through the example below carefully and more than once. The problem may appear to be complicated, but it relies on the same techniques you've already practiced. Keep your work well-organized to avoid making mistakes.

\section*{A GENERAL NOTE: DECOMPOSITION OF \(\frac{P(x)}{Q(x)}\) WHEN \(Q(X)\) HAS A REPEATED IRREDUCIBLE QUADRATIC FACTOR}

The partial fraction decomposition of \(\frac{P(x)}{Q(x)}\), when \(Q(x)\) has a repeated irreducible quadratic factor and the degree of \(P(x)\) is less than the degree of \(Q(x)\), is
\[
\frac{P(x)}{\left(a x^{2}+b x+c\right)^{n}}=\frac{A_{1} x+B_{1}}{\left(a x^{2}+b x+c\right)}+\frac{A_{2} x+B_{2}}{\left(a x^{2}+b x+c\right)^{2}}+\frac{A_{3} x+B_{3}}{\left(a x^{2}+b x+c\right)^{3}}+\cdots+\frac{A_{n} x+B_{n}}{\left(a x^{2}+b x+c\right)^{n}}
\]

Write the denominators in increasing powers.

\section*{HOW TO: GIVEN A RATIONAL EXPRESSION THAT HAS A REPEATED IRREDUCIBLE FACTOR, DECOMPOSE IT.}
1. Use variables like \(A, B\), or \(C\) for the constant numerators over linear factors, and linear expressions such as \(A_{1} x+B_{1}, A_{2} x+B_{2}\), etc., for the numerators of each quadratic factor in the denominator written in increasing powers, such as
\[
\frac{P(x)}{Q(x)}=\frac{A}{a x+b}+\frac{A_{1} x+B_{1}}{\left(a x^{2}+b x+c\right)}+\frac{A_{2} x+B_{2}}{\left(a x^{2}+b x+c\right)^{2}}+\cdots+\frac{A_{n}+B_{n}}{\left(a x^{2}+b x+c\right)^{n}}
\]
2. Multiply both sides of the equation by the common denominator to eliminate fractions.
3. Expand the right side of the equation and collect like terms.
4. Set coefficients of like terms from the left side of the equation equal to those on the right side to create a system of equations to solve for the numerators.

\section*{EXAMPLE: DECOMPOSING A RATIONAL FUNCTION WITH A REPEATED IRREDUCIBLE QUADRATIC FACTOR IN THE DENOMINATOR}

Decompose the given expression that has a repeated irreducible factor in the denominator.
\[
\frac{x^{4}+x^{3}+x^{2}-x+1}{x\left(x^{2}+1\right)^{2}}
\]

Answer
The factors of the denominator are \(x,\left(x^{2}+1\right)\), and \(\left(x^{2}+1\right)^{2}\). Recall that, when a factor in the denominator is a quadratic that includes at least two terms, the numerator must be of the linear form \(A x+B\). So, let's begin the decomposition.
\[
\frac{x^{4}+x^{3}+x^{2}-x+1}{x\left(x^{2}+1\right)^{2}}=\frac{A}{x}+\frac{B x+C}{\left(x^{2}+1\right)}+\frac{D x+E}{\left(x^{2}+1\right)^{2}}
\]

We eliminate the denominators by multiplying each term by \(x\left(x^{2}+1\right)^{2}\). Thus,
\(x^{4}+x^{3}+x^{2}-x+1=A\left(x^{2}+1\right)^{2}+(B x+C)(x)\left(x^{2}+1\right)+(D x+E)(x)\)
Expand the right side.
\[
\begin{aligned}
& x^{4}+x^{3}+x^{2}-x+1=A\left(x^{4}+2 x^{2}+1\right)+B x^{4}+B x^{2}+C x^{3}+C x+D x^{2}+E x \\
& =A x^{4}+2 A x^{2}+A+B x^{4}+B x^{2}+C x^{3}+C x+D x^{2}+E x
\end{aligned}
\]

Now we will collect like terms.
\(x^{4}+x^{3}+x^{2}-x+1=(A+B) x^{4}+(C) x^{3}+(2 A+B+D) x^{2}+(C+E) x+A\)
Set up the system of equations matching corresponding coefficients on each side of the equal sign.
\[
\begin{aligned}
A+B & =1 \\
C & =1
\end{aligned}
\]
\(2 A+B+D=1\)
\[
\begin{array}{r}
C+E=-1 \\
A=1
\end{array}
\]

We can use substitution from this point. Substitute \(A=1\) into the first equation.
\(1+B=1\)
\[
B=0
\]

Substitute \(A=1\) and \(B=0\) into the third equation.
\[
2(1)+0+D=1
\]
\[
D=-1
\]

Substitute \(C=1\) into the fourth equation.
\(1+E=-1\)
\[
E=-2
\]

Now we have solved for all of the unknowns on the right side of the equal sign. We have \(A=1, B=0\), \(C=1, D=-1\), and \(E=-2\). We can write the decomposition as follows:
\[
\frac{x^{4}+x^{3}+x^{2}-x+1}{x\left(x^{2}+1\right)^{2}}=\frac{1}{x}+\frac{1}{\left(x^{2}+1\right)}-\frac{x+2}{\left(x^{2}+1\right)^{2}}
\]

\section*{TRY IT}

Find the partial fraction decomposition of the expression with a repeated irreducible quadratic factor.
\(\frac{x^{3}-4 x^{2}+9 x-5}{\left(x^{2}-2 x+3\right)^{2}}\)
Answer
\(\frac{x-2}{x^{2}-2 x+3}+\frac{2 x+1}{\left(x^{2}-2 x+3\right)^{2}}\)


A YouTube element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=392

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\section*{SUMMARY: PARTIAL FRACTIONS: AN APPLICATION OF SYSTEMS}

\section*{Key Concepts}
- Decompose \(\frac{P(x)}{Q(x)}\) by writing the partial fractions as \(\frac{A}{a_{1} x+b_{1}}+\frac{B}{a_{2} x+b_{2}}\). Solve by clearing the fractions, expanding the right side, collecting like terms, and setting corresponding coefficients equal to each other, then setting up and solving a system of equations.
- The decomposition of \(\frac{P(x)}{Q(x)}\) with repeated linear factors must account for the factors of the denominator in increasing powers.
- The decomposition of \(\frac{P(x)}{Q(x)}\) with a nonrepeated irreducible quadratic factor needs a linear numerator over the quadratic factor, as in \(\frac{A}{x}+\frac{B x+C}{\left(a x^{2}+b x+c\right)}\).
- In the decomposition of \(\frac{P(x)}{Q(x)}\), where \(Q(x)\) has a repeated irreducible quadratic factor, when the irreducible quadratic factors are repeated, powers of the denominator factors must be represented in increasing powers as
\[
\frac{A x+B}{\left(a x^{2}+b x+c\right)}+\frac{A_{2} x+B_{2}}{\left(a x^{2}+b x+c\right)^{2}}+\cdots+\frac{A_{n} x+B_{n}}{\left(a x^{2}+b x+c\right)^{n}} .
\]

\section*{Glossary}
partial fractions the individual fractions that make up the sum or difference of a rational expression before combining them into a simplified rational expression
partial fraction decomposition the process of returning a simplified rational expression to its original form, a sum or difference of simpler rational expressions

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\section*{PUTTING IT TOGETHER: SYSTEMS OF EQUATIONS AND INEQUALITIES}

You've come a long way since the beginning of the module when you were considering open your first coffee shop. Now that you know about systems of equations, you can solve your problem of how much of each type of coffee bean to put in your signature mixture.


Recall that you were trying to make a 100-pound mixture of two kinds of coffee beans that would sell for \(\$ 9.80\) per pound. One bean sells for \(\$ 8\) per pound and the other sells for \(\$ 14\) per pound. You came up with two equations to describe your problem. Now you know that you have a system of equations.
\[
\begin{gathered}
x+y=100 \\
8 x+14 y=980
\end{gathered}
\]

You have learned several different methods of solving a system of equations. Although all of the methods will work, you can help decide which one to use by looking at the equations. The first equation can easily be rearranged to solve for \(x\) or \(y\). You may find it easiest to do this, and then solve by substitution.

Let's choose \(x\). Rearrange to solve for \(x\).
\(x=100-y\)
Now substitute for \(x\) in the second equation, and solve for \(y\).
\begin{tabular}{|l|l|}
\hline Start with the original equation. & \(8 x+14 y=980\) \\
\hline Replace \(x\) with \(100-y\). & \(8(100-y)+14 y=980\) \\
\hline Distribute. & \(800-8 y+14 y=980\) \\
\hline Combine like terms. & \(800+6 y=980\) \\
\hline Subtract 800 from both sides. & \(6 y=180\) \\
\hline Divide both sides by 6. & \(y=30\) \\
\hline
\end{tabular}

Now that you know the value of \(y\), you can solve for \(x\).
\begin{tabular}{|l|l|}
\hline Start with the original equation. & \(x+y=100\) \\
\hline Replace y with 30. & \(x+30=100\) \\
\hline Subtract 30 from both sides. & \(x=70\) \\
\hline
\end{tabular}

The solution to the system of equations is \((70,30)\). What does this solution mean? Remember that you were looking for the number of pounds of each type of coffee bean. The solution indicates that you should use 70 pounds of the \(\$ 8\)-beans and 30 pounds of the \(\$ 14\)-beans. Does this make sense? Take a moment to check.

70 pounds +30 pounds \(=100\) pounds
\(70(8)+30(14)=980\)
Yes, your answer makes sense. Is this the only possible combination? Making a graph is always a nice way to classify a system of equations to determine if there is only one solution. The graph below represents these equations.


As you can see, the lines intersect at a single point, which is exactly the solution you found. This system of equations is independent. The only way the solution will change is if you choose different coffee beans at different prices or change the price per pound of the mixture. That is something you just might do as your business grows and you want to increase your profits. But for now, you might want to concentrate on making a reasonably priced mixture that will bring in new customers.

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\title{
MODULE 14: SOLVE SYSTEMS WITH MATRICES
}

\section*{WHY IT MATTERS: SOLVING SYSTEMS WITH MATRICES}

\section*{Why Solve Systems with Matrices?}

In a previous module, Polynomial and Rational Expressions, we explored a scenario in which a furniture company produces chairs and tables. Now suppose that our company has two workshops that both produce the tables and chairs, one in Atlanta, and the other in Boston. Furthermore, the two workshops make the furniture at different rates. The table below shows the maximum number of chairs and tables that can be made each day at the two workshops.
\begin{tabular}{|l|c|c|}
\hline & Atlanta & Boston \\
\hline Chairs & 20 per day & 30 per day \\
\hline Tables & 15 per day & 12 per day \\
\hline
\end{tabular}

A large order comes in for 620 chairs and 360 tables. How many days should each workshop dedicate to table and chair production in order to fulfill the order most efficiently?


When we use the phrase most efficiently, we mean that the workshops should be operating at full capacity with no extra production. For example, if we allowed the Atlanta shop to produce all of the tables and chairs by itself, then after 34 days, there would be \(31 \times 20=620\) chairs and \(31 \times 15=465\) tables. That would fulfill the order, but then there would be \(465-360=105\) extra tables cluttering up the workshop. And if we did not make those extra 105 tables, then that would mean the workshop was not working at full capacity. In
a similar way, if the Boston workshop produced the entire order, there would either be extra furniture or inefficient use of resources there.

We expect to make use of both workshops. But how many days should each one be utilized for table and chair production? This calls for a system of equations! Let \(A\) stand for the number of days to run the Atlanta workshop, and let \(B\) stand for the number of days to run the Boston workshop. Then there will be two equations, one for chairs and one for tables:

Total chairs: \(20 A+30 B=620\)
Total tables: \(15 A+12 B=360\)
Of course, using methods from the previous module, Systems of Equations and Inequalities, you can easily solve this system. However, every time a new order for chairs and tables comes in, you would have to solve a similar linear system again. It would be better if we had a more general method that could be used to solve the system,
\(20 A+30 B=C\)
\(15 A+12 B=T\)
in which \(C\) and \(T\) could be plugged in later. Fortunately, there is a straightforward method for doing exactly that. In this module, you will learn a new method for solving systems, one that uses a table of numbers called a matrix (pl. matrices), very much like the array of numbers from the initial table. The coefficient matrix for this system is:
\(\left[\begin{array}{ll}20 & 30 \\ 15 & 12\end{array}\right]\)

Then you can write the system as an equation of matrices as shown below.
\(\left[\begin{array}{ll}20 & 30 \\ 15 & 12\end{array}\right]\left[\begin{array}{l}A \\ B\end{array}\right]=\left[\begin{array}{l}C \\ T\end{array}\right]\)
The methods for working with such an equation will be discussed shortly. We will revisit our thriving furniture business at the end of the module and see how matrices can be used to solve our table and chair problem once and for all.

\section*{LEARNING OUTCOMES}

\section*{Review Topics for Success}
- Identify and define the properties of real numbers

\section*{Matrices and Matrix Operations}
- Determine the dimensions of a matrix.
- Add and subtract two matrices.
- Multiply a matrix by a scalar, sum scalar multiples of matrices.
- Multiply two matrices together.
- Use a calculator to perform operations on matrices.

\section*{Gaussian Elimination}
- Write the augmented matrix for a system of equations.
- Perform row operations on an augmented matrix.
- Use Gaussian elimination to solve a systems of equations represented as an augmented matrix.
- Interpret the solution to a system of equations represented as an augmented matrix.

Solving Systems with Inverses
- Verify that multiplying a matrix by its inverse results in 1.
- Use matrix multiplication to find the inverse of a matrix.
- Find an inverse by augmenting with an identity matrix.
- Solve a \(2 \times 2\) system using an inverse.
- Solve a \(3 \times 3\) systems using an inverse.
- Solve a system with a calculator.

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\section*{REVIEW TOPICS FOR SUCCESS}

\section*{Learning Outcomes}

By the end of this section, you will be able to:
- Identify and define the properties of real numbers

This module introduces the matrix: a mathematical tool that may be brand new to you. We can use matrices to solve systems of linear equations, but there are much more to these powerfully efficient constructions. Matrix applications can be found in nearly every scientific field. They are used heavily in statistics, physics, electrical engineering, and computer science, to name a few. A deeper exploration of matrices and matrix operations appears in Linear Algebra, a branch of mathematics that is fundamental to nearly all mathematical areas.

In this module, you'll learn how to perform operations on matrices to combine them and scale them by a numerical multiplier. You'll also learn how to find and use the inverse of a matrix and how to use matrices to solve a linear system.

Before studying the properties of matrix operations (sums, differences, products, inverses, and identity) you should refresh your understanding of the properties of real numbers. Though matrix operations work a little differently than operations on real numbers, refreshing the terminology will be helpful.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

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\section*{PROPERTIES OF REAL NUMBERS}

\section*{Using Properties of Real Numbers}

For some activities we perform, the order of certain operations does not matter, but the order of other operations does. For example, it does not make a difference if we put on the right shoe before the left or vice-versa. However, it does matter whether we put on shoes or socks first. The same thing is true for operations in mathematics.

\section*{Commutative Properties}

The commutative property of addition states that numbers may be added in any order without affecting the sum.
\[
a+b=b+a
\]

We can better see this relationship when using real numbers.
\[
(-2)+7=5 \text { and } 7+(-2)=5
\]

Similarly, the commutative property of multiplication states that numbers may be multiplied in any order without affecting the product.
\[
a \cdot b=b \cdot a
\]

Again, consider an example with real numbers.
\[
(-11) \cdot(-4)=44 \text { and }(-4) \cdot(-11)=44
\]

It is important to note that neither subtraction nor division is commutative. For example, \(17-5\) is not the same as \(5-17\). Similarly, \(20 \div 5 \neq 5 \div 20\).

\section*{Associative Properties}

The associative property of multiplication tells us that it does not matter how we group numbers when multiplying. We can move the grouping symbols to make the calculation easier, and the product remains the same.
\[
a(b c)=(a b) c
\]

Consider this example.
\[
(3 \cdot 4) \cdot 5=60 \text { and } 3 \cdot(4 \cdot 5)=60
\]

The associative property of addition tells us that numbers may be grouped differently without affecting the sum.
\[
a+(b+c)=(a+b)+c
\]

This property can be especially helpful when dealing with negative integers. Consider this example.
\[
[15+(-9)]+23=29 \text { and } 15+[(-9)+23]=29
\]

Are subtraction and division associative? Review these examples.
\[
\begin{aligned}
8-(3-15) & \stackrel{?}{=}(8-3)-15 \\
8-(-12) & \stackrel{?}{=} 5-15 \\
20 & \neq 20-10
\end{aligned}
\]
\[
\begin{aligned}
64 \div(8 \div 4) & \stackrel{?}{=}(64 \div 8) \div 4 \\
64 \div 2 & \stackrel{?}{=} 8 \div 4 \\
32 & \neq 2
\end{aligned}
\]

As we can see, neither subtraction nor division is associative.

\section*{Distributive Property}

The distributive property states that the product of a factor times a sum is the sum of the factor times each term in the sum.
\[
a \cdot(b+c)=a \cdot b+a \cdot c
\]

This property combines both addition and multiplication (and is the only property to do so). Let us consider an example.


Note that 4 is outside the grouping symbols, so we distribute the 4 by multiplying it by 12 , multiplying it by 7 , and adding the products.

To be more precise when describing this property, we say that multiplication distributes over addition. The reverse is not true, as we can see in this example.
\[
\begin{aligned}
6+(3 \cdot 5) & \stackrel{?}{=}(6+3) \cdot(6+5) \\
6+(15) & \stackrel{?}{=}(9) \cdot(11) \\
21 & \neq 99
\end{aligned}
\]

Multiplication does not distribute over subtraction, and division distributes over neither addition nor subtraction.

A special case of the distributive property occurs when a sum of terms is subtracted.
\[
a-b=a+(-b)
\]

For example, consider the difference \(12-(5+3)\). We can rewrite the difference of the two terms 12 and \((5+3)\) by turning the subtraction expression into addition of the opposite. So instead of subtracting ( \(5+3\) ), we add the opposite.
\[
12+(-1) \cdot(5+3)
\]

Now, distribute -1 and simplify the result.
\[
\begin{aligned}
12+(-1) \cdot(5+3) & =12+[(-1) \cdot 5+(-1) \cdot 3] \\
& =12+(-5-3) \\
& =12+(-8) \\
& =4
\end{aligned}
\]

This seems like a lot of trouble for a simple sum, but it illustrates a powerful result that will be useful once we introduce algebraic terms. To subtract a sum of terms, change the sign of each term and add the results. With this in mind, we can rewrite the last example.
\[
\begin{aligned}
12-(5+3) & =12+(-5-3) \\
& =12+(-8) \\
& =4
\end{aligned}
\]

\section*{Identity Properties}

The identity property of addition states that there is a unique number, called the additive identity ( 0 ) that, when added to a number, results in the original number.
\[
a+0=a
\]

The identity property of multiplication states that there is a unique number, called the multiplicative identity (1) that, when multiplied by a number, results in the original number.
\[
a \cdot 1=a
\]

For example, we have \((-6)+0=-6\) and \(23 \cdot 1=23\). There are no exceptions for these properties; they work for every real number, including 0 and 1.

\section*{Inverse Properties}

The inverse property of addition states that, for every real number \(a\), there is a unique number, called the additive inverse (or opposite), denoted-a, that, when added to the original number, results in the additive identity, 0.
\[
a+(-a)=0
\]

For example, if \(a=-8\), the additive inverse is 8 , since \((-8)+8=0\).
The inverse property of multiplication holds for all real numbers except 0 because the reciprocal of 0 is not defined. The property states that, for every real number \(a\), there is a unique number, called the multiplicative inverse (or reciprocal), denoted \(\frac{1}{a}\), that, when multiplied by the original number, results in the multiplicative identity, 1.
\[
a \cdot \frac{1}{a}=1
\]

For example, if \(a=-\frac{2}{3}\), the reciprocal, denoted \(\frac{1}{a}\), is \(-\frac{3}{2}\) because
\[
a \cdot \frac{1}{a}=\left(-\frac{2}{3}\right) \cdot\left(-\frac{3}{2}\right)=1
\]

\section*{A GENERAL NOTE: PROPERTIES OF REAL NUMBERS}

The following properties hold for real numbers \(a, b\), and \(c\).
\begin{tabular}{|c|c|c|}
\hline & Addition & Multiplication \\
\hline Commutative Property & \(a+b=b+a\) & \(a \cdot b=b \cdot a\) \\
\hline Associative Property & \(a+(b+c)=(a+b)+c\) & \(a(b c)=(a b) c\) \\
\hline Distributive Property & \(a \cdot(b+c)=a \cdot b+a \cdot c\) & \\
\hline Identity Property & There exists a unique real number called the additive identity, 0 , such that, for any real number \(a\)
\[
a+0=a
\] & There exists a unique real number called the multiplicative identity, 1 , such that, for any real number a
\[
a \cdot 1=a
\] \\
\hline Inverse Property & Every real number a has an additive inverse, or opposite, denoted \(-a\), such that
\[
a+(-a)=0
\] & Every nonzero real number \(a\) has a multiplicative inverse, or reciprocal, denoted \(\frac{1}{a}\), such that
\[
a \cdot\left(\frac{1}{a}\right)=1
\] \\
\hline
\end{tabular}

\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
associative property of addition the sum of three numbers may be grouped differently without affecting the result; in symbols, \(a+(b+c)=(a+b)+c\)
associative property of multiplication the product of three numbers may be grouped differently without affecting the result; in symbols, \(a \cdot(b \cdot c)=(a \cdot b) \cdot c\)
commutative property of addition two numbers may be added in either order without affecting the result;
in symbols, \(a+b=b+a\)
commutative property of multiplication two numbers may be multiplied in any order without affecting the result; in symbols, \(a \cdot b=b \cdot a\)
distributive property the product of a factor times a sum is the sum of the factor times each term in the sum; in symbols, \(a \cdot(b+c)=a \cdot b+a \cdot c\)
identity property of addition there is a unique number, called the additive identity, 0 , which, when added to a number, results in the original number; in symbols, \(a+0=a\)
identity property of multiplication there is a unique number, called the multiplicative identity, 1 , which, when multiplied by a number, results in the original number; in symbols, \(a \cdot 1=a\)
inverse property of addition for every real number \(a\), there is a unique number, called the additive inverse (or opposite), denoted \(-a\), which, when added to the original number, results in the additive identity, 0 ; in symbols, \(a+(-a)=0\)
inverse property of multiplication for every non-zero real number \(a\), there is a unique number, called the multiplicative inverse (or reciprocal), denoted \(\frac{1}{a}\), which, when multiplied by the original number, results in the multiplicative identity, 1 ; in symbols, \(a \cdot \frac{1}{a}=1\)

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\section*{INTRODUCTION TO MATRICES AND MATRIX OPERATIONS}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Determine the dimensions of a matrix.
- Add and subtract two matrices.
- Multiply a matrix by a scalar, sum scalar multiples of matrices.
- Multiply two matrices together.
- Use a calculator to perform operations on matrices.


Two teams competing in a soccer game. (credit: "SD Dirk," Flickr)
Two club soccer teams, the Wildcats and the Mud Cats, are hoping to obtain new equipment for an upcoming season. The table shows the needs of both teams.
\begin{tabular}{|l|c|c|}
\hline & Wildcats & Mud Cats \\
\hline Goals & 6 & 10 \\
\hline Balls & 30 & 24 \\
\hline Jerseys & 14 & 20 \\
\hline
\end{tabular}

A goal costs \(\$ 300\); a ball costs \(\$ 10\); and a jersey costs \(\$ 30\). How can we find the total cost for the equipment needed for each team? In this section, we discover a method in which the data in the soccer equipment table can be displayed and used for calculating other information. Then, we will be able to calculate the cost of the equipment.

\section*{ADDING AND SUBTRACTING MATRICES}

\section*{LEARNING OUTCOMES}
- Determine the dimensions of a matrix.
- Add and subtract two matrices.

To solve a problem like the one described for the soccer teams, we can use a matrix, which is a rectangular array of numbers. A row in a matrix is a set of numbers that are aligned horizontally. A column in a matrix is a set of numbers that are aligned vertically. Each number is an entry, sometimes called an element, of the matrix. Matrices (plural) are enclosed in [ ] or ( ) and are usually named with capital letters. For example, three matrices named \(A, B\), and \(C\) are shown below.
\(A=\left[\begin{array}{ll}1 & 2 \\ 3 & 4\end{array}\right], B=\left[\begin{array}{rrr}1 & 2 & 7 \\ 0 & -5 & 6 \\ 7 & 8 & 2\end{array}\right], C=\left[\begin{array}{rr}-1 & 3 \\ 0 & 2 \\ 3 & 1\end{array}\right]\)

\section*{Describing Matrices}

A matrix is often referred to by its size or dimensions: \(m \times n\) indicating \(m\) rows and \(n\) columns. Matrix entries are defined first by row and then by column. For example, to locate the entry in matrix \(A\) identified as \(a_{i j}\), we look for the entry in row \(i\), column \(j\). In matrix \(A\) shown below, the entry in row 2 , column 3 is \(a_{23}\).
\(A=\left[\begin{array}{lll}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{array}\right]\)

\section*{HOW ARE THOSE SUBSCRIPTS PRONOUNCED?}

Since the subscripts in a matrix refer to row and column entry locations, the numbers are pronounced distinctly from one another. To refer to the 1st row, third column entry in the matrix \(A\) above, we say " \(a\) one three."

A square matrix is a matrix with dimensions \(n \times n\), meaning that it has the same number of rows as columns. The \(3 \times 3\) matrix above is an example of a square matrix.

A row matrix is a matrix consisting of one row with dimensions \(1 \times n\).
\(\left[\begin{array}{lll}a_{11} & a_{12} & a_{13}\end{array}\right]\)
A column matrix is a matrix consisting of one column with dimensions \(m \times 1\).
\(\left[\begin{array}{l}a_{11} \\ a_{21} \\ a_{31}\end{array}\right]\)
A matrix may be used to represent a system of equations. In these cases, the numbers represent the coefficients of the variables in the system. Matrices often make solving systems of equations easier because they are not encumbered with variables. We will investigate this idea further in the next section, but first we will look at basic matrix operations.

\section*{A GENERAL NOTE: MATRICES}

A matrix is a rectangular array of numbers that is usually named by a capital letter: \(A, B, C\), and so on. Each entry in a matrix is referred to as \(a_{i j}\), such that \(i\) represents the row and \(j\) represents the column. Matrices are often referred to by their dimensions: \(m \times n\) indicating \(m\) rows and \(n\) columns.

\section*{EXAMPLE: FINDING THE DIMENSIONS OF THE GIVEN MATRIX AND LOCATING ENTRIES}

Given matrix \(A\) :
1. What are the dimensions of matrix \(A\) ?
2. What are the entries at \(a_{31}\) and \(a_{22}\) ?
\(A=\left[\begin{array}{rrr}2 & 1 & 0 \\ 2 & 4 & 7 \\ 3 & 1 & -2\end{array}\right]\)
Answer
1. The dimensions are \(3 \times 3\) because there are three rows and three columns.
2. Entry \(a_{31}\) is the number at row 3 , column 1 which is 3 . The entry \(a_{22}\) is the number at row 2 , column 2 which is 4 . Remember, the row comes first, then the column.

\section*{TRY IT}

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\section*{Adding and Subtracting Matrices}

We use matrices to list data or to represent systems. Because the entries are numbers, we can perform operations on matrices. We add or subtract matrices by adding or subtracting corresponding entries.

In order to do this, the entries must correspond. Therefore, addition and subtraction of matrices is only possible when the matrices have the same dimensions. We can add or subtract a \(3 \times 3\) matrix and another \(3 \times 3\) matrix, but we cannot add or subtract a \(2 \times 3\) matrix and a \(3 \times 3\) matrix because some entries in one matrix will not have a corresponding entry in the other matrix.

\section*{A GENERAL NOTE: ADDING AND SUBTRACTING MATRICES}

Given matrices \(A\) and \(B\) of like dimensions, addition and subtraction of \(A\) and \(B\) will produce matrix \(C\) or matrix \(D\) of the same dimension.
\(A+B=C\) such that \(a_{i j}+b_{i j}=c_{i j}\)
\(A-B=D\) such that \(a_{i j}-b_{i j}=d_{i j}\)
Matrix addition is commutative.
\(A+B=B+A\)
It is also associative.
\((A+B)+C=A+(B+C)\)

\section*{EXAMPLE: FINDING THE SUM OF MATRICES}

Find the sum of \(A\) and \(B\) given
\(A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]\) and \(B=\left[\begin{array}{ll}e & f \\ g & h\end{array}\right]\)
Answer
Add corresponding entries.
\begin{tabular}{lll}
\(A+B\) & =Veft[a & \(b\) \\
\(c\) & dlright]+Veft[e] & \(f\) \\
\(g\) & hlright] &
\end{tabular}
\(=\) Veft \([a+e \quad b+f\)
\(c+g\)
d+hlright]

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\section*{EXAMPLE: ADDING MATRIX A AND MATRIX B}

Find the sum of \(A\) and \(B\).
\(A=\left[\begin{array}{ll}4 & 1 \\ 3 & 2\end{array}\right]\) and \(B=\left[\begin{array}{ll}5 & 9 \\ 0 & 7\end{array}\right]\)
Answer
Add corresponding entries. Add the entry in row 1, column 1, \(a_{11}\), of matrix \(A\) to the entry in row 1, column \(1, b_{11}\), of \(B\). Continue the pattern until all entries have been added.


\section*{EXAMPLE: FINDING THE DIFFERENCE OF TWO MATRICES}

Find the difference of \(A\) and \(B\).
\(A=\left[\begin{array}{rr}-2 & 3 \\ 0 & 1\end{array}\right]\) and \(B=\left[\begin{array}{ll}8 & 1 \\ 5 & 4\end{array}\right]\)
Answer
We subtract the corresponding entries of each matrix.
\begin{tabular}{|c|c|c|c|}
\hline \(A-B\) & =-veft[-2 & 3 & \\
\hline 0 & 1 1right]-Veft[8 & 1 & \\
\hline 5 & 4right] & & \\
\hline & =Veft[-2-8 & & 3-1 \\
\hline \(0-5\) & & 1-4 \right] & \\
\hline & =- left [-10 & & 2 \\
\hline -5 & & -3\right] & \\
\hline
\end{tabular}

\section*{TIP FOR SUCCESS}

Be sure to handle negative numbers carefully when finding the difference of two matrices. In the example using the \(3 \times 3\) matrix below, for instance, subtracting \(a_{13}-b_{13}\) results in \(-2-(-2)=0\)

\section*{EXAMPLE: FINDING THE SUM AND DIFFERENCE OF TWO 3 X 3 MATRICES}

Given \(A\) and \(B\) :
1. Find the sum.
2. Find the difference.
\(A=\left[\begin{array}{rrr}2 & -10 & -2 \\ 14 & 12 & 10 \\ 4 & -2 & 2\end{array}\right]\) and \(B=\left[\begin{array}{rrr}6 & 10 & -2 \\ 0 & -12 & -4 \\ -5 & 2 & -2\end{array}\right]\)

\section*{Answer}
1. Add the corresponding entries.

2. Subtract the corresponding entries.


\section*{TRY IT}

Add matrix \(A\) and matrix \(B\).
\(A=\left[\begin{array}{rr}2 & 6 \\ 1 & 0 \\ 1 & -3\end{array}\right]\) and \(B=\left[\begin{array}{rr}3 & -2 \\ 1 & 5 \\ -4 & 3\end{array}\right]\)
Answer
\(A+B=\left[\begin{array}{cc}2 & 6 \\ 1 & 0 \\ 1 & -3\end{array}\right]+\left[\begin{array}{cc}3 & -2 \\ 1 & 5 \\ -4 & 3\end{array}\right]=\left[\begin{array}{cc}2+3 & 6+(-2) \\ 1+1 & 0+5 \\ 1+(-4) & -3+3\end{array}\right]=\left[\begin{array}{cc}5 & 4 \\ 2 & 5 \\ -3 & 0\end{array}\right]\)
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\section*{PRODUCTS OF MATRICES}

\section*{LEARNING OUTCOMES}
- Multiply a matrix by a scalar, sum scalar multiples of matrices.
- Multiply two matrices together.
- Use a calculator to perform operations on matrices.

Besides adding and subtracting whole matrices, there are many situations in which we need to multiply a matrix by a constant called a scalar. A scalar is a real number quantity that has magnitude but not direction. For example, time, temperature, and distance are scalar quantities. The process of scalar multiplication involves multiplying each entry in a matrix by a scalar. A scalar multiple is any entry of a matrix that results from scalar multiplication.

Consider a real-world scenario in which a university needs to add to its inventory of computers, computer tables, and chairs in two of the campus labs due to increased enrollment from the fall 2013 semester to the fall of 2014. They estimate that \(15 \%\) more equipment is needed in both labs. The school's current inventory is displayed in the table below.
\begin{tabular}{|l|c|c|}
\hline & Lab A & Lab B \\
\hline Computers & 15 & 27 \\
\hline Computer Tables & 16 & 34 \\
\hline Chairs & 16 & 34 \\
\hline
\end{tabular}

Converting the data to a matrix, we have the computer inventory in fall 2013 given by
\(C_{2013}=\left[\begin{array}{ll}15 & 27 \\ 16 & 34 \\ 16 & 34\end{array}\right]\)
To calculate how much computer equipment will be needed in 2014, we multiply all entries in matrix \(C\) by 0.15.
\((0.15) C_{2013}=\left[\begin{array}{l}(0.15) 15(0.15) 27 \\ (0.15) 16(0.15) 34 \\ (0.15) 16(0.15) 34\end{array}\right]=\left[\begin{array}{cc}2.25 & 4.05 \\ 2.4 & 5.1 \\ 2.4 & 5.1\end{array}\right]\)
We must round up to the next integer, so the amount of new equipment needed is
\(\left[\begin{array}{ll}3 & 5 \\ 3 & 6 \\ 3 & 6\end{array}\right]\)
Adding the two matrices as shown below, we see the new inventory amounts.
\(\left[\begin{array}{ll}15 & 27 \\ 16 & 34 \\ 16 & 34\end{array}\right]+\left[\begin{array}{ll}3 & 5 \\ 3 & 6 \\ 3 & 6\end{array}\right]=\left[\begin{array}{ll}18 & 32 \\ 19 & 40 \\ 19 & 40\end{array}\right]\)
This means
\(C_{2014}=\left[\begin{array}{ll}18 & 32 \\ 19 & 40 \\ 19 & 40\end{array}\right]\)
Thus, Lab A will have 18 computers, 19 computer tables, and 19 chairs; Lab B will have 32 computers, 40 computer tables, and 40 chairs.

\section*{A GENERAL NOTE: SCALAR MULTIPLICATION}

Scalar multiplication involves finding the product of a constant by each entry in the matrix. Given
\(A=\left[\begin{array}{lll}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right]\)
the scalar multiple \(c A\) is
```

cA =c\left{{a}_{11} a
a21 {a}_{22}\right]
=\left[c{a}_{11}
cal2
ca21 c{a}_{22}\right]

```
\(a(A+B)=a A+a B\)
\((a+b) A=a A+b A\)

\section*{EXAMPLE: MULTIPLYING THE MATRIX BY A SCALAR}

Multiply matrix \(A\) by the scalar 3 .
\(A=\left[\begin{array}{ll}8 & 1 \\ 5 & 4\end{array}\right]\)
Answer
Multiply each entry in \(A\) by the scalar 3.
\(3 A \quad=3\) left \([8 \quad 1\)

5 4 4right]
= Veft[3]cdot \(8 \quad 3 \cdot 1\)
\(3 \cdot 5 \quad\) 3lcdot 4 4right]
\(=\) Veft[24 3

15 12\right]

\section*{TRY IT}

Given matrix \(B\), find \(-2 B\) where
\(B=\left[\begin{array}{ll}4 & 1 \\ 3 & 2\end{array}\right]\)
Answer
\(-2 B=\left[\begin{array}{ll}-8 & -2 \\ -6 & -4\end{array}\right]\)

\section*{Finding the Product of Two Matrices}

In addition to multiplying a matrix by a scalar, we can multiply two matrices. Finding the product of two matrices is only possible when the inner dimensions are the same, meaning that the number of columns of the first matrix is equal to the number of rows of the second matrix. If \(A\) is an \(m \times r\) matrix and \(B\) is an \(r \times n\) matrix, then the product matrix \(A B\) is an \(m \times n\) matrix. For example, the product \(A B\) is possible because the number of columns in \(A\) is the same as the number of rows in \(B\). If the inner dimensions do not match, the product is not defined.


\section*{EXAMPLE: FINDING THE SUM OF SCALAR MULTIPLES}

Find the sum \(3 A+2 B\).
\(A=\left[\begin{array}{rrr}1 & -2 & 0 \\ 0 & -1 & 2 \\ 4 & 3 & -6\end{array}\right]\) and \(B=\left[\begin{array}{rrr}-1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & 1 & -4\end{array}\right]\)
Answer
First, find \(3 A\), then \(2 B\).
\begin{tabular}{|c|c|c|c|}
\hline \(3 A\) & =- Veft[3lcdot 1 & \(3(-2)\) & \(3 \cdot 0\) \\
\hline \(3 \cdot 0\) & \(3(-1)\) & \(3 \cdot 2\) & \\
\hline \(3 \cdot 4\) & \(3 \cdot 3\) & 3\left(-6\right)\right] & \\
\hline & =- \(\mathrm{left}[3\) & -6 & 0 \\
\hline 0 & -3 & 6 & \\
\hline 12 & 9 & --18\right] & \\
\hline
\end{tabular}
\(2 B \quad=\) =left[2Vleft(-1 1 right) \(2 \cdot 2 \quad 2 \cdot 1\)
\begin{tabular}{llll}
\(2 \cdot 0\) & \(2(-3)\) & \(2 \cdot 2\) & \\
\(2 \cdot 0\) & \(2 \cdot 1\) & 2Veft(-4lright)\right] & \\
& =Veft[-2 & 4 & 2 \\
0 & -6 & 4 & \\
0 & 2 & -8 lright] &
\end{tabular}

Now, add \(3 A+2 B\).
\begin{tabular}{|c|c|c|c|}
\hline \(3 A+2 B\) & =Veft[3 & -6 & 0 \\
\hline 0 & -3 & 6 & \\
\hline 12 & 9 & -18\right]+\eft[-2 & 4 \\
\hline 0 & -6 & 4 & \\
\hline \multirow[t]{2}{*}{0} & 2 & -8\right] & \\
\hline & =Veft[3-2 & \(-6+4\) & \(0+2\) \\
\hline \(0+0\) & \(-3-6\) & \(6+4\) & \\
\hline \multirow[t]{2}{*}{\(12+0\)} & \(9+2\) & -18-8\right] & \\
\hline & -Veft[1 & -2 & 2 \\
\hline
\end{tabular}
\begin{tabular}{lll}
0 & -9 & 10 \\
12 & 11 & -26 light]
\end{tabular}

\section*{TRY IT}

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We multiply entries of \(A\) with entries of \(B\) according to a specific pattern as outlined below. The process of matrix multiplication becomes clearer when working a problem with real numbers.

\section*{TIP FOR SUCCESS}

Work through the example model of matrix multiplication below on paper, then apply the process to the example problem below. It may take more than once to gain familiarity with it. Don't be discouraged if you don't understand fully right away. Matrix multiplication is a new skill and it will take time and practice for it to feel comfortable.

To obtain the entries in row \(i\) of \(A B\), we multiply the entries in row \(i\) of \(A\) by column \(j\) in \(B\) and add. For example, given matrices \(A\) and \(B\), where the dimensions of \(A\) are \(2 \times 3\) and the dimensions of \(B\) are \(3 \times 3\), the product of \(A B\) will be a \(2 \times 3\) matrix.
\(A=\left[\begin{array}{lll}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23}\end{array}\right]\) and \(B=\left[\begin{array}{lll}b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33}\end{array}\right]\)
Multiply and add as follows to obtain the first entry of the product matrix \(A B\).
1. To obtain the entry in row 1 , column 1 of \(A B\), multiply the first row in \(A\) by the first column in \(B\) and add.
\[
\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13}
\end{array}\right] \cdot\left[\begin{array}{l}
b_{11} \\
b_{21} \\
b_{31}
\end{array}\right]=a_{11} \cdot b_{11}+a_{12} \cdot b_{21}+a_{13} \cdot b_{31}
\]
2. To obtain the entry in row 1 , column 2 of \(A B\), multiply the first row of \(A\) by the second column in \(B\) and add.
\[
\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13}
\end{array}\right] \cdot\left[\begin{array}{l}
b_{12} \\
b_{22} \\
b_{32}
\end{array}\right]=a_{11} \cdot b_{12}+a_{12} \cdot b_{22}+a_{13} \cdot b_{32}
\]
3. To obtain the entry in row 1 , column 3 of \(A B\), multiply the first row of \(A\) by the third column in \(B\) and add.
\[
\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13}
\end{array}\right] \cdot\left[\begin{array}{l}
b_{13} \\
b_{23} \\
b_{33}
\end{array}\right]=a_{11} \cdot b_{13}+a_{12} \cdot b_{23}+a_{13} \cdot b_{33}
\]

We proceed the same way to obtain the second row of \(A B\). In other words, row 2 of \(A\) times column 1 of \(B\); row 2 of \(A\) times column 2 of \(B\); row 2 of \(A\) times column 3 of \(B\). When complete, the product matrix will be
\(A B=\left[\begin{array}{llll}a_{11} \cdot b_{11}+a_{12} \cdot b_{21}+a_{13} \cdot b_{31} & a_{11} \cdot b_{12}+a_{12} \cdot b_{22}+a_{13} \cdot b_{32} & a_{11} \cdot b_{13}+a_{12} \cdot b_{23}+a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11}+a_{22} \cdot b_{21}+a_{23} \cdot b_{31} & a_{21} \cdot b_{12}+a_{22} \cdot b_{22}+a_{23} \cdot b_{32} & a_{21} \cdot b_{13}+a_{22} \cdot b_{23}+a_{23} \cdot b_{33}\end{array}\right]\)
A GENERAL NOTE: PROPERTIES OF MATRIX MULTIPLICATION

For the matrices \(A, B\), and \(C\) the following properties hold.
- Matrix multiplication is associative:
\((A B) C=A(B C)\)
- Matrix multiplication is distributive:
\[
\begin{aligned}
& C(A+B)=C A+C B, \\
& (A+B) C=A C+B C .
\end{aligned}
\]

Note that matrix multiplication is not commutative.

\section*{EXAMPLE: MULTIPLYING TWO MATRICES}

Multiply matrix \(A\) and matrix \(B\).
\(A=\left[\begin{array}{ll}1 & 2 \\ 3 & 4\end{array}\right]\) and \(B=\left[\begin{array}{ll}5 & 6 \\ 7 & 8\end{array}\right]\)
Answer
First, we check the dimensions of the matrices. Matrix \(A\) has dimensions \(2 \times 2\) and matrix \(B\) has dimensions \(2 \times 2\). The inner dimensions are the same so we can perform the multiplication. The product will have the dimensions \(2 \times 2\).
We perform the operations outlined previously.
\[
\begin{aligned}
A B & =\left[\begin{array}{ll}
1 & 2 \\
3 & 4
\end{array}\right] \cdot\left[\begin{array}{ll}
5 & 6 \\
7 & 8
\end{array}\right] \\
& =\left[\begin{array}{ll}
1(5)+2(7) & 1(6)+2(8) \\
3(5)+4(7) & 3(6)+4(8)
\end{array}\right] \\
& =\left[\begin{array}{ll}
19 & 22 \\
43 & 50
\end{array}\right]
\end{aligned}
\]

\section*{TRY IT}

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\section*{EXAMPLE: MULTIPLYING TWO MATRICES}

Given \(A\) and \(B\) :
1. Find \(A B\).
2. Find \(B A\).
\(A=\left[\begin{array}{rrr}-1 & 2 & 3 \\ 4 & 0 & 5\end{array}\right]\) and \(B=\left[\begin{array}{rr}5 & -1 \\ -4 & 0 \\ 2 & 3\end{array}\right]\)

\section*{Answer}
1. As the dimensions of \(A\) are \(2 \times 3\) and the dimensions of \(B\) are \(3 \times 2\), these matrices can be multiplied together because the number of columns in \(A\) matches the number of rows in \(B\). The resulting product will be a \(2 \times 2\) matrix, the number of rows in \(A\) by the number of columns in \(B\).
\begin{tabular}{|c|c|c|c|}
\hline \(A B\) & =Veft[-1 & 2 & 3 \\
\hline 4 & 0 & 5\right]\eft[5 & -1 \\
\hline -4 & 0 & & \\
\hline 2 & 3 3right] & & \\
\hline & =Veft[-1 Veft(5\right)+2\eft(-4\right)+3Veft(2Vright) & \(-1(-1)+2(0)+3(3)\) & \\
\hline \(4(5)+0(-4)+5(2)\) &  & & \\
\hline & =Veft[-7 & 10 & \\
\hline 30 & 11\right] & & \\
\hline
\end{tabular}
2. The dimensions of \(B\) are \(3 \times 2\) and the dimensions of \(A\) are \(2 \times 3\). The inner dimensions match so the product is defined and will be a \(3 \times 3\) matrix.
\begin{tabular}{|c|c|c|c|}
\hline \(B A\) & - Veft[5 & -1 & \\
\hline -4 & 0 & & \\
\hline 2 & 3 3right]\eft[-1 & 2 & 3 \\
\hline \multirow[t]{2}{*}{4} & 0 & 5 Sright] & \\
\hline & = \(=\) Veft[5Veft(-1 \right)+-1 \({ }^{\text {left(4Vright) }}\) & \(5(2)+-1(0)\) & \(5(3)+-1(5)\) \\
\hline \(-4(-1)+0(4)\) & \(-4(2)+0(0)\) & \(-4(3)+0(5)\) & \\
\hline \multirow[t]{2}{*}{\(2(-1)+3(4)\)} & \(2(2)+3(0)\) & 2Veft(3)right)+3Veft(5\right)\right] & \\
\hline & - Veft[-9 & 10 & 10 \\
\hline 4 & -8 & -12 & \\
\hline 10 & 4 & 21\right] & \\
\hline
\end{tabular}

Notice that the products \(A B\) and \(B A\) are not equal.
\(A B=\left[\begin{array}{cc}-7 & 10 \\ 30 & 11\end{array}\right] \neq\left[\begin{array}{ccc}-9 & 10 & 10 \\ 4 & -8 & -12 \\ 10 & 4 & 21\end{array}\right]=B A\)
This illustrates the fact that matrix multiplication is not commutative.

\section*{Q \& A}

Is it possible for \(A B\) to be defined but not BA?
Yes, consider a matrix \(A\) with dimension \(3 \times 4\) and matrix \(B\) with dimension \(4 \times 2\). For the product \(A B\) the inner dimensions are 4 and the product is defined, but for the product BA the inner dimensions are 2 and 3 so the product is undefined.

\section*{EXAMPLE: USING MATRICES IN REAL-WORLD PROBLEMS}

Let's return to the problem presented at the opening of this section. We have the table below, representing the equipment needs of two soccer teams.
\begin{tabular}{|l|c|c|c|}
\hline & \multicolumn{2}{|c|}{ Wildcats } & Mud Cats \\
\hline Goals & 6 & & 10 \\
\hline Balls & 30 & & 24 \\
\hline Jerseys & 14 & & 20 \\
\hline We are also given the prices of the equipment, as shown in the table below. & \\
\hline Goal & \(\$ 300\) & \\
\hline Ball & \(\$ 10\) & \\
\hline Jersey & \(\$ 30\) & \\
\hline
\end{tabular}

We will convert the data to matrices. Thus, the equipment need matrix is written as
\(E=\left[\begin{array}{cc}6 & 10 \\ 30 & 24 \\ 14 & 20\end{array}\right]\)
The cost matrix is written as
\(C=\left[\begin{array}{lll}300 & 10 & 30\end{array}\right]\)
We perform matrix multiplication to obtain costs for the equipment.
\(C E=\) =Veft[300
10
\(30 \quad 24\)
14 20\right]
=\eft[300\eft(6\right)+10\eft(30\right)+30\eft(14\right) 300\eft(10\right)+10\eft(24\right)+30\eft(20\right)\right]
\(=\operatorname{Veft}[2,520\) 3,840\right]

The total cost for equipment for the Wildcats is \(\$ 2,520\), and the total cost for equipment for the Mud Cats is \(\$ 3,840\).

\section*{HOW TO: GIVEN A MATRIX OPERATION, EVALUATE USING A CALCULATOR}
1. Save each matrix as a matrix variable
\[
[A],[B],[C], \ldots
\]
2. Enter the operation into the calculator, calling up each matrix variable as needed.
3. If the operation is defined, the calculator will present the solution matrix; if the operation is undefined, it will display an error message.

\section*{EXAMPLE: USING A CALCULATOR TO PERFORM MATRIX OPERATIONS}

Find \(A B-C\) given
\(A=\left[\begin{array}{rrr}-15 & 25 & 32 \\ 41 & -7 & -28 \\ 10 & 34 & -2\end{array}\right], B=\left[\begin{array}{rrr}45 & 21 & -37 \\ -24 & 52 & 19 \\ 6 & -48 & -31\end{array}\right]\), and \(C=\left[\begin{array}{rrr}-100 & -89 & -98 \\ 25 & -56 & 74 \\ -67 & 42 & -75\end{array}\right]\).
Answer
On the matrix page of the calculator, we enter matrix \(A\) above as the matrix variable \([A]\), matrix \(B\) above as the matrix variable \([B]\), and matrix \(C\) above as the matrix variable \([C]\).
On the home screen of the calculator, we type in the problem and call up each matrix variable as needed.
\([A] \times[B]-[C]\)
The calculator gives us the following matrix.
\(\left[\begin{array}{rrr}-983 & -462 & 136 \\ 1,820 & 1,897 & -856 \\ -311 & 2,032 & 413\end{array}\right]\)

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\section*{SUMMARY: MATRICES AND MATRIX OPERATIONS}

\section*{Key Concepts}
- A matrix is a rectangular array of numbers. Entries are arranged in rows and columns.
- The dimensions of a matrix refer to the number of rows and the number of columns. A \(3 \times 2\) matrix has three rows and two columns.
- We add and subtract matrices of equal dimensions by adding and subtracting corresponding entries of each matrix.
- Scalar multiplication involves multiplying each entry in a matrix by a constant.
- Scalar multiplication is often required before addition or subtraction can occur.
- Multiplying matrices is possible when inner dimensions are the same-the number of columns in the first matrix must match the number of rows in the second.
- The product of two matrices, \(A\) and \(B\), is obtained by multiplying each entry in row 1 of \(A\) by each entry in column 1 of \(B\); then multiply each entry of row 1 of \(A\) by each entry in columns 2 of \(B\), and so on.
- Many real-world problems can often be solved using matrices.
- We can use a calculator to perform matrix operations after saving each matrix as a matrix variable.

\section*{Glossary}
column a set of numbers aligned vertically in a matrix
entry an element, coefficient, or constant in a matrix
matrix a rectangular array of numbers
row a set of numbers aligned horizontally in a matrix
scalar multiple an entry of a matrix that has been multiplied by a scalar

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\section*{INTRODUCTION TO GAUSSIAN ELIMINATION}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Write the augmented matrix for a system of equations.
- Perform row operations on an augmented matrix.
- Use Gaussian elimination to solve a systems of equations represented as an augmented matrix.
- Interpret the solution to a system of equations represented as an augmented matrix.


German mathematician Carl Friedrich Gauss (1777-1855).
Carl Friedrich Gauss lived during the late 18th century and early 19th century, but he is still considered one of the most prolific mathematicians in history. His contributions to the science of mathematics and physics span fields such as algebra, number theory, analysis, differential geometry, astronomy, and optics, among
others. His discoveries regarding matrix theory changed the way mathematicians have worked for the last two centuries.

We first encountered Gaussian elimination in Systems of Linear Equations: Two Variables. In this section, we will revisit this technique for solving systems, this time using matrices.

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\section*{ROW OPERATIONS AND AUGMENTED MATRICES}

\section*{LEARNING OUTCOMES}
- Write the augmented matrix for a system of equations.
- Perform row operations on an augmented matrix.

A matrix can serve as a device for representing and solving a system of equations. To express a system in matrix form, we extract the coefficients of the variables and the constants, and these become the entries of the matrix. We use a vertical line to separate the coefficient entries from the constants, essentially replacing the equal signs. When a system is written in this form, we call it an augmented matrix.

For example, consider the following \(2 \times 2\) system of equations.
\[
\begin{aligned}
& 3 x+4 y=7 \\
& 4 x-2 y=5
\end{aligned}
\]

We can write this system as an augmented matrix:
\[
\left[\begin{array}{rr|r}
3 & 4 & 7 \\
4 & -2 & 5
\end{array}\right]
\]

We can also write a matrix containing just the coefficients. This is called the coefficient matrix.
\[
\left[\begin{array}{cc}
3 & 4 \\
4 & -2
\end{array}\right]
\]

A three-by-three system of equations such as
\[
\begin{aligned}
& 3 x-y-z=0 \\
& x+y=5 \\
& 2 x-3 z=2
\end{aligned}
\]
has a coefficient matrix
\(\left[\begin{array}{rrr}3 & -1 & -1 \\ 1 & 1 & 0 \\ 2 & 0 & -3\end{array}\right]\)
and is represented by the augmented matrix
\(\left[\begin{array}{rrr|r}3 & -1 & -1 & 0 \\ 1 & 1 & 0 & 5 \\ 2 & 0 & -3 & 2\end{array}\right]\)
Notice that the matrix is written so that the variables line up in their own columns: \(x\)-terms go in the first column, \(y\)-terms in the second column, and \(z\)-terms in the third column. It is very important that each equation is written in standard form \(a x+b y+c z=d\) so that the variables line up. When there is a missing variable term in an equation, the coefficient is 0 .

\section*{HOW TO: GIVEN A SYSTEM OF EQUATIONS, WRITE AN AUGMENTED MATRIX}
1. Write the coefficients of the \(x\)-terms as the numbers down the first column.
2. Write the coefficients of the \(y\)-terms as the numbers down the second column.
3. If there are \(z\)-terms, write the coefficients as the numbers down the third column.
4. Draw a vertical line and write the constants to the right of the line.

\section*{EXAMPLE: WRITING THE AUGMENTED MATRIX FOR A SYSTEM OF EQUATIONS}

Write the augmented matrix for the given system of equations.
\(x+2 y-z=3\)
\(2 x-y+2 z=6\)
\(x-3 y+3 z=4\)

\section*{Answer}

The augmented matrix displays the coefficients of the variables and an additional column for the constants.
\[
\left[\begin{array}{rrr|r}
1 & 2 & -1 & 3 \\
2 & -1 & 2 & 6 \\
1 & -3 & 3 & 4
\end{array}\right]
\]

\section*{TRY IT}

Write the augmented matrix of the given system of equations.
\(4 x-3 y=11\)
\(3 x+2 y=4\)
Answer

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\section*{Writing a System of Equations from an Augmented Matrix}

We can use augmented matrices to help us solve systems of equations because they simplify operations when the systems are not encumbered by the variables. However, it is important to understand how to move back and forth between formats in order to make finding solutions smoother and more intuitive. Here, we will use the information in an augmented matrix to write the system of equations in standard form.

\section*{EXAMPLE: WRITING A SYSTEM OF EQUATIONS FROM AN AUGMENTED MATRIX FORM}

Find the system of equations from the augmented matrix.
\[
\left[\begin{array}{rrr|r}
1 & -3 & -5 & -2 \\
2 & -5 & -4 & 5 \\
-3 & 5 & 4 & 6
\end{array}\right]
\]

\section*{Answer}

When the columns represent the variables \(x, y\), and \(z\),
\[
\left[\begin{array}{rrr|r}
1 & -3 & -5 & -2 \\
2 & -5 & -4 & 5 \\
-3 & 5 & 4 & 6
\end{array}\right] \rightarrow \begin{aligned}
& x-3 y-5 z=-2 \\
& 2 x-5 y-4 z=5 \\
& -3 x+5 y+4 z=6
\end{aligned}
\]

\section*{TRY IT}

Write the system of equations from the augmented matrix.
\[
\left[\begin{array}{rrr|r}
1 & -1 & 1 & 5 \\
2 & -1 & 3 & 1 \\
0 & 1 & 1 & -9
\end{array}\right]
\]

\section*{Answer}
\(x-y+z=5\)
\(2 x-y+3 z=1\)
\(y+z=-9\)

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\section*{Row Operations}

Now that we can write systems of equations in augmented matrix form, we will examine the various row operations that can be performed on a matrix, such as addition, multiplication by a constant, and interchanging rows.

Performing row operations on a matrix is the method we use for solving a system of equations. In order to solve the system of equations, we want to convert the matrix to row-echelon form, in which there are ones down the main diagonal from the upper left corner to the lower right corner and zeros in every position below the main diagonal as shown.
Row-echelon form
\begin{tabular}{lll}
\hline Neft[1 & \(a\) & \(b\) \\
0 & 1 & \(d\) \\
0 & 0 & 1/right]
\end{tabular}

We use row operations corresponding to equation operations to obtain a new matrix that is row-equivalent in a simpler form. Here are the guidelines to obtaining row-echelon form.
1. In any nonzero row, the first nonzero number is a 1 . It is called a leading 1 .
2. Any all-zero rows are placed at the bottom of the matrix.
3. Any leading 1 is below and to the right of a previous leading 1.
4. Any column containing a leading 1 has zeros in all other positions in the column.

To solve a system of equations we can perform the following row operations to convert the coefficient matrix to row-echelon form and do back-substitution to find the solution.
1. Interchange rows. (Notation: \(R_{i} \leftrightarrow R_{j}\) )
2. Multiply a row by a constant. (Notation: \(c R_{i}\) )
3. Add the product of a row multiplied by a constant to another row. (Notation: \(R_{i}+c R_{j}\) )

Each of the row operations corresponds to the operations we have already learned to solve systems of equations in three variables. With these operations, there are some key moves that will quickly achieve the goal of writing a matrix in row-echelon form. To obtain a matrix in row-echelon form for finding solutions, we use Gaussian elimination, a method that uses row operations to obtain a 1 as the first entry so that row 1 can be used to convert the remaining rows.

\section*{TIP FOR SUCCESS}

As with other challenging concepts you've encountered in this course, time and patient practice with the process of Gaussian elimination will aid familiarity.
If the instructions seem intimidating, work out the example below on paper once or twice, then try to apply the General Note and How To information to the example.
Work back and forth between them until you gain a firmer understanding. Then try the practice problems.

\section*{A GENERAL NOTE: GAUSSIAN ELIMINATION}

The Gaussian elimination method refers to a strategy used to obtain the row-echelon form of a matrix. The goal is to write matrix \(A\) with the number 1 as the entry down the main diagonal and have all zeros below.
\[
A=\left[\begin{array}{lll}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right] \stackrel{\text { After Gaussian elimination }}{\rightarrow} A=\left[\begin{array}{rrr}
1 & b_{12} & b_{13} \\
0 & 1 & b_{23} \\
0 & 0 & 1
\end{array}\right]
\]

The first step of the Gaussian strategy includes obtaining a 1 as the first entry, so that row 1 may be used to alter the rows below.

\section*{HOW TO: GIVEN AN AUGMENTED MATRIX, PERFORM ROW OPERATIONS TO ACHIEVE ROW-ECHELON FORM}
1. The first equation should have a leading coefficient of 1 . Interchange rows or multiply by a constant, if necessary.
2. Use row operations to obtain zeros down the first column below the first entry of 1 .
3. Use row operations to obtain a 1 in row 2 , column 2.
4. Use row operations to obtain zeros down column 2, below the entry of 1.
5. Use row operations to obtain a 1 in row 3 , column 3.
6. Continue this process for all rows until there is a 1 in every entry down the main diagonal and there are only zeros below.
7. If any rows contain all zeros, place them at the bottom.

\section*{EXAMPLE: PERFORMING ROW OPERATIONS ON A \(3 \times 3\) AUGMENTED MATRIX TO OBTAIN ROW-ECHELON FORM}

Perform row operations on the given matrix to obtain row-echelon form.
\[
\left[\begin{array}{rrr|r}
1 & -3 & 4 & 3 \\
2 & -5 & 6 & 6 \\
-3 & 3 & 4 & 6
\end{array}\right]
\]

\section*{Answer}

The first row already has a 1 in row 1 , column 1 . The next step is to multiply row 1 by -2 and add it to row 2. Then replace row 2 with the result.
\[
-2 R_{1}+R_{2}=R_{2} \rightarrow\left[\begin{array}{rrr|r}
1 & -3 & 4 & 3 \\
0 & 1 & -2 & 0 \\
-3 & 3 & 4 & 6
\end{array}\right]
\]

Next, obtain a zero in row 3 , column 1.
\[
3 R_{1}+R_{3}=R_{3} \rightarrow\left[\begin{array}{rrr|r}
1 & -3 & 4 & 3 \\
0 & 1 & -2 & 0 \\
0 & -6 & 16 & 15
\end{array}\right]
\]

Next, obtain a zero in row 3, column 2.
\[
6 R_{2}+R_{3}=R_{3} \rightarrow\left[\begin{array}{rrr|r}
1 & -3 & 4 & 3 \\
0 & 1 & -2 & 0 \\
0 & 0 & 4 & 15
\end{array}\right]
\]

The last step is to obtain a 1 in row 3 , column 3.
\[
\frac{1}{4} R_{3}=R_{3} \rightarrow\left[\begin{array}{rrr|r}
1 & -3 & 4 & 3 \\
0 & 1 & -2 & -6 \\
0 & 0 & 1 & \frac{15}{4}
\end{array}\right]
\]

\section*{TRY IT}

Write the system of equations in row-echelon form.
\(x-2 y+3 z=9\)
\[
-x+3 y=-4
\]
\[
2 x-5 y+5 z=17
\]

\section*{Answer}
\(\left[\begin{array}{rrr|r}1 & -\frac{5}{2} & \frac{5}{2} & \frac{17}{2} \\ 0 & 1 & 5 & 9 \\ 0 & 0 & 1 & 2\end{array}\right]\)

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\section*{SOLVING A SYSTEM WITH GAUSSIAN ELIMINATION}

\section*{LEARNING OUTCOMES}
- Use Gaussian elimination to solve a systems of equations represented as an augmented matrix.
- Interpret the solution to a system of equations represented as an augmented matrix.

We have seen how to write a system of equations with an augmented matrix and then how to use row operations and back-substitution to obtain row-echelon form. Now we will use Gaussian Elimination as a tool for solving a system written as an augmented matrix. In our first example, we will show you the process for using Gaussian Elimination on a system of two equations in two variables.

\section*{TIP FOR SUCCESS}

Gain practice by working out the given examples on paper, perhaps more than once or twice, before trying practice problems.
As with any challenging concept in mathematics, patient practice and effort aid understanding.
In our next example, we will solve a system of two equations in two variables that is dependent. Recall that a dependent system has an infinite number of solutions and the result of row operations on its augmented matrix will be an equation such as \(0=0\). We also review writing the general solution to a dependent system.

Now, we will take row-echelon form a step further to solve a 3 by 3 system of linear equations. The general

\section*{EXAMPLE: SOLVING A 2 X 2 SYSTEM BY GAUSSIAN ELIMINATION}

Solve the given system by Gaussian elimination.
\[
\begin{aligned}
& 2 x+3 y=6 \\
& x-y=\frac{1}{2}
\end{aligned}
\]

\section*{Answer}

First, we write this as an augmented matrix.
\[
\left[\begin{array}{rr|r}
2 & 3 & 6 \\
1 & -1 & \frac{1}{2}
\end{array}\right]
\]

We want a 1 in row 1 , column 1 . This can be accomplished by interchanging row 1 and row 2.
\[
R_{1} \leftrightarrow R_{2} \rightarrow\left[\begin{array}{rr|r}
1 & -1 & \frac{1}{2} \\
2 & 3 & 6
\end{array}\right]
\]

We now have a 1 as the first entry in row 1 , column 1 . Now let's obtain a 0 in row 2 , column 1 . This can be accomplished by multiplying row 1 by -2 and then adding the result to row 2 .
\(-2 R_{1}+R_{2}=R_{2} \rightarrow\left[\begin{array}{rr|r}1 & -1 & \frac{1}{2} \\ 0 & 5 & 5\end{array}\right]\)
We only have one more step, to multiply row 2 by \(\frac{1}{5}\).
\(\frac{1}{5} R_{2}=R_{2} \rightarrow\left[\begin{array}{rr|r}1 & -1 & \frac{1}{2} \\ 0 & 1 & 1\end{array}\right]\)
Use back-substitution. The second row of the matrix represents \(y=1\). Back-substitute \(y=1\) into the first equation.
\[
\begin{aligned}
& x-(1)=\frac{1}{2} \\
& x=\frac{3}{2}
\end{aligned}
\]

The solution is the point \(\left(\frac{3}{2}, 1\right)\).

\section*{TRY IT}

Solve the given system by Gaussian elimination.
\(4 x+3 y=11\)
\(x-3 y=-1\)

\section*{Answer}
\((2,1)\)
idea is to eliminate all but one variable using row operations and then back-substitute to solve for the other variables.

Recall that there are three possible outcomes for solutions to linear systems. In the previous example, the solution \((4,-3,1)\) represents a point in three dimensional space. This point represents the intersection of three planes. In the next example, we solve a system using row operations and find that it represents a dependent system. A dependent system in 3 dimensions can be represented by two planes that are identical, much like in 2 dimensions where a dependent system represents two lines that are identical.

\section*{EXAMPLE: SOLVING A DEPENDENT SYSTEM}

Solve the system of equations.
\[
\begin{aligned}
& 3 x+4 y=12 \\
& 6 x+8 y=24
\end{aligned}
\]

\section*{Answer}

Perform row operations on the augmented matrix to try and achieve row-echelon form.
\(A=\left[\begin{array}{ll|l}3 & 4 & 12 \\ 6 & 8 & 24\end{array}\right]\)
-lfrac\{1\}2\}R\}_\{2\}+\{R\}_\{1\}=\{R\}_\{1\}|to Veft|Dbegin\{array\}cclc\}0 00

6
Lright]
\(\{R\}\) _\{1\}\leftrightarrow \(\{R\}\) _\{2\}|to Veft|\begin\{array\}cclc\}6 }
0

00
Vright]
The matrix ends up with all zeros in the last row: \(0 y=0\). Thus, there are an infinite number of solutions and the system is classified as dependent. To find the generic solution, return to one of the original equations and solve for \(y\).
\(3 x+4 y=12\)
\(4 y=12-3 x\)
\(y=3-\frac{3}{4} x\)
So the solution to this system is \(\left(x, 3-\frac{3}{4} x\right)\).

\section*{Applications of Systems of Equations}

Now we will turn to the applications for which systems of equations are used. In the next example we determine how much money was invested at two different rates given the sum of the interest earned by both accounts.

\section*{EXAMPLE: SOLVING A SYSTEM OF LINEAR EQUATIONS USING MATRICES}

Solve the system of linear equations using matrices.
\[
\begin{aligned}
& x-y+z=8 \\
& 2 x+3 y-z=-2 \\
& 3 x-2 y-9 z=9
\end{aligned}
\]

\section*{Answer}

First, we write the augmented matrix.
\(\left[\begin{array}{rrr|r}1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9\end{array}\right]\)

Next, we perform row operations to obtain row-echelon form.
\begin{tabular}{|c|c|c|c|c|}
\hline -2\{R\}_\{1\}+\{R\}_\{2\}=\{R\}_\{2\}to Veft||begin\{array \({ }^{\text {cccclc }} \mathbf{1} 1\) & -1 & 1 & 8 & \\
\hline 0 & 5 & -3 & -18 & \\
\hline 3 & -2 & -9 & 9right] & -3\{R\}_\{1\}+\{R\}_\{3\}=\{R\}_\{3\}|to \e \\
\hline 0 & 5 & -3 & -18 & \\
\hline 0 & 1 & -12 & -15\right] & \\
\hline
\end{tabular}

The easiest way to obtain a 1 in row 2 of column 1 is to interchange \(R_{2}\) and \(R_{3}\).
Interchange \(R_{2}\) and \(R_{3} \rightarrow\left[\begin{array}{rrr|r}1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18\end{array}\right]\)
Then

\begin{tabular}{llll}
0 & 1 & -12 & -15 \\
0 & 0 & 57 & 57vright] \\
0 & 1 & -12 & -15 \\
0 & 0 & 1 & 1\right]
\end{tabular}
- -lfrac \(\{1\} 57\}\) R\}_\{3\}=\{R\}_\{3\}|to \(\}\)

The last matrix represents the equivalent system.
\[
\begin{aligned}
& x-y+z=8 \\
& y-12 z=-15 \\
& z=1
\end{aligned}
\]

Using back-substitution, we obtain the solution as \((4,-3,1)\).

\section*{EXAMPLE: SOLVING A 3 X 3 DEPENDENT SYSTEM}

Solve the following system of linear equations using Gaussian Elimination.
\[
\begin{aligned}
& -x-2 y+z=-1 \\
& 2 x+3 y=2 \\
& y-2 z=0
\end{aligned}
\]

\section*{Answer}

Write the augmented matrix.
\(\left[\begin{array}{rrr|r}-1 & -2 & 1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0\end{array}\right]\)
First, multiply row 1 by -1 to get a 1 in row 1 , column 1 . Then, perform row operations to obtain rowechelon form.
\(-R_{1} \rightarrow\left[\begin{array}{rrr|r}1 & 2 & -1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0\end{array}\right]\)
\(R_{2} \leftrightarrow R_{3} \rightarrow\left[\begin{array}{rrr|r}1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 2 & 3 & 0 & 2\end{array}\right]\)
\(-2 R_{1}+R_{3}=R_{3} \rightarrow\left[\begin{array}{rrr|r}1 & 2 & -1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0\end{array}\right]\)
\(R_{2}+R_{3}=R_{3} \rightarrow\left[\begin{array}{rrr|r}1 & 2 & -1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0\end{array}\right]\)
The last matrix represents the following system.
\(x+2 y-z=1\)
\(y-2 z=0\)
\(0=0\)
We see by the identity \(0=0\) that this is a dependent system with an infinite number of solutions. We then find the generic solution. By solving the second equation for \(y\) and substituting it into the first equation we can solve for \(z\) in terms of \(x\).
\[
\begin{aligned}
& x+2 y-z=1 \\
& y=2 z \\
& x+2(2 z)-z=1 \\
& x+3 z=1 \\
& z=\frac{1-x}{3}
\end{aligned}
\]

Now we substitute the expression for \(z\) into the second equation to solve for \(y\) in terms of \(x\).
\(y-2 z=0\)
\(z=\frac{1-x}{3}\)
\(y-2\left(\frac{1-x}{3}\right)=0\)
\(y=\frac{2-2 x}{3}\)
The generic solution is \(\left(x, \frac{2-2 x}{3}, \frac{1-x}{3}\right)\).

\section*{THE GENERAL SOLUTION TO A DEPENDENT 3 X 3 SYSTEM}

Recall that when you solve a dependent system of linear equations in two variables using elimination or substitution, you can write the solution \((x, y)\) in terms of x , because there are infinitely many ( \(\mathrm{x}, \mathrm{y}\) ) pairs that will satisfy a dependent system of equations, and they all fall on the line \((x, m x+b)\). Now that you are working in three dimensions, the solution will represent a plane, so you would write it in the general form ( \(x, m_{1} x+b_{1}, m_{2} x+b_{2}\) ).

\section*{TIP FOR SUCCESS}

Do you recall how you wrote the general solution to dependent linear systems in the previous module? Writing the general solution for these systems in terms of \(x\) is no different.

\section*{TRY IT}

Solve the system using Gaussian Elimination.
\(x+4 y-z=4\)
\(2 x+5 y+8 z=15\)
\(x+3 y-3 z=1\)

\section*{Answer}
\((1,1,1)\)
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\section*{Q \& A}

Can any system of linear equations be solved by Gaussian elimination?
Yes, a system of linear equations of any size can be solved by Gaussian elimination.

\section*{HOW TO: GIVEN A SYSTEM OF EQUATIONS, SOLVE WITH MATRICES USING A CALCULATOR}
1. Save the augmented matrix as a matrix variable \([A],[B],[C], \ldots\).
2. Use the ref( function in the calculator, calling up each matrix variable as needed.

\section*{EXAMPLE: SOLVING SYSTEMS OF EQUATIONS USING A CALCULATOR}

Solve the system of equations.
\[
\begin{aligned}
& 5 x+3 y+9 z=-1 \\
& -2 x+3 y-z=-2 \\
& -x-4 y+5 z=1
\end{aligned}
\]

\section*{Answer}

Write the augmented matrix for the system of equations.
\(\left[\begin{array}{rrr|r}5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1\end{array}\right]\)
On the matrix page of the calculator, enter the augmented matrix above as the matrix variable \([A]\).
\([A]=\left[\begin{array}{rrrr}5 & 3 & 9 & -1 \\ -2 & 3 & -1 & -2 \\ -1 & -4 & 5 & 1\end{array}\right]\)
Use the ref( function in the calculator, calling up the matrix variable \([A]\).
\(\operatorname{ref}([A])\)
Evaluate.
\(\left[\begin{array}{rrrr}1 & \frac{3}{5} & \frac{9}{5} & \frac{1}{5} \\ 0 & 1 & \frac{13}{21} & -\frac{4}{7} \\ 0 & 0 & 1 & -\frac{24}{187}\end{array}\right] \rightarrow \begin{aligned} & x+\frac{3}{5} y+\frac{9}{5} z=-\frac{1}{5} \\ & y+\frac{13}{21} z=-\frac{4}{7} \\ & z=-\frac{24}{187}\end{aligned}\)
Using back-substitution, the solution is \(\left(\frac{61}{187},-\frac{92}{187},-\frac{24}{187}\right)\).

\section*{TIP FOR SUCCESS}

Setting up a system of equations in the following examples uses the same ideas you have used before to write a system of linear equations to model a situation. The only difference is that now you'll use matrices and Gaussian elimination to solve the system.

\section*{EXAMPLE: APPLYING \(2 \times 2\) MATRICES TO FINANCE}

Carolyn invests a total of \(\$ 12,000\) in two municipal bonds, one paying \(10.5 \%\) interest and the other paying \(12 \%\) interest. The annual interest earned on the two investments last year was \(\$ 1,335\). How much was invested at each rate?
Answer
We have a system of two equations in two variables. Let \(x=\) the amount invested at \(10.5 \%\) interest, and \(y=\) the amount invested at \(12 \%\) interest.
\(x+y=12,000\)
\(0.105 x+0.12 y=1,335\)
As a matrix, we have
\(\left[\begin{array}{rr|r}1 & 1 & 12,000 \\ 0.105 & 0.12 & 1,335\end{array}\right]\)
Multiply row 1 by -0.105 and add the result to row 2 .
\(\left[\begin{array}{rr|r}1 & 1 & 12,000 \\ 0 & 0.015 & 75\end{array}\right]\)
Then,
\(0.015 y=75\)
\(y=5,000\)
So \(12,000-5,000=7,000\).
Thus, \(\$ 5,000\) was invested at \(12 \%\) interest and \(\$ 7,000\) at \(10.5 \%\) interest.

\section*{TRY IT}

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\section*{EXAMPLE: APPLYING \(3 \times 3\) MATRICES TO FINANCE}

Ava invests a total of \$10,000 in three accounts, one paying \(5 \%\) interest, another paying \(8 \%\) interest, and the third paying \(9 \%\) interest. The annual interest earned on the three investments last year was \(\$ 770\). The amount invested at \(9 \%\) was twice the amount invested at \(5 \%\). How much was invested at each rate?

\section*{Answer}

We have a system of three equations in three variables. Let \(x\) be the amount invested at \(5 \%\) interest, let \(y\) be the amount invested at \(8 \%\) interest, and let \(z\) be the amount invested at \(9 \%\) interest. Thus,
\[
\begin{aligned}
& x+y+z=10,000 \\
& 0.05 x+0.08 y+0.09 z=770 \\
& 2 x-z=0
\end{aligned}
\]

As a matrix, we have
\(\left[\begin{array}{rrr|r}1 & 1 & 1 & 10,000 \\ 0.05 & 0.08 & 0.09 & 770 \\ 2 & 0 & -1 & 0\end{array}\right]\)

Now, we perform Gaussian elimination to achieve row-echelon form.
\begin{tabular}{|c|c|c|c|}
\hline -0.05\{R\}_\{1\}+\{R\}_\{2\}=\{R\}_\{2\}\to Veft|[begin\{array \({ }^{\text {cccclc }\} 1}\) & 1 & 1 & 10,000 \\
\hline 0 & 0.03 & 0.04 & 270 \\
\hline 2 & 0 & -1 & OVright] \\
\hline  & 1 & 1 & 10, 000 \\
\hline 0 & 0.03 & 0.04 & 270 \\
\hline 0 & -2 & -3 & -20,000\right] \\
\hline  & 1 & 1 & 10,000 \\
\hline 0 & 1 & \(\frac{4}{3}\) & 9,000 \\
\hline 0 & -2 & -3 & -20,000\right] \\
\hline  & 1 & 1 & 10,000 \\
\hline 0 & 1 & \(\frac{4}{3}\) & 9,000 \\
\hline 0 & 0 & \(-\frac{1}{3}\) & -2,000\right] \\
\hline
\end{tabular}

The third row tells us \(-\frac{1}{3} z=-2,000\); thus \(z=6,000\).
The second row tells us \(y+\frac{4}{3} z=9,000\).
Substituting \(z=6,000\), we get
\[
\begin{aligned}
& y+\frac{4}{3}(6,000)=9,000 \\
& y+8,000=9,000 \\
& y=1,000
\end{aligned}
\]

The first row tells us \(x+y+z=10,000\). Substituting \(y=1,000\) and \(z=6,000\), we get
\(x+1,000+6,000=10,000\)
\(x=3,000\)

The answer is \$3,000 invested at \(5 \%\) interest, \(\$ 1,000\) invested at \(8 \%\), and \(\$ 6,000\) invested at \(9 \%\) interest.

\section*{TRY IT}

A small shoe company took out a loan of \(\$ 1,500,000\) to expand their inventory. Part of the money was borrowed at \(7 \%\), part was borrowed at \(8 \%\), and part was borrowed at \(10 \%\). The amount borrowed at \(10 \%\) was four times the amount borrowed at \(7 \%\), and the annual interest on all three loans was \(\$ 130,500\). Use matrices to find the amount borrowed at each rate.
Answer
\(\$ 150,000\) at \(7 \%, \$ 750,000\) at \(8 \%, \$ 600,000\) at \(10 \%\)

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\section*{SUMMARY: GAUSSIAN ELIMINATION}

\section*{Key Concepts}
- An augmented matrix is one that contains the coefficients and constants of a system of equations.
- A matrix augmented with the constant column can be represented as the original system of equations.
- Row operations include multiplying a row by a constant, adding one row to another row, and interchanging rows.
- We can use Gaussian elimination to solve a system of equations.
- Row operations are performed on matrices to obtain row-echelon form.
- To solve a system of equations, write it in augmented matrix form. Perform row operations to obtain rowechelon form. Back-substitute to find the solutions.
- A calculator can be used to solve systems of equations using matrices.
- Many real-world problems can be solved using augmented matrices.

\section*{Glossary}
augmented matrix a coefficient matrix adjoined with the constant column separated by a vertical line within the matrix brackets
coefficient matrix a matrix that contains only the coefficients from a system of equations
Gaussian elimination using elementary row operations to obtain a matrix in row-echelon form
main diagonal entries from the upper left corner diagonally to the lower right corner of a square matrix
row-echelon form after performing row operations, the matrix form that contains ones down the main diagonal and zeros at every space below the diagonal
row-equivalent two matrices \(A\) and \(B\) are row-equivalent if one can be obtained from the other by performing basic row operations
row operations adding one row to another row, multiplying a row by a constant, interchanging rows, and so on, with the goal of achieving row-echelon form

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\section*{INTRODUCTION TO SOLVING SYSTEMS WITH INVERSES}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Verify that multiplying a matrix by its inverse results in 1.
- Use matrix multiplication to find the inverse of a matrix.
- Find an inverse by augmenting with an identity matrix.
- Solve a \(2 \times 2\) system using an inverse.
- Solve a \(3 \times 3\) systems using an inverse.
- Solve a system with a calculator.

Nancy plans to invest \(\$ 10,500\) into two different bonds to spread out her risk. The first bond has an annual return of \(10 \%\), and the second bond has an annual return of \(6 \%\). In order to receive an \(8.5 \%\) return from the two bonds, how much should Nancy invest in each bond? What is the best method to solve this problem?

There are several ways we can solve this problem. As we have seen in previous sections, systems of equations and matrices are useful in solving real-world problems involving finance. After studying this section, we will have the tools to solve the bond problem using the inverse of a matrix.

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\section*{FIND THE INVERSE OF A MATRIX}

\section*{LEARNING OUTCOMES}
- Verify that multiplying a matrix by its inverse results in 1.
- Use matrix multiplication to find the inverse of a matrix.
- Find an inverse by augmenting with an identity matrix.

We know that the multiplicative inverse of a real number \(a\) is \(a^{-1}\) and \(a a^{-1}=a^{-1} a=\left(\frac{1}{a}\right) a=1\). For example, \(2^{-1}=\frac{1}{2}\) and \(\left(\frac{1}{2}\right) 2=1\). The multiplicative inverse of a matrix is similar in concept, except that the product of matrix \(A\) and its inverse \(A^{-1}\) equals the identity matrix. The identity matrix is a square matrix containing ones down the main diagonal and zeros everywhere else. We identify identity matrices by \(I_{n}\) where \(n\) represents the dimension of the matrix. The equations below are the identity matrices for a \(2 \times 2\) matrix and \(3 \times 3\) matrix, respectively.
\(I_{2}=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]\)
\(I_{3}=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]\)
The identity matrix acts as a 1 in matrix algebra. For example, \(A I=I A=A\).
A matrix that has a multiplicative inverse has the properties
\[
\begin{aligned}
& A A^{-1}=I \\
& A^{-1} A=I
\end{aligned}
\]

A matrix that has a multiplicative inverse is called an invertible matrix. Only a square matrix may have a multiplicative inverse, as the reversibility, \(A A^{-1}=A^{-1} A=I\), is a requirement. Not all square matrices have an inverse, but if \(A\) is invertible, then \(A^{-1}\) is unique. We will look at two methods for finding the inverse of a \(2 \times 2\) matrix and a third method that can be used on both \(2 \times 2\) and \(3 \times 3\) matrices.

\section*{A GENERAL NOTE: THE IDENTITY MATRIX AND MULTIPLICATIVE INVERSE}

The identity matrix, \(I_{n}\), is a square matrix containing ones down the main diagonal and zeros everywhere else.
\begin{tabular}{|c|c|c|c|c|c|}
\hline \(I_{2}\) & =\eft[1 & \multirow[t]{2}{*}{0} & \multirow[b]{2}{*}{\(I_{3}\)} & \multirow[b]{2}{*}{eft[1} & \multirow[b]{2}{*}{00} \\
\hline 0 & 1 1 right] & & & & \\
\hline 0 & 1 & 0 & & & \\
\hline 0 & 0 & 1 1 right] & & & \\
\hline & \(2 \times 2\) & & & & \\
\hline
\end{tabular}

\section*{TIP FOR SUCCESS}

Work through each of the examples below on paper, then apply the information in the General Note and How To boxes.
Repeating this process a few times will help you to gain familiarity with the material.
Remember, practice and patient effort enable understanding. Don't be discouraged if you don't "get it" right away!

\section*{EXAMPLE: SHOWING THAT THE IDENTITY MATRIX ACTS AS A 1}

Given matrix \(A\), show that \(A I=I A=A\).
\(A=\left[\begin{array}{cc}3 & 4 \\ -2 & 5\end{array}\right]\)

\section*{Answer}

Use matrix multiplication to show that the product of \(A\) and the identity is equal to the product of the identity and \(A\).
\[
\begin{aligned}
& A I=\left[\begin{array}{rr}
3 & 4 \\
-2 & 5
\end{array}\right]\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]=\left[\begin{array}{rr}
3 \cdot 1+4 \cdot 0 & 3 \cdot 0+4 \cdot 1 \\
-2 \cdot 1+5 \cdot 0 & -2 \cdot 0+5 \cdot 1
\end{array}\right]=\left[\begin{array}{rr}
3 & 4 \\
-2 & 5
\end{array}\right] \\
& A I=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{rr}
3 & 4 \\
-2 & 5
\end{array}\right]=\left[\begin{array}{ll}
1 \cdot 3+0 \cdot(-2) & 1 \cdot 4+0 \cdot 5 \\
0 \cdot 3+1 \cdot(-2) & 0 \cdot 4+1 \cdot 5
\end{array}\right]=\left[\begin{array}{rr}
3 & 4 \\
-2 & 5
\end{array}\right]
\end{aligned}
\]

\section*{HOW TO: GIVEN TWO MATRICES, SHOW THAT ONE IS THE MULTIPLICATIVE INVERSE OF THE OTHER}
1. Given matrix \(A\) of order \(n \times n\) and matrix \(B\) of order \(n \times n\) multiply \(A B\).
2. If \(A B=I\), then find the product \(B A\). If \(B A=I\), then \(B=A^{-1}\) and \(A=B^{-1}\).

\section*{Finding the Multiplicative Inverse Using Matrix Multiplication}

We can now determine whether two matrices are inverses, but how would we find the inverse of a given matrix? Since we know that the product of a matrix and its inverse is the identity matrix, we can find the inverse of a matrix by setting up an equation using matrix multiplication.

\section*{Finding the Multiplicative Inverse by Augmenting with the Identity}

Another way to find the multiplicative inverse is by augmenting with the identity. When matrix \(A\) is transformed into \(I\), the augmented matrix \(I\) transforms into \(A^{-1}\).

For example, given

\section*{EXAMPLE: SHOWING THAT MATRIX A IS THE MULTIPLICATIVE INVERSE OF MATRIX B}

Show that the given matrices are multiplicative inverses of each other.
\(A=\left[\begin{array}{rr}1 & 5 \\ -2 & -9\end{array}\right], B=\left[\begin{array}{rr}-9 & -5 \\ 2 & 1\end{array}\right]\)
Answer
Multiply \(A B\) and \(B A\). If both products equal the identity, then the two matrices are inverses of each other.
\begin{tabular}{|c|c|c|c|c|}
\hline \(A B\) & =- \(\mathrm{left}[1\) & & 5 & \\
\hline -2 & & -9\right]lcdot Veft[-9 & & -5 \\
\hline 2 & & 1 1right] & & \\
\hline & =-Veft[1Veft(-9\right)+5\eft(2Vright) & & \(1(-5)+5(1)\) & \\
\hline \(-2(-9)-9(2)\) & & -2Veft(-5Vright)-9Veft(1\right)\right] & & \\
\hline & =-veft[1 & & 0 & \\
\hline 0 & & 14right] & & \\
\hline \(B A \quad\) & -Veft[-9 & & -5 & \\
\hline 2 & & 1\right]/cdot \eft[1 & & 5 \\
\hline -2 & & -9\right] & & \\
\hline & =Veft[-9Veft(1 \right)-5Veft(-2Vright) & & \(-9(5)-5(-9)\) & \\
\hline \(2(1)+1(-2)\) & & 2Veft(-5\right)+1\eft(-9\right)\right] & & \\
\hline & =Veft[1 & & 0 & \\
\hline 0 & & 1 \right] & & \\
\hline \(A\) and \(B\) are inver & rses of each other. & & & \\
\hline
\end{tabular}

\section*{TRY IT}

Show that the following two matrices are inverses of each other.
\(A=\left[\begin{array}{rr}1 & 4 \\ -1 & -3\end{array}\right], B=\left[\begin{array}{rr}-3 & -4 \\ 1 & 1\end{array}\right]\)
Answer
\begin{tabular}{|c|c|c|}
\hline AB=Veft[1] & 4 & \\
\hline -1 & -3lright]\eft[-3 & -4 \\
\hline 1 &  & \(1(-4)+4(1)\) \\
\hline \(-1(-3)+-3(1)\) &  & 0 \\
\hline 0 & 1 \right] & \\
\hline \(B A=\) Veft[-3 & -4 & \\
\hline
\end{tabular}

1
\(-1\)
\(1(1)+1(-1)\)
0
4
-3 light] \(=\) Veft[-3\left(1 1 Vright)+-4 Veft(-1 1 right) 1Veft(4\right)+1\eft(-3\right)\right]=Veft[1 0 1 right]

\section*{TIP FOR SUCCESS}

The following sections offer examples of various efficient techniques that show the power of matrix operations. Again, work through them slowly and carefully as needed to gain familiarity.
These techniques, using matrix multiplication, augmenting a matrix with the identity then reducing, and using a formula to find the inverse matrix appear frequently in scientific application and are worth taking time to obtain.
\(A=\left[\begin{array}{ll}2 & 1 \\ 5 & 3\end{array}\right]\)
augment \(A\) with the identity
\(\left[\begin{array}{ll|ll}2 & 1 & 1 & 0 \\ 5 & 3 & 0 & 1\end{array}\right]\)
Perform row operations with the goal of turning \(A\) into the identity.
1. Switch row 1 and row 2.
\(\left[\begin{array}{ll|ll}5 & 3 & 0 & 1 \\ 2 & 1 & 1 & 0\end{array}\right]\)
2. Multiply row 2 by -2 and add to row 1 .
\(\left[\begin{array}{rr|rr}1 & 1 & -2 & 1 \\ 2 & 1 & 1 & 0\end{array}\right]\)
3. Multiply row 1 by -2 and add to row 2 .
\(\left[\begin{array}{rr|rr}1 & 1 & -2 & 1 \\ 0 & -1 & 5 & -2\end{array}\right]\)
4. Add row 2 to row 1.
\(\left[\begin{array}{rr|rr}1 & 0 & 3 & -1 \\ 0 & -1 & 5 & -2\end{array}\right]\)
5. Multiply row 2 by -1 .
\(\left[\begin{array}{rr|rr}1 & 0 & 3 & -1 \\ 0 & 1 & -5 & 2\end{array}\right]\)
The matrix we have found is \(A^{-1}\).
\(A^{-1}=\left[\begin{array}{rr}3 & -1 \\ -5 & 2\end{array}\right]\)

\section*{Finding the Multiplicative Inverse of \(2 \times 2\) Matrices Using a Formula}

\section*{EXAMPLE: FINDING THE MULTIPLICATIVE INVERSE USING MATRIX MULTIPLICATION}

Use matrix multiplication to find the inverse of the given matrix.
\(A=\left[\begin{array}{ll}1 & -2 \\ 2 & -3\end{array}\right]\)

\section*{Answer}

For this method, we multiply \(A\) by a matrix containing unknown constants and set it equal to the identity.
\(\left[\begin{array}{ll}1 & -2 \\ 2 & -3\end{array}\right]\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]\)
Find the product of the two matrices on the left side of the equal sign.
\(\left[\begin{array}{ll}1 & -2 \\ 2 & -3\end{array}\right]\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]=\left[\begin{array}{ll}1 a-2 c & 1 b-2 d \\ 2 a-3 c & 2 b-3 d\end{array}\right]\)
Next, set up a system of equations with the entry in row 1 , column 1 of the new matrix equal to the first entry of the identity, 1 . Set the entry in row 2 , column 1 of the new matrix equal to the corresponding entry of the identity, which is 0 .
\[
\begin{aligned}
& 1 a-2 c=1 R_{1} \\
& 2 a-3 c=0 R_{2}
\end{aligned}
\]

Using row operations, multiply and add as follows: \((-2) R_{1}+R_{2} \rightarrow R_{2}\). Add the equations, and solve for c.
\[
\begin{aligned}
& 1 a-2 c=1 \\
& 0+1 c=-2 \\
& c=-2
\end{aligned}
\]

Back-substitute to solve for \(a\).
\[
\begin{aligned}
& a-2(-2)=1 \\
& a+4=1 \\
& a=-3
\end{aligned}
\]

Write another system of equations setting the entry in row 1, column 2 of the new matrix equal to the corresponding entry of the identity, 0 . Set the entry in row 2 , column 2 equal to the corresponding entry of the identity.
\(1 b-2 d=0 \quad R_{1}\)
\(2 b-3 d=1 \quad R_{2}\)
Using row operations, multiply and add as follows: \((-2) R_{1}+R_{2}=R_{2}\). Add the two equations and solve for \(d\).
\[
\begin{aligned}
& 1 b-2 d=0 \\
& \frac{0+1 d=1}{d=1}
\end{aligned}
\]

Once more, back-substitute and solve for \(b\).
\[
\begin{aligned}
& b-2(1)=0 \\
& b-2=0 \\
& b=2 \\
& A^{-1}=\left[\begin{array}{ll}
-3 & 2 \\
-2 & 1
\end{array}\right]
\end{aligned}
\]

When we need to find the multiplicative inverse of a \(2 \times 2\) matrix, we can use a special formula instead of using matrix multiplication or augmenting with the identity.

If \(A\) is a \(2 \times 2\) matrix, such as
\(A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]\)
the multiplicative inverse of \(A\) is given by the formula
\(A^{-1}=\frac{1}{a d-b c}\left[\begin{array}{rr}d & -b \\ -c & a\end{array}\right]\)
where \(a d-b c \neq 0\). If \(a d-b c=0\), then \(A\) has no inverse.

\section*{Finding the Multiplicative Inverse of \(3 \times 3\) Matrices}

Unfortunately, we do not have a formula similar to the one for a \(2 \times 2\) matrix to find the inverse of a \(3 \times 3\) matrix. Instead, we will augment the original matrix with the identity matrix and use row operations to obtain the inverse.

Given a \(3 \times 3\) matrix
\(A=\left[\begin{array}{lll}2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1\end{array}\right]\)
augment \(A\) with the identity matrix
\(A \left\lvert\, I=\left[\begin{array}{lll|lll}2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1\end{array}\right]\right.\)
To begin, we write the augmented matrix with the identity on the right and \(A\) on the left. Performing elementary row operations so that the identity matrix appears on the left, we will obtain the inverse matrix on the right. We will find the inverse of this matrix in the next example.

\section*{EXAMPLE: USING THE FORMULA TO FIND THE MULTIPLICATIVE INVERSE OF MATRIX A}

Use the formula to find the multiplicative inverse of
\(A=\left[\begin{array}{ll}1 & -2 \\ 2 & -3\end{array}\right]\)
Answer
Using the formula, we have

-21 1right]
- \(=\) frac \(\{1\}-3+4\}\) left \([-3\)
-21 1right]
\(=\) Meft[-3 2
-21 1right]

Analysis of the Solution
We can check that our formula works by using one of the other methods to calculate the inverse. Let's augment \(A\) with the identity.
\(\left[\begin{array}{ll|ll}1 & -2 & 1 & 0 \\ 2 & -3 & 0 & 1\end{array}\right]\)
Perform row operations with the goal of turning \(A\) into the identity.
1. Multiply row 1 by -2 and add to row 2 .
\[
\left[\begin{array}{rr|rr}
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1
\end{array}\right]
\]
2. Multiply row 1 by 2 and add to row 1 .
\[
\left[\begin{array}{ll|ll}
1 & 0 & -3 & 2 \\
0 & 1 & -2 & 1
\end{array}\right]
\]

So, we have verified our original solution.
\(A^{-1}=\left[\begin{array}{ll}-3 & 2 \\ -2 & 1\end{array}\right]\)

\section*{TRY IT}

Use the formula to find the inverse of matrix \(A\). Verify your answer by augmenting with the identity matrix.
\(A=\left[\begin{array}{rr}1 & -1 \\ 2 & 3\end{array}\right]\)
Answer
\(A^{-1}=\left[\begin{array}{rr}\frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5}\end{array}\right]\)

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\section*{EXAMPLE: FINDING THE INVERSE OF THE MATRIX, IF IT EXISTS}

Find the inverse, if it exists, of the given matrix.
\(A=\left[\begin{array}{ll}3 & 6 \\ 1 & 2\end{array}\right]\)
Answer
We will use the method of augmenting with the identity.
\(\left[\begin{array}{ll|ll}3 & 6 & 1 & 0 \\ 1 & 2 & 0 & 1\end{array}\right]\)
1. Switch row 1 and row 2.
\(\left[\begin{array}{ll|ll}1 & 2 & 0 & 1 \\ 3 & 6 & 1 & 0\end{array}\right]\)
2. Multiply row 1 by -3 and add it to row 2 .
\(\left[\begin{array}{rr|rr}1 & 2 & 1 & 0 \\ 0 & 0 & 1 & -3\end{array}\right]\)
3. There is nothing further we can do. The zeros in row 2 indicate that this matrix has no inverse.

\section*{HOW TO: GIVEN A \(3 \times 3\) MATRIX, FIND THE INVERSE}
1. Write the original matrix augmented with the identity matrix on the right.
2. Use elementary row operations so that the identity appears on the left.
3. What is obtained on the right is the inverse of the original matrix.
4. Use matrix multiplication to show that \(A A^{-1}=I\) and \(A^{-1} A=I\).

\section*{EXAMPLE: FINDING THE INVERSE OF A \(3 \times 3\) MATRIX}

Given the \(3 \times 3\) matrix \(A\), find the inverse.
\(A=\left[\begin{array}{lll}2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1\end{array}\right]\)
Answer
Augment \(A\) with the identity matrix, and then begin row operations until the identity matrix replaces \(A\). The matrix on the right will be the inverse of \(A\).
\(\left[\begin{array}{lll|lll}2 & 3 & 1 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1\end{array}\right] \underset{\rightarrow}{\text { Interchange } R_{2}}\) and \(R_{1}\left[\begin{array}{ccc|ccc}3 & 3 & 1 & 0 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1\end{array}\right]\)
\(-R_{2}+R_{1}=R_{1} \rightarrow\left[\begin{array}{lll|rrr}1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 & 1\end{array}\right]\)
\(-R_{2}+R_{3}=R_{3} \rightarrow\left[\begin{array}{lll|rll}1 & 0 & 0 & -1 & 1 & 0 \\ 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1\end{array}\right]\)
\(R_{3} \leftrightarrow R_{2} \rightarrow\left[\begin{array}{lll|rll}1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 2 & 3 & 1 & 1 & 0 & 0\end{array}\right]\)
\(-2 R_{1}+R_{3}=R_{3} \rightarrow\left[\begin{array}{lll|rrr}1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & 1 & 3 & -2 & 0\end{array}\right]\)
\(-3 R_{2}+R_{3}=R_{3} \rightarrow\left[\begin{array}{lll|rrr}1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 6 & -2 & -3\end{array}\right]\)
Thus,
\(A^{-1}=B=\left[\begin{array}{ccc}-1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3\end{array}\right]\)
Analysis of the Solution
To prove that \(B=A^{-1}\), let's multiply the two matrices together to see if the product equals the identity, if \(A A^{-1}=I\) and \(A^{-1} A=I\).
\(A A^{-1}\)
\(=\) Veft[2
3
3
2
3
4
1 right]\eft[-1
\(-1\)
0
\begin{tabular}{|c|c|c|}
\hline \multirow[t]{2}{*}{6} & -2 & -3lright] \\
\hline & =Veft[2Veft(-1 \right)+3Veft(-1 \right)+1 Veft(6Vright) & \(2(1)+3(0)+1(-2)\) \\
\hline \(3(-1)+3(-1)+1(6)\) & \(3(1)+3(0)+1(-2)\) & \(3(0)+3(1)+1(-3)\) \\
\hline \multirow[t]{2}{*}{\(2(-1)+4(-1)+1(6)\)} & \(2(1)+4(0)+1(-2)\) & 2\eft(OVright)+4\eft(1\right)+1\eft(-3\ri¢ \\
\hline & =Veft[1 & 0 \\
\hline 0 & 1 & 0 \\
\hline 0 & 0 & 1 1right] \\
\hline \(A^{-1} A\) & - =efeft-1 & 1 \\
\hline -1 & 0 & 1 \\
\hline 6 & -2 & -3\right]\eft[2 \\
\hline 3 & 3 & 1 \\
\hline \multirow[t]{2}{*}{2} & 4 & 1 \right] \\
\hline & =Veft[-1 Veft(2\right)+1\eft(3\right)+0\eft(2Vright) & \(-1(3)+1(3)+0(4)\) \\
\hline \(-1(2)+0(3)+1(2)\) & \(-1(3)+0(3)+1(4)\) & \(-1(1)+0(1)+1(1)\) \\
\hline \multirow[t]{2}{*}{\(6(2)+-2(3)+-3(2)\)} & \(6(3)+-2(3)+-3(4)\) & 6Veft(1\right)+-2Veft(1 1 \right)+-3Veft(1洤 \\
\hline & =Veft[1 & 0 \\
\hline 0 & 1 & 0 \\
\hline 0 & 0 & 1 1 right] \\
\hline
\end{tabular}

\section*{TRY IT}

Find the inverse of the \(3 \times 3\) matrix.
\(A=\left[\begin{array}{rrr}2 & -17 & 11 \\ -1 & 11 & -7 \\ 0 & 3 & -2\end{array}\right]\)
Answer
\(A^{-1}=\left[\begin{array}{rrr}1 & 1 & 2 \\ 2 & 4 & -3 \\ 3 & 6 & -5\end{array}\right]\)
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=409

\section*{SOLVING A SYSTEM USING AN INVERSE}

\section*{LEARNING OUTCOMES}
- Solve a \(2 \times 2\) system using an inverse.
- Solve a \(3 \times 3\) systems using an inverse.
- Solve a system with a calculator.

Solving a system of linear equations using the inverse of a matrix requires the definition of two new matrices: \(X\) is the matrix representing the variables of the system, and \(B\) is the matrix representing the constants. Using matrix multiplication, we may define a system of equations with the same number of equations as variables as \(A X=B\)

To solve a system of linear equations using an inverse matrix, let \(A\) be the coefficient matrix, let \(X\) be the variable matrix, and let \(B\) be the constant matrix. Thus, we want to solve a system \(A X=B\). For example, look at the following system of equations.
\[
\begin{aligned}
& a_{1} x+b_{1} y=c_{1} \\
& a_{2} x+b_{2} y=c_{2}
\end{aligned}
\]

From this system, the coefficient matrix is
\(A=\left[\begin{array}{ll}a_{1} & b_{1} \\ a_{2} & b_{2}\end{array}\right]\)
The variable matrix is
\(X=\left[\begin{array}{l}x \\ y\end{array}\right]\)
And the constant matrix is
\(B=\left[\begin{array}{l}c_{1} \\ c_{2}\end{array}\right]\)
Then \(A X=B\) looks like
\(\left[\begin{array}{ll}a_{1} & b_{1} \\ a_{2} & b_{2}\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}c_{1} \\ c_{2}\end{array}\right]\)

Recall the discussion earlier in this section regarding multiplying a real number by its inverse, \(\left(2^{-1}\right) 2=\left(\frac{1}{2}\right) 2=1\). To solve a single linear equation \(a x=b\) for \(x\), we would simply multiply both sides of the equation by the multiplicative inverse (reciprocal) of \(a\). Thus,
\[
\begin{aligned}
& a x=b \\
& \left(\frac{1}{a}\right) a x=\left(\frac{1}{a}\right) b \\
& \left(a^{-1}\right) a x=\left(a^{-1}\right) b \\
& {\left[\left(a^{-1}\right) a\right] x=\left(a^{-1}\right) b} \\
& 1 x=\left(a^{-1}\right) b \\
& x=\left(a^{-1}\right) b
\end{aligned}
\]

The only difference between solving a linear equation and a system of equations written in matrix form is that finding the inverse of a matrix is more complicated, and matrix multiplication is a longer process. However, the goal is the same-to isolate the variable.

\section*{TIP FOR SUCCESS}

As you work through the following examples, keep in mind that matrix operations aren't bound to the same property characteristics as operations on real numbers. Since matrix multiplication is not commutative, order matters!
Otherwise, the analogy of isolating the variable in the equation is a good one. Comparing a new concept or process to similar familiar material is a good way to develop intuition.

We will investigate this idea in detail, but it is helpful to begin with a \(2 \times 2\) system and then move on to a \(3 \times 3\) system.

\section*{A GENERAL NOTE: SOLVING A SYSTEM OF EQUATIONS USING THE INVERSE OF A MATRIX}

Given a system of equations, write the coefficient matrix \(A\), the variable matrix \(X\), and the constant matrix \(B\). Then \(A X=B\). Multiply both sides by the inverse of \(A\) to obtain the solution.
\[
\begin{aligned}
& \left(A^{-1}\right) A X=\left(A^{-1}\right) B \\
& {\left[\left(A^{-1}\right) A\right] X=\left(A^{-1}\right) B} \\
& I X=\left(A^{-1}\right) B \\
& X=\left(A^{-1}\right) B
\end{aligned}
\]

\section*{Q \& A}

If the coefficient matrix does not have an inverse, does that mean the system has no solution?
No, if the coefficient matrix is not invertible, the system could be inconsistent and have no solution or be dependent and have infinitely many solutions.

\section*{EXAMPLE: SOLVING A \(2 \times 2\) SYSTEM USING THE INVERSE OF A MATRIX}

Solve the given system of equations using the inverse of a matrix.
\[
\begin{aligned}
& 3 x+8 y=5 \\
& 4 x+11 y=7
\end{aligned}
\]

\section*{Answer}

Write the system in terms of a coefficient matrix, a variable matrix, and a constant matrix.
\(A=\left[\begin{array}{cc}3 & 8 \\ 4 & 11\end{array}\right], X=\left[\begin{array}{l}x \\ y\end{array}\right], B=\left[\begin{array}{l}5 \\ 7\end{array}\right]\)
Then
\(\left[\begin{array}{cc}3 & 8 \\ 4 & 11\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}5 \\ 7\end{array}\right]\)
First, we need to calculate \(A^{-1}\). Using the formula to calculate the inverse of a 2 by 2 matrix, we have:
\(A^{-1}\) =lfrac\{1\}ad-bc\}lleft[d
\(-b\)
-c alright]
- \(=\) frac \(\{1\} 3\) \left(11 \right)-8Veft(4 4 right) \} \(\}\) left[11 \(\quad-8\)
-4 3lright]
=-frac\{1\}1\}\left[11
-4 3lright]

So,
\(A^{-1}=\left[\begin{array}{cc}11 & -8 \\ -4 & 3\end{array}\right]\)
Now we are ready to solve. Multiply both sides of the equation by \(A^{-1}\).
\(\left(A^{-1}\right) A X=\left(A^{-1}\right) B\)
left[11
\(-4\)
4
y 4 right] \(=\) Veft[11
-4
7 rright]
Veft[1
0
\[
-8
\]

3 Wright]left[3 8
11\right]\\eft[x
-8
3light]\Veft[5

0
14right]left[x
ylright]=Veft[11 Veft(5\right)+Veft(-8\right) 7
-4 Veft(5\right) +3 Veft(7right)\right]
Veft[x
y पright]=Veft[-1

The solution is \((-1,1)\).

\section*{Q \& A}

Can we solve for \(X\) by finding the product \(B A^{-1}\) ?
No, recall that matrix multiplication is not commutative, so \(A^{-1} B \neq B A^{-1}\). Consider our steps for solving the matrix equation.
\[
\begin{aligned}
& \left(A^{-1}\right) A X=\left(A^{-1}\right) B \\
& {\left[\left(A^{-1}\right) A\right] X=\left(A^{-1}\right) B} \\
& I X=\left(A^{-1}\right) B \\
& X=\left(A^{-1}\right) B
\end{aligned}
\]

Notice in the first step we multiplied both sides of the equation by \(A^{-1}\), but the \(A^{-1}\) was to the left of \(A\) on the left side and to the left of \(B\) on the right side. Because matrix multiplication is not commutative, order matters.

\section*{EXAMPLE: SOLVING A \(3 \times 3\) SYSTEM USING THE INVERSE OF A MATRIX}

Solve the following system using the inverse of a matrix.
\[
\begin{aligned}
& 5 x+15 y+56 z=35 \\
& -4 x-11 y-41 z=-26 \\
& -x-3 y-11 z=-7
\end{aligned}
\]

\section*{Answer}

Write the equation \(A X=B\).
\(\left[\begin{array}{ccc}5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11\end{array}\right]\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{r}35 \\ -26 \\ -7\end{array}\right]\)
First, we will find the inverse of \(A\) by augmenting with the identity.
\(\left[\begin{array}{rrr|rrr}5 & 15 & 56 & 1 & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1\end{array}\right]\)
Multiply row 1 by \(\frac{1}{5}\).
\(\left[\begin{array}{rrr|rrr}1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ -4 & -11 & -41 & 0 & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1\end{array}\right]\)
Multiply row 1 by 4 and add to row 2 .
\(\left[\begin{array}{ccc|ccc}1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ -1 & -3 & -11 & 0 & 0 & 1\end{array}\right]\)
Add row 1 to row 3.
\(\left[\begin{array}{ccc|ccc}1 & 3 & \frac{56}{5} & \frac{1}{5} & 0 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1\end{array}\right]\)
Multiply row 2 by -3 and add to row 1 .
\(\left[\begin{array}{rrr|rrr}1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & \frac{1}{5} & \frac{1}{5} & 0 & 1\end{array}\right]\)
Multiply row 3 by 5 .
\(\left[\begin{array}{rrr|rrr}1 & 0 & -\frac{1}{5} & -\frac{11}{5} & -3 & 0 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5\end{array}\right]\)
Multiply row 3 by \(\frac{1}{5}\) and add to row 1 .
\(\left[\begin{array}{rrr|rrr}1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & \frac{19}{5} & \frac{4}{5} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 5\end{array}\right]\)

Multiply row 3 by \(-\frac{19}{5}\) and add to row 2.
\(\left[\begin{array}{rrr|rrr}1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & 1 & -19 \\ 0 & 0 & 1 & 1 & 0 & 5\end{array}\right]\)
So,
\(A^{-1}=\left[\begin{array}{ccc}-2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5\end{array}\right]\)
Multiply both sides of the equation by \(A^{-1}\). We want
\(A^{-1} A X=A^{-1} B\) :
\(\left[\begin{array}{rrr}-2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5\end{array}\right]\left[\begin{array}{rrr}5 & 15 & 56 \\ -4 & -11 & -41 \\ -1 & -3 & -11\end{array}\right]\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{rrr}-2 & -3 & 1 \\ -3 & 1 & -19 \\ 1 & 0 & 5\end{array}\right]\left[\begin{array}{r}35 \\ -26 \\ -7\end{array}\right]\)
Thus,
\(A^{-1} B=\left[\begin{array}{r}-70+78-7 \\ -105-26+133 \\ 35+0-35\end{array}\right]=\left[\begin{array}{l}1 \\ 2 \\ 0\end{array}\right]\)
The solution is \((1,2,0)\).

\section*{TRY IT}

Solve the system using the inverse of the coefficient matrix.
\(2 x-17 y+11 z=0\)
\(-x+11 y-7 z=8\)
\(3 y-2 z=-2\)

Answer
\(X=\left[\begin{array}{c}4 \\ 38 \\ 58\end{array}\right]\)

\section*{HOW TO: GIVEN A SYSTEM OF EQUATIONS, SOLVE WITH MATRIX INVERSES USING A CALCULATOR}
1. Save the coefficient matrix and the constant matrix as matrix variables \([A]\) and \([B]\).
2. Enter the multiplication into the calculator, calling up each matrix variable as needed.
3. If the coefficient matrix is invertible, the calculator will present the solution matrix; if the coefficient matrix is not invertible, the calculator will present an error message.

\section*{EXAMPLE: USING A CALCULATOR TO SOLVE A SYSTEM OF EQUATIONS WITH MATRIX INVERSES}

Solve the system of equations with matrix inverses using a calculator
\[
\begin{aligned}
& 2 x+3 y+z=32 \\
& 3 x+3 y+z=-27 \\
& 2 x+4 y+z=-2
\end{aligned}
\]

\section*{Answer}

On the matrix page of the calculator, enter the coefficient matrix as the matrix variable \([A]\), and enter the constant matrix as the matrix variable \([B]\).
\([A]=\left[\begin{array}{lll}2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1\end{array}\right], \quad[B]=\left[\begin{array}{c}32 \\ -27 \\ -2\end{array}\right]\)
On the home screen of the calculator, type in the multiplication to solve for \(X\), calling up each matrix variable as needed.
\([A]^{-1} \times[B]\)
Evaluate the expression.
\(\left[\begin{array}{c}-59 \\ -34 \\ 252\end{array}\right]\)

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\section*{SUMMARY: SOLVING SYSTEMS WITH INVERSES}

\section*{Key Equations}

Identity matrix for a \(2 \times 2\) matrix
\[
I_{2}=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right]
\]

Identity matrix for a \(3 \times 3\) matrix
\[
I_{3}=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
\]

Multiplicative inverse of a \(2 \times 2\) matrix
\[
A^{-1}=\frac{1}{a d-b c}\left[\begin{array}{cc}
d & -b \\
-c & a
\end{array}\right], \text { where } a d-b c \neq 0
\]

\section*{Key Concepts}
- An identity matrix has the property \(A I=I A=A\).
- An invertible matrix has the property \(A A^{-1}=A^{-1} A=I\).
- Use matrix multiplication and the identity to find the inverse of a \(2 \times 2\) matrix.
- The multiplicative inverse can be found using a formula.
- Another method of finding the inverse is by augmenting with the identity.
- We can augment a \(3 \times 3\) matrix with the identity on the right and use row operations to turn the original matrix into the identity, and the matrix on the right becomes the inverse.
- Write the system of equations as \(A X=B\), and multiply both sides by the inverse of
\(A: A^{-1} A X=A^{-1} B\).
- We can also use a calculator to solve a system of equations with matrix inverses.

\section*{Glossary}
identity matrix a square matrix containing ones down the main diagonal and zeros everywhere else; it acts as a 1 in matrix algebra
multiplicative inverse of a matrix a matrix that, when multiplied by the original, equals the identity matrix
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\section*{PUTTING IT TOGETHER: SOLVING SYSTEMS WITH MATRICES}

Our furniture-making company has two workshops (Atlanta and Boston) that specialize in creating chairs and tables. Suppose we have received an order of 620 chairs and 360 tables. If the production rates for each workshop are given in the table below, how many days should each workshop commit to fulfilling the order keeping in mind that we are striving for efficiency?
\begin{tabular}{|l|c|c|}
\hline & Atlanta & Boston \\
\hline Chairs & 20 per day & 30 per day \\
\hline Tables & 15 per day & 12 per day \\
\hline
\end{tabular}

Maximum production rates for chairs and tables at each of the two workshops, Atlanta and Boston.

To make the problem more general, let's set it up for an arbitrary order of \(C\) chairs and \(T\) tables. As discussed at the beginning of this module, this scenario requires us to write a system of linear equations:
\(20 A+30 B=C\)
\(15 A+12 B=T\)

The system may then be expressed as an equation of matrices,
\(\left[\begin{array}{ll}20 & 30 \\ 15 & 12\end{array}\right]\left[\begin{array}{l}A \\ B\end{array}\right]=\left[\begin{array}{l}C \\ T\end{array}\right]\)

Now to solve the system for the unknowns \(A\) and \(B\), we need the inverse of the coefficient matrix. Because the coefficient matrix is 2 by 2 , we may use the special shortcut formula for the inverse.
\(\left[\begin{array}{ll}20 & 30 \\ 15 & 12\end{array}\right]^{-1}=\frac{1}{(20)(12)-(30)(15)}\left[\begin{array}{cc}12 & -30 \\ -15 & 20\end{array}\right]\)
\(=\frac{1}{-210}\left[\begin{array}{cc}12 & -30 \\ -15 & 20\end{array}\right]\)
\(=\left[\begin{array}{cc}-\frac{2}{35} & \frac{1}{7} \\ \frac{1}{14} & -\frac{2}{21}\end{array}\right]\)
This means that the required production levels ( \(A\) and \(B\) ) can be found by the matrix multiplication,
\(\left[\begin{array}{l}A \\ B\end{array}\right]=\left[\begin{array}{ll}20 & 30 \\ 15 & 12\end{array}\right]^{-1}\left[\begin{array}{l}C \\ T\end{array}\right]=\left[\begin{array}{cc}-\frac{2}{35} & \frac{1}{7} \\ \frac{1}{14} & -\frac{2}{21}\end{array}\right]\left[\begin{array}{l}C \\ T\end{array}\right]\)

For example, in the case that \(C=620\) and \(T=360\), we find:
\(\left[\begin{array}{l}A \\ B\end{array}\right]=\left[\begin{array}{cc}-\frac{2}{35} & \frac{1}{7} \\ \frac{1}{14} & -\frac{2}{21}\end{array}\right]\left[\begin{array}{l}620 \\ 360\end{array}\right]\)
\(=\left[\begin{array}{cc}-\frac{2}{35}(620) & \frac{1}{7}(360) \\ \frac{1}{14}(620) & -\frac{2}{21}(360)\end{array}\right]\)
\(=\left[\begin{array}{l}16 \\ 10\end{array}\right]\)
Therefore, \(A=16\) and \(B=10\), which means that the Atlanta workshop should work on producing chairs and tables for 16 days and the Boston workshop should take 10 days.

While this may seem like a lot of work just to solve a simple system of equations, the general equation in terms of \(C\) and \(T\) may now be used for the next order and the order after that! No matter what the particular numbers of chairs and tables required, the same matrix equation can determine how many hours each workshop should run to efficiently produce the furniture.
```

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## MODULE 15: CONIC SECTIONS

## WHY IT MATTERS: CONIC SECTIONS

What do the U.S Capitol Building in Washington, D.C., Grand Central Station in New York, and St. Paul's Cathedral in London all have in common? Aside from being architectural marvels, each of these buildings contains a whispering gallery. A whispering gallery is most often an oblong enclosure beneath a dome or arch. These galleries get their name because a whisper in one part of the gallery can be heard clearly on the opposite side. Many movies and television shows have made use of whispering galleries in their plots as the conversations of the main characters are overhead. And tour guides at the U.S. Capitol recount a story in which John Quincy Adams used the whispering gallery to eavesdrop on conversations of other House members in the 1800s.


A whispering gallery can be produced through architecture alone, or it can be created using whispering dishes, which are large, flat dishes that reflect sound. The key is knowing exactly where to place the dishes. Suppose you are asked to create a whispering gallery in a room with an arch that has a width of 100 feet and a height of 40 feet. Where can you place the whispering dishes to create the gallery?

To answer this question, you'll need to learn about conics. The arch represents one type of conic. At this end of this module, you can use what you've learned to create the whispering gallery.

## LEARNING OUTCOMES

## Review Topics for Success

- Complete the square to create a perfect square trinomial.


## The Ellipse

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.

The Hyperbola

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.


## The Parabola

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.


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## REVIEW TOPICS FOR SUCCESS

## Learning Outcomes

By the end of this section, you will be able to:

- Complete the square to form perfect square trinomials

This module looks at geometric objects: the ellipse, the hyperbola, and the parabola. You have seen a parabola before. It is the shape formed in the real plane by a second-degree polynomial in two variables, the quadratic. In this module, though, you'll explore its geometric characteristics along with those of the other conic sections. You'll encounter a slightly different form of the equation for a parabola than you've seen before and you'll see that they can open left and right as well as up and down. Since the conic sections are geometric objects, their graphs are not limited to function characteristics.

A key algebraic feature involved in writing the equation of conic sections is the technique of completing the square to write a perfect square trinomial. You last saw this method in the text when solving quadratic equations, so a refresher is probably in order. Practice the skill before beginning this module to help reduce the challenge in upcoming examples and practice problems.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

## RECALL FOR SUCCESS

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

## COMPLETE THE SQUARE

## Learning Outcome

- Complete the square to create a perfect square trinomial

We have seen the technique of completing the square before when solving quadratic equations. Using this method, we added or subtracted terms to both sides of the equation until we had a perfect square trinomial on one side of the equal sign. Here is an example demonstrated earlier in the text.

We will use the example $x^{2}+4 x+1=0$ to illustrate each step. To complete the square, the leading coefficient, $a$, must equal 1 . If it does not, then divide the entire equation by $a$. Then, we can use the following procedures to solve a quadratic equation by completing the square.

1. Given a quadratic equation that cannot be factored and with $a=1$, first add or subtract the constant term to the right sign of the equal sign.

$$
x^{2}+4 x=-1
$$

2. Multiply the $b$ term by $\frac{1}{2}$ and square it.

$$
\begin{aligned}
& \frac{1}{2}(4)=2 \\
& 2^{2}=4
\end{aligned}
$$

3. Add $\left(\frac{1}{2} b\right)^{2}$ to both sides of the equal sign and simplify the right side. We have

$$
\begin{aligned}
& x^{2}+4 x+4=-1+4 \\
& x^{2}+4 x+4=3
\end{aligned}
$$

4. The left side of the equation can now be factored as a perfect square.

$$
\begin{aligned}
& x^{2}+4 x+4=3 \\
& (x+2)^{2}=3
\end{aligned}
$$

5. Use the square root property and solve.

$$
\begin{aligned}
& \sqrt{(x+2)^{2}}= \pm \sqrt{3} \\
& x+2= \pm \sqrt{3} \\
& x=-2 \pm \sqrt{3}
\end{aligned}
$$

6. The solutions are $x=-2+\sqrt{3}, x=-2-\sqrt{3}$.

We will see when studying conic sections that the method of completing the square comes in handy when rewriting the equation of a conic section given in general form. In preparation to use the method in this manner, it will be good to practice it first.

## HOW TO: USE THE METHOD OF COMPLETE THE SQUARE TO WRITE A PERFECT SQUARE TRINOMIAL FROM AN EXPRESSION.

1. Given an expression of the form $a\left(x^{2}+b x\right)$, add $\left(\frac{b}{2}\right)^{2}$ inside the parentheses.
2. Then subtract $a\left(\frac{b}{2}\right)^{2}$ to counteract the change you made to the expression.
3. If completing the square on one side of an equation, you may either subtract the value of $a\left(\frac{b}{2}\right)^{2}$ from that side, or add it to the other to maintain equality.
4. Then factor the perfect square trinomial you created inside the original parentheses.

## THE RESULTING FORM WILL LOOK LIKE THIS:

Given

$$
a\left(x^{2}+b x\right)
$$

add $(b / 2)^{2}$ inside the parentheses and subtract $a(b / 2)^{2}$ to counteract the change you made to the expression
$=a\left(x^{2}+b x+\left(\frac{b}{2}\right)^{2}\right)-a\left(\frac{b}{2}\right)^{2}$
then factor the resulting perfect square trinomial
$=a\left(x+\frac{b}{2}\right)^{2}-a\left(\frac{b}{2}\right)^{2}$.

## EXAMPLE : CREATE A PERFECT SQUARE TRINOMIAL USING THE METHOD OF COMPLETE THE SQUARE

Complete the square on: $3\left(x^{2}-10 x\right)$.
Answer
Add $\left(\frac{b}{2}\right)^{2}$ inside the parentheses and subtract $a\left(\frac{b}{2}\right)^{2}$
$3\left(x^{2}-10 x+25\right)-3 \cdot 25$
Factor the perfect square trinomial and simplify
$3(x-5)^{2}-75$.

## Analysis

The resulting expression is equivalent to the original expression. To test this, substitute a small value for $x$, say $x=3$.
$3\left(3^{2}-10 \cdot 3\right)=-63=3(3-5)^{2}-75$. True.

## TRY IT

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## SUMMARY: REVIEW

## Key Concepts

- Completing the square may be used to rewrite an expression in the form $a\left(x^{2}+b x\right)$ as a perfect square trinomial of form $a(x+b)^{2}+c$


## Glossary

Perfect square trinoial a trinomial of form $x^{2}+2 b x+b^{2}$ that factors into $(x+b)^{2}$

## INTRODUCTION TO THE ELLIPSE

## LEARNING OUTCOMES

By the end of this section, you will be able to:

- Write equations of ellipses in standard form.
- Graph ellipses centered at the origin.
- Graph ellipses not centered at the origin.
- Solve applied problems involving ellipses.


Figure 1. The National Statuary Hall in Washington, D.C. (credit: Greg Palmer, Flickr)
Can you imagine standing at one end of a large room and still being able to hear a whisper from a person standing at the other end? The National Statuary Hall in Washington, D.C. is such a room. (Note: Architect of the Capitol. http://www.aoc.gov. Accessed April 15, 2014.) It is an oval-shaped room called a whispering chamber because the shape makes it possible for sound to travel along the walls. In this section, we will investigate the shape of this room and its real-world applications, including how far apart two people in Statuary Hall can stand and still hear each other whisper.

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## EQUATIONS OF ELLIPSES

- Identify the foci, vertices, axes, and center of an ellipse.
- Write equations of ellipses centered at the origin.
- Write equations of ellipses not centered at the origin.

A conic section, or conic, is a shape resulting from intersecting a right circular cone with a plane. The angle at which the plane intersects the cone determines the shape.


Ellipse


Hyperbola


Parabola

Conic sections can also be described by a set of points in the coordinate plane. Later in this chapter we will see that the graph of any quadratic equation in two variables is a conic section. The signs of the equations and the coefficients of the variable terms determine the shape. This section focuses on the four variations of the standard form of the equation for the ellipse. An ellipse is the set of all points $(x, y)$ in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus (plural: foci) of the ellipse.

We can draw an ellipse using a piece of cardboard, two thumbtacks, a pencil, and string. Place the thumbtacks in the cardboard to form the foci of the ellipse. Cut a piece of string longer than the distance between the two thumbtacks (the length of the string represents the constant in the definition). Tack each end of the string to the cardboard, and trace a curve with a pencil held taut against the string. The result is an ellipse.


Every ellipse has two axes of symmetry. The longer axis is called the major axis, and the shorter axis is called the minor axis. Each endpoint of the major axis is the vertex of the ellipse (plural: vertices), and each endpoint of the minor axis is a co-vertex of the ellipse. The center of an ellipse is the midpoint of both the major and minor axes. The axes are perpendicular at the center. The foci always lie on the major axis, and the sum of the distances from the foci to any point on the ellipse (the constant sum) is greater than the distance between the foci.


In this section we restrict ellipses to those that are positioned vertically or horizontally in the coordinate plane. That is, the axes will either lie on or be parallel to the $x$ - and $y$-axes. Later in the chapter, we will see ellipses that are rotated in the coordinate plane.

To work with horizontal and vertical ellipses in the coordinate plane, we consider two cases: those that are centered at the origin and those that are centered at a point other than the origin. First we will learn to derive the equations of ellipses, and then we will learn how to write the equations of ellipses in standard form. Later we will use what we learn to draw the graphs.

To derive the equation of an ellipse centered at the origin, we begin with the foci $(-c, 0)$ and $(-c, 0)$. The ellipse is the set of all points $(x, y)$ such that the sum of the distances from $(x, y)$ to the foci is constant, as shown in the figure below.


If $(a, 0)$ is a vertex of the ellipse, the distance from $(-c, 0)$ to $(a, 0)$ is $a-(-c)=a+c$. The distance from $(c, 0)$ to $(a, 0)$ is $a-c$. The sum of the distances from the foci to the vertex is
$(a+c)+(a-c)=2 a$
If $(x, y)$ is a point on the ellipse, then we can define the following variables:
$d_{1}=$ the distance from $(-c, 0)$ to $(x, y)$
$d_{2}=$ the distance from $(c, 0)$ to $(x, y)$
By the definition of an ellipse, $d_{1}+d_{2}$ is constant for any point $(x, y)$ on the ellipse. We know that the sum of these distances is $2 a$ for the vertex $(a, 0)$. It follows that $d_{1}+d_{2}=2 a$ for any point on the ellipse. The derivation of the standard form of the equation of an ellipse relies on this relationship and the distance formula. The derivation is beyond the scope of this course, but the equation is:
$\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$
for an ellipse centered at the origin with its major axis on the $X$-axis and
$\frac{x^{2}}{b^{2}}+\frac{y^{2}}{a^{2}}=1$
for an ellipse centered at the origin with its major axis on the $Y$-axis.

## Writing Equations of Ellipses Centered at the Origin in Standard Form

Standard forms of equations tell us about key features of graphs. Take a moment to recall some of the standard forms of equations we've worked with in the past: linear, quadratic, cubic, exponential, logarithmic, and so on. By learning to interpret standard forms of equations, we are bridging the relationship between algebraic and geometric representations of mathematical phenomena.

The key features of the ellipse are its center, vertices, co-vertices, foci, and lengths and positions of the major and minor axes. Just as with other equations, we can identify all of these features just by looking at the standard form of the equation. There are four variations of the standard form of the ellipse. These variations are categorized first by the location of the center (the origin or not the origin), and then by the position (horizontal or vertical). Each is presented along with a description of how the parts of the equation relate to the graph. Interpreting these parts allows us to form a mental picture of the ellipse.

The standard form of the equation of an ellipse with center $(0,0)$ and major axis parallel to the $x$-axis is $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$
where

- $a>b$
- the length of the major axis is $2 a$
- the coordinates of the vertices are $( \pm a, 0)$
- the length of the minor axis is $2 b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the coordinates of the foci are $( \pm c, 0)$, where $c^{2}=a^{2}-b^{2}$.

The standard form of the equation of an ellipse with center $(0,0)$ and major axis parallel to the $y$-axis is $\frac{x^{2}}{b^{2}}+\frac{y^{2}}{a^{2}}=1$
where

- $a>b$
- the length of the major axis is $2 a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the minor axis is $2 b$
- the coordinates of the co-vertices are $( \pm b, 0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^{2}=a^{2}-b^{2}$.

Note that the vertices, co-vertices, and foci are related by the equation $c^{2}=a^{2}-b^{2}$. When we are given the coordinates of the foci and vertices of an ellipse, we can use the relationship to find the equation of the ellipse in standard form.


Figure:
(a) Horizontal ellipse with center $(0,0)$,

(b) Vertical ellipse with center $(0,0)$

## HOW TO: GIVEN THE VERTICES AND FOCI OF AN ELLIPSE CENTERED AT THE ORIGIN, WRITE ITS EQUATION IN STANDARD FORM.

1. Determine whether the major axis is on the $x$ - or $y$-axis.
2. If the given coordinates of the vertices and foci have the form ( $\pm a, 0)$ and ( $\pm c, 0$ ) respectively, then the major axis is parallel to the $x$-axis. Use the standard form $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$.
3. If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(0, \pm c)$ respectively, then the major axis is parallel to the $y$-axis. Use the standard form $\frac{x^{2}}{b^{2}}+\frac{y^{2}}{a^{2}}=1$.
4. Use the equation $c^{2}=a^{2}-b^{2}$ along with the given coordinates of the vertices and foci, to solve for $b^{2}$.
5. Substitute the values for $a^{2}$ and $b^{2}$ into the standard form of the equation determined in Step 1.

## EXAMPLE: WRITING THE EQUATION OF AN ELLIPSE CENTERED AT THE ORIGIN IN STANDARD FORM

What is the standard form equation of the ellipse that has vertices $( \pm 8,0)$ and foci $( \pm 5,0)$ ?

## Answer

The foci are on the $x$-axis, so the major axis is the $x$-axis. Thus the equation will have the form:
$\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$
The vertices are $( \pm 8,0)$, so $a=8$ and $a^{2}=64$.
The foci are $( \pm 5,0)$, so $c=5$ and $c^{2}=25$.
We know that the vertices and foci are related by the equation $c^{2}=a^{2}-b^{2}$. Solving for $b^{2}$ we have
$c^{2}=a^{2}-b^{2}$
$25=64-b^{2} \quad$ Substitute for $c^{2}$ and $a^{2}$.
$b^{2}=39 \quad$ Solve for $b^{2}$.
Now we need only substitute $a^{2}=64$ and $b^{2}=39$ into the standard form of the equation. The equation of the ellipse is
$\frac{x^{2}}{64}+\frac{y^{2}}{39}=1$

## TRY IT

What is the standard form equation of the ellipse that has vertices $(0, \pm 8)$ and foci $(0, \pm \sqrt{5})$ ?
Answer
$x^{2}+\frac{y^{2}}{16}=1$
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=419

## Writing Equations of Ellipses Not Centered at the Origin

Like the graphs of other equations, the graph of an ellipse can be translated. If an ellipse is translated $h$ units horizontally and $k$ units vertically, the center of the ellipse will be $(h, k)$. This translation results in the standard form of the equation we saw previously, with $x$ replaced by $(x-h)$ and $y$ replaced by $(y-k)$.

## TIP FOR SUCCESS

Writing equations of ellipses not centered at the origin uses several concepts that you've built up for yourself over the entirety of this course such as transformations of graphs, rational equations, and the
midpoint formula, to name a few.
Remember that you can return to earlier module sections at any time if you need a refresher. As always, give yourself plenty of time to apply patient, repeated study of the examples and practice problems below.

## A GENERAL NOTE: STANDARD FORMS OF THE EQUATION OF AN ELLIPSE WITH CENTER $(H, K)$

The standard form of the equation of an ellipse with center ( $h, k$ ) and major axis parallel to the $x$-axis is $\frac{(x-h)^{2}}{a^{2}}+\frac{(y-k)^{2}}{b^{2}}=1$
where

- $a>b$
- the length of the major axis is $2 a$
- the coordinates of the vertices are $(h \pm a, k)$
- the length of the minor axis is $2 b$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^{2}=a^{2}-b^{2}$.

The standard form of the equation of an ellipse with center $(h, k)$ and major axis parallel to the $y$-axis is $\frac{(x-h)^{2}}{b^{2}}+\frac{(y-k)^{2}}{a^{2}}=1$
where

- $a>b$
- the length of the major axis is $2 a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the minor axis is $2 b$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^{2}=a^{2}-b^{2}$.

Just as with ellipses centered at the origin, ellipses that are centered at a point ( $h, k$ ) have vertices, covertices, and foci that are related by the equation $c^{2}=a^{2}-b^{2}$. We can use this relationship along with the midpoint and distance formulas to find the equation of the ellipse in standard form when the vertices and foci are given.

(a) Horizontal ellipse with center ( $h, k$ ) (b) Vertical ellipse with center $(h, k)$

## HOW TO: GIVEN THE VERTICES AND FOCI OF AN ELLIPSE NOT CENTERED AT THE ORIGIN, WRITE ITS EQUATION IN STANDARD FORM.

1. Determine whether the major axis is parallel to the $x$ - or $y$-axis.
2. If the $y$-coordinates of the given vertices and foci are the same, then the major axis is parallel to the $x$-axis. Use the standard form $\frac{(x-h)^{2}}{a^{2}}+\frac{(y-k)^{2}}{b^{2}}=1$.
3. If the $x$-coordinates of the given vertices and foci are the same, then the major axis is parallel to the $y$-axis. Use the standard form $\frac{(x-h)^{2}}{b^{2}}+\frac{(y-k)^{2}}{a^{2}}=1$.
4. Identify the center of the ellipse $(h, k)$ using the midpoint formula and the given coordinates for the vertices.
5. Find $a^{2}$ by solving for the length of the major axis, $2 a$, which is the distance between the given vertices.
6. Find $c^{2}$ using $h$ and $k$, found in Step 2, along with the given coordinates for the foci.
7. Solve for $b^{2}$ using the equation $c^{2}=a^{2}-b^{2}$.
8. Substitute the values for $h, k, a^{2}$, and $b^{2}$ into the standard form of the equation determined in Step 1.

## Solving Applied Problems Involving Ellipses

Many real-world situations can be represented by ellipses, including orbits of planets, satellites, moons and comets, and shapes of boat keels, rudders, and some airplane wings. A medical device called a lithotripter uses elliptical reflectors to break up kidney stones by generating sound waves. Some buildings, called whispering chambers, are designed with elliptical domes so that a person whispering at one focus can easily be heard by someone standing at the other focus. This occurs because of the acoustic properties of an ellipse. When a sound wave originates at one focus of a whispering chamber, the sound wave will be reflected off the elliptical dome and back to the other focus. In the whisper chamber at the Museum of Science and Industry in Chicago, two people standing at the foci-about 43 feet apart-can hear each other whisper.

## EXAMPLE: WRITING THE EQUATION OF AN ELLIPSE CENTERED AT A POINT OTHER THAN THE ORIGIN

What is the standard form equation of the ellipse that has vertices $(-2,-8)$ and $(-2,2)$ and foci $(-2,-7)$ and $(-2,1)$ ?
Answer
The $x$-coordinates of the vertices and foci are the same, so the major axis is parallel to the $y$-axis. Thus, the equation of the ellipse will have the form
$\frac{(x-h)^{2}}{b^{2}}+\frac{(y-k)^{2}}{a^{2}}=1$
First, we identify the center, $(h, k)$. The center is halfway between the vertices, $(-2,-8)$ and $(-2,2)$.
Applying the midpoint formula, we have:

$$
\begin{aligned}
(h, k) & =\left(\frac{-2+(-2)}{2}, \frac{-8+2}{2}\right) \\
& =(-2,-3)
\end{aligned}
$$

Next, we find $a^{2}$. The length of the major axis, $2 a$, is bounded by the vertices. We solve for $a$ by finding the distance between the $y$-coordinates of the vertices.
$2 a=2-(-8)$
$2 a=10$
$a=5$
So $a^{2}=25$.
Now we find $c^{2}$. The foci are given by $(h, k \pm c)$. So, $(h, k-c)=(-2,-7)$ and $(h, k+c)=(-2,1)$. We substitute $k=-3$ using either of these points to solve for $c$.

$$
\begin{gathered}
k+c=1 \\
-3+c=1 \\
c=4
\end{gathered}
$$

So $c^{2}=16$.
Next, we solve for $b^{2}$ using the equation $c^{2}=a^{2}-b^{2}$.
${ }^{2}=a^{2}-b^{2}$
$16=25-b^{2}$
$b^{2}=9$
Finally, we substitute the values found for $h, k, a^{2}$, and $b^{2}$ into the standard form equation for an ellipse:
$\frac{(x+2)^{2}}{9}+\frac{(y+3)^{2}}{25}=1$

## TIP FOR SUCCESS

These problems are too large to carry in your head all the way through. Be sure to work them out on paper. Try to list the steps in such a way that you can use a mnemonic to memorize the process.

## TRY IT

What is the standard form equation of the ellipse that has vertices $(-3,3)$ and $(5,3)$ and foci $(1-2 \sqrt{3}, 3)$ and $(1+2 \sqrt{3}, 3)$ ?
Answer

$$
\frac{(x-1)^{2}}{16}+\frac{(y-3)^{2}}{4}=1
$$

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## EXAMPLE: LOCATING THE FOCI OF A WHISPERING CHAMBER

The Statuary Hall in the Capitol Building in Washington, D.C. is a whispering chamber. Its dimensions are 46 feet wide by 96 feet long.
a. What is the standard form of the equation of the ellipse representing the outline of the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0,0)$.
b. If two senators standing at the foci of this room can hear each other whisper, how far apart are the senators? Round to the nearest foot.


Answer
a. We are assuming a horizontal ellipse with center $(0,0)$, so we need to find an equation of the form $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$, where $a>b$. We know that the length of the major axis, $2 a$, is longer than the length of the minor axis, $2 b$. So the length of the room, 96 , is represented by the major axis, and the width of the room, 46 , is represented by the minor axis.

- Solving for $a$, we have $2 a=96$, so $a=48$, and $a^{2}=2304$.
- Solving for $b$, we have $2 b=46$, so $b=23$, and $b^{2}=529$.

Therefore, the equation of the ellipse is $\frac{x^{2}}{2304}+\frac{y^{2}}{529}=1$.
b. To find the distance between the senators, we must find the distance between the foci, $( \pm c, 0)$, where $c^{2}=a^{2}-b^{2}$. Solving for $c$, we have:
$c^{2}=a^{2}-b^{2}$
$c^{2}=2304-529 \quad$ Substitute using the values found in part (a).
$c= \pm \sqrt{2304-529} \quad$ Take the square root of both sides.
$c= \pm \sqrt{1775} \quad$ Subtract.
$c \approx \pm 42 \quad$ Round to the nearest foot.
The points $( \pm 42,0)$ represent the foci. Thus, the distance between the senators is $2(42)=84$ feet.

## TIP FOR SUCCESS

Don't be discouraged if it takes several tries through the example and practice problem to understand the process. These problems are challenging, but you can achieve success with time and effort.

## TRY IT

Suppose a whispering chamber is 480 feet long and 320 feet wide.
a. What is the standard form of the equation of the ellipse representing the room? Hint: assume a horizontal ellipse, and let the center of the room be the point $(0,0)$.
b. If two people are standing at the foci of this room and can hear each other whisper, how far apart are the people? Round to the nearest foot.

## Answer

a. $\frac{x^{2}}{57,600}+\frac{y^{2}}{25,600}=1$
b. The people are standing 358 feet apart.

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## GRAPHS OF ELLIPSES

## LEARNING OUTCOMES

- Sketch a graph of an ellipse centered at the origin.
- Sketch a graph of an ellipse not centered at the origin.
- Express the equation of an ellipse in standard form given the equation in general form.

Just as we can write the equation for an ellipse given its graph, we can graph an ellipse given its equation. To graph ellipses centered at the origin, we use the standard form $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1, a>b$ for horizontal ellipses and $\frac{x^{2}}{b^{2}}+\frac{y^{2}}{a^{2}}=1, a>b$ for vertical ellipses.

## HOW TO: GIVEN THE STANDARD FORM OF AN EQUATION FOR AN ELLIPSE CENTERED AT $(0,0)$, SKETCH THE GRAPH.

- Use the standard forms of the equations of an ellipse to determine the major axis, vertices, covertices, and foci.
- If the equation is in the form $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$, where $a>b$, then
- the major axis is the $x$-axis
- the coordinates of the vertices are $( \pm a, 0)$
- the coordinates of the co-vertices are $(0, \pm b)$
- the coordinates of the foci are $( \pm c, 0)$
- If the equation is in the form $\frac{x^{2}}{b^{2}}+\frac{y^{2}}{a^{2}}=1$, where $a>b$, then
- the major axis is the $y$-axis
- the coordinates of the vertices are $(0, \pm a)$
- the coordinates of the co-vertices are $( \pm b, 0)$
- the coordinates of the foci are $(0, \pm c)$
- Solve for $c$ using the equation $c^{2}=a^{2}-b^{2}$.
- Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.


## RECALL THE SQUARE ROOT PROPERTY

With the $x^{2}$ term isolated, the square root propty states that:
if $x^{2}=k$, then $x= \pm \sqrt{k}$
Don't forget to take the plus or minus!

## Graphing Ellipses Not Centered at the Origin

When an ellipse is not centered at the origin, we can still use the standard forms to find the key features of the graph. When the ellipse is centered at some point, $(h, k)$, we use the standard forms $\frac{(x-h)^{2}}{a^{2}}+\frac{(y-k)^{2}}{b^{2}}=1, a>b$ for horizontal ellipses and $\frac{(x-h)^{2}}{b^{2}}+\frac{(y-k)^{2}}{a^{2}}=1, a>b$ for vertical ellipses. From these standard equations, we can easily determine the center, vertices, co-vertices, foci, and positions of the major and minor axes.

## EXAMPLE: GRAPHING AN ELLIPSE CENTERED AT THE ORIGIN

Graph the ellipse given by the equation, $\frac{x^{2}}{9}+\frac{y^{2}}{25}=1$. Identify and label the center, vertices, co-vertices, and foci.

## Answer

First, we determine the position of the major axis. Because $25>9$, the major axis is on the $y$-axis.
Therefore, the equation is in the form $\frac{x^{2}}{b^{2}}+\frac{y^{2}}{a^{2}}=1$, where $b^{2}=9$ and $a^{2}=25$. It follows that:

- the center of the ellipse is $(0,0)$
- the coordinates of the vertices are $(0, \pm a)=(0, \pm \sqrt{25})=(0, \pm 5)$
- the coordinates of the co-vertices are $( \pm b, 0)=( \pm \sqrt{9}, 0)=( \pm 3,0)$
- the coordinates of the foci are $(0, \pm c)$, where $c^{2}=a^{2}-b^{2}$ Solving for $c$, we have:

$$
\begin{aligned}
c & = \pm \sqrt{a^{2}-b^{2}} \\
& = \pm \sqrt{25-9} \\
& = \pm \sqrt{16} \\
& = \pm 4
\end{aligned}
$$

Therefore, the coordinates of the foci are $(0, \pm 4)$.
Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.


## TRY IT

Graph the ellipse given by the equation $\frac{x^{2}}{36}+\frac{y^{2}}{4}=1$. Identify and label the center, vertices, co-vertices, and foci.
Answer
center: $(0,0)$; vertices: $( \pm 6,0)$; co-vertices: $(0, \pm 2)$; foci: $( \pm 4 \sqrt{2}, 0)$


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## EXAMPLE: GRAPHING AN ELLIPSE CENTERED AT THE ORIGIN FROM AN EQUATION NOT IN STANDARD FORM

Graph the ellipse given by the equation $4 x^{2}+25 y^{2}=100$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.
Answer
First, use algebra to rewrite the equation in standard form.

$$
\begin{gathered}
4 x^{2}+25 y^{2}=100 \\
\frac{4 x^{2}}{100}+\frac{25 y^{2}}{100}=\frac{100}{100} \\
\frac{x^{2}}{25}+\frac{y^{2}}{4}=1
\end{gathered}
$$

Next, we determine the position of the major axis. Because $25>4$, the major axis is on the $x$-axis.
Therefore, the equation is in the form $\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1$, where $a^{2}=25$ and $b^{2}=4$. It follows that:

- the center of the ellipse is $(0,0)$
- the coordinates of the vertices are $( \pm a, 0)=( \pm \sqrt{25}, 0)=( \pm 5,0)$
- the coordinates of the co-vertices are $(0, \pm b)=(0, \pm \sqrt{4})=(0, \pm 2)$
- the coordinates of the foci are $( \pm c, 0)$, where $c^{2}=a^{2}-b^{2}$. Solving for $c$, we have:

$$
\begin{aligned}
c & = \pm \sqrt{a^{2}-b^{2}} \\
& = \pm \sqrt{25-4} \\
& = \pm \sqrt{21}
\end{aligned}
$$

Therefore the coordinates of the foci are $( \pm \sqrt{21}, 0)$.
Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.


## TRY IT

Graph the ellipse given by the equation $49 x^{2}+16 y^{2}=784$. Rewrite the equation in standard form. Then identify and label the center, vertices, co-vertices, and foci.
Answer

Standard form: $\frac{x^{2}}{16}+\frac{y^{2}}{49}=1$; center: $(0,0)$; vertices: $(0, \pm 7)$; co-vertices: $( \pm 4,0)$; foci: $(0, \pm \sqrt{33})$


## HOW TO: GIVEN THE STANDARD FORM OF AN EQUATION FOR AN ELLIPSE CENTERED AT $(h, k)$, SKETCH THE GRAPH.

- Use the standard forms of the equations of an ellipse to determine the center, position of the major axis, vertices, co-vertices, and foci.
- If the equation is in the form $\frac{(x-h)^{2}}{a^{2}}+\frac{(y-k)^{2}}{b^{2}}=1$, where $a>b$, then
- the center is $(h, k)$
- the major axis is parallel to the $x$-axis
- the coordinates of the vertices are $(h \pm a, k)$
- the coordinates of the co-vertices are $(h, k \pm b)$
- the coordinates of the foci are ( $h \pm c, k$ )
- If the equation is in the form $\frac{(x-h)^{2}}{b^{2}}+\frac{(y-k)^{2}}{a^{2}}=1$, where $a>b$, then
- the center is $(h, k)$
- the major axis is parallel to the $y$-axis
- the coordinates of the vertices are $(h, k \pm a)$
- the coordinates of the co-vertices are ( $h \pm b, k$ )
- the coordinates of the foci are ( $h, k \pm c$ )
- Solve for $c$ using the equation $c^{2}=a^{2}-b^{2}$.
- Plot the center, vertices, co-vertices, and foci in the coordinate plane, and draw a smooth curve to form the ellipse.


## EXAMPLE: GRAPHING AN ELLIPSE CENTERED AT $(H, K)$

Graph the ellipse given by the equation, $\frac{(x+2)^{2}}{4}+\frac{(y-5)^{2}}{9}=1$. Identify and label the center, vertices, co-vertices, and foci.

## Answer

First, we determine the position of the major axis. Because $9>4$, the major axis is parallel to the $y$-axis. Therefore, the equation is in the form $\frac{(x-h)^{2}}{b^{2}}+\frac{(y-k)^{2}}{a^{2}}=1$, where $b^{2}=4$ and $a^{2}=9$. It follows that:

- the center of the ellipse is $(h, k)=(-2,5)$
- the coordinates of the vertices are $(h, k \pm a)=(-2,5 \pm \sqrt{9})=(-2,5 \pm 3)$, or $(-2,2)$ and $(-2,8)$
- the coordinates of the co-vertices are $(h \pm b, k)=(-2 \pm \sqrt{4}, 5)=(-2 \pm 2,5)$, or $(-4,5)$ and $(0,5)$
- the coordinates of the foci are $(h, k \pm c)$, where $c^{2}=a^{2}-b^{2}$. Solving for $c$, we have:

$$
\begin{aligned}
c & = \pm \sqrt{a^{2}-b^{2}} \\
& = \pm \sqrt{9-4} \\
& = \pm \sqrt{5}
\end{aligned}
$$

Therefore, the coordinates of the foci are $(-2,5-\sqrt{5})$ and $(-2,5+\sqrt{5})$.
Next, we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.


## TRY IT

Graph the ellipse given by the equation $\frac{(x-4)^{2}}{36}+\frac{(y-2)^{2}}{20}=1$. Identify and label the center, vertices, co-vertices, and foci.

Answer
Center: $(4,2)$; vertices: $(-2,2)$ and $(10,2)$; co-vertices: $(4,2-2 \sqrt{5})$ and $(4,2+2 \sqrt{5})$; foci: $(0,2)$ and $(8,2)$


## HOW TO: GIVEN THE GENERAL FORM OF AN EQUATION FOR AN ELLIPSE CENTERED AT $(H, K)$, EXPRESS THE EQUATION IN STANDARD FORM.

- Recognize that an ellipse described by an equation in the form $a x^{2}+b y^{2}+c x+d y+e=0$ is in general form.
- Rearrange the equation by grouping terms that contain the same variable. Move the constant term to the opposite side of the equation.
- Factor out the coefficients of the $x^{2}$ and $y^{2}$ terms in preparation for completing the square.
- Complete the square for each variable to rewrite the equation in the form of the sum of multiples of two binomials squared set equal to a constant, $m_{1}(x-h)^{2}+m_{2}(y-k)^{2}=m_{3}$, where $m_{1}, m_{2}$, and $m_{3}$ are constants.
- Divide both sides of the equation by the constant term to express the equation in standard form.


## RECALL COMPLETE THE SQUARE

To write the equation of an ellipse in standard form, you'll need to complete the square two times.
Given an expression of the form $a\left(x^{2}+b x\right)$, add $\left(\frac{b}{2}\right)^{2}$ inside the parentheses, then subtract $a\left(\frac{b}{2}\right)^{2}$ to counteract the change you made.

If completing the square on one side of an equation, you may either subtract the value of $a\left(\frac{b}{2}\right)^{2}$ from that side, or add it to the other to maintain equality.
Then factor the perfect square trinomial you created inside the original parentheses.
Example

$$
a\left(x^{2}+b x\right)
$$

$=a\left(x^{2}+b x+\left(\frac{b}{2}\right)^{2}\right)-a\left(\frac{b}{2}\right)^{2}$
$=a\left(x+\frac{b}{2}\right)^{2}-a\left(\frac{b}{2}\right)^{2}$

## EXAMPLE: GRAPHING AN ELLIPSE CENTERED AT $(H, K)$ BY FIRST WRITING IT IN STANDARD FORM

Graph the ellipse given by the equation $4 x^{2}+9 y^{2}-40 x+36 y+100=0$. Identify and label the center, vertices, co-vertices, and foci.
Answer
We must begin by rewriting the equation in standard form.
$4 x^{2}+9 y^{2}-40 x+36 y+100=0$
Group terms that contain the same variable, and move the constant to the opposite side of the equation.
$\left(4 x^{2}-40 x\right)+\left(9 y^{2}+36 y\right)=-100$
Factor out the coefficients of the squared terms.
$4\left(x^{2}-10 x\right)+9\left(y^{2}+4 y\right)=-100$
Complete the square twice. Remember to balance the equation by adding the same constants to each side.
$4\left(x^{2}-10 x+25\right)+9\left(y^{2}+4 y+4\right)=-100+100+36$
Rewrite as perfect squares.
$4(x-5)^{2}+9(y+2)^{2}=36$
Divide both sides by the constant term to place the equation in standard form.
$\frac{(x-5)^{2}}{9}+\frac{(y+2)^{2}}{4}=1$
Now that the equation is in standard form, we can determine the position of the major axis. Because $9>4$, the major axis is parallel to the $x$-axis. Therefore, the equation is in the form
$\frac{(x-h)^{2}}{a^{2}}+\frac{(y-k)^{2}}{b^{2}}=1$, where $a^{2}=9$ and $b^{2}=4$. It follows that:

- the center of the ellipse is $(h, k)=(5,-2)$
- the coordinates of the vertices are $(h \pm a, k)=(5 \pm \sqrt{9},-2)=(5 \pm 3,-2)$, or $(2,-2)$ and $(8,-2)$
- the coordinates of the co-vertices are $(h, k \pm b)=(5,-2 \pm \sqrt{4})=(5,-2 \pm 2)$, or $(5,-4)$ and $(5,0)$
- the coordinates of the foci are $(h \pm c, k)$, where $c^{2}=a^{2}-b^{2}$. Solving for $c$, we have:

$$
\begin{aligned}
c & = \pm \sqrt{a^{2}-b^{2}} \\
& = \pm \sqrt{9-4} \\
& = \pm \sqrt{5}
\end{aligned}
$$

Therefore, the coordinates of the foci are $(5-\sqrt{5},-2)$ and $(5+\sqrt{5},-2)$.
Next we plot and label the center, vertices, co-vertices, and foci, and draw a smooth curve to form the ellipse.


## TRY IT

Express the equation of the ellipse given in standard form. Identify the center, vertices, co-vertices, and foci of the ellipse.
$4 x^{2}+y^{2}-24 x+2 y+21=0$
Answer
$\frac{(x-3)^{2}}{4}+\frac{(y+1)^{2}}{16}=1$; center: $(3,-1)$; vertices: $(3,-5)$ and $(3,3)$; co-vertices: $(1,-1)$ and $(5,-1)$;
foci: $(3,-1-2 \sqrt{3})$ and $(3,-1+2 \sqrt{3})$
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## SUMMARY: THE ELLIPSE

## Key Equations

Horizontal ellipse, center at origin

$$
\begin{aligned}
& \frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1, a>b \\
& \frac{x^{2}}{b^{2}}+\frac{y^{2}}{a^{2}}=1, a>b \\
& \frac{(x-h)^{2}}{a^{2}}+\frac{(y-k)^{2}}{b^{2}}=1, a>b \\
& \frac{(x-h)^{2}}{b^{2}}+\frac{(y-k)^{2}}{a^{2}}=1, a>b
\end{aligned}
$$

## Key Concepts

- An ellipse is the set of all points $(x, y)$ in a plane such that the sum of their distances from two fixed points is a constant. Each fixed point is called a focus (plural: foci).
- When given the coordinates of the foci and vertices of an ellipse, we can write the equation of the ellipse in standard form.
- When given an equation for an ellipse centered at the origin in standard form, we can identify its vertices, co-vertices, foci, and the lengths and positions of the major and minor axes in order to graph the ellipse.
- When given the equation for an ellipse centered at some point other than the origin, we can identify its key features and graph the ellipse.
- Real-world situations can be modeled using the standard equations of ellipses and then evaluated to find key features, such as lengths of axes and distance between foci.


## Glossary

center of an ellipse the midpoint of both the major and minor axes
conic section any shape resulting from the intersection of a right circular cone with a plane
ellipse the set of all points $(x, y)$ in a plane such that the sum of their distances from two fixed points is a constant
foci plural of focus
focus (of an ellipse) one of the two fixed points on the major axis of an ellipse such that the sum of the distances from these points to any point $(x, y)$ on the ellipse is a constant
major axis the longer of the two axes of an ellipse
minor axis the shorter of the two axes of an ellipse

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## INTRODUCTION TO THE HYPERBOLA

## LEARNING OUTCOMES

By the end of this section, you will be able to:

- Locate a hyperbola's vertices and foci.
- Write equations of hyperbolas in standard form.
- Graph hyperbolas centered at the origin.
- Graph hyperbolas not centered at the origin.
- Solve applied problems involving hyperbolas.

What do paths of comets, supersonic booms, ancient Grecian pillars, and natural draft cooling towers have in common? They can all be modeled by the same type of conic. For instance, when something moves faster than the speed of sound, a shock wave in the form of a cone is created. A portion of a conic is formed when the wave intersects the ground, resulting in a sonic boom.


A shock wave intersecting the ground forms a portion of a conic and results in a sonic boom.
Most people are familiar with the sonic boom created by supersonic aircraft, but humans were breaking the sound barrier long before the first supersonic flight. The crack of a whip occurs because the tip is exceeding the speed of sound. The bullets shot from many firearms also break the sound barrier, although the bang of the gun usually supersedes the sound of the sonic boom.

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## LEARNING OUTCOMES

- Derive an equation for a hyperbola centered at the origin
- Write an equation for a hyperbola centered at the origin
- Solve an applied problem involving hyperbolas

In analytic geometry a hyperbola is a conic section formed by intersecting a right circular cone with a plane at an angle such that both halves of the cone are intersected. This intersection produces two separate unbounded curves that are mirror images of each other.


## A hyperbola

Like the ellipse, the hyperbola can also be defined as a set of points in the coordinate plane. A hyperbola is the set of all points $(x, y)$ in a plane such that the difference of the distances between $(x, y)$ and the foci is a positive constant.

Notice that the definition of a hyperbola is very similar to that of an ellipse. The distinction is that the hyperbola is defined in terms of the difference of two distances, whereas the ellipse is defined in terms of the
sum of two distances.
As with the ellipse, every hyperbola has two axes of symmetry. The transverse axis is a line segment that passes through the center of the hyperbola and has vertices as its endpoints. The foci lie on the line that contains the transverse axis. The conjugate axis is perpendicular to the transverse axis and has the covertices as its endpoints. The center of a hyperbola is the midpoint of both the transverse and conjugate axes, where they intersect. Every hyperbola also has two asymptotes that pass through its center. As a hyperbola recedes from the center, its branches approach these asymptotes. The central rectangle of the hyperbola is centered at the origin with sides that pass through each vertex and co-vertex; it is a useful tool for graphing the hyperbola and its asymptotes. To sketch the asymptotes of the hyperbola, simply sketch and extend the diagonals of the central rectangle.


Key features of the hyperbola
In this section we will limit our discussion to hyperbolas that are positioned vertically or horizontally in the coordinate plane; the axes will either lie on or be parallel to the $x$ - and $y$-axes. We will consider two cases: those that are centered at the origin, and those that are centered at a point other than the origin.

## Standard Form of the Equation of a Hyperbola Centered at the Origin

Let $(-c, 0)$ and $(c, 0)$ be the foci of a hyperbola centered at the origin. The hyperbola is the set of all points $(x, y)$ such that the difference of the distances from $(x, y)$ to the foci is constant.


If $(a, 0)$ is a vertex of the hyperbola, the distance from $(-c, 0)$ to $(a, 0)$ is $a-(-c)=a+c$. The distance from $(c, 0)$ to $(a, 0)$ is $c-a$. The difference of the distances from the foci to the vertex is
$(a+c)-(c-a)=2 a$
If $(x, y)$ is a point on the hyperbola, we can define the following variables:
$d_{2}=$ the distance from $(-c, 0)$ to $(x, y)$
$d_{1}=$ the distance from $(c, 0)$ to $(x, y)$
By definition of a hyperbola, $\left|d_{2}-d_{1}\right|$ is constant for any point $(x, y)$ on the hyperbola. We know that the difference of these distances is $2 a$ for the vertex $(a, 0)$. It follows that $\left|d_{2}-d_{1}\right|=2 a$ for any point on the hyperbola. The derivation of the equation of a hyperbola is based on applying the distance formula, but is again beyond the scope of this text. The standard form of an equation of a hyperbola centered at the origin with vertices $( \pm a, 0)$ and co-vertices $(0 \pm b)$ is $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$.

## A GENERAL NOTE: STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER $(0,0)$

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the $x$-axis is $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$
where

- the length of the transverse axis is $2 a$
- the coordinates of the vertices are $( \pm a, 0)$
- the length of the conjugate axis is $2 b$
- the coordinates of the co-vertices are $(0, \pm b)$
- the distance between the foci is $2 c$, where $c^{2}=a^{2}+b^{2}$
- the coordinates of the foci are ( $\pm c, 0$ )
- the equations of the asymptotes are $y= \pm \frac{b}{a} x$

The standard form of the equation of a hyperbola with center $(0,0)$ and transverse axis on the $y$-axis is $\frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1$
where

- the length of the transverse axis is $2 a$
- the coordinates of the vertices are $(0, \pm a)$
- the length of the conjugate axis is $2 b$
- the coordinates of the co-vertices are $( \pm b, 0)$
- the distance between the foci is $2 c$, where $c^{2}=a^{2}+b^{2}$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y= \pm \frac{a}{b} x$

Note that the vertices, co-vertices, and foci are related by the equation $c^{2}=a^{2}+b^{2}$. When we are given the equation of a hyperbola, we can use this relationship to identify its vertices and foci.

(a) Horizontal hyperbola with center $(0,0)$ (b) Vertical hyperbola with center $(0,0)$

## HOW TO: GIVEN THE EQUATION OF A HYPERBOLA IN STANDARD FORM, LOCATE ITS VERTICES AND FOCI.

1. Determine whether the transverse axis lies on the $x$ - or $y$-axis. Notice that $a^{2}$ is always under the variable with the positive coefficient. So, if you set the other variable equal to zero, you can easily find the intercepts. In the case where the hyperbola is centered at the origin, the intercepts coincide with the vertices.
a. If the equation has the form $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$, then the transverse axis lies on the $x$-axis. The vertices are located at $( \pm a, 0)$, and the foci are located at $( \pm c, 0)$.
b. If the equation has the form $\frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1$, then the transverse axis lies on the $y$-axis. The vertices are located at $(0, \pm a)$, and the foci are located at $(0, \pm c)$.
2. Solve for $a$ using the equation $a=\sqrt{a^{2}}$.
3. Solve for $c$ using the equation $c=\sqrt{a^{2}+b^{2}}$.

## Writing Equations of Hyperbolas in Standard Form

Just as with ellipses, writing the equation for a hyperbola in standard form allows us to calculate the key

## EXAMPLE: LOCATING A HYPERBOLA'S VERTICES AND FOCI

Identify the vertices and foci of the hyperbola with equation $\frac{y^{2}}{49}-\frac{x^{2}}{32}=1$.

## Answer

The equation has the form $\frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1$, so the transverse axis lies on the $y$-axis. The hyperbola is centered at the origin, so the vertices serve as the $y$-intercepts of the graph. To find the vertices, set $x=0$, and solve for $y$.

$$
\begin{array}{r}
1=\frac{y^{2}}{49}-\frac{x^{2}}{32} \\
1=\frac{y^{2}}{49}-\frac{0^{2}}{32} \\
1=\frac{y^{2}}{49} \\
y^{2}=49 \\
y= \pm \sqrt{49}= \pm 7
\end{array}
$$

The foci are located at $(0, \pm c)$. Solving for $c$,
$c=\sqrt{a^{2}+b^{2}}=\sqrt{49+32}=\sqrt{81}=9$
Therefore, the vertices are located at $(0, \pm 7)$, and the foci are located at $(0,9)$.

## RECALL THE SQUARE ROOT PROPERTY

With the $x^{2}$ term isolated, the square root propty states that:
if $x^{2}=k$, then $x= \pm \sqrt{k}$
Don't forget to take the plus or minus!

## TRY IT

Identify the vertices and foci of the hyperbola with equation $\frac{x^{2}}{9}-\frac{y^{2}}{25}=1$.
Answer
Vertices: $( \pm 3,0)$; Foci: $( \pm \sqrt{34}, 0)$
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features: its center, vertices, co-vertices, foci, asymptotes, and the lengths and positions of the transverse and conjugate axes. Conversely, an equation for a hyperbola can be found given its key features. We begin by finding standard equations for hyperbolas centered at the origin. Then we will turn our attention to finding standard equations for hyperbolas centered at some point other than the origin.

## Hyperbolas Centered at the Origin

Reviewing the standard forms given for hyperbolas centered at $(0,0)$, we see that the vertices, co-vertices, and foci are related by the equation $c^{2}=a^{2}+b^{2}$. Note that this equation can also be rewritten as $b^{2}=c^{2}-a^{2}$. This relationship is used to write the equation for a hyperbola when given the coordinates of its foci and vertices.

## HOW TO: GIVEN THE VERTICES AND FOCI OF A HYPERBOLA CENTERED AT $(0,0)$, WRITE ITS EQUATION IN STANDARD FORM.

1. Determine whether the transverse axis lies on the $x$ - or $y$-axis.
2. If the given coordinates of the vertices and foci have the form ( $\pm a, 0)$ and $( \pm c, 0)$, respectively, then the transverse axis is the $x$-axis. Use the standard form $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$.
3. If the given coordinates of the vertices and foci have the form $(0, \pm a)$ and $(0, \pm c)$, respectively, then the transverse axis is the $y$-axis. Use the standard form $\frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1$.
4. Find $b^{2}$ using the equation $b^{2}=c^{2}-a^{2}$.
5. Substitute the values for $a^{2}$ and $b^{2}$ into the standard form of the equation determined in Step 1.

## Hyperbolas Not Centered at the Origin

Like the graphs for other equations, the graph of a hyperbola can be translated. If a hyperbola is translated $h$ units horizontally and $k$ units vertically, the center of the hyperbola will be $(h, k)$. This translation results in the standard form of the equation we saw previously, with $x$ replaced by $(x-h)$ and $y$ replaced by $(y-k)$.

The asymptotes of the hyperbola coincide with the diagonals of the central rectangle. The length of the rectangle is $2 a$ and its width is $2 b$. The slopes of the diagonals are $\pm \frac{b}{a}$, and each diagonal passes through the center $(h, k)$. Using the point-slope formula, it is simple to show that the equations of the asymptotes are $y= \pm \frac{b}{a}(x-h)+k$.

The standard form of the equation of a hyperbola with center $(h, k)$ and transverse axis parallel to the $y$-axis is
$\frac{(y-k)^{2}}{a^{2}}-\frac{(x-h)^{2}}{b^{2}}=1$
where

- the length of the transverse axis is $2 a$
- the coordinates of the vertices are $(h, k \pm a)$
- the length of the conjugate axis is $2 b$
- the coordinates of the co-vertices are ( $h \pm b, k$ )
- the distance between the foci is $2 c$, where $c^{2}=a^{2}+b^{2}$
- the coordinates of the foci are ( $h, k \pm c$ )

Using the reasoning above, the equations of the asymptotes are $y= \pm \frac{a}{b}(x-h)+k$.

## EXAMPLE: FINDING THE EQUATION OF A HYPERBOLA CENTERED AT $(0,0)$ GIVEN ITS FOCI AND VERTICES

What is the standard form equation of the hyperbola that has vertices $( \pm 6,0)$ and foci $( \pm 2 \sqrt{10}, 0)$ ?
Answer
The vertices and foci are on the $x$-axis. Thus, the equation for the hyperbola will have the form $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$.
The vertices are $( \pm 6,0)$, so $a=6$ and $a^{2}=36$.
The foci are $( \pm 2 \sqrt{10}, 0)$, so $c=2 \sqrt{10}$ and $c^{2}=40$.
Solving for $b^{2}$, we have
$b^{2}=c^{2}-a^{2}$
$b^{2}=40-36 \quad$ Substitute for $c^{2}$ and $a^{2}$.
$b^{2}=4 \quad$ Subtract.
Finally, we substitute $a^{2}=36$ and $b^{2}=4$ into the standard form of the equation, $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$. The equation of the hyperbola is $\frac{x^{2}}{36}-\frac{y^{2}}{4}=1$.


## TRY IT

What is the standard form equation of the hyperbola that has vertices $(0, \pm 2)$ and foci $(0, \pm 2 \sqrt{5})$ ?
Answer
$\frac{y^{2}}{4}-\frac{x^{2}}{16}=1$
An interactive or media element has been excluded from this version of the text. You can view it online here: https://courses.lumenlearning.com/waymakercollegealgebracorequisite/?p=423

## A GENERAL NOTE: STANDARD FORMS OF THE EQUATION OF A HYPERBOLA WITH CENTER $(H, K)$

The standard form of the equation of a hyperbola with center $(h, k)$ and transverse axis parallel to the $x$ axis is
$\frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1$
where

- the length of the transverse axis is $2 a$
- the coordinates of the vertices are ( $h \pm a, k$ )
- the length of the conjugate axis is $2 b$
- the coordinates of the co-vertices are ( $h, k \pm b$ )
- the distance between the foci is $2 c$, where $c^{2}=a^{2}+b^{2}$
- the coordinates of the foci are ( $h \pm c, k$ )

(a) Horizontal hyperbola with center ( $h, k$ ) (b) Vertical hyperbola with center $(h, k)$

Like hyperbolas centered at the origin, hyperbolas centered at a point ( $h, k$ ) have vertices, co-vertices, and foci that are related by the equation $c^{2}=a^{2}+b^{2}$. We can use this relationship along with the midpoint and distance formulas to find the standard equation of a hyperbola when the vertices and foci are given.

## HOW TO: GIVEN THE VERTICES AND FOCI OF A HYPERBOLA CENTERED AT $(h, k)$, WRITE ITS EQUATION IN STANDARD FORM.

1. Determine whether the transverse axis is parallel to the $x$ - or $y$-axis.
2. If the $y$-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the $x$-axis. Use the standard form $\frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1$.
3. If the $x$-coordinates of the given vertices and foci are the same, then the transverse axis is parallel to the $y$-axis. Use the standard form $\frac{(y-k)^{2}}{a^{2}}-\frac{(x-h)^{2}}{b^{2}}=1$.
4. Identify the center of the hyperbola, $(h, k)$, using the midpoint formula and the given coordinates for the vertices.
5. Find $a^{2}$ by solving for the length of the transverse axis, $2 a$, which is the distance between the given vertices.
6. Find $c^{2}$ using $h$ and $k$ found in Step 2 along with the given coordinates for the foci.
7. Solve for $b^{2}$ using the equation $b^{2}=c^{2}-a^{2}$.
8. Substitute the values for $h, k, a^{2}$, and $b^{2}$ into the standard form of the equation determined in Step 1.

## EXAMPLE: FINDING THE EQUATION OF A HYPERBOLA CENTERED AT ( $H$, K) GIVEN ITS FOCI AND VERTICES

What is the standard form equation of the hyperbola that has vertices at $(0,-2)$ and $(6,-2)$ and foci at $(-2,-2)$ and $(8,-2)$ ?
Answer
The $y$-coordinates of the vertices and foci are the same, so the transverse axis is parallel to the $x$-axis. Thus, the equation of the hyperbola will have the form

$$
\frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1
$$

First, we identify the center, $(h, k)$. The center is halfway between the vertices $(0,-2)$ and $(6,-2)$.
Applying the midpoint formula, we have
$(h, k)=\left(\frac{0+6}{2}, \frac{-2+(-2)}{2}\right)=(3,-2)$
Next, we find $a^{2}$. The length of the transverse axis, $2 a$, is bounded by the vertices. So, we can find $a^{2}$ by finding the distance between the $x$-coordinates of the vertices.

$$
\begin{gathered}
2 a=|0-6| \\
2 a=6 \\
a=3 \\
a^{2}=9
\end{gathered}
$$

Now we need to find $c^{2}$. The coordinates of the foci are $(h \pm c, k)$. So $(h-c, k)=(-2,-2)$ and $(h+c, k)=(8,-2)$. We can use the $x$-coordinate from either of these points to solve for $c$. Using the point $(8,-2)$, and substituting $h=3$,
$h+c=8$
$3+c=8$
$c=5$
$c^{2}=25$
Next, solve for $b^{2}$ using the equation $b^{2}=c^{2}-a^{2}$ :

$$
\begin{aligned}
b^{2} & =c^{2}-a^{2} \\
& =25-9
\end{aligned}
$$

$$
=16
$$

Finally, substitute the values found for $h, k, a^{2}$, and $b^{2}$ into the standard form of the equation.

$$
\frac{(x-3)^{2}}{9}-\frac{(y+2)^{2}}{16}=1
$$

## TRY IT

What is the standard form equation of the hyperbola that has vertices $(1,-2)$ and $(1,8)$ and foci $(1,-10)$ and $(1,16)$ ?
Answer

$$
\frac{(y-3)^{2}}{25}+\frac{(x-1)^{2}}{144}=1
$$

## Solving Applied Problems Involving Hyperbolas

As we discussed at the beginning of this section, hyperbolas have real-world applications in many fields, such as astronomy, physics, engineering, and architecture. The design efficiency of hyperbolic cooling towers is particularly interesting. Cooling towers are used to transfer waste heat to the atmosphere and are often touted for their ability to generate power efficiently. Because of their hyperbolic form, these structures are able to withstand extreme winds while requiring less material than any other forms of their size and strength. For example a 500 -foot tower can be made of a reinforced concrete shell only 6 or 8 inches wide!


Cooling towers at the Drax power station in North Yorkshire, United Kingdom (credit: Les Haines, Flickr)
The first hyperbolic towers were designed in 1914 and were 35 meters high. Today, the tallest cooling towers are in France, standing a remarkable 170 meters tall. In Example 6 we will use the design layout of a cooling tower to find a hyperbolic equation that models its sides.

## EXAMPLE: SOLVING APPLIED PROBLEMS INVOLVING HYPERBOLAS

The design layout of a cooling tower is shown below. The tower stands 179.6 meters tall. The diameter of the top is 72 meters. At their closest, the sides of the tower are 60 meters apart.


Project design for a natural draft cooling tower
Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola-indicated by the intersection of dashed perpendicular lines in the figure-is the origin of the coordinate plane. Round final values to four decimal places.

## Answer

We are assuming the center of the tower is at the origin, so we can use the standard form of a horizontal hyperbola centered at the origin: $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$, where the branches of the hyperbola form the sides of the cooling tower. We must find the values of $a^{2}$ and $b^{2}$ to complete the model.
First, we find $a^{2}$. Recall that the length of the transverse axis of a hyperbola is $2 a$. This length is represented by the distance where the sides are closest, which is given as 65.3 meters. So, $2 a=60$. Therefore, $a=30$ and $a^{2}=900$.
To solve for $b^{2}$, we need to substitute for $x$ and $y$ in our equation using a known point. To do this, we can use the dimensions of the tower to find some point $(x, y)$ that lies on the hyperbola. We will use the top right corner of the tower to represent that point. Since the $y$-axis bisects the tower, our $x$-value can be
represented by the radius of the top, or 36 meters. The $y$-value is represented by the distance from the origin to the top, which is given as 79.6 meters. Therefore,

$$
\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1 \quad \text { Standard form of horizontal hyperbola. }
$$

$$
b^{2}=\frac{y^{2}}{\frac{x^{2}}{a^{2}}-1} \quad \text { Isolate } b^{2}
$$

$$
=\frac{(79.6)^{2}}{\frac{(36)^{2}}{900}-1} \quad \text { Substitute for } a^{2}, x, \text { and } y
$$

$\approx 14400.3636$ Round to four decimal places
The sides of the tower can be modeled by the hyperbolic equation

$$
\frac{x^{2}}{900}-\frac{y^{2}}{14400.3636}=1, \text { or } \frac{x^{2}}{30^{2}}-\frac{y^{2}}{120.0015^{2}}=1
$$

## TIP FOR SUCCESS

Don't be discouraged if it takes several tries through the example and practice problem to understand the process. Keep trying!

## TRY IT

A design for a cooling tower project is shown below. Find the equation of the hyperbola that models the sides of the cooling tower. Assume that the center of the hyperbola-indicated by the intersection of dashed perpendicular lines in the figure-is the origin of the coordinate plane. Round final values to four decimal places.


## Answer

The sides of the tower can be modeled by the hyperbolic equation. $\frac{x^{2}}{400}-\frac{y^{2}}{3600}=1$ or $\frac{x^{2}}{20^{2}}-\frac{y^{2}}{60^{2}}=1$.

## GRAPH HYPERBOLAS

## LEARNING OUTCOMES

- Graph a hyperbola centered at the origin.
- Graph a hyperbola not centered at the origin.

When we have an equation in standard form for a hyperbola centered at the origin, we can interpret its parts to identify the key features of its graph: the center, vertices, co-vertices, asymptotes, foci, and lengths and positions of the transverse and conjugate axes. To graph hyperbolas centered at the origin, we use the standard form $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$ for horizontal hyperbolas and the standard form $\frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1$ for vertical hyperbolas.

## HOW TO: GIVEN A STANDARD FORM EQUATION FOR A HYPERBOLA CENTERED AT $(0,0)$, SKETCH THE GRAPH.

- Determine which of the standard forms applies to the given equation.
- Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the vertices, co-vertices, and foci; and the equations for the asymptotes.
- If the equation is in the form $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$, then
- the transverse axis is on the $x$-axis
- the coordinates of the vertices are $( \pm a, 0)$
- the coordinates of the co-vertices are $(0, \pm b)$
- the coordinates of the foci are $( \pm c, 0)$
- the equations of the asymptotes are $y= \pm \frac{b}{a} x$
- If the equation is in the form $\frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1$, then
- the transverse axis is on the $y$-axis
- the coordinates of the vertices are $(0, \pm a)$
- the coordinates of the co-vertices are $( \pm b, 0)$
- the coordinates of the foci are $(0, \pm c)$
- the equations of the asymptotes are $y= \pm \frac{a}{b} x$
- Solve for the coordinates of the foci using the equation $c= \pm \sqrt{a^{2}+b^{2}}$.
- Plot the vertices, co-vertices, foci, and asymptotes in the coordinate plane, and draw a smooth curve to form the hyperbola.


## EXAMPLE: GRAPHING A HYPERBOLA CENTERED AT $(0,0)$ GIVEN AN EQUATION IN STANDARD FORM

Graph the hyperbola given by the equation $\frac{y^{2}}{64}-\frac{x^{2}}{36}=1$. Identify and label the vertices, co-vertices, foci, and asymptotes.

## Answer

The standard form that applies to the given equation is $\frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1$. Thus, the transverse axis is on the $y$-axis
The coordinates of the vertices are $(0, \pm a)=(0, \pm \sqrt{64})=(0, \pm 8)$
The coordinates of the co-vertices are $( \pm b, 0)=( \pm \sqrt{36}, 0)=( \pm 6,0)$
The coordinates of the foci are $(0, \pm c)$, where $c= \pm \sqrt{a^{2}+b^{2}}$. Solving for $c$ we have

$$
\begin{aligned}
c & = \pm \sqrt{a^{2}+b^{2}} \\
& = \pm \sqrt{64+36} \\
& = \pm \sqrt{100} \\
& = \pm 10
\end{aligned}
$$

Therefore, the coordinates of the foci are $(0, \pm 10)$
The equations of the asymptotes are
$y= \pm \frac{a}{b} x$
$y= \pm \frac{8}{6} x$
$y= \pm \frac{4}{3} x$
Plot and label the vertices and co-vertices, and then sketch the central rectangle. Sides of the rectangle are parallel to the axes and pass through the vertices and co-vertices. Sketch and extend the diagonals of the central rectangle to show the asymptotes. The central rectangle and asymptotes provide the framework needed to sketch an accurate graph of the hyperbola. Label the foci and asymptotes, and draw a smooth curve to form the hyperbola.


## TRY IT

Graph the hyperbola given by the equation $\frac{x^{2}}{144}-\frac{y^{2}}{81}=1$. Identify and label the vertices, co-vertices, foci, and asymptotes.
Answer
vertices: $( \pm 12,0)$; co-vertices: $(0, \pm 9)$; foci: $( \pm 15,0)$; asymptotes: $y= \pm \frac{3}{4} x$;


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## TRY IT

Use an online graphing tool to plot the equation $\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1$. Adjust the values you use for $a, b$ to values between 1,20 . Your task in this exercise is to graph a hyperbola and then calculate and add the following features to the graph:

- vertices
- co-vertices
- foci
- asymptotes


## Graphing Hyperbolas Not Centered at the Origin

Graphing hyperbolas centered at a point $(h, k)$ other than the origin is similar to graphing ellipses centered at a point other than the origin. We use the standard forms $\frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1$ for horizontal hyperbolas, and $\frac{(y-k)^{2}}{a^{2}}-\frac{(x-h)^{2}}{b^{2}}=1$ for vertical hyperbolas. From these standard form equations we can easily calculate and plot key features of the graph: the coordinates of its center, vertices, co-vertices, and foci; the equations of its asymptotes; and the positions of the transverse and conjugate axes.

## TIP FOR SUCCESS

Writing the equation of a hyperbola not centered at the origin uses graph transformation techniques in the same way that writing the equation of an ellipse does. The center is represented by $(h, k)$ as it was in the ellipse.
Look for similarities and differences in the construction of ellipses, hyperbolas, and parabolas to help you build your intuition about these objects.

## HOW TO: GIVEN A GENERAL FORM FOR A HYPERBOLA CENTERED AT $(h, k)$, SKETCH THE GRAPH.

- Convert the general form to that standard form. Determine which of the standard forms applies to the given equation.
- Use the standard form identified in Step 1 to determine the position of the transverse axis; coordinates for the center, vertices, co-vertices, foci; and equations for the asymptotes.
- If the equation is in the form $\frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1$, then
- the transverse axis is parallel to the $x$-axis
- the center is $(h, k)$
- the coordinates of the vertices are ( $h \pm a, k$ )
- the coordinates of the co-vertices are ( $h, k \pm b$ )
- the coordinates of the foci are ( $h \pm c, k$ )
- the equations of the asymptotes are $y= \pm \frac{b}{a}(x-h)+k$
- If the equation is in the form $\frac{(y-k)^{2}}{a^{2}}-\frac{(x-h)^{2}}{b^{2}}=1$, then
- the transverse axis is parallel to the $y$-axis
- the center is $(h, k)$
- the coordinates of the vertices are $(h, k \pm a)$
- the coordinates of the co-vertices are $(h \pm b, k)$
- the coordinates of the foci are $(h, k \pm c)$
- the equations of the asymptotes are $y= \pm \frac{a}{b}(x-h)+k$
- Solve for the coordinates of the foci using the equation $c= \pm \sqrt{a^{2}+b^{2}}$.
- Plot the center, vertices, co-vertices, foci, and asymptotes in the coordinate plane and draw a smooth curve to form the hyperbola.


## RECALL COMPLETE THE SQUARE

You will need to complete the square twice when rewriting the general form of a hyperbola in the same way you did with the ellipse. Here's the form to help you.
Completing the Square
Given an expression of the form $a\left(x^{2}+b x\right)$, add $\left(\frac{b}{2}\right)^{2}$ inside the parentheses, then subtract $a\left(\frac{b}{2}\right)^{2}$ to counteract the change you made.
If completing the square on one side of an equation, you may either subtract the value of $a\left(\frac{b}{2}\right)^{2}$ from that side, or add it to the other to maintain equality.
Then factor the perfect square trinomial you created inside the original parentheses.
Example
$a\left(x^{2}+b x\right)$
$=a\left(x^{2}+b x+\left(\frac{b}{2}\right)^{2}\right)-a\left(\frac{b}{2}\right)^{2}$
$=a\left(x+\frac{b}{2}\right)^{2}-a\left(\frac{b}{2}\right)^{2}$

## EXAMPLE: GRAPHING A HYPERBOLA CENTERED AT $(H, K)$ GIVEN AN EQUATION IN GENERAL FORM

Graph the hyperbola given by the equation $9 x^{2}-4 y^{2}-36 x-40 y-388=0$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.
Answer
Start by expressing the equation in standard form. Group terms that contain the same variable, and move the constant to the opposite side of the equation.
$\left(9 x^{2}-36 x\right)-\left(4 y^{2}+40 y\right)=388$
Factor the leading coefficient of each expression.
$9\left(x^{2}-4 x\right)-4\left(y^{2}+10 y\right)=388$
Complete the square twice. Remember to balance the equation by adding the same constants to each side.
$9\left(x^{2}-4 x+4\right)-4\left(y^{2}+10 y+25\right)=388+9 \cdot 4-4 \cdot 25$
Rewrite as perfect squares.
$9(x-2)^{2}-4(y+5)^{2}=324$
Divide both sides by the constant term to place the equation in standard form.
$\frac{(x-2)^{2}}{36}-\frac{(y+5)^{2}}{81}=1$
The standard form that applies to the given equation is $\frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1$, where $a^{2}=36$ and $b^{2}=81$, or $a=6$ and $b=9$. Thus, the transverse axis is parallel to the $x$-axis. It follows that: the center of the ellipse is $(h, k)=(2,-5)$

- the coordinates of the vertices are $(h \pm a, k)=(2 \pm 6,-5)$, or $(-4,-5)$ and $(8,-5)$
- the coordinates of the co-vertices are $(h, k \pm b)=(2,-5 \pm 9)$, or $(2,-14)$ and $(2,4)$
- the coordinates of the foci are $(h \pm c, k)$, where $c= \pm \sqrt{a^{2}+b^{2}}$. Solving for $c$, we have
$c= \pm \sqrt{36+81}= \pm \sqrt{117}= \pm 3 \sqrt{13}$
Therefore, the coordinates of the foci are $(2-3 \sqrt{13},-5)$ and $(2+3 \sqrt{13},-5)$.
The equations of the asymptotes are $y= \pm \frac{b}{a}(x-h)+k= \pm \frac{3}{2}(x-2)-5$.
Next, we plot and label the center, vertices, co-vertices, foci, and asymptotes and draw smooth curves to form the hyperbola.



## TRY IT

Graph the hyperbola given by the standard form of an equation $\frac{(y+4)^{2}}{100}-\frac{(x-3)^{2}}{64}=1$. Identify and label the center, vertices, co-vertices, foci, and asymptotes.
Answer
center: $(3,-4)$; vertices: $(3,-14)$ and $(3,6)$; co-vertices: $(-5,-4)$; and $(11,-4)$; foci: $(3,-4-2 \sqrt{41})$ and $(3,-4+2 \sqrt{41})$; asymptotes: $y= \pm \frac{5}{4}(x-3)-4$


## TRY IT

Using an online graphing calculator, plot a hyperbola not centered at the origin.
The equation used to generate the graph is $\frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1$.
Adjust the values you use for $a, b$ to values between 1,20 , and the variables $h, k$ to numbers between 10,10 . Your task in this exercise is to graph the hyperbola and then calculate and add the following features to the graph:

- center
- vertices
- co-vertices
- foci
- asymptotes


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## SUMMARY: THE HYPERBOLA

## Key Equations

Hyperbola, center at origin, transverse axis on $x$-axis

Hyperbola, center at origin, transverse axis on $y$-axis

$$
\begin{aligned}
& \frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1 \\
& \frac{y^{2}}{a^{2}}-\frac{x^{2}}{b^{2}}=1 \\
& \frac{(x-h)^{2}}{a^{2}}-\frac{(y-k)^{2}}{b^{2}}=1 \\
& \frac{(y-k)^{2}}{a^{2}}-\frac{(x-h)^{2}}{b^{2}}=1
\end{aligned}
$$

## Key Concepts

- A hyperbola is the set of all points $(x, y)$ in a plane such that the difference of the distances between $(x, y)$ and the foci is a positive constant.
- The standard form of a hyperbola can be used to locate its vertices and foci.
- When given the coordinates of the foci and vertices of a hyperbola, we can write the equation of the hyperbola in standard form.
- When given an equation for a hyperbola, we can identify its vertices, co-vertices, foci, asymptotes, and lengths and positions of the transverse and conjugate axes in order to graph the hyperbola.
- Real-world situations can be modeled using the standard equations of hyperbolas. For instance, given the dimensions of a natural draft cooling tower, we can find a hyperbolic equation that models its sides.


## Glossary

center of a hyperbola the midpoint of both the transverse and conjugate axes of a hyperbola
conjugate axis the axis of a hyperbola that is perpendicular to the transverse axis and has the co-vertices as its endpoints
hyperbola the set of all points $(x, y)$ in a plane such that the difference of the distances between $(x, y)$ and the foci is a positive constant
transverse axis the axis of a hyperbola that includes the foci and has the vertices as its endpoints
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## INTRODUCTION TO THE PARABOLA

## LEARNING OUTCOMES

By the end of this section, you will be able to:

- Graph parabolas with vertices at the origin.
- Write equations of parabolas in standard form.
- Graph parabolas with vertices not at the origin.
- Solve applied problems involving parabolas.


The Olympic torch concludes its journey around the world when it is used to light the Olympic cauldron during the opening ceremony. (credit: Ken Hackman, U.S. Air Force)

Did you know that the Olympic torch is lit several months before the start of the games? The ceremonial method for lighting the flame is the same as in ancient times. The ceremony takes place at the Temple of Hera in Olympia, Greece, and is rooted in Greek mythology, paying tribute to Prometheus, who stole fire from Zeus to give to all humans. One of eleven acting priestesses places the torch at the focus of a parabolic mirror, which focuses light rays from the sun to ignite the flame.

Parabolic mirrors (or reflectors) are able to capture energy and focus it to a single point. The advantages of this property are evidenced by the vast list of parabolic objects we use every day: satellite dishes, suspension bridges, telescopes, microphones, spotlights, and car headlights, to name a few. Parabolic reflectors are also used in alternative energy devices, such as solar cookers and water heaters, because they are inexpensive to manufacture and need little maintenance. In this section we will explore the parabola and its uses, including low-cost, energy-efficient solar designs.

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## PARABOLAS WITH VERTICES AT THE ORIGIN

## LEARNING OUTCOMES

- Identify and label the focus, directrix, and endpoints of the focal diameter of a parabola.
- Write the equation of a parabola given a focus and directrix.

In The Ellipse we saw that an ellipse is formed when a plane cuts through a right circular cone. If the plane is parallel to the edge of the cone, an unbounded curve is formed. This curve is a parabola.


## Parabola

## TIP FOR SUCCESS

You've seen parabolas before as the set of all points satisfying a quadratic function. We'll look at the geometric form of a parabola in this section. It still describes a set of points that satisfy an equation in two variables, but without the need to qualify as a function, it can open left and right as well as up and down. It will be necessary to use another form of its equation to take all of the characteristics of this object into consideration. You'll learn new terminology for the parts of a parabola just as you did with the ellipse and hyperbola as well.

Like the ellipse and hyperbola, the parabola can also be defined by a set of points in the coordinate plane. A parabola is the set of all points $(x, y)$ in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.

We previously learned about a parabola's vertex and axis of symmetry. Now we extend the discussion to include other key features of the parabola. Notice that the axis of symmetry passes through the focus and vertex and is perpendicular to the directrix. The vertex is the midpoint between the directrix and the focus.

The line segment that passes through the focus and is parallel to the directrix is called the latus rectum, also called the focal diameter. The endpoints of the focal diameter lie on the curve. By definition, the distance $d$ from the focus to any point $P$ on the parabola is equal to the distance from $P$ to directrix.


Key features of the parabola
To work with parabolas in the coordinate plane, we consider two cases: those with a vertex at the origin and those with a vertex at a point other than the origin. We begin with the former.


Let $(x, y)$ be a point on the parabola with vertex $(0,0)$, focus $(0, p)$, and directrix $y=-p$ as shown in Figure 4. The distance $d$ from point $(x, y)$ to point $(x,-p)$ on the directrix is the difference of the $y$-values: $d=y+p$ . The distance from the focus $(0, p)$ to the point $(x, y)$ is also equal to $d$ and can be expressed using the distance formula.

$$
\begin{aligned}
d & =\sqrt{(x-0)^{2}+(y-p)^{2}} \\
& =\sqrt{x^{2}+(y-p)^{2}}
\end{aligned}
$$

Set the two expressions for $d$ equal to each other and solve for $y$ to derive the equation of the parabola. We do this because the distance from $(x, y)$ to $(0, p)$ equals the distance from $(x, y)$ to $(x,-p)$.
$\sqrt{x^{2}+(y-p)^{2}}=y+p$
We then square both sides of the equation, expand the squared terms, and simplify by combining like terms.

$$
\begin{gathered}
x^{2}+(y-p)^{2}=(y+p)^{2} \\
x^{2}+y^{2}-2 p y+p^{2}=y^{2}+2 p y+p^{2} \\
x^{2}-2 p y=2 p y \\
x^{2}=4 p y
\end{gathered}
$$

The equations of parabolas with vertex $(0,0)$ are $y^{2}=4 p x$ when the $x$-axis is the axis of symmetry and $x^{2}=4 p y$ when the $y$-axis is the axis of symmetry. These standard forms are given below, along with their general graphs and key features.

## A GENERAL NOTE: STANDARD FORMS OF PARABOLAS WITH VERTEX (0, <br> 0)

The table below summarizes the standard features of parabolas with a vertex at the origin.

| Axis of Symmetry | Equation | Focus | Directrix | Endpoints of Focal Diameter |
| :--- | :--- | :--- | :--- | :--- |
| $x$-axis | $y^{2}=4 p x$ | $(p, 0)$ | $x=-p$ | $(p, \pm 2 p)$ |
| $y$-axis | $x^{2}=4 p y$ | $(0, p)$ | $y=-p$ | $( \pm 2 p, p)$ |


(a) When $p>0$ and the axis of symmetry is the $x$-axis, the parabola opens right. (b) When $p<0$ and the axis of symmetry is the $x$-axis, the parabola opens left. (c) When $p<0$ and the axis of symmetry is the $y$-axis, the parabola opens up. (d) When $p<0$ and the axis of symmetry is the $y$-axis, the parabola opens down.

The key features of a parabola are its vertex, axis of symmetry, focus, directrix, and focal diameter. When given a standard equation for a parabola centered at the origin, we can easily identify the key features to graph the parabola.

A line is said to be tangent to a curve if it intersects the curve at exactly one point. If we sketch lines tangent to the parabola at the endpoints of the focal diameter, these lines intersect on the axis of symmetry.


## HOW TO: GIVEN A STANDARD FORM EQUATION FOR A PARABOLA CENTERED AT ( 0,0 ), SKETCH THE GRAPH.

- Determine which of the standard forms applies to the given equation: $y^{2}=4 p x$ or $x^{2}=4 p y$.
- Use the standard form identified in Step 1 to determine the axis of symmetry, focus, equation of the directrix, and endpoints of the focal diameter.
- If the equation is in the form $y^{2}=4 p x$, then
- the axis of symmetry is the $x$-axis, $y=0$
- set $4 p$ equal to the coefficient of $x$ in the given equation to solve for $p$. If $p>0$, the parabola opens right. If $p<0$, the parabola opens left.
- use $p$ to find the coordinates of the focus, $(p, 0)$
- use $p$ to find the equation of the directrix, $x=-p$
- use $p$ to find the endpoints of the focal diameter, ( $p, \pm 2 p$ ). Alternately, substitute $x=p$ into the original equation.
- If the equation is in the form $x^{2}=4 p y$, then
- the axis of symmetry is the $y$-axis, $x=0$
- set $4 p$ equal to the coefficient of $y$ in the given equation to solve for $p$. If $p>0$, the parabola opens up. If $p<0$, the parabola opens down.
- use $p$ to find the coordinates of the focus, $(0, p)$
- use $p$ to find equation of the directrix, $y=-p$
- use $p$ to find the endpoints of the focal diameter, $( \pm 2 p, p)$
- Plot the focus, directrix, and focal diameter, and draw a smooth curve to form the parabola.


## Writing Equations of Parabolas in Standard Form

In the previous examples we used the standard form equation of a parabola to calculate the locations of its key features. We can also use the calculations in reverse to write an equation for a parabola when given its key features.

## EXAMPLE: GRAPHING A PARABOLA WITH VERTEX $(0,0)$ AND THE X-AXIS AS THE AXIS OF SYMMETRY

Graph $y^{2}=24 x$. Identify and label the focus, directrix, and endpoints of the focal diameter.
Answer
The standard form that applies to the given equation is $y^{2}=4 p x$. Thus, the axis of symmetry is the $x$-axis. It follows that:

- $24=4 p$, so $p=6$. Since $p>0$, the parabola opens right the coordinates of the focus are $(p, 0)=(6,0)$
- the equation of the directrix is $x=-p=-6$
- the endpoints of the focal diameter have the same $x$-coordinate at the focus. To find the endpoints, substitute $x=6$ into the original equation: $(6, \pm 12)$

Next we plot the focus, directrix, and focal diameter, and draw a smooth curve to form the parabola.


## TRY IT

Graph $y^{2}=-16 x$. Identify and label the focus, directrix, and endpoints of the focal diameter.

## Answer

Focus: $(-4,0)$; Directrix: $x=4$; Endpoints of the latus rectum: $(-4, \pm 8)$


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## EXAMPLE: GRAPHING A PARABOLA WITH VERTEX $(0,0)$ AND THE $Y$-AXIS AS THE AXIS OF SYMMETRY

Graph $x^{2}=-6 y$. Identify and label the focus, directrix, and endpoints of the focal diameter.
Answer
The standard form that applies to the given equation is $x^{2}=4 p y$. Thus, the axis of symmetry is the $y$-axis. It follows that:

- $-6=4 p$, so $p=-\frac{3}{2}$. Since $p<0$, the parabola opens down.
- the coordinates of the focus are $(0, p)=\left(0,-\frac{3}{2}\right)$
- the equation of the directrix is $y=-p=\frac{3}{2}$
- the endpoints of the focal diameter can be found by substituting $y=\frac{3}{2}$ into the original equation, $\left( \pm 3,-\frac{3}{2}\right)$

Next we plot the focus, directrix, and latus rectum, and draw a smooth curve to form the parabola.


## TRY IT

Graph $x^{2}=8 y$. Identify and label the focus, directrix, and endpoints of the focal diameter.

## Answer

Focus: $(0,2)$; Directrix: $y=-2$; Endpoints of the latus rectum: $( \pm 4,2)$.


## TRY IT

Use an online graphing tool to plot the following parabola whose axis of symmetry is the x -axis, $y^{2}=4 p x$ Adjust the free variable $p$ to values between $-10,10$. Your task in this exercise is to add the focus, directrix, and endpoints of the focal diameter in terms of the free variable, $p$.
For example, to add the focus, you would define a point, $(p, 0)$.
Answer


## TIP FOR SUCCESS

In this section, we will write the equation of a parabola in standard form, as opposed to the equation of a quadratic or second degree polynomial. The language we use when discussing the object is specific. It is true that a quadratic function forms a parabola when graphed in the plane, but here we are using the phrase standard form of the equation of a parabola to indicate that we wish to describe the geometric object. When talking about this object in this context, we would naturally use the equations described below.

## HOW TO: GIVEN ITS FOCUS AND DIRECTRIX, WRITE THE EQUATION FOR A PARABOLA IN STANDARD FORM.

- Determine whether the axis of symmetry is the $x$ - or $y$-axis.
- If the given coordinates of the focus have the form $(p, 0)$, then the axis of symmetry is the $x$-axis. Use the standard form $y^{2}=4 p x$.
- If the given coordinates of the focus have the form $(0, p)$, then the axis of symmetry is the $y$-axis. Use the standard form $x^{2}=4 p y$.
- Multiply $4 p$.
- Substitute the value from Step 2 into the equation determined in Step 1.


## EXAMPLE: WRITING THE EQUATION OF A PARABOLA IN STANDARD FORM GIVEN ITS FOCUS AND DIRECTRIX

What is the equation for the parabola with focus $\left(-\frac{1}{2}, 0\right)$ and directrix $x=\frac{1}{2}$ ?
Answer
The focus has the form $(p, 0)$, so the equation will have the form $y^{2}=4 p x$.
Multiplying $4 p$, we have $4 p=4\left(-\frac{1}{2}\right)=-2$. Substituting for $4 p$, we have $y^{2}=4 p x=-2 x$.
Therefore, the equation for the parabola is $y^{2}=-2 x$.

## TRY IT

What is the equation for the parabola with focus $\left(0, \frac{7}{2}\right)$ and directrix $y=-\frac{7}{2}$ ?

## Answer

$x^{2}=14 y$
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## PARABOLAS WITH VERTICES NOT AT THE ORIGIN

## LEARNING OUTCOMES

- Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the focal diameter given the equation of a parabola in standard form.
- Find the equation of a parabolic shaped object given dimensions.

Like other graphs we've worked with, the graph of a parabola can be translated. If a parabola is translated $h$ units horizontally and $k$ units vertically, the vertex will be $(h, k)$. This translation results in the standard form of the equation we saw previously with $x$ replaced by $(x-h)$ and $y$ replaced by $(y-k)$.

## TIP FOR SUCCESS

Again, note the familiar transformations of graphs being applied to the translation of the parabola.
To graph parabolas with a vertex $(h, k)$ other than the origin, we use the standard form $(y-k)^{2}=4 p(x-h)$ for parabolas that have an axis of symmetry parallel to the $x$-axis, and $(x-h)^{2}=4 p(y-k)$ for parabolas that have an axis of symmetry parallel to the $y$-axis. These standard forms are given below, along with their general graphs and key features.

## A GENERAL NOTE: STANDARD FORMS OF PARABOLAS WITH VERTEX ( $H$, $K)$

The table summarizes the standard features of parabolas with a vertex at a point $(h, k)$.

| Axis of Symmetry | Equation | Focus | Directrix | Endpoints of focal diameter |
| :--- | :--- | :--- | :--- | :--- |
| $y=k$ | $(y-k)^{2}=4 p(x-h)$ | $(h+p, k)$ | $x=h-p$ | $(h+p, k \pm 2 p)$ |
| $x=h$ | $(x-h)^{2}=4 p(y-k)$ | $(h, k+p)$ | $y=k-p$ | $(h \pm 2 p, k+p)$ |


(a) When $p>0$, the parabola opens right. (b) When $p<0$, the parabola opens left. (c) When $p>0$, the parabola opens up. (d) When $p<0$, the parabola opens down.

## HOW TO: GIVEN A STANDARD FORM EQUATION FOR A PARABOLA CENTERED AT $(H, K)$, SKETCH THE GRAPH.

1. Determine which of the standard forms applies to the given equation: $(y-k)^{2}=4 p(x-h)$ or $(x-h)^{2}=4 p(y-k)$.
2. Use the standard form identified in Step 1 to determine the vertex, axis of symmetry, focus, equation of the directrix, and endpoints of the focal diameter.
3. If the equation is in the form $(y-k)^{2}=4 p(x-h)$, then:

- use the given equation to identify $h$ and $k$ for the vertex, $(h, k)$
- use the value of $k$ to determine the axis of symmetry, $y=k$
- set $4 p$ equal to the coefficient of $(x-h)$ in the given equation to solve for $p$. If $p>0$, the parabola opens right. If $p<0$, the parabola opens left.
- use $h, k$, and $p$ to find the coordinates of the focus, $(h+p, k)$
- use $h$ and $p$ to find the equation of the directrix, $x=h-p$
- use $h, k$, and $p$ to find the endpoints of the focal diameter, $(h+p, k \pm 2 p)$

2. If the equation is in the form $(x-h)^{2}=4 p(y-k)$, then:

- use the given equation to identify $h$ and $k$ for the vertex, $(h, k)$
- use the value of $h$ to determine the axis of symmetry, $x=h$
- set $4 p$ equal to the coefficient of $(y-k)$ in the given equation to solve for $p$. If $p>0$, the parabola opens up. If $p<0$, the parabola opens down.
- use $h, k$, and $p$ to find the coordinates of the focus, $(h, k+p)$
- use $k$ and $p$ to find the equation of the directrix, $y=k-p$
- use $h, k$, and $p$ to find the endpoints of the focal diameter, $(h \pm 2 p, k+p)$

3. Plot the vertex, axis of symmetry, focus, directrix, and focal diameter, and draw a smooth curve to form the parabola.

## EXAMPLE: GRAPHING A PARABOLA WITH VERTEX $(H, K)$ AND AXIS OF SYMMETRY PARALLEL TO THE X-AXIS

Graph $(y-1)^{2}=-16(x+3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the focal diameter.
Answer
The standard form that applies to the given equation is $(y-k)^{2}=4 p(x-h)$. Thus, the axis of symmetry is parallel to the $x$-axis. It follows that:

- the vertex is $(h, k)=(-3,1)$
- the axis of symmetry is $y=k=1$
- $-16=4 p$, so $p=-4$. Since $p<0$, the parabola opens left.
- the coordinates of the focus are $(h+p, k)=(-3+(-4), 1)=(-7,1)$
- the equation of the directrix is $x=h-p=-3-(-4)=1$
- the endpoints of the focal diameter are $(h+p, k \pm 2 p)=(-3+(-4), 1 \pm 2(-4))$, or $(-7,-7)$ and $(-7,9)$

Next we plot the vertex, axis of symmetry, focus, directrix, and focal diameter, and draw a smooth curve to form the parabola.


Graph $(y+1)^{2}=4(x-8)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the focal diameter.

## Answer

Vertex: $(8,-1)$; Axis of symmetry: $y=-1$; Focus: $(9,-1)$; Directrix: $x=7$; Endpoints of the latus rectum: $(9,-3)$ and $(9,1)$.


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## RECALL COMPLETE THE SQUARE

You will need to complete the square when rewriting the general form of a parabola in the same way you did with the ellipse and hyperbola. Here's the form to help you.
Completing the Square
Given an expression of the form $a\left(x^{2}+b x\right)$, add $\left(\frac{b}{2}\right)^{2}$ inside the parentheses, then subtract $a\left(\frac{b}{2}\right)^{2}$ to counteract the change you made.
If completing the square on one side of an equation, you may either subtract the value of $a\left(\frac{b}{2}\right)^{2}$ from that side, or add it to the other to maintain equality.
Then factor the perfect square trinomial you created inside the original parentheses.
Example
$\begin{aligned} & a\left(x^{2}+b x\right) \\ = & a\left(x^{2}+b x+\left(\frac{b}{2}\right)^{2}\right)-a\left(\frac{b}{2}\right)^{2} \\ = & a\left(x+\frac{b}{2}\right)^{2}-a\left(\frac{b}{2}\right)^{2}\end{aligned}$

## EXAMPLE: GRAPHING A PARABOLA FROM AN EQUATION GIVEN IN GENERAL FORM

Graph $x^{2}-8 x-28 y-208=0$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the focal diameter.
Answer
Start by writing the equation of the parabola in standard form. The standard form that applies to the given equation is $(x-h)^{2}=4 p(y-k)$. Thus, the axis of symmetry is parallel to the $y$-axis. To express the equation of the parabola in this form, we begin by isolating the terms that contain the variable $x$ in order to complete the square.

$$
\begin{gathered}
x^{2}-8 x-28 y-208=0 \\
x^{2}-8 x=28 y+208 \\
x^{2}-8 x+16=28 y+208+16 \\
(x-4)^{2}=28 y+224 \\
(x-4)^{2}=28(y+8) \\
(x-4)^{2}=4 \cdot 7 \cdot(y+8)
\end{gathered}
$$

It follows that:

- the vertex is $(h, k)=(4,-8)$
- the axis of symmetry is $x=h=4$
- since $p=7, p>0$ and so the parabola opens up
- the coordinates of the focus are $(h, k+p)=(4,-8+7)=(4,-1)$
- the equation of the directrix is $y=k-p=-8-7=-15$
- the endpoints of the focal diameter are $(h \pm 2 p, k+p)=(4 \pm 2(7),-8+7)$, or $(-10,-1)$ and $(18,-1)$

Next we plot the vertex, axis of symmetry, focus, directrix, and focal diameter, and draw a smooth curve to form the parabola.


## TRY IT

Graph $(x+2)^{2}=-20(y-3)$. Identify and label the vertex, axis of symmetry, focus, directrix, and endpoints of the focal diameter.
Answer
Vertex: $(-2,3)$; Axis of symmetry: $x=-2$; Focus: $(-2,-2)$; Directrix: $y=8$; Endpoints of the latus rectum: $(-12,-2)$ and $(8,-2)$.


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## TRY IT

Use an online graphing tool to graph a parabola whose vertex is $(h, k)$.
The equation used to generate the parabola is $(y-k)^{2}=4 p(x-h)$.
Your task in this exercise is to plot the vertex, axis of symmetry, focus, directrix, and endpoints of the focal diameter in terms of the free variables $h, k, p$.
For example, to plot the vertex, you would create a point $(h, k)$.
Answer
One possible graph:


## TIP FOR SUCCESS

Remember to look for familiar methods and techniques and to find similarities between different types of problems to help you build problem solving skills and intuition.

As we mentioned at the beginning of the section, parabolas are used to design many objects we use every day, such as telescopes, suspension bridges, microphones, and radar equipment. Parabolic mirrors, such as the one used to light the Olympic torch, have a very unique reflecting property. When rays of light parallel to the parabola's axis of symmetry are directed toward any surface of the mirror, the light is reflected directly to the focus. This is why the Olympic torch is ignited when it is held at the focus of the parabolic mirror.


Reflecting property of parabolas
Parabolic mirrors have the ability to focus the sun's energy to a single point, raising the temperature hundreds of degrees in a matter of seconds. Thus parabolic mirrors are featured in many low-cost, energy efficient solar products, such as solar cookers, solar heaters, and even travel-sized fire starters.

## EXAMPLE: SOLVING APPLIED PROBLEMS INVOLVING PARABOLAS

A cross-section of a design for a travel-sized solar fire starter. The sun's rays reflect off the parabolic mirror toward an object attached to the igniter. Because the igniter is located at the focus of the parabola, the reflected rays cause the object to burn in just seconds.

1. Find the equation of the parabola that models the fire starter. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane.
2. Use the equation found in part (a) to find the depth of the fire starter.


Cross-section of a travel-sized solar fire starter

## Answer

The vertex of the dish is the origin of the coordinate plane, so the parabola will take the standard form $x^{2}=4 p y$, where $p>0$. The igniter, which is the focus, is 1.7 inches above the vertex of the dish. Thus we have $p=1.7$.
$x^{2}=4 p y \quad$ Standard form of upward-facing parabola with vertex $(0,0)$
$x^{2}=4(1.7) y \quad$ Substitute 1.7 for $p$.
$x^{2}=6.8 y \quad$ Multiply.
The dish extends $\frac{4.5}{2}=2.25$ inches on either side of the origin. We can substitute 2.25 for $x$ in the equation from part (a) to find the depth of the dish.
$x^{2}=6.8 y \quad$ Equation found in part (a).
$(2.25)^{2}=6.8 y \quad$ Substitute 2.25 for $x$.
$y \approx 0.74 \quad$ Solve for $y$.
The dish is about 0.74 inches deep.

## TRY IT

Balcony-sized solar cookers have been designed for families living in India. The top of a dish has a diameter of 1600 mm . The sun's rays reflect off the parabolic mirror toward the "cooker," which is placed 320 mm from the base.
a. Find an equation that models a cross-section of the solar cooker. Assume that the vertex of the parabolic mirror is the origin of the coordinate plane, and that the parabola opens to the right (i.e., has
the $x$-axis as its axis of symmetry).
b. Use the equation found in part (a) to find the depth of the cooker.

## Answer

a. $y^{2}=1280 x$
b. The depth of the cooker is 500 mm

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\section*{SUMMARY: THE PARABOLA}

\section*{Key Equations}
\begin{tabular}{|l|l|}
\hline Parabola, vertex at origin, axis of symmetry on \(x\)-axis & \(y^{2}=4 p x\) \\
\hline Parabola, vertex at origin, axis of symmetry on \(y\)-axis & \(x^{2}=4 p y\) \\
\hline Parabola, vertex at \((h, k)\), axis of symmetry on \(x\)-axis & \((y-k)^{2}=4 p(x-h)\) \\
\hline Parabola, vertex at \((h, k)\), axis of symmetry on \(y\)-axis & \((x-h)^{2}=4 p(y-k)\) \\
\hline
\end{tabular}

\section*{Key Concepts}
- A parabola is the set of all points \((x, y)\) in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix.
- The standard form of a parabola with vertex \((0,0)\) and the \(x\)-axis as its axis of symmetry can be used to graph the parabola. If \(p>0\), the parabola opens right. If \(p<0\), the parabola opens left.
- The standard form of a parabola with vertex \((0,0)\) and the \(y\)-axis as its axis of symmetry can be used to graph the parabola. If \(p>0\), the parabola opens up. If \(p<0\), the parabola opens down.
- When given the focus and directrix of a parabola, we can write its equation in standard form.
- The standard form of a parabola with vertex \((h, k)\) and axis of symmetry parallel to the \(x\)-axis can be used to graph the parabola. If \(p>0\), the parabola opens right. If \(p<0\), the parabola opens left.
- The standard form of a parabola with vertex \((h, k)\) and axis of symmetry parallel to the \(y\)-axis can be used to graph the parabola. If \(p>0\), the parabola opens up. If \(p<0\), the parabola opens down.
- Real-world situations can be modeled using the standard equations of parabolas. For instance, given the diameter and focus of a cross-section of a parabolic reflector, we can find an equation that models its sides.

\section*{Glossary}
directrix a line perpendicular to the axis of symmetry of a parabola; a line such that the ratio of the distance between the points on the conic and the focus to the distance to the directrix is constant
focus (of a parabola) a fixed point in the interior of a parabola that lies on the axis of symmetry
focal diameter (latus rectum) the line segment that passes through the focus of a parabola parallel to the directrix, with endpoints on the parabola
parabola the set of all points \((x, y)\) in a plane that are the same distance from a fixed line, called the directrix, and a fixed point (the focus) not on the directrix

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\section*{PUTTING IT TOGETHER: CONIC SECTIONS}

At the beginning of this module, you were presented with the challenge of creating a whispering gallery. Now that you know about conics, you can do just this. It turns out that a whispering gallery forms under an elliptical arch.


You can create the gallery by placing the whispering dishes at the foci of the ellipse. Let's place the center of the arch above the origin. The equation for an ellipse centered at the origin is
\(\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}=1\)
The width of the arch is 100 feet, so \(a=50\). The height of the arch is 40 feet, so \(b=40\). Now you can rewrite the ellipse equation.
\[
\frac{x^{2}}{2500}+\frac{y^{2}}{1600}=1
\]

To find the distance, \(c\), from the origin to the foci of the ellipse, use
\[
\begin{gathered}
c^{2}=a^{2}-b^{2} \\
c^{2}=50^{2}-40^{2} \\
c^{2}=900 \\
c=30
\end{gathered}
\]

So you need to place the whispering dishes 30 feet from the center of the room.

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What if the height of the arch were lowered to 30 feet? Could you still make it a whispering gallery?
If this happens, the width of the arch is still 100 feet, so \(a=50\). But the height of the arch is 30 feet, so \(b=30\). Now you can rewrite the ellipse equation.
\[
\frac{x^{2}}{2500}+\frac{y^{2}}{900}=1
\]

To find the distance, c , from the origin to the foci of the ellipse, use
\[
\begin{gathered}
c^{2}=a^{2}-b^{2} \\
c^{2}=50^{2}-30^{2} \\
c^{2}=1600 \\
c=40
\end{gathered}
\]

You'll have to move the dishes out another 10 feet on each side so that they are 40 feet from the center. Thanks to your knowledge of conics, you can use the dimensions of the arch to place the dishes where they belong to create a whispering gallery. Now, just be careful not to share any secrets you don't want anyone to hear.
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\title{
MODULE 16: SEQUENCES AND SERIES
}

\section*{WHY IT MATTERS: SEQUENCES AND SERIES}

\section*{Why Study Sequences and Series?}

A sequence is simply a list of numbers, and a series is the sum of a list of numbers. So any time you have data arranged in a list, you may require methods from sequences and series to analyze the data.

For example, suppose you take out a small student loan for \(\$ 10,000\). When the loan is in repayment, it will accrue \(6 \%\) annual interest, compounded each month. Suppose that the monthly payments were fixed at \(\$ 300\) by the lender. How long will it take to pay off the loan? How much interest will you end up paying?

To answer these questions quickly, we would have to know about certain loan formulas. However we can still answer the questions by finding out exactly what is still left to pay after each month. That is, we will examine the sequence of balances.

Let's work out the first month carefully. First of all, the bank will tack on interest. Each month, the bank charges \(0.5 \%\) interest \((6 \% \div 12)\), or as a decimal: 0.005 . et \(I_{n}\) stand for the interest payment in month \(n\).
\(I_{1}=\$ 10,000 \times 0.005=\$ 50\)
Therefore out of the first \(\$ 300\), only \(\$ 250\) will go towards paying off the loan. Let \(A_{n}\) be the balance after month \(n\). By convention, we write \(A_{0}=10,000\) for the initial amount of the loan (let's not worry about the dollar signs going forward). So after the first month, the new balance would be:
\(A_{1}=10,000-250=9,750\)
To compute the balance after the second month, we repeat the same steps. The new interest payment is:
\(I_{2}=9750 \times 0.005=48.75\)
That would leave \(300-48.75=251.25\) for paying off the loan. Altogether then, the new balance will be:
\(A_{2}=9750-251.25=9498.75\)
But now in order to automate the process, let's write everything in terms of \(I_{n}\) and \(A_{n}\). First, the interest payment is always equal to the previous balance times 0.005 .
\(I_{n}=A_{n-1} \times 0.005\)
Then the new payment is \(300-I_{n}\). This leads to the formula for the new balance (which we can simplify algebraically):
\[
\begin{aligned}
A_{n} & =A_{n-1}-\left(300-I_{n}\right) \\
& =A_{n-1}-300+I_{n} \\
& =A_{n-1}-300+A_{n-1} \times(0.005) \\
& =(1.005) A_{n-1}-300
\end{aligned}
\]

Although it seems complicated at first, the formula for \(A_{n}\), which is called a recursive definition for the sequence, helps us to compute all of the balances very quickly. The reason is that each balance calculation is found by a simple formula from the previous. In fact, spreadsheets are very good at this kind of work. Here is a table of the first six months of balances and interest payments.
\begin{tabular}{|l|l|l|}
\hline Month & Balance & Interest Paid \\
\hline 0 & \(\$ 10,000\) & - \\
\hline 1 & \(\$ 9750\) & \(\$ 50\) \\
\hline 2 & \(\$ 9498.75\) & \(\$ 48.75\) \\
\hline 3 & \(\$ 9246.24\) & \(\$ 47.49\) \\
\hline 4 & \(\$ 8992.47\) & \(\$ 46.23\) \\
\hline 5 & \(\$ 8737.43\) & \(\$ 44.96\) \\
\hline 6 & \(\$ 8481.12\) & \(\$ 43.69\) \\
\hline
\end{tabular}

Try filling out more of the table. You may find that after about 3 years, the student loan will be paid off.
How much interest did you pay over the life of the loan? In this module you will also learn about sums of sequences, or series. Just add the terms of the series \(I_{1}+I_{2}+I_{3}+\cdots\) in order to figure out how much total interest was paid. That \(6 \%\) interest really adds up, doesn't it?

\section*{LEARNING OUTCOMES}

\section*{Sequences and Their Notations}
- Write the terms of a sequence defined by an explicit formula.
- Write the terms of a sequence defined by a recursive formula.
- Use factorial notation.

\section*{Arithmetic Sequences}
- Find the common difference for an arithmetic sequence.
- Write terms of an arithmetic sequence.
- Use a recursive formula for an arithmetic sequence.
- Use an explicit formula for an arithmetic sequence.

\section*{Geometric Sequences}
- Find the common ratio for a geometric sequence.
- List the terms of a geometric sequence.
- Use a recursive formula for a geometric sequence.
- Use an explicit formula for a geometric sequence.

\section*{Series and Their Notations}
- Use summation notation.
- Use the formula for the sum of the first \(n\) terms of an arithmetic series.
- Use the formula for the sum of the first \(n\) terms of a geometric series.
- Use the formula for the sum of an infinite geometric series.
- Solve annuity problems.

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\section*{REVIEW TOPICS FOR SUCCESS}

\section*{Learning Outcomes}

By the end of this section, you will be able to:
- Recall the rules for simplifying expressions containing exponents
- Recall the slope-intercept form of a linear function
- Recall the processes for operations on fractions by hand

In this module, you'll be introduced to a new operator and new notation, formulas, and processes. Of course, since topics in mathematics build upon what you've learned before, it will be helpful to brush up on a few concepts before beginning.

Recall the rules for simplifying expressions containing exponents. They appeared initially in Module 1 : Algebra Essentials and you encountered them again as part of the Review for Success in Module 11. Feel free to return for a refresher as needed.

You'll also need a good understanding of the slope-intercept form of a linear equation since sequences can be visualized as discrete inputs and outputs on the real plane. These appeared in Module 3: The Rectangular Coordinate System and Equations of Lines.

Finally, skill with operations on fractions by hand will help the upcoming module be less challenging. You saw these in the Review Topics for Success in Module 1: Algebra Essentials and they have reappeared several times again throughout the course.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

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\title{
REFRESHER TOPICS: EXPONENTS, SLOPEINTERCEPT FORM, OPERATIONS ON FRACTIONS
}

\section*{Learning Outcome}
- Recall and refresh: Simplifying expressions containing exponents
- Recall and refresh: Slope-intercept form of a linear function
- Recall and refresh: Operations on fractions by hand

\section*{Exponent Rules}

Product Rule \(a^{m} \cdot a^{n}=a^{m+n}\)
Quotient Rule \(\frac{a^{m}}{a^{n}}=a^{m-n}\)
Power Rule \(\left(a^{m}\right)^{n}=a^{m \cdot n}\)
Zero Exponent \(a^{0}=1\)
Negative Exponent \(a^{-n}=\frac{1}{a^{n}}\) and \(a^{n}=\frac{1}{a^{-n}}\)
Power of a Product \((a b)^{n}=a^{n} b^{n}\)
Power of a Quotient \(\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}\)

\section*{Slope-Intercept Form}

Perhaps the most familiar form of a linear equation is slope-intercept form written as \(y=m x+b\), where \(m=\) slope and \(b=y\)-intercept.

\section*{Fractions}

When simplifying algebraic expressions, we may sometimes need to add, subtract, simplify, multiply, or divide fractions. It is important to be able to do these operations on the fractions without converting them to decimals.
- To multiply fractions, multiply the numerators and place them over the product of the denominators.
- \(\frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}\)
- To divide fractions, multiply the first by the reciprocal of the second.
- \(\frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}=\frac{a d}{b c}\)
- To simplify fractions, find common factors in the numerator and denominator that cancel.
- \(\frac{24}{32}=\frac{2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}=\frac{3}{2 \cdot 2}=\frac{3}{4}\)

To add or subtract fractions, first rewrite each fraction as an equivalent fraction such that each has a common denominator, then add or subtract the numerators and place the result over the common denominator.
- \(\frac{a}{b} \pm \frac{c}{d}=\frac{a d \pm b c}{b d}\)

\section*{SUMMARY: REVIEW}

Key Equations and formulas
- Negative Exponent \(a^{-n}=\frac{1}{a^{n}}\) and \(a^{n}=\frac{1}{a^{-n}}\)
- \(y=m x+b\), where \(m=\) slope and \(b=y\)-intercept.
- \(\frac{a}{b} \cdot \frac{c}{d}=\frac{a c}{b d}\)
- \(\frac{a}{b} \div \frac{c}{d}=\frac{a}{b} \cdot \frac{d}{c}=\frac{a d}{b c}\)
- \(\frac{24}{32}=\frac{2 \cdot 2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}=\frac{3}{2 \cdot 2}=\frac{3}{4}\)
- \(\frac{a}{b} \pm \frac{c}{d}=\frac{a d \pm b c}{b d}\)

\section*{INTRODUCTION TO SEQUENCES AND THEIR NOTATIONS}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Write the terms of a sequence defined by an explicit formula.
- Write the terms of a sequence defined by a recursive formula.
- Use factorial notation.

A video game company launches an exciting new advertising campaign. They predict the number of online visits to their website, or hits, will double each day. The model they are using shows 2 hits the first day, 4 hits the second day, 8 hits the third day, and so on.
\begin{tabular}{|l|l|l|l|l|l|l}
\hline Day & 1 & 2 & 3 & 4 & 5 & \(\ldots\) \\
\hline Hits & 2 & 4 & 8 & 16 & 32 & \(\ldots\) \\
\hline
\end{tabular}

If their model continues, how many hits will there be at the end of the month? To answer this question, we'll first need to know how to determine a list of numbers written in a specific order. In this section we will explore these kinds of ordered lists.
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\section*{SEQUENCES DEFINED BY AN EXPLICIT FORMULA}

\section*{LEARNING OUTCOMES}
- Write the terms of a sequence defined by an explicit formula
- Write the formula for the nth term of a sequence
- Find the terms of an alternating sequence

\section*{TIP FOR SUCCESS}

We'll learn quite a lot of new terminology in this module that will be frequently used in context. Making a glossary for yourself and using flashcards to memorize the terms will help reduce the challenge in this module. Work on memorizing the terms and using them in context each day so that they will be familiar to you on quizzes and tests.

One way to describe an ordered list of numbers is as a sequence. A sequence is a function whose domain is a subset of the counting numbers. The sequence established by the number of hits on the website is
\[
\{2,4,8,16,32, \ldots\}
\]

The ellipsis (...) indicates that the sequence continues indefinitely. Each number in the sequence is called a term. The first five terms of this sequence are \(2,4,8,16\), and 32 .

Listing all of the terms for a sequence can be cumbersome. For example, finding the number of hits on the website at the end of the month would require listing out as many as 31 terms. A more efficient way to determine a specific term is by writing a formula to define the sequence.

One type of formula is an explicit formula, which defines the terms of a sequence using their position in the sequence. Explicit formulas are helpful if we want to find a specific term of a sequence without finding all of the previous terms. We can use the formula to find the \(n\)th term of the sequence, where \(n\) is any positive number. In our example, each number in the sequence is double the previous number, so we can use powers of 2 to write a formula for the \(n\)th term.


The first term of the sequence is \(2^{1}=2\), the second term is \(2^{2}=4\), the third term is \(2^{3}=8\), and so on. The \(n\)th term of the sequence can be found by raising 2 to the \(n\)th power. An explicit formula for a sequence is named by a lower case letter \(a, b, c\)... with the subscript \(n\). The explicit formula for this sequence is
\(a_{n}=2^{n}\)
Now that we have a formula for the \(n\)th term of the sequence, we can answer the question posed at the beginning of this section. We were asked to find the number of hits at the end of the month, which we will take to be 31 days. To find the number of hits on the last day of the month, we need to find the \(31^{\text {st }}\) term of the sequence. We will substitute 31 for \(n\) in the formula.
\[
\begin{aligned}
a_{31} & =2^{31} \\
& =2,147,483,648
\end{aligned}
\]

If the doubling trend continues, the company will get \(2,147,483,648\) hits on the last day of the month. That is over 2.1 billion hits! The huge number is probably a little unrealistic because it does not take consumer interest and competition into account. It does, however, give the company a starting point from which to consider business decisions.

Another way to represent the sequence is by using a table. The first five terms of the sequence and the \(n\)th term of the sequence are shown in the table.
\begin{tabular}{|l|l|l|l|l|l|l|}
\hline\(n\) & 1 & 2 & 3 & 4 & 5 & \(n\) \\
\hline\(n\)th term of the sequence, \(a_{n}\) & 2 & 4 & 8 & 16 & 32 & \(2^{n}\) \\
\hline
\end{tabular}

Graphing provides a visual representation of the sequence as a set of distinct points. We can see from the graph below that the number of hits is rising at an exponential rate. This particular sequence forms an exponential function.


Lastly, we can write this particular sequence as
\(\left\{2,4,8,16,32, \ldots, 2^{n}, \ldots\right\}\).
A sequence that continues indefinitely is called an infinite sequence. The domain of an infinite sequence is the set of counting numbers. If we consider only the first 10 terms of the sequence, we could write
\[
\{2,4,8,16,32, \ldots, 1024\}
\]

This sequence is called a finite sequence because it does not continue indefinitely.

\section*{A GENERAL NOTE: SEQUENCE}

A sequence is a function whose domain is the set of positive integers. A finite sequence is a sequence whose domain consists of only the first \(n\) positive integers. The numbers in a sequence are called terms. The variable \(a\) with a number subscript is used to represent the terms in a sequence and to indicate the position of the term in the sequence.
\(a_{1}, a_{2}, a_{3}, \ldots, a_{n}, \ldots\)
We call \(a_{1}\) the first term of the sequence, \(a_{2}\) the second term of the sequence, \(a_{3}\) the third term of the sequence, and so on. The term \(a_{n}\) is called the \(n\)th term of the sequence, or the general term of the sequence. An explicit formula defines the \(n\)th term of a sequence using the position of the term. A sequence that continues indefinitely is an infinite sequence.

\section*{DOES A SEQUENCE ALWAYS HAVE TO BEGIN WITH \(a_{1}\) ?}

No. In certain problems, it may be useful to define the initial term as \(a_{0}\) instead of \(a_{1}\). In these problems, the domain of the function includes 0 .

\section*{HOW TO: GIVEN AN EXPLICIT FORMULA, WRITE THE FIRST \(n\) TERMS OF A SEQUENCE.}
1. Substitute each value of \(n\) into the formula. Begin with \(n=1\) to find the first term, \(a_{1}\).
2. To find the second term, \(a_{2}\), use \(n=2\).
3. Continue in the same manner until you have identified all \(n\) terms.

\section*{Investigating Explicit Formulas}

We've learned that sequences are functions whose domain is over the positive integers. This is true for other types of functions, including some piecewise functions. Recall that a piecewise function is a function defined by multiple subsections. A different formula might represent each individual subsection.

\section*{Finding an Explicit Formula}

Thus far, we have been given the explicit formula and asked to find a number of terms of the sequence. Sometimes, the explicit formula for the \(n\)th term of a sequence is not given. Instead, we are given several terms from the sequence. When this happens, we can work in reverse to find an explicit formula from the first few terms of a sequence. The key to finding an explicit formula is to look for a pattern in the terms. Keep in mind that the pattern may involve alternating terms, formulas for numerators, formulas for denominators, exponents, or bases.

\section*{EXAMPLE: WRITING THE TERMS OF A SEQUENCE DEFINED BY AN EXPLICIT FORMULA}

Write the first five terms of the sequence defined by the explicit formula \(a_{n}=-3 n+8\).
Answer
Substitute \(n=1\) into the formula. Repeat with values 2 through 5 for \(n\).
\(n=1 \quad a_{1}=-3(1)+8=5\)
\(n=2 \quad a_{2}=-3(2)+8=2\)
\(n=3 \quad a_{3}=-3(3)+8=-1\)
\(n=4 \quad a_{4}=-3(4)+8=-4\)
\(n=5 \quad a_{5}=-3(5)+8=-7\)
The first five terms are \(\{5,2,-1,-4,-7\}\).

\section*{Analysis of the Solution}

The sequence values can be listed in a table. A table is a convenient way to input the function into a graphing utility.
\begin{tabular}{|l|l|l|l|l|l|}
\hline\(n\) & 1 & 2 & 3 & 4 & 5 \\
\hline\(a_{n}\) & 5 & 2 & -1 & -4 & -7 \\
\hline
\end{tabular}

A graph can be made from this table of values. From the graph in Figure 2, we can see that this sequence represents a linear function, but notice the graph is not continuous because the domain is over the positive integers only.


\section*{TRY IT}

Write the first five terms of the sequence defined by the explicit formula \(t_{n}=5 n-4\).
Answer
The first five terms are \(\{1,6,11,16,21\}\)
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\section*{RECALL PIECEWISE FUNCTIONS}

You saw piecewise-defined functions in Module 5: Function Basics. Feel free to return there for a refresher at anytime.

\section*{HOW TO: GIVEN AN EXPLICIT FORMULA FOR A PIECEWISE FUNCTION, WRITE THE FIRST \(n\) TERMS OF A SEQUENCE}
1. Identify the formula to which \(n=1\) applies.
2. To find the first term, \(a_{1}\), use \(n=1\) in the appropriate formula.
3. Identify the formula to which \(n=2\) applies.
4. To find the second term, \(a_{2}\), use \(n=2\) in the appropriate formula.
5. Continue in the same manner until you have identified all \(n\) terms.

Sometimes sequences have terms that alternate in sign. The steps to finding terms of the sequence are the same as if the signs did not alternate. Let's take a look at the following sequence.
\[
\{2,-4,6,-8\}
\]

Notice the first term is greater than the second term, the second term is less than the third term, and the third term is greater than the fourth term. This trend continues forever. Do not rearrange the terms in numerical order to interpret the sequence.

\section*{EXAMPLE: WRITING THE TERMS OF A SEQUENCE DEFINED BY A PIECEWISE EXPLICIT FORMULA}

Write the first six terms of the sequence.
\(a_{n}= \begin{cases}n^{2} & \text { if } n \text { is not divisible by } 3 \\ \frac{n}{3} & \text { if } n \text { is divisible by } 3\end{cases}\)

\section*{Answer}

Substitute \(n=1, n=2\), and so on in the appropriate formula. Use \(n^{2}\) when \(n\) is not a multiple of 3 . Use \(\frac{n}{3}\) when \(n\) is a multiple of 3 .
\(a_{1}=1^{2}=1 \quad 1\) is not a multiple of 3 . Use \(n^{2}\).
\(a_{2}=2^{2}=4 \quad 2\) is not a multiple of 3 . Use \(n^{2}\).
\(a_{3}=\frac{3}{3}=1 \quad 3\) is a multiple of 3 . Use \(\frac{n}{3}\).
\(a_{4}=4^{2}=16 \quad 4\) is not a multiple of 3 . Use \(n^{2}\).
\(a_{5}=5^{2}=25 \quad 5\) is not a multiple of 3 . Use \(n^{2}\).
\(a_{6}=\frac{6}{3}=2 \quad 6\) is a multiple of 3 . Use \(\frac{n}{3}\).
The first six terms are \(\{1,4,1,16,25,2\}\).

Analysis of the Solution
Every third point on the graph shown in Figure 5 stands out from the two nearby points. This occurs because the sequence was defined by a piecewise function.


\section*{TRY IT}

Write the first six terms of the sequence.
\(a_{n}= \begin{cases}2 n^{3} & \text { if } n \text { is odd } \\ \frac{5 n}{2} & \text { if } n \text { is even }\end{cases}\)

\section*{Answer}

The first six terms are \(\{2,5,54,10,250,15\}\).

\section*{TIP FOR SUCCESS}

Finding an explicit formula for a sequence involves creativity and diligence. Sometimes it can be difficult to find! Do work through all the examples to gain experience with as many patterns as possible.

\section*{HOW TO: GIVEN THE FIRST FEW TERMS OF A SEQUENCE, FIND AN EXPLICIT FORMULA FOR THE SEQUENCE.}
1. Look for a pattern among the terms.
2. If the terms are fractions, look for a separate pattern among the numerators and denominators.
3. Look for a pattern among the signs of the terms.
4. Write a formula for \(a_{n}\) in terms of \(n\). Test your formula for \(n=1, n=2\), and \(n=3\).

\section*{TIP FOR SUCCESS}

Try to find the patterns in the example below before looking at the answers.

\section*{EXAMPLE: WRITING AN EXPLICIT FORMULA FOR THE nth term ofa sequence}

Write an explicit formula for the \(n\)th term of each sequence.
1. \(\left\{-\frac{2}{11}, \frac{3}{13},-\frac{4}{15}, \frac{5}{17},-\frac{6}{19}, \ldots\right\}\)
2. \(\left\{-\frac{2}{25},-\frac{2}{125},-\frac{2}{625},-\frac{2}{3,125},-\frac{2}{15,625}, \ldots\right\}\)
3. \(\left\{e^{4}, e^{5}, e^{6}, e^{7}, e^{8}, \ldots\right\}\)

\section*{Answer}

Look for the pattern in each sequence.
1. The terms alternate between positive and negative. We can use \((-1)^{n}\) to make the terms alternate. The numerator can be represented by \(n+1\). The denominator can be represented by \(2 n+9\).
\[
a_{n}=\frac{(-1)^{n}(n+1)}{2 n+9}
\]
2. The terms are all negative.
\(\left\{-\frac{2}{25},-\frac{2}{125},-\frac{2}{625},-\frac{2}{3,125},-\frac{2}{15,625}, \ldots\right\} \quad\) The numerator is 2.
\(\left\{-\frac{2}{5^{2}},-\frac{2}{5^{3}},-\frac{2}{5^{4}},-\frac{2}{5^{5}},-\frac{2}{5^{6}}, \ldots,-\frac{2}{5^{n}}, \ldots\right\} \quad\) The denominators are increasing powers of 5.
So we know that the fraction is negative, the numerator is 2 , and the denominator can be represented by \(5^{n+1}\).
\[
a_{n}=-\frac{2}{5^{n+1}}
\]
3. The terms are powers of \(e\). For \(n=1\), the first term is \(e^{4}\) so the exponent must be \(n+3\).
\[
a_{n}=e^{n+3}
\]

\section*{TRY IT}

Write an explicit formula for the \(n\)th term of the sequence.
\(\{9 ;-81,729 ;-6,561 ; 59,049\}\)
Answer
\(a_{n}=(-1)^{n+1} 9^{n}\)
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\section*{TRY IT}

Write an explicit formula for the \(n\)th term of the sequence.
\[
\left\{-\frac{3}{4},-\frac{9}{8},-\frac{27}{12},-\frac{81}{16},-\frac{243}{20}, \ldots\right\}
\]

Answer
\(a_{n}=-\frac{3^{n}}{4 n}\)
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\section*{TRY IT}

Write an explicit formula for the \(n\)th term of the sequence.
\[
\left\{\frac{1}{e^{2}}, \frac{1}{e}, 1, e, e^{2}, \ldots\right\}
\]

Answer
\(a_{n}=e^{n-3}\)

\section*{HOW TO: GIVEN AN EXPLICIT FORMULA WITH ALTERNATING TERMS, WRITE THE FIRST \(n\) TERMS OF A SEQUENCE.}
1. Substitute each value of \(n\) into the formula. Begin with \(n=1\) to find the first term, \(a_{1}\). The sign of the term in the explicit formula is given by the \((-1)^{n}\) if the first term is negative and \((-1)^{n-1}\) if the first term is positive.
2. To find the second term, \(a_{2}\), use \(n=2\).
3. Continue in the same manner until you have identified all \(n\) terms.

\section*{EXAMPLE: WRITING THE TERMS OF AN ALTERNATING SEQUENCE DEFINED BY AN EXPLICIT FORMULA}

Write the first five terms of the sequence.
\[
a_{n}=\frac{(-1)^{n} n^{2}}{n+1}
\]

Answer
Substitute \(n=1, n=2\), and so on in the formula.
\[
\begin{array}{ll}
n=1 & a_{1}=\frac{(-1)^{1} 2^{2}}{1+1}=-\frac{1}{2} \\
n=2 & a_{2}=\frac{(-1)^{2} 2^{2}}{2+1}=\frac{4}{3} \\
n=3 & a_{3}=\frac{(-1)^{3} 3^{2}}{3+1}=-\frac{9}{4} \\
n=4 & a_{4}=\frac{(-1)^{4} 4^{2}}{4+1}=\frac{16}{5} \\
n=5 & a_{5}=\frac{(-1)^{5} 5^{2}}{5+1}=-\frac{25}{6}
\end{array}
\]

The first five terms are \(\left\{-\frac{1}{2}, \frac{4}{3},-\frac{9}{4}, \frac{16}{5},-\frac{25}{6}\right\}\).

\section*{Analysis of the Solution}

The graph of this function looks different from the ones we have seen previously in this section because the terms of the sequence alternate between positive and negative values.


In Example: Writing the Terms of an Alternating Sequence Defined by an Explicit Formula, does the \((-1)\) to the power of \(n\) account for the oscillations of signs?

Yes, the power might be \(n, n+1, n-1\), and so on, but any odd powers will result in a negative term, and any even power will result in a positive term.

\section*{TRY IT}

Write the first five terms of the sequence:
\[
a_{n}=\frac{4 n}{(-2)^{n}}
\]

Answer
The first five terms are \(\left\{-2,2,-\frac{3}{2}, 1,-\frac{5}{8}\right\}\).
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\section*{SEQUENCES DEFINED BY A RECURSIVE FORMULA}

\section*{LEARNING OUTCOMES}
- Write terms of a sequence defined by a recursive formula
- Write terms of a sequence using factorial notation

Sequences occur naturally in the growth patterns of nautilus shells, pinecones, tree branches, and many other natural structures. We may see the sequence in the leaf or branch arrangement, the number of petals of a flower, or the pattern of the chambers in a nautilus shell. Their growth follows the Fibonacci sequence, a famous sequence in which each term can be found by adding the preceding two terms. The numbers in the
sequence are \(1,1,2,3,5,8,13,21,34, \ldots\). Other examples from the natural world that exhibit the Fibonacci sequence are the Calla Lily, which has just one petal, the Black-Eyed Susan with 13 petals, and different varieties of daisies that may have 21 or 34 petals.

Each term of the Fibonacci sequence depends on the terms that come before it. The Fibonacci sequence cannot easily be written using an explicit formula. Instead, we describe the sequence using a recursive formula, a formula that defines the terms of a sequence using previous terms.

A recursive formula always has two parts: the value of an initial term (or terms), and an equation defining \(a_{n}\) in terms of preceding terms. For example, suppose we know the following:
\(a_{1}=3\)
\(a_{n}=2 a_{n-1}-1\), for \(n \geq 2\)
We can find the subsequent terms of the sequence using the first term.
\(a_{1}=3\)
\(a_{2}=2 a_{1}-1=2(3)-1=5\)
\(a_{3}=2 a_{2}-1=2(5)-1=9\)
\(a_{4}=2 a_{3}-1=2(9)-1=17\)
So the first four terms of the sequence are \(\{3,5,9,17\}\).
The recursive formula for the Fibonacci sequence states the first two terms and defines each successive term as the sum of the preceding two terms.
\(a_{1}=1\)
\(a_{2}=1\)
\(a_{n}=a_{n-1}+a_{n-2}\), for \(n \geq 3\)
To find the tenth term of the sequence, for example, we would need to add the eighth and ninth terms. We saw above that the eighth and ninth terms are 21 and 34 , so
\(a_{10}=a_{9}+a_{8}=34+21=55\)

\section*{A GENERAL NOTE: RECURSIVE FORMULA}

A recursive formula is a formula that defines each term of a sequence using preceding term(s).
Recursive formulas must always state the initial term, or terms, of the sequence.

\section*{Q \& A}

Must the first two terms always be given in a recursive formula?

No. The Fibonacci sequence defines each term using the two preceding terms, but many recursive formulas define each term using only one preceding term. These sequences need only the first term to be defined.

\section*{HOW TO: GIVEN A RECURSIVE FORMULA WITH ONLY THE FIRST TERM PROVIDED, WRITE THE FIRST \(n\) TERMS OF A SEQUENCE.}
1. Identify the initial term, \(a_{1}\), which is given as part of the formula. This is the first term.
2. To find the second term, \(a_{2}\), substitute the initial term into the formula for \(a_{n-1}\). Solve.
3. To find the third term, \(a_{3}\), substitute the second term into the formula. Solve.
4. Repeat until you have solved for the \(n\)th term.

\section*{EXAMPLE: WRITING THE TERMS OF A SEQUENCE DEFINED BY A RECURSIVE FORMULA}

Write the first five terms of the sequence defined by the recursive formula.
\(a_{1}=9\)
\(a_{n}=3 a_{n-1}-20\), for \(n \geq 2\)

\section*{Answer}

The first term is given in the formula. For each subsequent term, we replace \(a_{n-1}\) with the value of the preceding term.
\(n=1 \quad a_{1}=9\)
\(n=2 \quad a_{2}=3 a_{1}-20=3(9)-20=27-20=7\)
\(n=3 \quad a_{3}=3 a_{2}-20=3(7)-20=21-20=1\)
\(n=4 \quad a_{4}=3 a_{3}-20=3(1)-20=3-20=-17\)
\(n=5 \quad a_{5}=3 a_{4}-20=3(-17)-20=-51-20=-71\)
The first five terms are \(\{9,7,1,-17,-71\}\)

\section*{TRY IT}

Write the first five terms of the sequence defined by the recursive formula.
\(a_{1}=2\)
\(a_{n}=2 a_{n-1}+1\), for \(n \geq 2\)

\section*{Answer}
\(\{2,5,11,23,47\}\)
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\section*{Using Factorial Notation}

\section*{A NEW OPERATOR: FACTORIAL}

We saw a new operator introduced earlier while studying functions, the composition operator, o, which indicated we should compose two functions.
Now we see another new operator, factorial, !. It doesn't mean we should get excited about its pronunciation, though! Factorial asks us to multiply together all the positive integers that come before the number in front of the symbol. See the examples below.
Factorial is used heavily in Combinatorics, a branch of mathematics that is concerned with methods for counting sets.

The formulas for some sequences include products of consecutive positive integers. \(n\) factorial, written as \(n!\), is the product of the positive integers from 1 to \(n\). For example,
\[
\begin{aligned}
& 4!=4 \cdot 3 \cdot 2 \cdot 1=24 \\
& 5!=5 \cdot 4 \cdot 3 \cdot 2 \cdot 1=120
\end{aligned}
\]

An example of formula containing a factorial is \(a_{n}=(n+1)\) !. The sixth term of the sequence can be found by substituting 6 for \(n\).
\[
a_{6}=(6+1)!=7!=7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1=5040
\]

The factorial of any whole number \(n\) is \(n(n-1)\) ! We can therefore also think of 5 ! as \(5 \cdot 4\) !.

\section*{A GENERAL NOTE: FACTORIAL}
\(\boldsymbol{n}\) factorial is a mathematical operation that can be defined using a recursive formula. The factorial of \(n\), denoted \(n\) !, is defined for a positive integer \(n\) as:
\(0!=1\)
\(1!=1\)
\(n!=n(n-1)(n-2) \cdots(2)(1)\), for \(n \geq 2\)
The special case \(0!\) is defined as \(0!=1\).

\section*{TRY IT}

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\section*{Q \& A}

\section*{CAN FACTORIALS ALWAYS BE FOUND USING A CALCULATOR?}

No. Factorials get large very quickly-faster than even exponential functions! When the output gets too large for the calculator, it will not be able to calculate the factorial.

\section*{EXAMPLE: WRITING THE TERMS OF A SEQUENCE USING FACTORIALS}

Write the first five terms of the sequence defined by the explicit formula \(a_{n}=\frac{5 n}{(n+2)!}\).
Answer
Substitute \(n=1, n=2\), and so on in the formula.
\[
\begin{array}{ll}
n=1 & a_{1}=\frac{5(1)}{(1+2)!}=\frac{5}{3!}=\frac{5}{3 \cdot 2 \cdot 1}=\frac{5}{6} \\
n=2 & a_{2}=\frac{5(2)}{(2+2)!}=\frac{10}{4!}=\frac{10}{4 \cdot 3 \cdot 2 \cdot 1}=\frac{5}{12} \\
n=3 & a_{3}=\frac{5(3)}{(3+2)!}=\frac{15}{5!}=\frac{15}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}=\frac{1}{8} \\
n=4 & a_{4}=\frac{5(4)}{(4+2)!}=\frac{20}{6!}=\frac{20}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}=\frac{1}{36} \\
n=5 & a_{5}=\frac{5(5)}{(5+2)!}=\frac{25}{7!}=\frac{25}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}=\frac{5}{1,008}
\end{array}
\]

The first five terms are \(\left\{\frac{5}{6}, \frac{5}{12}, \frac{1}{8}, \frac{1}{36}, \frac{5}{1,008}\right\}\).

\section*{Analysis of the Solution}

The figure below shows the graph of the sequence. Notice that, since factorials grow very quickly, the presence of the factorial term in the denominator results in the denominator becoming much larger than the numerator as \(n\) increases. This means the quotient gets smaller and, as the plot of the terms shows, the terms are decreasing and nearing zero.


\section*{TRY IT}

Write the first five terms of the sequence defined by the explicit formula \(a_{n}=\frac{(n+1)!}{2 n}\).

\section*{Answer}

The first five terms are \(\left\{1, \frac{3}{2}, 4,15,72\right\}\)
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\section*{SUMMARY: SEQUENCES AND THEIR NOTATIONS}

\section*{Key Equations}
\begin{tabular}{|l|l|}
\hline & \(0!=1\) \\
Formula for a factorial & \(1!=1\) \\
& \(n!=n(n-1)(n-2) \cdots(2)(1)\), for \(n \geq 2\)
\end{tabular}

\section*{Key Concepts}
- A sequence is a list of numbers, called terms, written in a specific order.
- Explicit formulas define each term of a sequence using the position of the term.
- An explicit formula for the \(n\)th term of a sequence can be written by analyzing the pattern of several terms.
- Recursive formulas define each term of a sequence using previous terms.
- Recursive formulas must state the initial term, or terms, of a sequence.
- A set of terms can be written by using a recursive formula.
- A factorial is a mathematical operation that can be defined recursively.
- The factorial of \(n\) is the product of all integers from 1 to \(n\)

\section*{Glossary}
explicit formula a formula that defines each term of a sequence in terms of its position in the sequence
finite sequence a function whose domain consists of a finite subset of the positive integers \(\{1,2, \ldots n\}\) for some positive integer \(n\)
infinite sequence a function whose domain is the set of positive integers
\(\mathbf{n}\) factorial the product of all the positive integers from 1 to \(n\)
nth term of a sequence a formula for the general term of a sequence
recursive formula a formula that defines each term of a sequence using previous term(s)

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\section*{INTRODUCTION TO ARITHMETIC SEQUENCES}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Find the common difference for an arithmetic sequence.
- Write terms of an arithmetic sequence.
- Use a recursive formula for an arithmetic sequence.
- Use an explicit formula for an arithmetic sequence.

Companies often make large purchases, such as computers and vehicles, for business use. The book-value of these supplies decreases each year for tax purposes. This decrease in value is called depreciation. One method of calculating depreciation is straight-line depreciation, in which the value of the asset decreases by the same amount each year.

As an example, consider a woman who starts a small contracting business. She purchases a new truck for \(\$ 25,000\). After five years she estimates that she will be able to sell the truck for \(\$ 8,000\). The loss in value of the truck will therefore be \(\$ 17,000\), which is \(\$ 3,400\) per year for five years. The truck will be worth \(\$ 21,600\) after the first year; \$18,200 after two years; \$14,800 after three years; \$11,400 after four years; and \$8,000 at the end of five years. In this section we will consider specific kinds of sequences that will allow us to calculate depreciation, such as the truck's value.
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\section*{TERMS OF AN ARITHMETIC SEQUENCE}

\section*{LEARNING OUTCOMES}
- Find the common difference for an arithmetic sequence
- Find terms of an arithmetic sequence

The values of the truck in the example on the previous page form an arithmetic sequence because they change by a constant amount each year. Each term increases or decreases by the same constant value called the common difference of the sequence. For this sequence the common difference is \(-3,400\).


The sequence below is another example of an arithmetic sequence. In this case the constant difference is 3 . You can choose any term of the sequence, and add 3 to find the subsequent term.


\section*{A GENERAL NOTE: ARITHMETIC SEQUENCE}

An arithmetic sequence is a sequence that has the property that the difference between any two consecutive terms is a constant. This constant is called the common difference. If \(a_{1}\) is the first term of an arithmetic sequence and \(d\) is the common difference, the sequence will be:
\(\left\{a_{n}\right\}=\left\{a_{1}, a_{1}+d, a_{1}+2 d, a_{1}+3 d, \ldots\right\}\)

\section*{Writing Terms of Arithmetic Sequences}

Now that we can recognize an arithmetic sequence, we will find the terms if we are given the first term and the common difference. The terms can be found by beginning with the first term and adding the common difference repeatedly. In addition, any term can also be found by plugging in the values of \(n\) and \(d\) into formula below.
\(a_{n}=a_{1}+(n-1) d\)

\section*{EXAMPLE: FINDING COMMON DIFFERENCES}

Is each sequence arithmetic? If so, find the common difference.
1. \(\{1,2,4,8,16, \ldots\}\)
2. \(\{-3,1,5,9,13, \ldots\}\)

Answer
Subtract each term from the subsequent term to determine whether a common difference exists.
1. The sequence is not arithmetic because there is no common difference.
\[
2-1=1 \quad 4-2=2 \quad 8-4=4 \quad 16-8=8
\]
2. The sequence is arithmetic because there is a common difference. The common difference is 4 .
\[
1-(-3)=4 \quad 5-1=4 \quad 9-5=4 \quad 13-9=4
\]

\section*{Analysis of the Solution}

The graph of each of these sequences is shown in Figure 1. We can see from the graphs that, although both sequences show growth, \(a\) is not linear whereas \(b\) is linear. Arithmetic sequences have a constant rate of change so their graphs will always be points on a line.

(a)

(b)

\section*{Q \& A}

If we are told that a sequence is arithmetic, do we have to subtract every term from the following term to find the common difference?

No. If we know that the sequence is arithmetic, we can choose any one term in the sequence, and subtract it from the subsequent term to find the common difference.

\section*{TRY IT}

Is the given sequence arithmetic? If so, find the common difference.
\(\{18,16,14,12,10, \ldots\}\)

\section*{Answer}

The sequence is arithmetic. The common difference is -2 .

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\section*{TRY IT}

Is the given sequence arithmetic? If so, find the common difference.
\(\{1,3,6,10,15, \ldots\}\)
Answer
The sequence is not arithmetic because \(3-1 \neq 6-3\).

\section*{TIP FOR SUCCESS}

The formula given above is a handy tool for calculating any term of an arithmetic sequence. As you work to memorize it, do work out the terms in the examples below individually as well to build you intuition for why the formula works.

\section*{HOW TO: GIVEN THE FIRST TERM AND THE COMMON DIFFERENCE OF AN ARITHMETIC SEQUENCE, FIND THE FIRST SEVERAL TERMS.}
1. Add the common difference to the first term to find the second term.
2. Add the common difference to the second term to find the third term.
3. Continue until all of the desired terms are identified.
4. Write the terms separated by commas within brackets.

\section*{EXAMPLE: WRITING TERMS OF ARITHMETIC SEQUENCES}

Write the first five terms of the arithmetic sequence with \(a_{1}=17\) and \(d=-3\).
Answer
Adding -3 is the same as subtracting 3 . Beginning with the first term, subtract 3 from each term to find the next term.
The first five terms are \(\{17,14,11,8,5\}\)

Analysis of the Solution
As expected, the graph of the sequence consists of points on a line.


\section*{TRY IT}

List the first five terms of the arithmetic sequence with \(a_{1}=1\) and \(d=5\).
Answer
\(\{1,6,11,16,21\}\)
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\section*{HOW TO: GIVEN ANY THE FIRST TERM AND ANY OTHER TERM IN AN ARITHMETIC SEQUENCE, FIND A GIVEN TERM.}
1. Substitute the values given for \(a_{1}, a_{n}, n\) into the formula \(a_{n}=a_{1}+(n-1) d\) to solve for \(d\).
2. Find a given term by substituting the appropriate values for \(a_{1}, n\), and \(d\) into the formula \(a_{n}=a_{1}+(n-1) d\).

\section*{EXAMPLE: WRITING TERMS OF ARITHMETIC SEQUENCES}

Given \(a_{1}=8\) and \(a_{4}=14\), find \(a_{5}\).

\section*{Answer}

The sequence can be written in terms of the initial term 8 and the common difference \(d\).
\(\{8,8+d, 8+2 d, 8+3 d\}\)
We know the fourth term equals 14 ; we know the fourth term has the form \(a_{1}+3 d=8+3 d\).
We can find the common difference \(d\).
\(a_{n}=a_{1}+(n-1) d\)
\(a_{4}=a_{1}+3 d\)
\(a_{4}=8+3 d \quad\) Write the fourth term of the sequence in terms of \(a_{1}\) and \(d\).
\(14=8+3 d \quad\) Substitute 14 for \(a_{4}\).
\(d=2 \quad\) Solve for the common difference.
Find the fifth term by adding the common difference to the fourth term.
\(a_{5}=a_{4}+2=16\)

\section*{Analysis of the Solution}

Notice that the common difference is added to the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by adding the common difference to the first term nine times or by using the equation \(a_{n}=a_{1}+(n-1) d\).

\section*{TRY IT}

Given \(a_{3}=7\) and \(a_{5}=17\), find \(a_{2}\).
Answer
\(a_{2}=2\)
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\section*{FORMULAS FOR ARITHMETIC SEQUENCES}

\section*{LEARNING OUTCOMES}
- Write an explicit formula for an arithmetic sequence.
- Write a recursive formula for the arithmetic sequence.

\section*{Using Explicit Formulas for Arithmetic Sequences}

We can think of an arithmetic sequence as a function on the domain of the natural numbers; it is a linear function because it has a constant rate of change. The common difference is the constant rate of change, or the slope of the function. We can construct the linear function if we know the slope and the vertical intercept.
\(a_{n}=a_{1}+d(n-1)\)
To find the \(y\)-intercept of the function, we can subtract the common difference from the first term of the sequence. Consider the following sequence.


The common difference is -50 , so the sequence represents a linear function with a slope of -50 . To find the \(y\)-intercept, we subtract -50 from 200:200-(-50)=200+50=250. You can also find the \(y\) intercept by graphing the function and determining where a line that connects the points would intersect the vertical axis.


Recall the slope-intercept form of a line is \(y=m x+b\). When dealing with sequences, we use \(a_{n}\) in place of \(y\) and \(n\) in place of \(x\). If we know the slope and vertical intercept of the function, we can substitute them for \(m\) and \(b\) in the slope-intercept form of a line. Substituting -50 for the slope and 250 for the vertical intercept, we get the following equation:
\(a_{n}=-50 n+250\)

\section*{SEQUENCES AS LINEAR FUNCTIONS}

We've seen several graphs of sequence terms in this module so far. If you think of \(n\) representing the input of the function of an arithmetic sequence and \(a_{n}\) as the output of the function, it may help you to better visualize the arithmetic sequence as a linear function of the form \(y=m x+b\), or using sequence notation, \(a_{n}=d n+a_{0}\) where each point on the graph is of the form ( \(n, a_{n}\) ) and the common difference gives us the slope of the line.
Note that if we let \(n=0\) in the explicit form \(a_{n}=a_{1}+d(n-1)\), we obtain the statement \(a_{0}=a_{1}-d\). That statement tells us that the vertical intercept \(a_{0}\) can be found by subtracting the common difference from the first term.

We do not need to find the vertical intercept to write an explicit formula for an arithmetic sequence. Another explicit formula for this sequence is \(a_{n}=200-50(n-1)\), which simplifies to \(a_{n}=-50 n+250\).

Some arithmetic sequences are defined in terms of the previous term using a recursive formula. The

\section*{A GENERAL NOTE: EXPLICIT FORMULA FOR AN ARITHMETIC SEQUENCE}

An explicit formula for the \(n\)th term of an arithmetic sequence is given by \(a_{n}=a_{1}+d(n-1)\)

\section*{HOW TO: GIVEN THE FIRST SEVERAL TERMS FOR AN ARITHMETIC SEQUENCE, WRITE AN EXPLICIT FORMULA.}
1. Find the common difference, \(a_{2}-a_{1}\).
2. Substitute the common difference and the first term into \(a_{n}=a_{1}+d(n-1)\).

\section*{EXAMPLE: WRITING THE NTH TERM EXPLICIT FORMULA FOR AN ARITHMETIC SEQUENCE}

Write an explicit formula for the arithmetic sequence.
\(\{2,12,22,32,42, \ldots\}\)
Answer
The common difference can be found by subtracting the first term from the second term.
\[
\begin{aligned}
d & =a_{2}-a_{1} \\
& =12-2 \\
& =10
\end{aligned}
\]

The common difference is 10 . Substitute the common difference and the first term of the sequence into the formula and simplify.
\(a_{n}=2+10(n-1)\)
\(a_{n}=10 n-8\)

Analysis of the Solution
The graph of this sequence shows a slope of 10 and a vertical intercept of -8 .


Write an explicit formula for the following arithmetic sequence.
\(\{50,47,44,41, \ldots\}\)
Answer
\(a_{n}=53-3 n\)
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formula provides an algebraic rule for determining the terms of the sequence. A recursive formula allows us to find any term of an arithmetic sequence using a function of the preceding term. Each term is the sum of the previous term and the common difference. For example, if the common difference is 5 , then each term is the previous term plus 5 . As with any recursive formula, the first term must be given.
\(a_{n}=a_{n-1}+d \quad n \geq 2\)

\section*{A GENERAL NOTE: RECURSIVE FORMULA FOR AN ARITHMETIC SEQUENCE}

The recursive formula for an arithmetic sequence with common difference \(d\) is:
\[
a_{n}=a_{n-1}+d \quad n \geq 2
\]

\section*{HOW TO: GIVEN AN ARITHMETIC SEQUENCE, WRITE ITS RECURSIVE FORMULA.}
1. Subtract any term from the subsequent term to find the common difference.
2. State the initial term and substitute the common difference into the recursive formula for arithmetic sequences.

\section*{TIP FOR SUCCESS}

Either the explicit or the recursive form may be used to describe an arithmetic sequence in which the first term is known. Practice using both of them on the examples in this section.

\section*{Find the Number of Terms in an Arithmetic Sequence}

Explicit formulas can be used to determine the number of terms in a finite arithmetic sequence. We need to find the common difference, and then determine how many times the common difference must be added to the first term to obtain the final term of the sequence.

In the following video lesson, we present a recap of some of the concepts presented about arithmetic sequences up to this point.

\section*{Solving Application Problems with Arithmetic Sequences}

In many application problems, it often makes sense to use an initial term of \(a_{0}\) instead of \(a_{1}\). In these problems we alter the explicit formula slightly to account for the difference in initial terms. We use the following formula:
\(a_{n}=a_{0}+d n\)

\section*{EXAMPLE: WRITING A RECURSIVE FORMULA FOR AN ARITHMETIC SEQUENCE}

Write a recursive formula for the arithmetic sequence.
\(\{-18,-7,4,15,26, \ldots\}\)
Answer
The first term is given as -18 . The common difference can be found by subtracting the first term from the second term.
\(d=-7-(-18)=11\)
Substitute the initial term and the common difference into the recursive formula for arithmetic sequences.
\(a_{1}=-18\)
\(a_{n}=a_{n-1}+11\), for \(n \geq 2\)

\section*{Analysis of the Solution}

We see that the common difference is the slope of the line formed when we graph the terms of the sequence, as shown in Figure 3. The growth pattern of the sequence shows the constant difference of 11 units.


\section*{HOW TO: DO WE HAVE TO SUBTRACT THE FIRST TERM FROM THE SECOND TERM TO FIND THE COMMON DIFFERENCE?}

No. We can subtract any term in the sequence from the subsequent term. It is, however, most common to subtract the first term from the second term because it is often the easiest method of finding the common difference.

\section*{TRY IT}

Write a recursive formula for the arithmetic sequence.
\(\{25,37,49,61, \ldots\}\)
Answer
\(a_{1}=25\)
\(a_{n}=a_{n-1}+12\), for \(n \geq 2\)
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\section*{HOW TO: GIVEN THE FIRST THREE TERMS AND THE LAST TERM OF A FINITE ARITHMETIC SEQUENCE, FIND THE TOTAL NUMBER OF TERMS.}
1. Find the common difference \(d\).
2. Substitute the common difference and the first term into \(a_{n}=a_{1}+d(n-1)\).
3. Substitute the last term for \(a_{n}\) and solve for \(n\).

\section*{TIP FOR SUCCESS}

Recall that we only need one equation in one unknown to solve for it. Given the explicit form of an arithmetic sequence, \(a_{n}=a_{1}+d(n-1)\), if we can substitute known values in for all but one component, we can solve for the missing one.
In this case, we are given that a certain finite sequence is arithmetic and we know the first term \(a_{1}\) and the final term \(a_{n}\). We are able to calculate the common difference from any two consecutive terms, which we are given. Substituting these known values into the explicit formula allows us to solve for the unknown value for the number of terms \(n\) without having to generate them.

\section*{EXAMPLE: FINDING THE NUMBER OF TERMS IN A FINITE ARITHMETIC SEQUENCE}

Find the number of terms in the finite arithmetic sequence.
\(\{8,1,-6, \ldots,-41\}\)
Answer
The common difference can be found by subtracting the first term from the second term.
\(1-8=-7\)
The common difference is -7 . Substitute the common difference and the initial term of the sequence into the
\(n\)th term formula and simplify.
\(a_{n}=a_{1}+d(n-1)\)
\(a_{n}=8+-7(n-1)\)
\(a_{n}=15-7 n\)
Substitute -41 for \(a_{n}\) and solve for \(n\)
\(-41=15-7 n\)
\[
8=n
\]

There are eight terms in the sequence.

\section*{TRY IT}

Find the number of terms in the finite arithmetic sequence.
\(\{6,11,16, \ldots, 56\}\)

\section*{Answer}

There are 11 terms in the sequence.

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\section*{TIP FOR SUCCESS}

See the red box SEQUENCES AS LINEAR FUNCTIONS at the start of this section for a derivation and explanation of this formula.
Keep in mind as you work through the example and practice problems that an arithmetic sequence may be represented as a linear function with input \(n\), output \(a_{n}\), and common difference (or slope) of \(d\).

\section*{EXAMPLE: SOLVING APPLICATION PROBLEMS WITH ARITHMETIC SEQUENCES}

A five-year old child receives an allowance of \(\$ 1\) each week. His parents promise him an annual increase of \(\$ 2\) per week.
1. Write a formula for the child's weekly allowance in a given year.
2. What will the child's allowance be when he is 16 years old?

\section*{Answer}
1. The situation can be modeled by an arithmetic sequence with an initial term of 1 and a common difference of 2. Let \(A\) be the amount of the allowance and \(n\) be the number of years after age 5 . Using the altered explicit formula for an arithmetic sequence we get:
\(A_{n}=1+2 n\)
2. We can find the number of years since age 5 by subtracting.
\(16-5=11\)
We are looking for the child's allowance after 11 years. Substitute 11 into the formula to find the child's allowance at age 16.
\(A_{11}=1+2(11)=23\)
The child's allowance at age 16 will be \(\$ 23\) per week.

A woman decides to go for a 10-minute run every day this week and plans to increase the time of her daily run by 4 minutes each week. Write a formula for the time of her run after \(n\) weeks. How long will her daily run be 8 weeks from today?
Answer
The formula is \(T_{n}=10+4 n\), and it will take her 42 minutes.
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\section*{SUMMARY: ARITHMETIC SEQUENCES}

\section*{Key Equations}
\begin{tabular}{|l|l|}
\hline recursive formula for nth term of an arithmetic sequence & \(a_{n}=a_{n-1}+d\) for \(n \geq 2\) \\
\hline explicit formula for nth term of an arithmetic sequence & \(a_{n}=a_{1}+d(n-1)\) \\
\hline
\end{tabular}

\section*{Key Concepts}
- An arithmetic sequence is a sequence where the difference between any two consecutive terms is a constant.
- The constant between two consecutive terms is called the common difference.
- The common difference is the number added to any one term of an arithmetic sequence that generates the subsequent term.
- The terms of an arithmetic sequence can be found by beginning with the initial term and adding the common difference repeatedly.
- A recursive formula for an arithmetic sequence with common difference \(d\) is given by \(a_{n}=a_{n-1}+d, n \geq 2\).
- As with any recursive formula, the initial term of the sequence must be given.
- An explicit formula for an arithmetic sequence with common difference \(d\) is given by \(a_{n}=a_{1}+d(n-1)\)
- An explicit formula can be used to find the number of terms in a sequence.
- In application problems, we sometimes alter the explicit formula slightly to \(a_{n}=a_{0}+d n\).

\section*{Glossary}
arithmetic sequence a sequence in which the difference between any two consecutive terms is a constant common difference the difference between any two consecutive terms in an arithmetic sequence

\section*{INTRODUCTION TO GEOMETRIC SEQUENCES}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Find the common ratio for a geometric sequence.
- List the terms of a geometric sequence.
- Use a recursive formula for a geometric sequence.
- Use an explicit formula for a geometric sequence.

Many jobs offer an annual cost-of-living increase to keep salaries consistent with inflation. Suppose, for example, a recent college graduate finds a position as a sales manager earning an annual salary of \(\$ 26,000\). He is promised a \(2 \%\) cost of living increase each year. His annual salary in any given year can be found by multiplying his salary from the previous year by \(102 \%\). His salary will be \(\$ 26,520\) after one year; \(\$ 27,050.40\) after two years; \(\$ 27,591.41\) after three years; and so on. When a salary increases by a constant rate each year, the salary grows by a constant factor. In this section we will review sequences that grow in this way.

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\section*{TERMS OF GEOMETRIC SEQUENCES}

\section*{LEARNING OUTCOMES}
- Find the common ratio of a geometric sequence
- List terms of a geometric sequence given the first term and the common ratio

\section*{Finding Common Ratios}

The yearly salary values described form a geometric sequence because they change by a constant factor each year. Each term of a geometric sequence increases or decreases by a constant factor called the common ratio. The sequence below is an example of a geometric sequence because each term increases by a constant factor of 6 . Multiplying any term of the sequence by the common ratio 6 generates the subsequent term.


\section*{A GENERAL NOTE: DEFINITION OF A GEOMETRIC SEQUENCE}

A geometric sequence is one in which any term divided by the previous term is a constant. This constant is called the common ratio of the sequence. The common ratio can be found by dividing any term in the sequence by the previous term. If \(a_{1}\) is the initial term of a geometric sequence and \(r\) is the common ratio, the sequence will be
\(\left\{a_{1}, a_{1} r, a_{1} r^{2}, a_{1} r^{3}, \ldots\right\}\).

\section*{HOW TO: GIVEN A SET OF NUMBERS, DETERMINE IF THEY REPRESENT A GEOMETRIC SEQUENCE.}
1. Divide each term by the previous term.
2. Compare the quotients. If they are the same, a common ratio exists and the sequence is geometric.

\section*{Writing Terms of Geometric Sequences}

Now that we can identify a geometric sequence, we will learn how to find the terms of a geometric sequence if we are given the first term and the common ratio. The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly. For instance, if the first term of a geometric sequence is \(a_{1}=-2\) and the common ratio is \(r=4\), we can find subsequent terms by multiplying \(-2 \cdot 4\) to get -8 then multiplying the result \(-8 \cdot 4\) to get -32 and so on.
\[
\begin{aligned}
& a_{1}=-2 \\
& a_{2}=(-2 \cdot 4)=-8 \\
& a_{3}=(-8 \cdot 4)=-32 \\
& a_{4}=(-32 \cdot 4)=-128
\end{aligned}
\]

The first four terms are \(\{-2,-8,-32,-128\}\).

\section*{EXAMPLE: FINDING COMMON RATIOS}

Is the sequence geometric? If so, find the common ratio.
1. \(1,2,4,8,16, \ldots\)
2. \(48,12,4,2, \ldots\)

Answer
Divide each term by the previous term to determine whether a common ratio exists.
1. \(\frac{2}{1}=2 \quad \frac{4}{2}=2 \quad \frac{8}{4}=2 \quad \frac{16}{8}=2\)

The sequence is geometric because there is a common ratio. The common ratio is 2 .
2. \(\frac{12}{48}=\frac{1}{4} \quad \frac{4}{12}=\frac{1}{3} \quad \frac{2}{4}=\frac{1}{2}\)

The sequence is not geometric because there is not a common ratio.

\section*{Analysis of the Solution}

The graph of each sequence is shown in Figure 1. It seems from the graphs that both (a) and (b) appear have the form of the graph of an exponential function in this viewing window. However, we know that (a) is geometric and so this interpretation holds, but (b) is not.


\section*{Q \& A}

If you are told that a sequence is geometric, do you have to divide every term by the previous term to find the common ratio?

No. If you know that the sequence is geometric, you can choose any one term in the sequence and divide it by the previous term to find the common ratio.

\section*{TRY IT}

Is the sequence geometric? If so, find the common ratio.
\(5,10,15,20, \ldots\)
Answer

The sequence is not geometric because \(\frac{10}{5} \neq \frac{15}{10}\).
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\section*{TRY IT}

Is the sequence geometric? If so, find the common ratio.
\(100,20,4, \frac{4}{5}, \ldots\)
Answer
The sequence is geometric. The common ratio is \(\frac{1}{5}\).

\section*{HOW TO: GIVEN THE FIRST TERM AND THE COMMON FACTOR, FIND THE FIRST FOUR TERMS OF A GEOMETRIC SEQUENCE.}
1. Multiply the initial term, \(a_{1}\), by the common ratio to find the next term, \(a_{2}\).
2. Repeat the process, using \(a_{n}=a_{2}\) to find \(a_{3}\) and then \(a_{3}\) to find \(a_{4}\), until all four terms have been identified.
3 . Write the terms separated by commons within brackets.

\section*{TIP FOR SUCCESS}

You can also use the definition of a geometric sequence, given at the beginning of the section, to write out several terms.
\(\left\{a_{1}, a_{1} r, a_{1} r^{2}, a_{1} r^{3}, \ldots\right\}\)
Try it on the example and practice problems below.

\section*{EXAMPLE: WRITING THE TERMS OF A GEOMETRIC SEQUENCE}

List the first four terms of the geometric sequence with \(a_{1}=5\) and \(r=-2\).

\section*{Answer}

Multiply \(a_{1}\) by -2 to find \(a_{2}\). Repeat the process, using \(a_{2}\) to find \(a_{3}\), and so on.
\(a_{1}=5\)
\(a_{2}=-2 a_{1}=-10\)
\(a_{3}=-2 a_{2}=20\)
\(a_{4}=-2 a_{3}=-40\)
The first four terms are \(\{5,-10,20,-40\}\).

\section*{TRY IT}

List the first five terms of the geometric sequence with \(a_{1}=18\) and \(r=\frac{1}{3}\).

Answer
\(\left\{18,6,2, \frac{2}{3}, \frac{2}{9}\right\}\)
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\section*{EXPLICIT FORMULAS FOR GEOMETRIC SEQUENCES}

\section*{LEARNING OUTCOMES}
- Write a recursive formula given a sequence of numbers.
- Given two terms in a geometric sequence, find a third.
- Solve an application problem using a geometric sequence.

\section*{Using Recursive Formulas for Geometric Sequences}

A recursive formula allows us to find any term of a geometric sequence by using the previous term. Each term is the product of the common ratio and the previous term. For example, suppose the common ratio is 9 . Then each term is nine times the previous term. As with any recursive formula, the initial term must be given.

\section*{A GENERAL NOTE: RECURSIVE FORMULA FOR A GEOMETRIC SEQUENCE}

The recursive formula for a geometric sequence with common ratio \(r\) and first term \(a_{1}\) is \(a_{n}=r \cdot a_{n-1}, n \geq 2\)

\section*{HOW TO: GIVEN THE FIRST SEVERAL TERMS OF A GEOMETRIC SEQUENCE, WRITE ITS RECURSIVE FORMULA.}
1. State the initial term.
2. Find the common ratio by dividing any term by the preceding term.
3. Substitute the common ratio into the recursive formula for a geometric sequence.

\section*{EXAMPLE: USING RECURSIVE FORMULAS FOR GEOMETRIC SEQUENCES}

Write a recursive formula for the following geometric sequence.
\(\{6,9,13.5,20.25, \ldots\}\)

\section*{Answer}

The first term is given as 6 . The common ratio can be found by dividing the second term by the first term. \(r=\frac{9}{6}=1.5\)
Substitute the common ratio into the recursive formula for geometric sequences and define \(a_{1}\).
\(a_{n}=r \cdot a_{n-1}\)
\(a_{n}=1.5 \cdot a_{n-1}\) for \(n \geq 2\)
\(a_{1}=6\)

\section*{Analysis of the Solution}

The sequence of data points follows an exponential pattern. The common ratio is also the base of an exponential function.


\section*{THE COMMON RATIO AS THE BASE OF AN EXPONENTIAL FUNCTION}

Using the definition of the geometric sequence \(\left\{a_{1}, a_{1} r, a_{1} r^{2}, a_{1} r^{3}, \ldots\right\}\), we have \(a_{n}=a_{1}(r)^{n-1}\). Recall the form of an exponential function \(f(x)=a(b)^{x}\). In this case we have \(a_{n}=6(1.5)^{n-1}\).
\[
\mathrm{Q} \& \mathrm{~A}
\]

Do we have to divide the second term by the first term to find the common ratio?
No. We can divide any term in the sequence by the previous term. It is, however, most common to divide the second term by the first term because it is often the easiest method of finding the common ratio.

\section*{TRY IT}

Write a recursive formula for the following geometric sequence.
\[
\left\{2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots\right\}
\]

Answer
\(a_{1}=2\)
\(a_{n}=\frac{2}{3} \cdot a_{n-1}\) for \(n \geq 2\)

\section*{Using Explicit Formulas for Geometric Sequences}

Because a geometric sequence is an exponential function whose domain is the set of positive integers, and the common ratio is the base of the function, we can write explicit formulas that allow us to find particular terms.
\(a_{n}=a_{1} r^{n-1}\)
Let's take a look at the sequence \(\{18,36,72,144,288, \ldots\}\). This is a geometric sequence with a common ratio of 2 and an exponential function with a base of 2 . An explicit formula for this sequence is
\(a_{n}=18 \cdot 2^{n-1}\)


\section*{A GENERAL NOTE: EXPLICIT FORMULA FOR A GEOMETRIC SEQUENCE}

The \(n\)th term of a geometric sequence is given by the explicit formula:
\(a_{n}=a_{1} r^{n-1}\)
In real-world scenarios involving arithmetic sequences, we may need to use an initial term of \(a_{0}\) instead of \(a_{1}\) . In these problems we can alter the explicit formula slightly by using the following formula:
\(a_{n}=a_{0} r^{n}\)
The following video provides a short lesson on some of the topics covered in this lesson.

\section*{EXAMPLE: WRITING TERMS OF GEOMETRIC SEQUENCES USING THE EXPLICIT FORMULA}

Given a geometric sequence with \(a_{1}=3\) and \(a_{4}=24\), find \(a_{2}\).

\section*{Answer}

The sequence can be written in terms of the initial term and the common ratio \(r\).
\(3,3 r, 3 r^{2}, 3 r^{3}, \ldots\)
Find the common ratio using the given fourth term.
\(a_{n}=a_{1} r^{n-1}\)
\(a_{4}=3 r^{3} \quad\) Write the fourth term of sequence in terms of \(a_{1}\) and \(r\)
\(24=3 r^{3} \quad\) Substitute 24 for \(a_{4}\)
\(8=r^{3} \quad\) Divide
\(r=2 \quad\) Solve for the common ratio
Find the second term by multiplying the first term by the common ratio.
\[
\begin{aligned}
a_{2} & =2 a_{1} \\
& =2(3) \\
& =6
\end{aligned}
\]

\section*{Analysis of the Solution}

The common ratio is multiplied by the first term once to find the second term, twice to find the third term, three times to find the fourth term, and so on. The tenth term could be found by multiplying the first term by the common ratio nine times or by multiplying by the common ratio raised to the ninth power.

\section*{TRY IT}

Given a geometric sequence with \(a_{2}=4\) and \(a_{3}=32\), find \(a_{6}\).
Answer
\(a_{6}=16,384\)
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\section*{EXAMPLE: WRITING AN EXPLICIT FORMULA FOR THE NTH TERM OF A GEOMETRIC SEQUENCE}

Write an explicit formula for the \(n\)th term of the following geometric sequence.
\(\{2,10,50,250, \ldots\}\)

\section*{Answer}

The first term is 2 . The common ratio can be found by dividing the second term by the first term.
\(\frac{10}{2}=5\)
The common ratio is 5 . Substitute the common ratio and the first term of the sequence into the formula.
\(a_{n}=a_{1} r^{(n-1)}\)
\(a_{n}=2 \cdot 5^{n-1}\)
The graph of this sequence shows an exponential pattern.


\section*{TRY IT}

Write an explicit formula for the following geometric sequence.
\(\{-1,3,-9,27, \ldots\}\)
Answer
\(a_{n}=-(-3)^{n-1}\)
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\section*{RECALL EXPONENTIAL APPLICATIONS}

You have solved problems such as the following before in Module 11: Exponential and Logarithmic Functions. Feel free to return to that material anytime for a refresher.

\section*{EXAMPLE: SOLVING APPLICATION PROBLEMS WITH GEOMETRIC SEQUENCES}

In 2013, the number of students in a small school is 284 . It is estimated that the student population will increase by 4\% each year.
1. Write a formula for the student population.
2. Estimate the student population in 2020.

\section*{Answer}
1. The situation can be modeled by a geometric sequence with an initial term of 284 . The student population will be \(104 \%\) of the prior year, so the common ratio is 1.04 . Let \(P\) be the student population and \(n\) be the number of years after 2013. Using the explicit formula for a geometric sequence we get \(P_{n}=284 \cdot 1.04^{n}\)
2. We can find the number of years since 2013 by subtracting.
\(2020-2013=7\)
We are looking for the population after 7 years. We can substitute 7 for \(n\) to estimate the population in 2020.
\(P_{7}=284 \cdot 1.04^{7} \approx 374\)
The student population will be about 374 in 2020.

\section*{TRY IT}

A business starts a new website. Initially the number of hits is 293 due to the curiosity factor. The business estimates the number of hits will increase by \(2.6 \%\) per week.
a. Write a formula for the number of hits.
b. Estimate the number of hits in 5 weeks.

\section*{Answer}
a. \(P_{n}=293 \cdot 1.026 a^{n}\)
b. The number of hits will be about 333 .


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\section*{SUMMARY: GEOMETRIC SEQUENCES}

\section*{Key Equations}
\[
\begin{array}{l|l}
\hline \text { recursive formula for } n t h \text { term of a geometric sequence } & a_{n}=r a_{n-1}, n \geq 2 \\
\hline \text { explicit formula for } n t h \text { term of a geometric sequence } & a_{n}=a_{1} r^{n-1}
\end{array}
\]

\section*{Key Concepts}
- A geometric sequence is a sequence in which the ratio between any two consecutive terms is a constant.
- The constant ratio between two consecutive terms is called the common ratio.
- The common ratio can be found by dividing any term in the sequence by the previous term.
- The terms of a geometric sequence can be found by beginning with the first term and multiplying by the common ratio repeatedly.
- A recursive formula for a geometric sequence with common ratio \(r\) is given by \(a_{n}=r a_{n-1}\) for \(n \geq 2\).
- As with any recursive formula, the initial term of the sequence must be given.
- An explicit formula for a geometric sequence with common ratio \(r\) is given by \(a_{n}=a_{1} r^{n-1}\).
- In application problems, we sometimes alter the explicit formula slightly to \(a_{n}=a_{0} r^{n}\).

\section*{Glossary}
common ratio the ratio between any two consecutive terms in a geometric sequence
geometric sequence a sequence in which the ratio of a term to a previous term is a constant

\section*{INTRODUCTION TO SERIES AND THEIR NOTATIONS}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Use summation notation.
- Use the formula for the sum of the first \(n\) terms of an arithmetic series.
- Use the formula for the sum of the first \(n\) terms of a geometric series.
- Use the formula for the sum of an infinite geometric series.
- Solve annuity problems.

A couple decides to start a college fund for their daughter. They plan to invest \(\$ 50\) in the fund each month. The fund pays \(6 \%\) annual interest, compounded monthly. How much money will they have saved when their daughter is ready to start college in 6 years? In this section we will learn how to answer this question. To do so we need to consider the amount of money invested and the amount of interest earned.

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\section*{ARITHMETIC SERIES}

\section*{LEARNING OUTCOMES}
- Evaluate a sum given in summation notation.
- Find the partial sum of an arithmetic series.
- Solve an application problem using an arithmetic series.

\section*{Using Summation Notation}

To find the total amount of money in the college fund and the sum of the amounts deposited, we need to add the amounts deposited each month and the amounts earned monthly. The sum of the terms of a sequence is called a series. Consider, for example, the following series.

The \(n\)th partial sum of a series is the sum of a finite number of consecutive terms beginning with the first term. The notation
\(S_{n}\) represents the partial sum.
\(S_{1}=3\)
\(S_{2}=3+7=10\)
\(S_{3}=3+7+11=21\)
\(S_{4}=3+7+11+15=36\)

\section*{TIP FOR SUCCESS}

We've been introduced to new notation several times in this course. Summation notation is a little bit more complicated than other notations we've seen before (arrow notation, interval notation, etc.). Don't be discouraged if it looks intimidating at first. The parts each stand for something specific so you will become familiar with it through practice and repetition. Try writing out form given in the example below a few times to start. Then, keep your notes nearby as you work through the examples. In time, you won't need your notes to complete problems.

Summation notation is used to represent series. Summation notation is often known as sigma notation because it uses the Greek capital letter sigma, \(\Sigma\), to represent the sum. Summation notation includes an explicit formula and specifies the first and last terms in the series. An explicit formula for each term of the series is given to the right of the sigma. A variable called the index of summation is written below the sigma. The index of summation is set equal to the lower limit of summation, which is the number used to generate the first term in the series. The number above the sigma, called the upper limit of summation, is the number used to generate the last term in a series.


If we interpret the given notation, we see that it asks us to find the sum of the terms in the series \(a_{k}=2 k\) for \(k=1\) through \(k=5\). We can begin by substituting the terms for \(k\) and listing out the terms of this series.
\(a_{1}=2(1)=2\)
\(a_{2}=2(2)=4\)
\(a_{3}=2(3)=6\)
\(a_{4}=2(4)=8\)
\(a_{5}=2(5)=10\)
We can find the sum of the series by adding the terms:
\(\sum_{k=1}^{5} 2 k=2+4+6+8+10=30\)

\section*{A GENERAL NOTE: SUMMATION NOTATION}

The sum of the first \(n\) terms of a series can be expressed in summation notation as follows:
\(\sum_{k=1}^{n} a_{k}\)
This notation tells us to find the sum of \(a_{k}\) from
\(k=1\) to \(k=n\).
\(k\) is called the index of summation, 1 is the lower limit of summation, and \(n\) is the upper limit of summation.

\section*{Q \& A}

Does the lower limit of summation have to be 1 ?

No. The lower limit of summation can be any number, but 1 is frequently used. We will look at examples with lower limits of summation other than 1.

\section*{HOW TO: GIVEN SUMMATION NOTATION FOR A SERIES, EVALUATE THE VALUE.}
1. Identify the lower limit of summation.
2. Identify the upper limit of summation.
3. Substitute each value of \(k\) from the lower limit to the upper limit into the formula.
4. Add to find the sum.

\section*{EXAMPLE: USING SUMMATION NOTATION}

Evaluate \(\sum_{k=3}^{7} k^{2}\).

\section*{Answer}

According to the notation, the lower limit of summation is 3 and the upper limit is 7 . So we need to find the sum of \(k^{2}\) from \(k=3\) to \(k=7\). We find the terms of the series by substituting \(k=3,4,5,6\), and 7 into the function \(k^{2}\). We add the terms to find the sum.
\[
\begin{aligned}
\sum_{k=3}^{7} k^{2} & =3^{2}+4^{2}+5^{2}+6^{2}+7^{2} \\
& =9+16+25+36+49 \\
& =135
\end{aligned}
\]

\section*{TRY IT}

Evaluate \(\sum_{k=2}^{5}(3 k-1)\).
Answer
38
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\section*{Arithmetic Series}

Just as we studied special types of sequences, we will look at special types of series. Recall that an arithmetic sequence is a sequence in which the difference between any two consecutive terms is the common difference, \(d\). The sum of the terms of an arithmetic sequence is called an arithmetic series. We can write the sum of the first \(n\) terms of an arithmetic series as:
\(S_{n}=a_{1}+\left(a_{1}+d\right)+\left(a_{1}+2 d\right)+\ldots+\left(a_{n}-d\right)+a_{n}\).
We can also reverse the order of the terms and write the sum as
\(S_{n}=a_{n}+\left(a_{n}-d\right)+\left(a_{n}-2 d\right)+\ldots+\left(a_{1}+d\right)+a_{1}\).
If we add these two expressions for the sum of the first \(n\) terms of an arithmetic series, we can derive a formula for the sum of the first \(n\) terms of any arithmetic series.
\[
\begin{aligned}
S_{n} & =a_{1}+\left(a_{1}+d\right)+\left(a_{1}+2 d\right)+\ldots+\left(a_{n}-d\right)+a_{n} \\
+S_{n} & =a_{n}+\left(a_{n}-d\right)+\left(a_{n}-2 d\right)+\ldots+\left(a_{1}+d\right)+a_{1} \\
\hline 2 S_{n} & =\left(a_{1}+a_{n}\right)+\left(a_{1}+a_{n}\right)+\ldots+\left(a_{1}+a_{n}\right)
\end{aligned}
\]

Because there are \(n\) terms in the series, we can simplify this sum to
\(2 S_{n}=n\left(a_{1}+a_{n}\right)\).
We divide by 2 to find the formula for the sum of the first \(n\) terms of an arithmetic series.
\(S_{n}=\frac{n\left(a_{1}+a_{n}\right)}{2}\)
This is generally referred to as the Partial Sum of the series.

\section*{TIP FOR SUCCESS}

The formula for finding the partial sum of an arithmetic sequence is handy and should be memorized, but it is important to understand how it was derived.

\section*{A GENERAL NOTE: FORMULA FOR THE PARTIAL SUM OF AN ARITHMETIC SERIES}

An arithmetic series is the sum of the terms of an arithmetic sequence. The formula for the partial sum of an arithmetic sequence is
\(S_{n}=\frac{n\left(a_{1}+a_{n}\right)}{2}\)

\section*{HOW TO: GIVEN TERMS OF AN ARITHMETIC SERIES, FIND THE PARTIAL SUM}
1. Identify \(a_{1}\) and \(a_{n}\).
2. Determine \(n\).
3. Substitute values for \(a_{1}, a_{n}\), and \(n\) into the formula \(S_{n}=\frac{n\left(a_{1}+a_{n}\right)}{2}\).
4. Simplify to find \(S_{n}\).

\section*{TIP FOR SUCCESS}

The examples and problems below may require more than one step to reach the correct conclusion. Some may seem as though they don't give enough information. But you have all the tools your need to solve them in this and in previous sections of this module.
Try working them out for yourself on paper before looking at the answers, then analyze your work by comparing to the solutions.

\section*{EXAMPLE: FINDING THE PARTIAL SUM OF AN ARITHMETIC SERIES}

Find the partial sum of each arithmetic series.
1. \(5+8+11+14+17+20+23+26+29+32\)
2. \(20+15+10+\cdots+-50\)
3. \(\sum_{k=1}^{12} 3 k-8\)

Answer
1. We are given \(a_{1}=5\) and \(a_{n}=32\). Count the number of terms in the sequence to find \(n=10\).Substitute values for \(a_{1}, a_{n}\), and \(n\) into the formula and simplify.
\[
\begin{aligned}
& S_{n}=\frac{n\left(a_{1}+a_{n}\right)}{2} \\
& S_{10}=\frac{10(5+32)}{2}=185
\end{aligned}
\]
2. We are given \(a_{1}=20\) and \(a_{n}=-50\). Use the formula for the general term of an arithmetic sequence to find \(n\).
\[
\begin{aligned}
a_{n} & =a_{1}+(n-1) d \\
-50 & =20+(n-1)(-5) \\
-70 & =(n-1)(-5) \\
14 & =n-1 \\
15 & =n
\end{aligned}
\]

Substitute values for \(a_{1}, a_{n}, n\) into the formula and simplify.
\[
\begin{aligned}
& S_{n}=\frac{n\left(a_{1}+a_{n}\right)}{2} \\
& S_{15}=\frac{15(20-50)}{2}=-225
\end{aligned}
\]
3. To find \(a_{1}\), substitute \(k=1\) into the given explicit formula.
\[
\begin{aligned}
& a_{k}=3 k-8 \\
& a_{1}=3(1)-8=-5
\end{aligned}
\]

We are given that \(n=12\). To find \(a_{12}\), substitute \(k=12\) into the given explicit formula.
\[
\begin{aligned}
a_{k} & =3 k-8 \\
a_{12} & =3(12)-8=28
\end{aligned}
\]

Substitute values for \(a_{1}, a_{n}\), and \(n\) into the formula and simplify.
\(S_{n}=\frac{n\left(a_{1}+a_{n}\right)}{2}\)
\(S_{12}=\frac{12(-5+28)}{2}=138\)

\section*{TRY IT}

Use the formula to find the partial sum of each arithmetic series.
a. \(1.4+1.6+1.8+2.0+2.2+2.4+2.6+2.8+3.0+3.2+3.4\)

\section*{Answer}
26.4
b. \(12+21+29 \cdots+69\)

Answer
328
c. \(\sum_{k=1}^{10} 5-6 k\)

Answer
-280
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\section*{EXAMPLE: SOLVING APPLICATION PROBLEMS WITH ARITHMETIC SERIES}

On the Sunday after a minor surgery, a woman is able to walk a half-mile. Each Sunday, she walks an additional quarter-mile. After 8 weeks, what will be the total number of miles she has walked?
Answer
This problem can be modeled by an arithmetic series with \(a_{1}=\frac{1}{2}\) and \(d=\frac{1}{4}\). We are looking for the total number of miles walked after 8 weeks, so we know that \(n=8\), and we are looking for \(S_{8}\). To find \(a_{8}\), we can use the explicit formula for an arithmetic sequence.
\(a_{n}=a_{1}+d(n-1)\)
\(a_{8}=\frac{1}{2}+\frac{1}{4}(8-1)=\frac{9}{4}\)
We can now use the formula for arithmetic series.
\(S_{n}=\frac{n\left(a_{1}+a_{n}\right)}{2}\)
\(S_{8}=\frac{8\left(\frac{1}{2}+\frac{9}{4}\right)}{2}=11\)
She will have walked a total of 11 miles.

\section*{TRY IT}

A man earns \(\$ 100\) in the first week of June. Each week, he earns \(\$ 12.50\) more than the previous week. After 12 weeks, how much has he earned?
Answer
\$2,025

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\section*{GEOMETRIC SERIES}

\section*{LEARNING OUTCOMES}
- Write the first n terms of a geometric sequence.
- Determine whether the sum of an infinite geometric series exists.
- Give the sum of a convergent infinite geometric series.
- Solve an annuity problem using a geometric series.

Just as the sum of the terms of an arithmetic sequence is called an arithmetic series, the sum of the terms in a geometric sequence is called a geometric series. Recall that a geometric sequence is a sequence in which the ratio of any two consecutive terms is the common ratio, \(r\). We can write the sum of the first \(n\) terms of a geometric series as
\(S_{n}=a_{1}+a_{1} r+a_{1} r^{2}+\ldots+a_{1} r^{n-1}\).
Just as with arithmetic series, we can do some algebraic manipulation to derive a formula for the sum of the first \(n\) terms of a geometric series. We will begin by multiplying both sides of the equation by \(r\).
\(r S_{n}=a_{1} r+a_{1} r^{2}+a_{1} r^{3}+\ldots+a_{1} r^{n}\)
Next, we subtract this equation from the original equation.
\[
\begin{aligned}
S_{n} & =a_{1}+a_{1} r+a_{1} r^{2}+\ldots+a_{1} r^{n-1} \\
-r S_{n} & =-\left(a_{1} r+a_{1} r^{2}+a_{1} r^{3}+\ldots+a_{1} r^{n}\right) \\
\hline-r) S_{n} & =a_{1}-a_{1} r^{n}
\end{aligned}
\]

Notice that when we subtract, all but the first term of the top equation and the last term of the bottom equation cancel out. To obtain a formula for \(S_{n}\), factor \(a_{1}\) on the right hand side and divide both sides by \((1-r)\).
\[
S_{n}=\frac{a_{1}\left(1-r^{n}\right)}{1-r} \mathrm{r} \neq 1
\]

\section*{A GENERAL NOTE: FORMULA FOR THE SUM OF THE FIRST NTERMS OF A GEOMETRIC SERIES}

A geometric series is the sum of the terms in a geometric sequence. The formula for the sum of the first \(n\) terms of a geometric sequence is represented as
\[
S_{n}=\frac{a_{1}\left(1-r^{n}\right)}{1-r} \mathrm{r} \neq 1
\]

\section*{HOW TO: GIVEN A GEOMETRIC SERIES, FIND THE SUM OF THE FIRST \(N\) TERMS.}
1. Identify \(a_{1}, r\), and \(n\).
2. Substitute values for \(a_{1}, r\), and \(n\) into the formula \(S_{n}=\frac{a_{1}\left(1-r^{n}\right)}{1-r}\).
3. Simplify to find \(S_{n}\).

\section*{Using the Formula for the Sum of an Infinite Geometric Series}

Thus far, we have looked only at finite series. Sometimes, however, we are interested in the sum of the terms of an infinite sequence rather than the sum of only the first \(n\) terms. An infinite series is the sum of the terms of an infinite sequence. An example of an infinite series is \(2+4+6+8+\ldots\).

This series can also be written in summation notation as \(\sum_{k=1}^{\infty} 2 k\), where the upper limit of summation is infinity. Because the terms are not tending to zero, the sum of the series increases without bound as we add more terms. Therefore, the sum of this infinite series is not defined. When the sum is not a real number, we say the series diverges.

\section*{Determining Whether the Sum of an Infinite Geometric Series is Defined}

If the terms of an infinite geometric series approach 0 , the sum of an infinite geometric series can be defined. The terms in this series approach 0 :
\(1+0.2+0.04+0.008+0.0016+\ldots\)
The common ratio is \(r=0.2\). As \(n\) gets large, the values of of \(r^{n}\) get very small and approach 0 . Each successive term affects the sum less than the preceding term. As each succeeding term gets closer to 0 , the sum of the terms approaches a finite value. The terms of any infinite geometric series with \(-1<r<1\) approach 0 ; the sum of a geometric series is defined when \(-1<r<1\).

\section*{Finding Sums of Infinite Series}

When the sum of an infinite geometric series exists, we can calculate the sum. The formula for the sum of an infinite series is related to the formula for the sum of the first \(n\) terms of a geometric series.
\(S_{n}=\frac{a_{1}\left(1-r^{n}\right)}{1-r}\)

\section*{EXAMPLE: FINDING THE FIRST NTERMS OF A GEOMETRIC SERIES}

Use the formula to find the indicated partial sum of each geometric series.
1. \(S_{11}\) for the series \(8+-4+2+\ldots\)
2. \(\sum_{k=1}^{6} 3 \cdot 2^{k}\)

Answer
1. \(a_{1}=8\), and we are given that \(n=11\). We can find \(r\) by dividing the second term of the series by the first.
\(r=\frac{-4}{8}=-\frac{1}{2}\)
Substitute values for \(a_{1}, r\), and \(n\) into the formula and simplify.
\[
\begin{aligned}
& S_{n}=\frac{a_{1}\left(1-r^{n}\right)}{1-r} \\
& S_{11}=\frac{8\left(1-\left(-\frac{1}{2}\right)^{11}\right)}{1-\left(-\frac{1}{2}\right)} \approx 5.336
\end{aligned}
\]
2. Find \(a_{1}\) by substituting \(k=1\) into the given explicit formula.
\(a_{1}=3 \cdot 2^{1}=6\)
We can see from the given explicit formula that \(r=2\). The upper limit of summation is 6 , so \(n=6\).Substitute values for \(a_{1}, r\), and \(n\) into the formula, and simplify.
\(S_{n}=\frac{a_{1}\left(1-r^{n}\right)}{1-r}\)
\(S_{6}=\frac{6\left(1-2^{6}\right)}{1-2}=378\)

\section*{TRY IT}

Use the formula to find the indicated partial sum of each geometric series.
\(S_{20}\) for the series \(1,000+500+250+\ldots\)
Answer
\(\approx 2,000.00\)
Use the formula to determine the sum \(\sum_{k=1}^{8} 3^{k}\)
Answer
9,840
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We will examine an infinite series with \(r=\frac{1}{2}\). What happens to \(r^{n}\) as \(n\) increases?

\section*{EXAMPLE: SOLVING AN APPLICATION PROBLEM WITH A GEOMETRIC SERIES}

At a new job, an employee's starting salary is \(\$ 26,750\). He receives a \(1.6 \%\) annual raise. Find his total earnings at the end of 5 years.
Answer
The problem can be represented by a geometric series with \(a_{1}=26,750 ; n=5\); and \(r=1.016\).
Substitute values for \(a_{1}, r\), and \(n\) into the formula and simplify to find the total amount earned at the end of 5 years.
\(S_{n}=\frac{a_{1}\left(1-r^{n}\right)}{1-r}\)
\(S_{5}=\frac{26,750\left(1-1.016^{5}\right)}{1-1.016} \approx 138,099.03\)
He will have earned a total of \(\$ 138,099.03\) by the end of 5 years.

\section*{TRY IT}

At a new job, an employee's starting salary is \(\$ 32,100\). She receives a \(2 \%\) annual raise. How much will she have earned by the end of 8 years?
Answer
\$275,513.31
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\section*{DETERMINING WHETHER THE SUM OF AN INFINITE GEOMETRIC SERIES IS DEFINED}

The sum of an infinite series is defined if the series is geometric and \(-1<r<1\).

\section*{HOW TO: GIVEN THE FIRST SEVERAL TERMS OF AN INFINITE SERIES, DETERMINE IF THE SUM OF THE SERIES EXISTS.}
1. Find the ratio of the second term to the first term.
2. Find the ratio of the third term to the second term.
3. Continue this process to ensure the ratio of a term to the preceding term is constant throughout. If so, the series is geometric.
4. If a common ratio, \(r\), was found in step 3 , check to see if \(-1<r<1\). If so, the sum is defined. If not, the sum is not defined.

\section*{TIP FOR SUCCESS}

Remember to try the examples below before looking at the answers!

\section*{EXAMPLE: DETERMINING WHETHER THE SUM OF AN INFINITE SERIES IS DEFINED}

Determine whether the sum of each infinite series is defined.
1. \(12+8+4+\ldots\)
2. \(\frac{3}{4}+\frac{1}{2}+\frac{1}{3}+\ldots\)
3. \(\sum_{k=1}^{\infty} 27 \cdot\left(\frac{1}{3}\right)^{k}\)
4. \(\sum_{k=1}^{\infty} 5 k\)

\section*{Answer}
1. The ratio of the second term to the first is \(\frac{2}{3}\), which is not the same as the ratio of the third term to the second, \(\frac{1}{2}\). The series is not geometric.
2. The ratio of the second term to the first is the same as the ratio of the third term to the second. The series is geometric with a common ratio of \(\frac{2}{3}\). The sum of the infinite series is defined.
3. The given formula is exponential with a base of \(\frac{1}{3}\); the series is geometric with a common ratio of \(\frac{1}{3}\). The sum of the infinite series is defined.
4. The given formula is not exponential. The series is arithmetic, not geometric and so cannot yield a finite sum.

\section*{TRY IT}

Determine whether the sum of the infinite series is defined.
1. \(\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+\frac{9}{8}+\cdots\)
2. \(24+(-12)+6+(-3)+\ldots\)
3. \(\sum_{k=1}^{\infty} 15 \cdot(-0.3)^{k}\)

\section*{Answer}
1. The series is geometric, but \(r=\frac{3}{2}>1\). The sum is not defined.
2. The series is geometric with \(r=-\frac{1}{2}\). The sum is defined.
3. The series is geometric with \(r=-0.3\). The sum is defined.
\[
\begin{aligned}
& \left(\frac{1}{2}\right)^{2}=\frac{1}{4} \\
& \left(\frac{1}{2}\right)^{3}=\frac{1}{8} \\
& \left(\frac{1}{2}\right)^{4}=\frac{1}{16}
\end{aligned}
\]

The value of \(r^{n}\) decreases rapidly. What happens for greater values of \(n\) ?
\[
\begin{aligned}
& \left(\frac{1}{2}\right)^{10}=\frac{1}{1,024} \\
& \left(\frac{1}{2}\right)^{20}=\frac{1}{1,048,576} \\
& \left(\frac{1}{2}\right)^{30}=\frac{1}{1,073,741,824}
\end{aligned}
\]

As \(n\) gets large, \(r^{n}\) gets very small. We say that as \(n\) increases without bound, \(r^{n}\) approaches 0 . As \(r^{n}\) approaches \(0,1-r^{n}\) approaches 1 . When this happens the numerator approaches \(a_{1}\). This gives us the formula for the sum of an infinite geometric series.

\section*{TIP FOR SUCCESS}

This is another very handy formula that should be memorized. It is important to understand, though, how it is derived from the formula for the sum of the first \(n\) terms of a geometric series for \(n\) increasing without bound.

\section*{A GENERAL NOTE: FORMULA FOR THE SUM OF AN INFINITE GEOMETRIC SERIES}

The formula for the sum of an infinite geometric series with \(-1<r<1\) is:
\(S=\frac{a_{1}}{1-r}\)

\section*{HOW TO: GIVEN AN INFINITE GEOMETRIC SERIES, FIND ITS SUM.}
1. Identify \(a_{1}\) and \(r\).
2. Confirm that \(-1<r<1\).
3. Substitute values for \(a_{1}\) and \(r\) into the formula, \(S=\frac{a_{1}}{1-r}\).
4. Simplify to find \(S\).

\section*{Annuities}

At the beginning of the section, we looked at a problem in which a couple invested a set amount of money each month into a college fund for six years. An annuity is an investment in which the purchaser makes a sequence of periodic, equal payments. To find the amount of an annuity, we need to find the sum of all the payments and the interest earned. In the example the couple invests \(\$ 50\) each month. This is the value of the initial deposit. The account paid \(6 \%\) annual interest, compounded monthly. To find the interest rate per payment period, we need to divide the \(6 \%\) annual percentage interest (APR) rate by 12 . So the monthly interest rate is \(0.5 \%\). We can multiply the amount in the account each month by \(100.5 \%\) to find the value of the account after interest has been added.

We can find the value of the annuity right after the last deposit by using a geometric series with \(a_{1}=50\) and \(r=100.5 \%=1.005\). After the first deposit, the value of the annuity will be \(\$ 50\). Let us see if we can determine the amount in the college fund and the interest earned.

We can find the value of the annuity after \(n\) deposits using the formula for the sum of the first \(n\) terms of a geometric series. In 6 years, there are 72 months, so \(n=72\). We can substitute \(a_{1}=50, r=1.005\), and \(n=72\) into the formula, and simplify to find the value of the annuity after 6 years.

\section*{EXAMPLE: FINDING THE SUM OF AN INFINITE GEOMETRIC SERIES}

Find the sum, if it exists, for the following:
1. \(10+9+8+7+\ldots\)
2. \(248.6+99.44+39.776+\ldots\)
3. \(\sum_{k=1}^{\infty} 4,374 \cdot\left(-\frac{1}{3}\right)^{k-1}\)
4. \(\sum_{k=1}^{\infty} \frac{1}{9} \cdot\left(\frac{4}{3}\right)^{k}\)

\section*{Answer}
1. There is not a constant ratio; the series is not geometric.
2. There is a constant ratio; the series is geometric. \(a_{1}=248.6\) and \(r=\frac{99.44}{248.6}=0.4\), so the sum exists. Substitute \(a_{1}=248.6\) and \(r=0.4\) into the formula and simplify to find the sum.
\[
\begin{aligned}
& S=\frac{a_{1}}{1-r} \\
& S=\frac{248.6}{1-0.4}=\frac{1243}{3}
\end{aligned}
\]
3. The formula is exponential, so the series is geometric with \(r=-\frac{1}{3}\). Find \(a_{1}\) by substituting \(k=1\) into the given explicit formula.
\[
a_{1}=4,374 \cdot\left(-\frac{1}{3}\right)^{1-1}=4,374
\]

Substitute 4,374 and \(r=-\frac{1}{3}\) into the formula, and simplify to find the sum.
\[
\begin{aligned}
& S=\frac{a_{1}}{1-r} \\
& S=\frac{4,374}{1-\left(-\frac{1}{3}\right)}=3,280.5
\end{aligned}
\]
4. The formula is exponential, so the series is geometric, but \(r>1\). The sum does not exist.
\(S_{72}=\frac{50\left(1-1.005^{72}\right)}{1-1.005} \approx 4,320.44\)
After the last deposit, the couple will have a total of \(\$ 4,320.44\) in the account. Notice, the couple made 72 payments of \(\$ 50\) each for a total of \(72(50)=\$ 3,600\). This means that because of the annuity, the couple earned \(\$ 720.44\) interest in their college fund.

\section*{EXAMPLE: FINDING AN EQUIVALENT FRACTION FOR A REPEATING \\ DECIMAL}

Find the equivalent fraction for the repeating decimal \(0 . \overline{3}\).

\section*{Answer}

We notice the repeating decimal \(0 . \overline{3}=0.333 \ldots\) so we can rewrite the repeating decimal as a sum of terms.
\(0 . \overline{3}=0.3+0.03+0.003+\ldots\)
Looking for a pattern, we rewrite the sum, noticing that we see the first term multiplied to 0.1 in the second term, and the second term multiplied to 0.1 in the third term.
\[
\begin{aligned}
0 . \overline{3} & =0.3+0.3 \cdot(0.1)+0.3 \cdot(0.01)+0.3 \cdot(0.001)+\ldots \\
& =0.3+0.3 \cdot(0.1)+0.3 \cdot(0.1)^{2}+0.3 \cdot(0.1)^{3}+\ldots
\end{aligned}
\]

Notice the pattern; we multiply each consecutive term by a common ratio of 0.1 starting with the first term of 0.3 . So, substituting into our formula for an infinite geometric sum, we have
\(S=\frac{a_{1}}{1-r}=\frac{0.3}{1-0.1}=\frac{0.3}{0.9}=\frac{1}{3}\)

\section*{TRY IT}

Find the sum if it exists.
1. \(2+\frac{2}{3}+\frac{2}{9}+\ldots\)
2. \(\sum_{k=1}^{\infty} 0.76 k+1\)
3. \(\sum_{k=1}^{\infty}\left(-\frac{3}{8}\right)^{k}\)

\section*{Answer}
1. 3
2. The series is arithmetic. The sum does not exist.
3. \(-\frac{3}{11}\)

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\section*{HOW TO: GIVEN AN INITIAL DEPOSIT AND AN INTEREST RATE, FIND THE VALUE OF AN ANNUITY.}
1. Determine \(a_{1}\), the value of the initial deposit.
2. Determine \(n\), the number of deposits.
3. Determine \(r\).
1. Divide the annual interest rate by the number of times per year that interest is compounded.
2. Add 1 to this amount to find \(r\).
4. Substitute values for \(a_{1}, r\), and \(n\)
into the formula for the sum of the first \(n\) terms of a geometric series, \(S_{n}=\frac{a_{1}\left(1-r^{n}\right)}{1-r}\).
5. Simplify to find \(S_{n}\), the value of the annuity after \(n\) deposits.

\section*{EXAMPLE: SOLVING AN ANNUITY PROBLEM}

A deposit of \(\$ 100\) is placed into a college fund at the beginning of every month for 10 years. The fund earns \(9 \%\) annual interest, compounded monthly, and paid at the end of the month. How much is in the account right after the last deposit?

\section*{Answer}

The value of the initial deposit is \(\$ 100\), so \(a_{1}=100\). A total of 120 monthly deposits are made in the 10 years, so \(n=120\). To find \(r\), divide the annual interest rate by 12 to find the monthly interest rate and add 1 to represent the new monthly deposit.
\(r=1+\frac{0.09}{12}=1.0075\)
Substitute \(a_{1}=100, r=1.0075\), and \(n=120\) into the formula for the sum of the first \(n\) terms of a geometric series, and simplify to find the value of the annuity.
\(S_{120}=\frac{100\left(1-1.0075^{120}\right)}{1-1.0075} \approx 19,351.43\)
So the account has \(\$ 19,351.43\) after the last deposit is made.

\section*{TRY IT}

At the beginning of each month, \(\$ 200\) is deposited into a retirement fund. The fund earns \(6 \%\) annual interest, compounded monthly, and paid into the account at the end of the month. How much is in the account if deposits are made for 10 years?

\section*{Answer}
\$92,408.18
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\section*{SUMMARY: SERIES AND THEIR NOTATIONS}

\section*{Key Equations}
\begin{tabular}{l|l}
\begin{tabular}{l} 
sum of the first \(n\) \\
terms of an arithmetic series
\end{tabular} & \(S_{n}=\frac{n\left(a_{1}+a_{n}\right)}{2}\) \\
\hline \begin{tabular}{l} 
sum of the first \(n\) \\
terms of a geometric series
\end{tabular} & \(S_{n}=\frac{a_{1}\left(1-r^{n}\right)}{1-r}, r \neq 1\) \\
\hline sum of an infinite geometric series with \(-1<r<1\) & \(S_{n}=\frac{a_{1}}{1-r}\)
\end{tabular}

\section*{Key Concepts}
- The sum of the terms in a sequence is called a series.
- A common notation for series is called summation notation, which uses the Greek letter sigma to represent the sum.
- The sum of the terms in an arithmetic sequence is called an arithmetic series.
- The sum of the first \(n\) terms of an arithmetic series can be found using a formula.
- The sum of the terms in a geometric sequence is called a geometric series.
- The sum of the first \(n\) terms of a geometric series can be found using a formula.
- The sum of an infinite series exists if the series is geometric with \(-1<r<1\).
- If the sum of an infinite series exists, it can be found using a formula.
- An annuity is an account into which the investor makes a series of regularly scheduled payments. The value of an annuity can be found using geometric series.

\section*{Glossary}
annuity an investment in which the purchaser makes a sequence of periodic, equal payments
arithmetic series the sum of the terms in an arithmetic sequence
diverge a series is said to diverge if the sum is not a real number
geometric series the sum of the terms in a geometric sequence
index of summation in summation notation, the variable used in the explicit formula for the terms of a series and written below the sigma with the lower limit of summation
infinite series the sum of the terms in an infinite sequence
lower limit of summation the number used in the explicit formula to find the first term in a series
\(n\)th partial sum the sum of the first \(n\) terms of a sequence
series the sum of the terms in a sequence
summation notation a notation for series using the Greek letter sigma; it includes an explicit formula and specifies the first and last terms in the series
upper limit of summation the number used in the explicit formula to find the last term in a series

\title{
PUTTING IT TOGETHER: SEQUENCES AND SERIES
}

The Greek philosopher Zeno of Elea (c. 490-450 BC) described certain everyday scenarios, such as walking down a path or shooting an arrow, in ways that made them seem impossible to accomplish. These stories are called Zeno's Paradoxes. The word paradox itself means selfcontradictory statement, and Zeno's original intent in telling these paradoxes was to argue that motion is somehow an illusion. However we can use what we have learned in this module to show that there is no self-contradiction at all.

Suppose you are at the beginning of a mile long path. In order to walk the whole mile, certainly you will have to make it to the halfway point first. After traversing that first half-mile, in order to complete the rest of the journey you will have to make it halfway through the next half-mile in other words, you will have to walk an additional quarter-mile. Now that you have walked three quarters of a mile total, you'll need to make it to the next halfway point, which is half of a quarter mile away, another \(1 / 8^{\text {th }}\) of a mile. The journey continues as you cover an additional \(1 / 16^{\text {th }}\), then \(1 / 32^{\text {nd }}\), and then \(1 / 64^{\text {th }}\) of a mile, and so on. Every time you get to


Marble bust of Zeno of Citium. the next halfway point, there will be a new halfway point looming ahead.

In other words, there are infinitely many halfway points to get past, and as Zeno argued, since no one can accomplish an infinite number of tasks, there is simply no way to walk the entire mile! In fact, the same reasoning applies to any distance, no matter how short. Motion itself seems to be impossible because the object in motion would have to pass an infinite number of halfway points before reaching its destination.

Now obviously people can and do walk various distances all the time. Where does Zeno's argument go wrong? It was in his misunderstanding of an infinite series. Let's see what the journey looks like at each stage, adding up all of the distances in turn. We'll keep track of our progress on a number line.



As you can see, at every stage the total distance only grows by smaller and smaller amounts. In fact, we are adding the terms of a geometric series whose common ratio is equal to \(r=\frac{1}{2}\).

But the process doesn't really stop, right? What would be the sum of infinitely many terms of the series? Let \(S=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\cdots\), which represents the sum of all stages of the journey.
Here, we have initial term \(a_{1}=\frac{1}{2}\), and because \(-1<r<1\), we can use the formula for the sum of an infinite geometric series.
\(S=\frac{a_{1}}{1-r}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=\frac{\left(\frac{1}{2}\right)}{\left(\frac{1}{2}\right)}=1\).

Notice that the formula correctly sums the infinitely many distances that make up the full mile. What Zeno did not realize is that an infinite series could have a finite sum!
```

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\title{
MODULE 17: PROBABILITY AND COUNTING PRINCIPLES
}

\section*{WHY IT MATTERS: PROBABILITY AND COUNTING PRINCIPLES}

According to the news, the lottery jackpot is climbing by the hour. Long lines of dreamers are forming wherever lottery tickets are sold. While you don't usually buy lottery tickets, it is getting tempting. Just imagine what you could do with \(\$ 100\) million dollars. Perhaps you could retire early or never even go to work. Maybe visit a beautiful island. All you need to do is pick the correct numbers, and the jackpot is all yours.


It sounds easy enough; just six simple numbers. But how likely are you to win? And could you increase the likelihood of winning by purchasing more lottery tickets?

To answer these questions, you need to know about permutations and combinations. So learn about them as you complete this module, and then we'll return to the lottery at the end. Then you'll be able to decide whether you want to stand in line to purchase a ticket.


\section*{LEARNING OUTCOMES}

\section*{Counting Principles}
- Solve counting problems using the Addition Principle and the Multiplication Principle.
- Solve counting problems using permutations and combinations involving \(n\) distinct objects.
- Find the number of subsets of a given set.
- Solve counting problems using permutations involving n non-distinct objects.
- Apply the Binomial Theorem.

\section*{Probability}
- Construct probability models.
- Compute probabilities of equally likely outcomes.
- Compute probabilities of the union of two events.
- Use the complement rule to find probabilities.
- Compute probability using counting theory.

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\section*{REVIEW TOPICS FOR SUCCESS}

\section*{Learning Outcomes}

By the end of this section, you will be able to:
- Identify and use the Addition Principle of Counting
- Identify and use the Multiplication Principle of Counting (Fundamental Counting Principle)
- Find the number of permutations of \(n\) distinct objects

This module will introduce probability and counting principles and the binomial theorem. The material is challenging and draws on the mathematical maturity you've built up throughout the course to piece together new formulas and ideas in application problems.

You may find certain concepts are still rusty as you work through the module, especially operations on fractions. If so, you can find many of the fundamental concepts used in this course in Module 1: Algebra Essentials. Feel free to return there as needed.

In preparation to explore probability and counting principles, it will be a good idea to familiarize yourself with some of the fundamental principles of counting first. You'll see these topics again right after the review section, which will hopefully help you to get a good, strong start.

Warm up for this module by refreshing important concepts and skills you'll need for success. As you study these review topics, recall that you can also return to Algebra Essentials any time you need to refresh the
basics.

\section*{RECALL FOR SUCCESS}

Look for red boxes like this one throughout the text. They'll show up just in time to give helpful reminders of the math you'll need, right where you'll need it.

\section*{FUNDAMENTAL PRINCIPLES OF COUNTING}

\section*{Learning Outcome}
- Identify and use the Addition Principle of Counting
- Identify and use the Multiplication Principle of Counting (Fundamental Counting Principle)
- Find the number of permutations of \(n\) distinct objects

\section*{The Addition Principle of Counting}

The Addition Principle states that if two sets of items are distinct from one another (there is no overlapping), then the sum of the union of the sets is obtained by adding the sum of each set together. This probably appears to be a rather straightforward statement, and it is. But without it, counting anything would not be possible. Counting begins here with examples like the following that include mutually exclusive sets.

Ex. On a certain used car lot, there are 9 blue cars and 12 green cars. There are no cars that are both blue and green at the same time. To determine the total number of cars, we do not have to count them individually. We can simply sum up the sets of cars: \(9+12=21\) total cars.

Ex. In a certain college algebra class, there are 18 freshmen and 7 sophomores. \(18+7=25\) students.
Ex. We can use mathematical notation to illustrate the principle. Let the probability of a certain event \(U\) be .09 and the probability of a separate, mutually exclusive event \(V\) be 0.34 . Then the sum of the two probabilities can be given by \(P(U)\) or \(P(V)\). That is, the probability of either event happening is \(P(P\) or \(V)=0.09+.34=0.43\).

\section*{A GENERAL NOTE: THE ADDITION PRINCIPLE}

According to the Addition Principle, if one event can occur in \(m\) ways and a second event with no common outcomes can occur in \(n\) ways, then the first or second event can occur in \(m+n\) ways.

\section*{The Multiplication Principle of Counting}

The Multiplication Principle, also called the Fundamental Counting Principle, states that if there are so many ways one event can occur after another has already occurred, the total number of ways the two can occur together can be found by multiplying. A classic example presents the choice made at a lunch counter.

Ex. A certain lunch special permits one sandwich, two sides, and a drink. There are five sandwiches, seven sides, and four drinks to choose from. How many possible lunch combinations can be made? The Multiplication Principle states that there are \(5 * 7 * 4=140\) possible combinations.

\section*{A GENERAL NOTE: THE MULTIPLICATION PRINCIPLE}

According to the Multiplication Principle, if one event can occur in \(m\) ways and a second event can occur in \(n\) ways after the first event has occurred, then the two events can occur in \(m \times n\) ways. This is also known as the Fundamental Counting Principle.

\section*{Find the Number of Permutations of \(n\) Distinct Objects}
we can use the Multiplication Principle to find the number of ways to arrange items or people in a specific order. For example, in a race with 6 contestants, how many possible orders are there (all things being equal) in which they may cross the finish line? The Multiplication Principle states that we may multiply together the possible number of contestants available to fill each position.

Before the first person crosses the finish line, there are 9 possibilities of who can take that spot. But after the winner crosses the line, there are only 8 remaining possibilities to take second place, and once that position has been claimed, there are only 7 remaining, and so on until only 1 person is left to take the final position. Using the Multiplication Principle, we can take the product to find that there are \(9 * 8 * 7 * 6 * 5 * 4 * 3 * 2 *=362,880\) possible ways.

If there are only four positions available to take out of the nine runners, how many possible ways can the four positions be filled? Take the product of the possibilities until the positions have been filled. There are \(9 * 8 * 7 * 6=3024\) possible ways for four of the nine racers to finish in the top four positions.

\section*{HOW TO: GIVEN \(n\) DISTINCT OPTIONS, DETERMINE HOW MANY PERMUTATIONS THERE ARE.}
1. Determine how many options there are for the first situation.
2. Determine how many options are left for the second situation.
3. Continue until all of the spots are filled.
4. Multiply the numbers together.

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\section*{SUMMARY: REVIEW}

\section*{Key Concepts}
- According to the Addition Principle, if one event can occur in \(m\) ways and a second event with no common outcomes can occur in \(n\) ways, then the first or second event can occur in \(m+n\) ways.
- According to the Multiplication Principle, if one event can occur in \(m\) ways and a second event can occur in \(n\) ways after the first event has occurred, then the two events can occur in \(m \times n\) ways. This is
- To determine how many permutations there are given \(n\) distinct options,
- Determine how many options there are for the first situation.
- Determine how many options are left for the second situation.
- Continue until all of the spots are filled.
- Multiply the numbers together.

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\section*{INTRODUCTION TO COUNTING PRINCIPLES}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Solve counting problems using the Addition Principle.
- Solve counting problems using the Multiplication Principle.
- Solve counting problems using permutations involving n distinct objects.
- Solve counting problems using combinations.
- Find the number of subsets of a given set.
- Solve counting problems using permutations involving n non-distinct objects.
- Apply the Binomial Theorem.

A new company sells customizable cases for tablets and smartphones. Each case comes in a variety of colors and can be personalized for an additional fee with images or a monogram. A customer can choose not to personalize or could choose to have one, two, or three images or a monogram. The customer can choose the order of the images and the letters in the monogram. The company is working with an agency to develop a marketing campaign with a focus on the huge number of options they offer. Counting the possibilities is challenging!

We encounter a wide variety of counting problems every day. There is a branch of mathematics devoted to the study of counting problems such as this one. Other applications of counting include secure passwords, horse racing outcomes, and college scheduling choices. We will examine this type of mathematics in this section.

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\section*{PERMUTATIONS}

\section*{LEARNING OUTCOMES}
- Use the addition principle to determine the total number of options for a given scenario.
- Use the multiplication principle to find the number of permutation of \(n\) distinct objects.
- Find the number of permutations of \(n\) distinct objects using a formula.
- Find the Number of Permutations of \(n\) Non-Distinct Objects.

\section*{TIP FOR SUCCESS}

It is important to try each of the examples in this section on paper and on your calculator as indicated to build up a good intuition for basic counting principles. The principles may seem fairly straightforward as you read through them, but future concepts will be more challenging without a good understanding of these.

\section*{Using the Addition Principle}

The company that sells customizable cases offers cases for tablets and smartphones. There are 3 supported tablet models and 5 supported smartphone models. The Addition Principle tells us that we can add the number of tablet options to the number of smartphone options to find the total number of options. By the Addition Principle there are 8 total options.


\section*{A GENERAL NOTE: THE ADDITION PRINCIPLE}

According to the Addition Principle, if one event can occur in \(m\) ways and a second event with no common outcomes can occur in \(n\) ways, then the first or second event can occur in \(m+n\) ways.

\section*{EXAMPLE: USING THE ADDITION PRINCIPLE}

There are 2 vegetarian entrée options and 5 meat entrée options on a dinner menu. What is the total number of entrée options?

\section*{Answer}

We can add the number of vegetarian options to the number of meat options to find the total number of entrée options.


There are 7 total options.

\section*{TRY IT}

A student is shopping for a new computer. He is deciding among 3 desktop computers and 4 laptop computers. What is the total number of computer options?
Answer
7
The Multiplication Principle applies when we are making more than one selection. Suppose we are choosing an appetizer, an entrée, and a dessert. If there are 2 appetizer options, 3 entrée options, and 2 dessert options on a fixed-price dinner menu, there are a total of 12 possible choices of one each as shown in the tree diagram.


The possible choices are:
1. soup, chicken, cake
2. soup, chicken, pudding
3. soup, fish, cake
4. soup, fish, pudding
5. soup, steak, cake
6. soup, steak, pudding
7. salad, chicken, cake
8. salad, chicken, pudding
9. salad, fish, cake
10. salad, fish, pudding
11. salad, steak, cake
12. salad, steak, pudding

We can also find the total number of possible dinners by multiplying.

We could also conclude that there are 12 possible dinner choices simply by applying the Multiplication Principle.
\begin{tabular}{cccccc} 
\# of appetizer options & \(\times\) & \# of entree options & \(\times\) & \# of dessert options & \\
2 & \(\times\) & 3 & \(\times\) & 2 & \(=12\)
\end{tabular}

\section*{A GENERAL NOTE: THE MULTIPLICATION PRINCIPLE}

According to the Multiplication Principle, if one event can occur in \(m\) ways and a second event can occur in \(n\) ways after the first event has occurred, then the two events can occur in \(m \times n\) ways. This is also known as the Fundamental Counting Principle.

\section*{EXAMPLE: USING THE MULTIPLICATION PRINCIPLE}

Diane packed 2 skirts, 4 blouses, and a sweater for her business trip. She will need to choose a skirt and a blouse for each outfit and decide whether to wear the sweater. Use the Multiplication Principle to find the total number of possible outfits.

\section*{Answer}

To find the total number of outfits, find the product of the number of skirt options, the number of blouse options, and the number of sweater options.
\begin{tabular}{ccccc}
\begin{tabular}{c}
\(\#\) of skirt \\
options
\end{tabular} & \(\times\) & \begin{tabular}{c} 
\# of blouse \\
options
\end{tabular} & \(\times\) & \begin{tabular}{c} 
\# of sweater \\
options
\end{tabular} \\
2 & \(\times\) & 4 & \(\times\) & 2
\end{tabular}

There are 16 possible outfits.

\section*{TRY IT}

A restaurant offers a breakfast special that includes a breakfast sandwich, a side dish, and a beverage. There are 3 types of breakfast sandwiches, 4 side dish options, and 5 beverage choices. Find the total number of possible breakfast specials.
Answer
There are 60 possible breakfast specials.
The Multiplication Principle can be used to solve a variety of problem types. One type of problem involves placing objects in order. We arrange letters into words and digits into numbers, line up for photographs, decorate rooms, and more. An ordering of objects is called a permutation.

\section*{Finding the Number of Permutations of \(n\) Distinct Objects Using the Multiplication Principle}

To solve permutation problems, it is often helpful to draw line segments for each option. That enables us to determine the number of each option so we can multiply. For instance, suppose we have four paintings, and we want to find the number of ways we can hang three of the paintings in order on the wall. We can draw three lines to represent the three places on the wall.
\(\qquad\) \(\times\) \(\qquad\) \(\times\) \(\qquad\)
There are four options for the first place, so we write a 4 on the first line.
\(\qquad\)
\(\qquad\)
After the first place has been filled, there are three options for the second place so we write a 3 on the second line.
\(\qquad\)
After the second place has been filled, there are two options for the third place so we write a 2 on the third line. Finally, we find the product.
\[
4 \times \underline{3} \times \underline{2}=24
\]

There are 24 possible permutations of the paintings.

\section*{HOW TO: GIVEN \(n\) DISTINCT OPTIONS, DETERMINE HOW MANY PERMUTATIONS THERE ARE.}
1. Determine how many options there are for the first situation.
2. Determine how many options are left for the second situation.
3. Continue until all of the spots are filled.
4. Multiply the numbers together.

\section*{Finding the Number of Permutations of \(n\) Distinct Objects Using a Formula}

For some permutation problems, it is inconvenient to use the Multiplication Principle because there are so many numbers to multiply. Fortunately, we can solve these problems using a formula. Before we learn the formula, let's look at two common notations for permutations. If we have a set of \(n\) objects and we want to choose \(r\) objects from the set in order, we write \(P(n, r)\). Another way to write this is \({ }_{n} P_{r}\), a notation commonly seen on computers and calculators. To calculate \(P(n, r)\), we begin by finding \(n!\), the number of ways to line up all \(n\) objects. We then divide by \((n-r)\) ! to cancel out the \((n-r)\) items that we do not wish to line up.

Let's see how this works with a simple example. Imagine a club of six people. They need to elect a president, a vice president, and a treasurer. Six people can be elected president, any one of the five remaining people can be elected vice president, and any of the remaining four people could be elected treasurer. The number of ways this may be done is \(6 \times 5 \times 4=120\). Using factorials, we get the same result.
\[
\frac{6!}{3!}=\frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!}=6 \cdot 5 \cdot 4=120
\]

There are 120 ways to select 3 officers in order from a club with 6 members. We refer to this as a permutation of 6 taken 3 at a time. The general formula is as follows.
\[
P(n, r)=\frac{n!}{(n-r)!}
\]

Note that the formula stills works if we are choosing all \(n\) objects and placing them in order. In that case we would be dividing by \((n-n)\) ! or 0 !, which we said earlier is equal to 1 . So the number of permutations of \(n\) objects taken \(n\) at a time is \(\frac{n!}{1}\) or just \(n!\).

\section*{Finding the Number of Permutations of \(n\) Non-Distinct Objects}

\section*{EXAMPLE: FINDING THE NUMBER OF PERMUTATIONS USING THE MULTIPLICATION PRINCIPLE}

At a swimming competition, nine swimmers compete in a race.
1. How many ways can they place first, second, and third?
2. How many ways can they place first, second, and third if a swimmer named Ariel wins first place? (Assume there is only one contestant named Ariel.)
3. How many ways can all nine swimmers line up for a photo?

\section*{Answer}
1. Draw lines for each place.
\[
\text { options for } 1^{\text {st }} \text { place } \times \text { options for 2nd place } \times \text { options for } 3^{\text {rd }} \text { place }
\]

There are 9 options for first place. Once someone has won first place, there are 8 remaining options for second place. Once first and second place have been won, there are 7 remaining options for third place.
\[
\underline{9} \times \underline{8} \times \underline{7}=504
\]

Multiply to find that there are 504 ways for the swimmers to place.
2. Draw lines for describing each place.
\[
\text { options for } 1^{\text {st }} \text { place } \times \text { options for } 2^{\text {nd }} \text { place } \times \text { options for } 3^{\text {rd }} \text { place }
\]

We know Ariel must win first place, so there is only 1 option for first place. There are 8 remaining options for second place, and then 7 remaining options for third place.
\[
1 \times \underline{8} \times \begin{aligned}
& 1 \\
&
\end{aligned} \times 56
\]

Multiply to find that there are 56 ways for the swimmers to place if Ariel wins first.
3. Draw lines for describing each place in the photo.
\(\qquad\) \(\times\) \(\qquad\) \(\times\) \(\qquad\) \(\times\) \(\qquad\) \(\times\) \(\qquad\) \(\times\) \(\qquad\) \(\times\) \(\qquad\) \(\times\) \(\qquad\) \(\times\) \(\qquad\)
There are 9 choices for the first spot, then 8 for the second, 7 for the third, 6 for the fourth, and so on until only 1 person remains for the last spot.
\(\qquad\) \(\times \quad 8\) \(\times\) \(\times\) \(\times\) \(\times\) \(=362,880\)

There are 362,880 possible permutations for the swimmers to line up.

\section*{Analysis of the Solution}

Note that in part c, we found there were 9 ! ways for 9 people to line up. The number of permutations of \(n\) distinct objects can always be found by \(n!\).

\section*{TRY IT}

A family of five is having portraits taken. Use the Multiplication Principle to find the following. How many ways can the family line up for the portrait?
Answer

120
How many ways can the photographer line up 3 family members?

\section*{Answer}

60
How many ways can the family line up for the portrait if the parents are required to stand on each end?

\section*{Answer}

12

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\section*{TIP FOR SUCCESS}

In mathematics, we sometimes use the same notation to mean different things in different contexts. In the notation given above, note that \(P(n, r)\) indicates we wish to calculate the Permutation of \(r\) objects out of a set of \(n\) objects. It looks similar to an ordered pair, but does not indicate an (input, output) relationship between \(n\) and \(r\).

\section*{A GENERAL NOTE: FORMULA FOR PERMUTATIONS OF N DISTINCT OBJECTS}

Given \(n\) distinct objects, the number of ways to select \(r\) objects from the set in order is
\[
P(n, r)=\frac{n!}{(n-r)!}
\]

\section*{HOW TO: GIVEN A WORD PROBLEM, EVALUATE THE POSSIBLE PERMUTATIONS.}
1. Identify \(n\) from the given information.
2. Identify \(r\) from the given information.
3. Replace \(n\) and \(r\) in the formula with the given values.
4. Evaluate.

We have studied permutations where all of the objects involved were distinct. What happens if some of the objects are indistinguishable? For example, suppose there is a sheet of 12 stickers. If all of the stickers were distinct, there would be 12 ! ways to order the stickers. However, 4 of the stickers are identical stars, and 3 are identical moons. Because all of the objects are not distinct, many of the 12 ! permutations we counted are duplicates. The general formula for this situation is as follows.
\(\frac{n!}{r_{1}!r_{2}!\ldots r_{k}!}\)
In this example, we need to divide by the number of ways to order the 4 stars and the ways to order the 3 moons to find the number of unique permutations of the stickers. There are 4 ! ways to order the stars and 3 ! ways to order the moon.
\(\frac{12!}{4!3!}=3,326,400\)

\section*{EXAMPLE: FINDING THE NUMBER OF PERMUTATIONS USING THE FORMULA}

A professor is creating an exam of 9 questions from a test bank of 12 questions. How many ways can she select and arrange the questions?

\section*{Answer}

Substitute \(n=12\) and \(r=9\) into the permutation formula and simplify.
\(P(n, r)=\frac{n!}{(n-r)!}\)
\(P(12,9)=\frac{12!}{(12-9)!}=\frac{12!}{3!}=79,833,600\)
There are \(79,833,600\) possible permutations of exam questions!

\section*{Analysis of the Solution}

We can also use a calculator to find permutations. For this problem, we would enter 12, press the \({ }_{n} P_{r}\) function, enter 9, and then press Enter. The \({ }_{n} P_{r}\) function may be located under the MATH menu with probability commands.

\section*{Q \& A}

Could we have solved using the Multiplication Principle?

Yes. We could have multiplied \(15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4\) to find the same answer.

\section*{TRY IT}

A play has a cast of 7 actors preparing to make their curtain call. Use the permutation formula to find the following.
How many ways can the 7 actors line up?
Answer
\(P(7,7)=5,040\)
How many ways can 5 of the 7 actors be chosen to line up?
Answer
\(P(7,5)=2,520\)
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There are 3,326,400 ways to order the sheet of stickers.

\section*{A GENERAL NOTE: FORMULA FOR FINDING THE NUMBER OF PERMUTATIONS OF N NON-DISTINCT OBJECTS}

If there are \(n\) elements in a set and \(r_{1}\) are alike, \(r_{2}\) are alike, \(r_{3}\) are alike, and so on through \(r_{k}\), the number of permutations can be found by
\[
\frac{n!}{r_{1}!r_{2}!\ldots r_{k}!}
\]

\section*{EXAMPLE: FINDING THE NUMBER OF PERMUTATIONS OF N NONDISTINCT OBJECTS}

Find the number of rearrangements of the letters in the word DISTINCT.
Answer
There are 8 letters. Both I and T are repeated 2 times. Substitute \(n=8, r_{1}=2\), and \(r_{2}=2\) into the formula.
\[
\frac{8!}{2!2!}=10,080
\]

There are 10,080 arrangements.

\section*{TRY IT}

Find the number of rearrangements of the letters in the word CARRIER.

\section*{Answer}

840
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\section*{COMBINATIONS}

\section*{LEARNING OUTCOMES}
- Find the number of combinations of n distinct choices.

So far, we have looked at problems asking us to put objects in order. There are many problems in which we want to select a few objects from a group of objects, but we do not care about the order. When we are selecting objects and the order does not matter, we are dealing with combinations. A selection of \(r\) objects
from a set of \(n\) objects where the order does not matter can be written as \(C(n, r)\). Just as with permutations, \(\mathrm{C}(n, r)\) can also be written as \({ }_{n} C_{r}\). In this case, the general formula is as follows.
\(\mathrm{C}(n, r)=\frac{n!}{r!(n-r)!}\)
An earlier problem considered choosing 3 of 4 possible paintings to hang on a wall. We found that there were 24 ways to select 3 of the 4 paintings in order. But what if we did not care about the order? We would expect a smaller number because selecting paintings \(1,2,3\) would be the same as selecting paintings 2,3 , 1. To find the number of ways to select 3 of the 4 paintings, disregarding the order of the paintings, divide the number of permutations by the number of ways to order 3 paintings. There are \(3!=3 \cdot 2 \cdot 1=6\) ways to order 3 paintings. There are \(\frac{24}{6}\), or 4 ways to select 3 of the 4 paintings. This number makes sense because every time we are selecting 3 paintings, we are not selecting 1 painting. There are 4 paintings we could choose not to select, so there are 4 ways to select 3 of the 4 paintings.

\section*{TIP FOR SUCCESS}

Note the similarity and difference between the formulas for permutations and combinations:
Permutations (order matters), \(P(n, r)=\frac{n!}{(n-r)!}\)
Combinations (order does not matter), \(C(n, r)=\frac{n!}{r!(n-r)!}\)
The formula for combinations is the formula for permutations with the number of ways to order \(r\) objects divided away from the result.

\section*{A GENERAL NOTE: FORMULA FOR COMBINATIONS OF N DISTINCT OBJECTS}

Given \(n\) distinct objects, the number of ways to select \(r\) objects from the set is
\(\mathrm{C}(n, r)=\frac{n!}{r!(n-r)!}\)

\section*{HOW TO: GIVEN A NUMBER OF OPTIONS, DETERMINE THE POSSIBLE NUMBER OF COMBINATIONS.}
1. Identify \(n\) from the given information.
2. Identify \(r\) from the given information.
3. Replace \(n\) and \(r\) in the formula with the given values.
4. Evaluate.

\section*{Finding the Number of Subsets of a Set}

We have looked only at combination problems in which we chose exactly \(r\) objects. In some problems, we want to consider choosing every possible number of objects. Consider, for example, a pizza restaurant that offers 5 toppings. Any number of toppings can be ordered. How many different pizzas are possible?

To answer this question, we need to consider pizzas with any number of toppings. There is \(C(5,0)=1\) way to order a pizza with no toppings. There are \(C(5,1)=5\) ways to order a pizza with exactly one topping. If we continue this process, we get

\section*{EXAMPLE: FINDING THE NUMBER OF COMBINATIONS USING THE FORMULA}

A fast food restaurant offers five side dish options. Your meal comes with two side dishes.
1. How many ways can you select your side dishes?
2. How many ways can you select 3 side dishes?

\section*{Answer}
1. We want to choose 2 side dishes from 5 options.
\[
\mathrm{C}(5,2)=\frac{5!}{2!(5-2)!}=10
\]
2. We want to choose 3 side dishes from 5 options.
\[
\mathrm{C}(5,3)=\frac{5!}{3!(5-3)!}=10
\]

\section*{Analysis of the Solution}

We can also use a graphing calculator to find combinations. Enter 5, then press \({ }_{n} C_{r}\), enter 3, and then press the equal sign. The \({ }_{n} C_{r}\), function may be located under the MATH menu with probability commands.

\section*{Q \& A}

Is it a coincidence that parts (a) and (b) in Example 4 have the same answers?
No. When we choose robjects from n objects, we are not choosing \((n-r)\) objects. Therefore, \(C(n, r)=C(n, n-r)\).

\section*{TRY IT}

An ice cream shop offers 10 flavors of ice cream. How many ways are there to choose 3 flavors for a banana split?
Answer
\(C(10,3)=120\)
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\(C(5,0)+C(5,1)+C(5,2)+C(5,3)+C(5,4)+C(5,5)=32\)
There are 32 possible pizzas. This result is equal to \(2^{5}\).
We are presented with a sequence of choices. For each of the \(n\) objects we have two choices: include it in the subset or not. So for the whole subset we have made \(n\) choices, each with two options. So there are a total of \(2 \cdot 2 \cdot 2 \cdots \cdots 2\) possible resulting subsets, all the way from the empty subset, which we obtain when we say "no" each time, to the original set itself, which we obtain when we say "yes" each time.

\section*{TIP FOR SUCCESS}

Did you notice a pattern when you calculated the 32 possible pizzas long-hand?
\(C(5,0)+C(5,1)+C(5,2)+C(5,3)+C(5,4)+C(5,5)=1+5+10+10+5+1=32\)
That is not a coincidence! We'll look more deeply at this phenomenon in the next section.

\section*{A GENERAL NOTE: FORMULA FOR THE NUMBER OF SUBSETS OF A SET}

A set containing \(n\) distinct objects has \(2^{n}\) subsets.

\section*{EXAMPLE: FINDING THE NUMBER OF SUBSETS OF A SET}

A restaurant offers butter, cheese, chives, and sour cream as toppings for a baked potato. How many different ways are there to order a potato?
Answer
We are looking for the number of subsets of a set with 4 objects. Substitute \(n=4\) into the formula.
\(2^{n}=2^{4}=16\)
There are 16 possible ways to order a potato.

\section*{TRY IT}

A sundae bar at a wedding has 6 toppings to choose from. Any number of toppings can be chosen. How many different sundaes are possible?
Answer
64 sundaes
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\section*{USE THE BINOMIAL THEOREM}
- Identify binomial coefficients given the formula for a combination.
- Expand a binomial using the binomial theorem.
- Use the binomial to find a single term in a binomial.

\section*{EXPANDING \((x+y)^{n}\)}

Earlier, we learned how to use the distributive property to multiply binomials together. We found that certain binomial products resulted in a nice pattern such as \((a+b)^{2}=a^{2}+2 a b+b^{2}\). But multiplying more than three binomials together was a tedious process.
We will learn now that there is a pattern we can apply to greater powers of binomial factors.
Previously, we studied combinations. In the shortcut to finding \((x+y)^{n}\), we will need to use combinations to find the coefficients that will appear in the expansion of the binomial. In this case, we use the notation \(\binom{n}{r}\) instead of \(C(n, r)\), but it can be calculated in the same way. So
\(\binom{n}{r}=C(n, r)=\frac{n!}{r!(n-r)!}\)
The combination \(\binom{n}{r}\) is called a binomial coefficient. An example of a binomial coefficient is \(\binom{5}{2}=C(5,2)=10\).

\section*{A GENERAL NOTE: BINOMIAL COEFFICIENTS}

If \(n\) and \(r\) are integers greater than or equal to 0 with \(n \geq r\), then the binomial coefficient is
\[
\binom{n}{r}=C(n, r)=\frac{n!}{r!(n-r)!}
\]

\section*{Q \& A}

Is a binomial coefficient always a whole number?
Yes. Just as the number of combinations must always be a whole number, a binomial coefficient will always be a whole number.

When we expand \((x+y)^{n}\) by multiplying, the result is called a binomial expansion, and it includes binomial coefficients. If we wanted to expand \((x+y)^{52}\), we might multiply \((x+y)\) by itself fifty-two times. This could take hours! If we examine some simple binomial expansions, we can find patterns that will lead us to a shortcut for finding more complicated binomial expansions.
\[
\begin{aligned}
& (x+y)^{2}=x^{2}+2 x y+y^{2} \\
& (x+y)^{3}=x^{3}+3 x^{2} y+3 x y^{2}+y^{3} \\
& (x+y)^{4}=x^{4}+4 x^{3} y+6 x^{2} y^{2}+4 x y^{3}+y^{4}
\end{aligned}
\]

First, let's examine the exponents. With each successive term, the exponent for \(x\) decreases and the exponent for \(y\) increases. The sum of the two exponents is \(n\) for each term.

Next, let's examine the coefficients. Notice that the coefficients increase and then decrease in a symmetrical

\section*{EXAMPLE: FINDING BINOMIAL COEFFICIENTS}

Find each binomial coefficient.
1. \(\binom{5}{3}\)
2. \(\binom{9}{2}\)
3. \(\binom{9}{7}\)

\section*{Answer}

Use the formula to calculate each binomial coefficient. You can also use the \(n C_{r}\) function on your calculator.
\(\binom{n}{r}=C(n, r)=\frac{n!}{r!(n-r)!}\)
1. \(\binom{5}{3}=\frac{5!}{3!(5-3)!}=\frac{5 \cdot 4 \cdot 3!}{3!2!}=10\)
2. \(\binom{9}{2}=\frac{9!}{2!(9-2)!}=\frac{9 \cdot 8 \cdot 7!}{2!7!}=36\)
3. \(\binom{9}{7}=\frac{9!}{7!(9-7)!}=\frac{9 \cdot 8 \cdot 7!}{7!2!}=36\)

\section*{Analysis of the Solution}

Notice that we obtained the same result for parts (b) and (c). If you look closely at the solution for these two parts, you will see that you end up with the same two factorials in the denominator, but the order is reversed, just as with combinations.
\[
\binom{n}{r}=\binom{n}{n-r}
\]

\section*{TRY IT}

Find each binomial coefficient.
a. \(\binom{7}{3}\)
b. \(\binom{11}{4}\)

Answer
a. 35
b. 330

\section*{TIP FOR SUCCESS}

The patterns that emerge from calculating binomial coefficients and that are present in Pascal's Triangle are handy and should be memorized over time as mathematical facts much in the same way that you just "know" 4 and 3 make 7 . Of course, that will take a lot of time and patient practice. If you are continuing in mathematics beyond this course, it will be well worth the effort.
Start by working through the next several paragraphs and the examples that follow on paper, perhaps more than once or twice, to begin to get familiar with the patterns and processes. Don't be discouraged if it takes time to become clear. It is a challenging concept.
pattern. The coefficients follow a pattern:
\(\binom{n}{0},\binom{n}{1},\binom{n}{2}, \ldots,\binom{n}{n}\).
These patterns lead us to the Binomial Theorem, which can be used to expand any binomial.
\[
\begin{aligned}
(x+y)^{n} & =\sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k} \\
& =x^{n}+\binom{n}{1} x^{n-1} y+\binom{c n}{2} x^{n-2} y^{2}+\cdots+\binom{n}{n-1} x y^{n-1}+y^{n}
\end{aligned}
\]

Another way to see the coefficients is to examine the expansion of a binomial in general form, \(x+y\), to successive powers \(1,2,3\), and 4 .
\[
\begin{aligned}
& (x+y)^{1}=x+y \\
& (x+y)^{2}=x^{2}+2 x y+y^{2} \\
& (x+y)^{3}=x^{3}+3 x^{2} y+3 x y^{2}+y^{3} \\
& (x+y)^{4}=x^{4}+4 x^{3} y+6 x^{2} y^{2}+4 x y^{3}+y^{4}
\end{aligned}
\]

Can you guess the next expansion for the binomial \((x+y)^{5}\) ?

\section*{Pascal's Triangle}


Above is an illustration the following:
- There are \(n+1\) terms in the expansion of \((x+y)^{n}\).
- The degree (or sum of the exponents) for each term is \(n\).
- The powers on \(x\) begin with \(n\) and decrease to 0 .
- The powers on \(y\) begin with 0 and increase to \(n\).
- The coefficients are symmetric.

To determine the expansion on \((x+y)^{5}\), we see \(n=5\), thus, there will be \(5+1=6\) terms. Each term has a combined degree of 5 . In descending order for powers of \(x\), the pattern is as follows:
- Introduce \(x^{5}\), and then for each successive term reduce the exponent on \(x\) by 1 until \(x^{0}=1\) is reached.
- Introduce \(y^{0}=1\), and then increase the exponent on \(y\) by 1 until \(y^{5}\) is reached.
\[
x^{5}, x^{4} y, x^{3} y^{2}, x^{2} y^{3}, x y^{4}, y^{5}
\]

The next expansion would be
\((x+y)^{5}=x^{5}+5 x^{4} y+10 x^{3} y^{2}+10 x^{2} y^{3}+5 x y^{4}+y^{5}\).
But where do those coefficients come from? The binomial coefficients are symmetric. We can see these coefficients in an array known as Pascal's Triangle.

Pascal's Triangle
1


To generate Pascal's Triangle, we start by writing a 1. In the row below, row 2 , we write two 1 's. In the \(3^{\text {rd }}\) row, flank the ends of the rows with 1's, and add \(1+1\) to find the middle number, 2 . In the \(n\)th row, flank the ends of the row with 1's. Each element in the triangle is the sum of the two elements immediately above it.

To see the connection between Pascal's Triangle and binomial coefficients, let us revisit the expansion of the binomials in general form.


\section*{A GENERAL NOTE: THE BINOMIAL THEOREM}

The Binomial Theorem is a formula that can be used to expand any binomial.
\[
\begin{aligned}
(x+y)^{n} & =\sum_{k=0}^{n}\binom{n}{k} x^{n-k} y^{k} \\
& =x^{n}+\binom{n}{1} x^{n-1} y+\binom{c n}{2} x^{n-2} y^{2}+\cdots+\binom{n}{n-1} x y^{n-1}+y^{n}
\end{aligned}
\]

\section*{HOW TO: GIVEN A BINOMIAL, WRITE IT IN EXPANDED FORM.}
1. Determine the value of \(n\) according to the exponent.
2. Evaluate the \(k=0\) through \(k=n\) using the Binomial Theorem formula.
3. Simplify.

\section*{TIP FOR SUCCESS}

Remember to try the examples and practice problems before looking at the answers!

\section*{EXAMPLE: EXPANDING A BINOMIAL}

Write in expanded form.
1. \((x+y)^{5}\)
2. \((3 x-y)^{4}\)

Answer
1. Substitute \(n=5\) into the formula. Evaluate the \(k=0\) through \(k=5\) terms. Simplify.
\[
\begin{aligned}
& (x+y)^{5}=\binom{5}{0} x^{5} y^{0}+\binom{5}{1} x^{4} y^{1}+\binom{5}{2} x^{3} y^{2}+\binom{5}{3} x^{2} y^{3}+\binom{5}{4} x^{1} y^{4}+\binom{5}{5} x^{0} y^{5} \\
& (x+y)^{5}=x^{5}+5 x^{4} y+10 x^{3} y^{2}+10 x^{2} y^{3}+5 x y^{4}+y^{5}
\end{aligned}
\]
2. Substitute \(n=4\) into the formula. Evaluate the \(k=0\) through \(k=4\) terms. Notice that \(3 x\) is in the place that was occupied by \(x\) and that \(-y\) is in the place that was occupied by \(y\). So we substitute them. Simplify.
\[
\begin{aligned}
& (3 x-y)^{4}=\binom{4}{0}(3 x)^{4}(-y)^{0}+\binom{4}{1}(3 x)^{3}(-y)^{1}+\binom{4}{2}(3 x)^{2}(-y)^{2}+\binom{4}{3}(3 x)^{1}(-y)^{3}+\binom{4}{4} \\
& (3 x-y)^{4}=81 x^{4}-108 x^{3} y+54 x^{2} y^{2}-12 x y^{3}+y^{4}
\end{aligned}
\]

\section*{Analysis of the Solution}

Notice the alternating signs in part b. This happens because \((-y)\) raised to odd powers is negative, but \((-y)\) raised to even powers is positive. This will occur whenever the binomial contains a subtraction sign.

\section*{TRY IT}

Write in expanded form.
a. \((x-y)^{5}\)
b. \((2 x+5 y)^{3}\)

Answer
a. \(x^{5}-5 x^{4} y+10 x^{3} y^{2}-10 x^{2} y^{3}+5 x y^{4}-y^{5}\)
b. \(8 x^{3}+60 x^{2} y+150 x y^{2}+125 y^{3}\)

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Expanding a binomial with a high exponent such as \((x+2 y)^{16}\) can be a lengthy process.
Sometimes we are interested only in a certain term of a binomial expansion. We do not need to fully expand a binomial to find a single specific term.

Note the pattern of coefficients in the expansion of \((x+y)^{5}\).
\((x+y)^{5}=x^{5}+\binom{5}{1} x^{4} y+\binom{5}{2} x^{3} y^{2}+\binom{5}{3} x^{2} y^{3}+\binom{5}{4} x y^{4}+y^{5}\)
The second term is \(\binom{5}{1} x^{4} y\). The third term is \(\binom{5}{2} x^{3} y^{2}\). We can generalize this result.
\(\binom{n}{r} x^{n-r} y^{r}\)

\section*{A GENERAL NOTE: THE (R+1)TH TERM OF A BINOMIAL EXPANSION}

The \((r+1)\) th term of the binomial expansion of \((x+y)^{n}\) is:
\(\binom{n}{r} x^{n-r} y^{r}\)

\section*{HOW TO: GIVEN A BINOMIAL, WRITE A SPECIFIC TERM WITHOUT FULLY EXPANDING.}
1. Determine the value of \(n\) according to the exponent.
2. Determine \((r+1)\).
3. Determine \(r\).
4. Replace \(r\) in the formula for the \((r+1)\) th term of the binomial expansion.

\section*{TIP FOR SUCCESS}

No peeking! Remember to try the examples and practice problems before looking at the answers.

\section*{EXAMPLE: WRITING A GIVEN TERM OF A BINOMIAL EXPANSION}

Find the tenth term of \((x+2 y)^{16}\) without fully expanding the binomial.

\section*{Answer}

Because we are looking for the tenth term, \(r+1=10\), we will use \(r=9\) in our calculations.
\(\binom{n}{r} x^{n-r} y^{r}\)
\(\binom{16}{9} x^{16-9}(2 y)^{9}=5,857,280 x^{7} y^{9}\)

Find the sixth term of \((3 x-y)^{9}\) without fully expanding the binomial.
Answer
\(-10,206 x^{4} y^{5}\)

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\section*{SUMMARY: COUNTING PRINCIPLES}

\section*{Key Equations}
number of permutations of \(n\) distinct objects taken \(r\) at a time
number of combinations of \(n\) distinct objects taken \(r\) at a time
number of permutations of \(n\) non-distinct objects
\[
\begin{aligned}
& P(n, r)=\frac{n!}{(n-r)!} \\
& C(n, r)=\frac{n!}{r!(n-r)!} \\
& \frac{n!}{r_{1}!r_{2}!\ldots r_{k}!}
\end{aligned}
\]

Binomial Theorem
\[
(x+y)^{n}=\sum_{k-0}^{n}\binom{n}{k} x^{n-k} y^{k}
\]
\((r+1) t h\) term of a binomial expansion
\[
\binom{n}{r} x^{n-r} y^{r}
\]

\section*{Key Concepts}
- If one event can occur in \(m\) ways and a second event with no common outcomes can occur in \(n\) ways, then the first or second event can occur in \(m+n\) ways.
- If one event can occur in \(m\) ways and a second event can occur in \(n\) ways after the first event has occurred, then the two events can occur in \(m \times n\) ways.
- A permutation is an ordering of \(n\) objects.
- If we have a set of \(n\) objects and we want to choose \(r\) objects from the set in order, we write \(P(n, r)\).
- Permutation problems can be solved using the Multiplication Principle or the formula for \(P(n, r)\).
- A selection of objects where the order does not matter is a combination.
- Given \(n\) distinct objects, the number of ways to select \(r\) objects from the set is \(\mathrm{C}(n, r)\) and can be found using a formula.
- A set containing \(n\) distinct objects has \(2^{n}\) subsets.
- For counting problems involving non-distinct objects, we need to divide to avoid counting duplicate permutations.
- \(\binom{n}{r}\) is called a binomial coefficient and is equal to \(C(n, r)\).
- The Binomial Theorem allows us to expand binomials without multiplying.
- We can find a given term of a binomial expansion without fully expanding the binomial.

\section*{Glossary}

Addition Principle if one event can occur in \(m\) ways and a second event with no common outcomes can occur in \(n\) ways, then the first or second event can occur in \(m+n\) ways
binomial coefficient the number of ways to choose robjects from \(n\) objects where order does not matter; equivalent to \(C(n, r)\), denoted \(\binom{n}{r}\)
binomial expansion the result of expanding \((x+y)^{n}\) by multiplying
Binomial Theorem a formula that can be used to expand any binomial
combination a selection of objects in which order does not matter
Fundamental Counting Principle if one event can occur in \(m\) ways and a second event can occur in \(n\) ways after the first event has occurred, then the two events can occur in \(m \times n\) ways; also known as the Multiplication Principle

Multiplication Principle if one event can occur in \(m\) ways and a second event can occur in \(n\) ways after the first event has occurred, then the two events can occur in \(m \times n\) ways; also known as the Fundamental Counting Principle
permutation a selection of objects in which order matters

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\section*{INTRODUCTION TO PROBABILITY}

\section*{LEARNING OUTCOMES}

By the end of this section, you will be able to:
- Construct probability models.
- Compute probabilities of equally likely outcomes.
- Compute probabilities of the union of two events.
- Use the complement rule to find probabilities.
- Compute probability using counting theory.

An example of a "spaghetti model," which can be used to predict possible paths of a tropical storm.
Residents of the Southeastern United States are all too familiar with charts, known as spaghetti models, such as the one above. (Note: The figure is for illustrative purposes only and does not model any particular storm.) They combine a collection of weather data to predict the most likely path of a hurricane. Each colored line represents one possible path. The group of squiggly lines can begin to resemble strands of spaghetti, hence the name. In this section, we will investigate methods for making these types of predictions.
```

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## CONSTRUCT PROBABILITY MODELS

## LEARNING OUTCOMES

- Construct a probability model that assigns the probability of each outcome in a sample space.
- Compute the probability of an event with equally likely outcomes.

Suppose we roll a six-sided number cube. Rolling a number cube is an example of an experiment, or an activity with an observable result. The numbers on the cube are possible results, or outcomes, of this experiment. The set of all possible outcomes of an experiment is called the sample space of the experiment. The sample space for this experiment is $\{1,2,3,4,5,6\}$. An event is any subset of a sample space.

The likelihood of an event is known as probability. The probability of an event $p$ is a number that always satisfies $0 \leq p \leq 1$, where 0 indicates an impossible event and 1 indicates a certain event. A probability model is a mathematical description of an experiment listing all possible outcomes and their associated probabilities. For instance, if there is a $1 \%$ chance of winning a raffle and a $99 \%$ chance of losing the raffle, a probability model would look much like the table below.

Outcome
Winning the raffle
Losing the raffle

## Probability

1\%
99\%

The sum of the probabilities listed in a probability model must equal 1, or $100 \%$.

## HOW TO: GIVEN A PROBABILITY EVENT WHERE EACH EVENT IS EQUALLY LIKELY, CONSTRUCT A PROBABILITY MODEL.

1. Identify every outcome.
2. Determine the total number of possible outcomes.
3. Compare each outcome to the total number of possible outcomes.

## TIP FOR SUCCESS

Be sure to try the examples and practice problems in this section on paper to build up your intuition for the next section's material.

## EXAMPLE: CONSTRUCTING A PROBABILITY MODEL

Construct a probability model for rolling a single, fair die, with the event being the number shown on the die.

## Answer

Begin by making a list of all possible outcomes for the experiment. The possible outcomes are the numbers that can be rolled: $1,2,3,4,5$, and 6 . There are six possible outcomes that make up the sample space.
Assign probabilities to each outcome in the sample space by determining a ratio of the outcome to the number of possible outcomes. There is one of each of the six numbers on the cube, and there is no reason to think that any particular face is more likely to show up than any other one, so the probability of rolling any number is $\frac{1}{6}$.

| Outcome | Roll of 1 | Roll of 2 | Roll of 3 | Roll of 4 | Roll of 5 | Roll of 6 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Probability | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

No. Probabilities can be expressed as fractions, decimals, or percents. Probability must always be a number between 0 and 1, inclusive of 0 and 1.

## TRY IT

Construct a probability model for tossing a fair coin.
Answer

| Outcome | Probability |
| :--- | :--- |
| Heads | $\frac{1}{2}$ |
| Tails | $\frac{1}{2}$ |

## Computing Probabilities of Equally Likely Outcomes

Let $S$ be a sample space for an experiment. When investigating probability, an event is any subset of $S$. When the outcomes of an experiment are all equally likely, we can find the probability of an event by dividing the number of outcomes in the event by the total number of outcomes in $S$. Suppose a number cube is rolled, and we are interested in finding the probability of the event "rolling a number less than or equal to 4 ." There are 4 possible outcomes in the event and 6 possible outcomes in $S$, so the probability of the event is $\frac{4}{6}=\frac{2}{3}$.

## A GENERAL NOTE: COMPUTING THE PROBABILITY OF AN EVENT WITH EQUALLY LIKELY OUTCOMES

The probability of an event $E$ in an experiment with sample space $S$ with equally likely outcomes is given by
$P(E)=\frac{\text { number of elements in } E}{\text { number of elements in } S}=\frac{n(E)}{n(S)}$
$E$ is a subset of $S$, so it is always true that $0 \leq P(E) \leq 1$.

## EXAMPLE: COMPUTING THE PROBABILITY OF AN EVENT WITH EQUALLY LIKELY OUTCOMES

A number cube is rolled. Find the probability of rolling an odd number.

## Answer

The event "rolling an odd number" contains three outcomes. There are 6 equally likely outcomes in the sample space. Divide to find the probability of the event.
$P(E)=\frac{3}{6}=\frac{1}{2}$

## TRY IT

A number cube is rolled. Find the probability of rolling a number greater than 2.

Answer
2

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## PROBABILITY FOR MULTIPLE EVENTS

## LEARNING OUTCOMES

- Find the probability of a union of two events.
- Find the probability of two events that share no common outcomes.
- Find the probability that an event will not happen.
- Find the number of events in a sample space that that includes many choices.

We are often interested in finding the probability that one of multiple events occurs. Suppose we are playing a card game, and we will win if the next card drawn is either a heart or a king. We would be interested in finding the probability of the next card being a heart or a king. The union of two events $E$ and $F$, written $E \cup F$, is the event that occurs if either or both events occur.
$P(E \cup F)=P(E)+P(F)-P(E \cap F)$
Suppose the spinner below is spun. We want to find the probability of spinning orange or spinning a $b$.


There are a total of 6 sections, and 3 of them are orange. So the probability of spinning orange is $\frac{3}{6}=\frac{1}{2}$. There are a total of 6 sections, and 2 of them have a $b$. So the probability of spinning a $b$ is $\frac{2}{6}=\frac{1}{3}$. If we added these two probabilities, we would be counting the sector that is both orange and a $b$ twice. To find the probability of spinning an orange or a $b$, we need to subtract the probability that the sector is both orange and has a $b$.
$\frac{1}{2}+\frac{1}{3}-\frac{1}{6}=\frac{2}{3}$
The probability of spinning orange or a $b$ is $\frac{2}{3}$.

## A GENERAL NOTE: PROBABILITY OF THE UNION OF TWO EVENTS

The probability of the union of two events $E$ and $F$ (written $E \cup F$ ) equals the sum of the probability of $E$ and the probability of $F$ minus the probability of $E$ and $F$ occurring together (which is called the intersection of $E$ and $F$ and is written as $E \cap F$ ).
$P(E \cup F)=P(E)+P(F)-P(E \cap F)$

## TIP FOR SUCCESS

The union symbol given above $\cup$ is the same symbol you used in the past to express the union of two intervals. Here we use it to represent the union of two events. Mathematically, it represents the word or. The union of $E$ and $F$ includes all the elements that could be present in $E$ or in $F$, one or the other.
The intersection symbol given above $\cap$ may be new notation for you. It is used to express the intersection of events. Mathematically, it represents the word and.
The intersection of $E$ and $F$ includes all the elements present in both $E$ and $F$, and not in just one or the other.
Try the example and practice problem below on paper to get familiar with the formula.

## EXAMPLE: COMPUTING THE PROBABILITY OF THE UNION OF TWO EVENTS

A card is drawn from a standard deck. Find the probability of drawing a heart or a 7.

## Answer

A standard deck contains an equal number of hearts, diamonds, clubs, and spades. So the probability of drawing a heart is $\frac{1}{4}$. There are four 7 s in a standard deck, and there are a total of 52 cards. So the probability of drawing a 7 is $\frac{1}{13}$.
The only card in the deck that is both a heart and a 7 is the 7 of hearts, so the probability of drawing both a heart and a 7 is $\frac{1}{52}$.
Substitute $P(H)=\frac{1}{4}, P(7)=\frac{1}{13}$, and $P(H \cap 7)=\frac{1}{52}$ into the formula.

$$
\begin{aligned}
P(E \cup F) & =P(E)+P(F)-P(E \cap F) \\
& =\frac{1}{4}+\frac{1}{13}-\frac{1}{52} \\
& =\frac{4}{13}
\end{aligned}
$$

The probability of drawing a heart or a 7 is $\frac{4}{13}$.

A card is drawn from a standard deck. Find the probability of drawing a red card or an ace.
Answer
7
13

## Computing the Probability of Mutually Exclusive Events

Suppose the spinner from earlier is spun again, but this time we are interested in the probability of spinning an orange or a $d$. There are no sectors that are both orange and contain a $d$, so these two events have no outcomes in common. Events are said to be mutually exclusive events when they have no outcomes in common. Because there is no overlap, there is nothing to subtract, so the general formula is
$P(E \cup F)=P(E)+P(F)$
Notice that with mutually exclusive events, the intersection of $E$ and $F$ is the empty set. The probability of spinning an orange is $\frac{3}{6}=\frac{1}{2}$ and the probability of spinning a $d$ is $\frac{1}{6}$. We can find the probability of spinning an orange or a $d$ simply by adding the two probabilities.

$$
\begin{aligned}
P(E \cup F) & =P(E)+P(F) \\
& =\frac{1}{2}+\frac{1}{6} \\
& =\frac{2}{3}
\end{aligned}
$$

The probability of spinning an orange or a $d$ is $\frac{2}{3}$.

## A GENERAL NOTE: PROBABILITY OF THE UNION OF MUTUALLY EXCLUSIVE EVENTS

The probability of the union of two mutually exclusive events $E$ and $F$ is given by
$P(E \cup F)=P(E)+P(F)$

## HOW TO: GIVEN A SET OF EVENTS, COMPUTE THE PROBABILITY OF THE UNION OF MUTUALLY EXCLUSIVE EVENTS.

1. Determine the total number of outcomes for the first event.
2. Find the probability of the first event.
3. Determine the total number of outcomes for the second event.
4. Find the probability of the second event.
5. Add the probabilities.

## Find the Probability That an Even Will Not Happen

We have discussed how to calculate the probability that an event will happen. Sometimes, we are interested in finding the probability that an event will not happen. The complement of an event $E$, denoted $E^{\prime}$, is the set of outcomes in the sample space that are not in $E$. For example, suppose we are interested in the probability that a horse will lose a race. If event $W$ is the horse winning the race, then the complement of

## EXAMPLE: COMPUTING THE PROBABILITY OF THE UNION OF MUTUALLY EXCLUSIVE EVENTS

A card is drawn from a standard deck. Find the probability of drawing a heart or a spade.

## Answer

The events "drawing a heart" and "drawing a spade" are mutually exclusive because they cannot occur at the same time. The probability of drawing a heart is $\frac{1}{4}$, and the probability of drawing a spade is also $\frac{1}{4}$, so the probability of drawing a heart or a spade is
$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$

## TRY IT

A card is drawn from a standard deck. Find the probability of drawing an ace or a king.
Answer
2
$\overline{13}$
event $W$ is the horse losing the race.
To find the probability that the horse loses the race, we need to use the fact that the sum of all probabilities in a probability model must be 1 .

$$
P\left(E^{\prime}\right)=1-P(E)
$$

The probability of the horse winning added to the probability of the horse losing must be equal to 1 .
Therefore, if the probability of the horse winning the race is $\frac{1}{9}$, the probability of the horse losing the race is simply
$1-\frac{1}{9}=\frac{8}{9}$

## A GENERAL NOTE: THE COMPLEMENT RULE

The probability that the complement of an event will occur is given by
$P\left(E^{\prime}\right)=1-P(E)$

## Computing Probability Using Counting Theory

Many interesting probability problems involve counting principles, permutations, and combinations. In these problems, we will use permutations and combinations to find the number of elements in events and sample spaces. These problems can be complicated, but they can be made easier by breaking them down into smaller counting problems.

Assume, for example, that a store has 8 cellular phones and that 3 of those are defective. We might want to find the probability that a couple purchasing 2 phones receives 2 phones that are not defective. To solve this problem, we need to calculate all of the ways to select 2 phones that are not defective as well as all of the ways to select 2 phones. There are 5 phones that are not defective, so there are $C(5,2)$ ways to select 2 phones that are not defective. There are 8 phones, so there are $C(8,2)$ ways to select 2 phones. The probability of selecting 2 phones that are not defective is:

## EXAMPLE: USING THE COMPLEMENT RULE TO CALCULATE PROBABILITIES

Two six-sided number cubes are rolled.

1. Find the probability that the sum of the numbers rolled is less than or equal to 3 .
2. Find the probability that the sum of the numbers rolled is greater than 3.

## Answer

The first step is to identify the sample space, which consists of all the possible outcomes. There are two number cubes, and each number cube has six possible outcomes. Using the Multiplication Principle, we find that there are $6 \times 6$, or 36 total possible outcomes. So, for example, 1-1 represents a 1 rolled on each number cube.

| $1-1$ | $1-2$ | $1-3$ | $1-4$ | $1-5$ | $1-6$ |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $2-1$ | $2-2$ | $2-3$ | $2-4$ | $2-5$ | $2-6$ |
| $3-1$ | $3-2$ | $3-3$ | $3-4$ | $3-5$ | $3-6$ |
| $4-1$ | $4-2$ | $4-3$ | $4-4$ | $4-5$ | $4-6$ |
| $5-1$ | $5-2$ | $5-3$ | $5-4$ | $5-5$ | $5-6$ |
| $6-1$ | $6-2$ | $6-3$ | $6-4$ | $6-5$ | $6-6$ |

1. We need to count the number of ways to roll a sum of 3 or less. These would include the following outcomes: 1-1, 1-2, and 2-1. So there are only three ways to roll a sum of 3 or less. The probability is

$$
\frac{3}{36}=\frac{1}{12}
$$

2. Rather than listing all the possibilities, we can use the Complement Rule. Because we have already found the probability of the complement of this event, we can simply subtract that probability from 1 to find the probability that the sum of the numbers rolled is greater than 3.

$$
\begin{aligned}
P\left(E^{\prime}\right) & =1-P(E) \\
& =1-\frac{1}{12} \\
& =\frac{11}{12}
\end{aligned}
$$

## TRY IT

Two number cubes are rolled. Use the Complement Rule to find the probability that the sum is less than 10.

Answer
5
$\overline{6}$
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$\frac{\text { ways to select } 2 \text { phones that are not defective }}{\text { ways to select } 2 \text { phones }}=\frac{C(5,2)}{C(8,2)}$
$=\frac{10}{28}$
$=\frac{5}{14}$

## TIP FOR SUCCESS

The following are challenging problems. Work them out on paper as many times as you need to gain insight. You can click to Try Another Version of This Question in the green Try It box as needed. Don't be discouraged if it takes some time!

## EXAMPLE: COMPUTING PROBABILITY USING COUNTING THEORY

A child randomly selects 5 toys from a bin containing 3 bunnies, 5 dogs, and 6 bears.
a. Find the probability that only bears are chosen.
b. Find the probability that 2 bears and 3 dogs are chosen.
c. Find the probability that at least 2 dogs are chosen.

## Answer

a. We need to count the number of ways to choose only bears and the total number of possible ways to select 5 toys. There are 6 bears, so there are $C(6,5)$ ways to choose 5 bears. There are 14 toys, so there are $C(14,5)$ ways to choose any 5 toys.

$$
\frac{C(6,5)}{C(14,5)}=\frac{6}{2,002}=\frac{3}{1,001}
$$

b. We need to count the number of ways to choose 2 bears and 3 dogs and the total number of possible ways to select 5 toys. There are 6 bears, so there are $C(6,2)$ ways to choose 2 bears. There are 5 dogs, so there are $C(5,3)$ ways to choose 3 dogs. Since we are choosing both bears and dogs at the same time, we will use the Multiplication Principle. There are $C(6,2) \cdot C(5,3)$ ways to choose 2 bears and 3 dogs. We can use this result to find the probability.

$$
\frac{C(6,2) C(5,3)}{C(14,5)}=\frac{15 \cdot 10}{2,002}=\frac{75}{1,001}
$$

c. It is often easiest to solve "at least" problems using the Complement Rule. We will begin by finding the probability that fewer than 2 dogs are chosen. If less than 2 dogs are chosen, then either no dogs could be chosen, or 1 dog could be chosen. When no dogs are chosen, all 5 toys come from the 9 toys that are not dogs. There are $C(9,5)$ ways to choose toys from the 9 toys that are not dogs. Since there are 14 toys, there are $C(14,5)$ ways to choose the 5 toys from all of the toys.

$$
\frac{C(9,5)}{C(14,5)}=\frac{63}{1,001}
$$

If there is 1 dog chosen, then 4 toys must come from the 9 toys that are not dogs, and 1 must come from the 5 dogs. Since we are choosing both dogs and other toys at the same time, we will use the Multiplication Principle. There are $C(5,1) \cdot C(9,4)$ ways to choose 1 dog and 1 other toy.

$$
\frac{C(5,1) C(9,4)}{C(14,5)}=\frac{5 \cdot 126}{2,002}=\frac{315}{1,001}
$$

Because these events would not occur together and are therefore mutually exclusive, we add the probabilities to find the probability that fewer than 2 dogs are chosen.

$$
\frac{63}{1,001}+\frac{315}{1,001}=\frac{378}{1,001}
$$

We then subtract that probability from 1 to find the probability that at least 2 dogs are chosen.

$$
1-\frac{378}{1,001}=\frac{623}{1,001}
$$

## TRY IT

A child randomly selects 3 gumballs from a container holding 4 purple gumballs, 8 yellow gumballs, and 2 green gumballs.
a. Find the probability that all 3 gumballs selected are purple.
b. Find the probability that no yellow gumballs are selected.
c. Find the probability that at least 1 yellow gumball is selected.

Answer
a. $\frac{1}{91}$
b. $\frac{5}{91}$
c. $\frac{86}{91}$

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## SUMMARY: PROBABILITY

## Key Equations

| probability of an event with equally likely outcomes | $P(E)=\frac{n(E)}{n(S)}$ |
| :--- | :--- |
| probability of the union of two events | $P(E \cup F)=P(E)+P(F)-P(E \cap F)$ |
| probability of the union of mutually exclusive events | $P(E \cup F)=P(E)+P(F)$ |
| probability of the complement of an event | $P\left(E^{\bullet}\right)=1-P(E)$ |

## Key Concepts

- Probability is always a number between 0 and 1 , where 0 means an event is impossible and 1 means an event is certain.
- The probabilities in a probability model must sum to 1 .
- When the outcomes of an experiment are all equally likely, we can find the probability of an event by dividing the number of outcomes in the event by the total number of outcomes in the sample space for the experiment.
- To find the probability of the union of two events, we add the probabilities of the two events and subtract the probability that both events occur simultaneously.
- To find the probability of the union of two mutually exclusive events, we add the probabilities of each of the events.
- The probability of the complement of an event is the difference between 1 and the probability that the event occurs.
- In some probability problems, we need to use permutations and combinations to find the number of elements in events and sample spaces.


## Glossary

complement of an event the set of outcomes in the sample space that are not in the event $E$ event any subset of a sample space
experiment an activity with an observable result
mutually exclusive events events that have no outcomes in common
outcomes the possible results of an experiment
probability a number from 0 to 1 indicating the likelihood of an event
probability model a mathematical description of an experiment listing all possible outcomes and their associated probabilities
sample space the set of all possible outcomes of an experiment
union of two events the event that occurs if either or both events occur
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## PUTTING IT TOGETHER: PROBABILITY AND COUNTING PRINCIPLES

The lottery jackpot has continued to climb as you completed this module. Now it is time to determine how likely you are to win.

Let's first assume that you not only need to pick six specific numbers from $1-49$, but you need to pick them in the correct order. If this is the case, you know you need to use a permutation to figure out the size of the sample space.
$P(n, r)=\frac{n!}{(n-r)!}$
In this case, $n$ is the possible numbers, which is 49 , and $r$ is the number of choices you make, which is 6 .
$P(49,6)=\frac{49!}{(49-6)!}$
$P(49,6)=\frac{49!}{43!}=10,068,347,520$
This tells you that there is one way out of about 10 billion to win. Your chances are not good at all.

Fortunately, most lottery winnings do not depend on order so you can use a combination instead.
$C(n, r)=\frac{n!}{r!(n-r)!}$
$C(49,6)=\frac{49!}{6!(49-6)!}$
$C(49,6)=\frac{49!}{6!(43)!}$
$C(49,6)=\frac{49!}{6!(43)!}=13,983,816$
Notice that the sample space has been greatly reduced from about 10 billion to about 14 million. So the likelihood of you winning is much greater than before, but still very slim.

What would happen to your chances of winning if you bought more than one ticket? Suppose you bought 100 tickets and chose a different combination of six numbers on each ticket. You could compare the number of tickets to sample space to determine your probability.
$\frac{100}{14 \text { million }}=0.0000071=0.00071 \%$
That's much less than a $1 \%$ chance of winning. Still not very good. So suppose you gather up some cash and buy 1,000 tickets.
$\frac{1,000}{14 \text { million }}=0.000071=0.0071 \%$
Now you are out $\$ 1000$, assuming each ticket costs $\$ 1$, and your chances are still less than a $1 \%$ chance.

Okay, maybe you are ready to go for broke. You and a group of friends gather your funds to purchase 1 million tickets.
$\frac{1 \text { million }}{14 \text { million }}=0.071=7.1 \%$

So even after purchasing 1 million tickets, which might cost $\$ 1$ million, your probability of winning the big jackpot is only about $7 \%$. To raise your probability to just $50 \%$, you would have to purchase 7 million tickets. It's up to you do decide how lucky you feel. Maybe just buy one ticket and see what happens. Good luck!

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